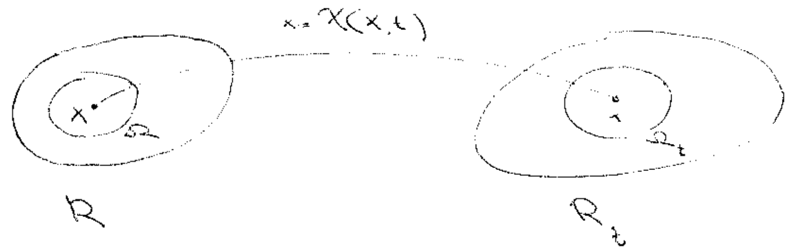
Ch. 1 EQUATIONS OF MOTION

Bodies. Motion. Control volumes

Objective to describe a mechanical process (or thermomechanical process)



R reference conf.
R_t current conf.

X labels a point in reference

x current position

$x = \chi(X,t)$. motion

$\chi(\cdot,t)$ trajectory of label X

Rmk For "mechanical" reason we need to impose some assumption on χ

one-to-one, onto.

continuous, continuous inverse

globally invertible (no interpenetration of matter). $\boxed{\det \nabla_X \chi > 0}$

Kinematic variables $V = \frac{\partial x}{\partial t}$ velocity

$A = \frac{\partial^2 x}{\partial t^2}$ acceleration

$F = \nabla_X \chi$ deformation gradient

Given any quantity "G" we can describe it by

$G(X,t)$ or $g(x,t)$. $G(X,t) = g(\chi(X,t), t)$

Lagrangian Eulerian

Mass $\rho_0(X)$ reference density (continuum hypothesis)

$\rho(x,t)$ current density

i). Balance of mass $m(\Omega) = m(\Omega_t)$

$$\int_\Omega \rho_0(X)\,dX = \int_{\Omega_t} \rho(x,t)\,dx = \int_\Omega \rho(X(x,t),t)\,\underbrace{\det \nabla_x X}_{J}\,dX \qquad \forall\,\Omega$$

$$\boxed{\rho_0 = \rho J}$$

$$\frac{d}{dt}\int_\Omega \rho(X(x,t),t)\,J\,dX = \int_\Omega \left(\nabla_x\rho\cdot\frac{\partial X}{\partial t} + \rho_t\right)J + \rho J_t \,dX$$

$$\overset{J_t = J\,\mathrm{div}\,v}{=} \int_\Omega \left(\nabla_x\rho\cdot v + \rho_t\right)J + \rho J\,\mathrm{div}\,v\,dX$$

$$= \int_{\Omega_t} \rho_t + \mathrm{div}(\rho v)\,dx = 0.$$

Balance of mass $\rho_0 = \rho J$ Lagrangian form

$\rho_t + \mathrm{div}\,\rho v = 0$ Eulerian form

ii) Transport theorem.

$$\frac{d}{dt}\int_{\Omega_t} g(x,t)\,\rho(x,t)\,dx = \int_{\Omega_t} \rho(x,t)\left(\frac{\partial g}{\partial t} + v\cdot\nabla_x g\right)dx$$

Pf. $$\frac{d}{dt}\int_{\Omega_t} g\rho\,dx = \frac{d}{dt}\int_\Omega g\rho J\,dX = \frac{d}{dt}\int_\Omega g(X(x,t),t)\,\rho_0(X)\,dX$$

$$= \int_\Omega \left(v\cdot\nabla_x g + g_t\right)\rho_0(X)\,dX = \int_{\Omega_t} \rho\left(\frac{\partial g}{\partial t} + v\cdot\nabla_x g\right)dx$$

$V(x,t)$ $\qquad$ $v(x,t)$

$$v(\mathcal{X}(x,t),t) = V(x,t) = \frac{\partial \mathcal{X}}{\partial t}(x,t).$$

$$J = \det \nabla_x \mathcal{X}.$$

<u>Lemma</u> $\quad J_t = J \operatorname{div}_x v$

$$J = \det \nabla \mathcal{X} = \det \begin{pmatrix} \text{---}\ \nabla \mathcal{X}^1\ \text{---} \\ \vdots \\ \text{---}\ \nabla \mathcal{X}^n\ \text{---} \end{pmatrix} \qquad \mathcal{X} = \begin{pmatrix} \mathcal{X}^1 \\ \vdots \\ \mathcal{X}^n \end{pmatrix}$$

$$J_t = \partial_t \det \nabla_x \mathcal{X}$$

$$= \underbrace{\det \begin{pmatrix} \text{---}\ \partial_t \nabla \mathcal{X}^1\ \text{---} \\ \text{---}\ \nabla \mathcal{X}^2\ \text{---} \\ \vdots \\ \text{---}\ \nabla \mathcal{X}^n\ \text{---} \end{pmatrix}}_{I^1} + \cdots + \underbrace{\det \begin{pmatrix} \text{---}\ \nabla \mathcal{X}^1\ \text{---} \\ \text{---}\ \nabla \mathcal{X}^2\ \text{---} \\ \vdots \\ \text{---}\ \partial_t \nabla \mathcal{X}^n\ \text{---} \end{pmatrix}}_{I^n}$$

$$\partial_t \nabla_x \mathcal{X}^i = \nabla \partial_t \mathcal{X}^i = \nabla V^i$$

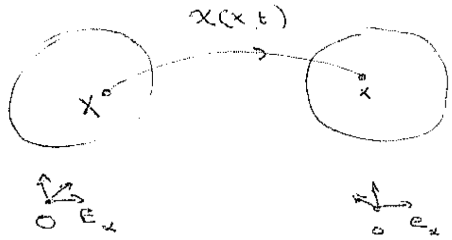
$$(\partial_t \nabla \mathcal{X}^i)_j = (\nabla V^i)_j = \frac{\partial V^i}{\partial x_j} = \frac{\partial}{\partial x_j} v^i(\mathcal{X}(x,t),t) = \sum_k \frac{\partial v^i}{\partial x_k} \frac{\partial \mathcal{X}^k}{\partial x_j}$$

$$I_i = \det\begin{pmatrix} — \nabla \chi^1 — \\ \vdots \\ — \partial_t \nabla \chi^i — \\ \vdots \\ — \nabla \chi^n — \end{pmatrix} = \det\begin{pmatrix} \frac{\partial \chi^1}{\partial x_1} & \frac{\partial \chi^1}{\partial x_2} & \cdots & \frac{\partial \chi^1}{\partial x_n} \\ \vdots & & & \\ \sum_k \frac{\partial v^i}{\partial x_k}\frac{\partial \chi^k}{\partial x_1} & \cdots & & \sum_k \frac{\partial v^i}{\partial x_k}\frac{\partial \chi^k}{\partial x_n} \\ \vdots & & & \\ \frac{\partial \chi^n}{\partial x_1} & \frac{\partial \chi^n}{\partial x_2} & \cdots & \frac{\partial \chi^n}{\partial x_n} \end{pmatrix}$$

$$= \sum_k \frac{\partial v^i}{\partial x_k} \begin{pmatrix} \frac{\partial \chi^1}{\partial x_1} & \cdots & \frac{\partial \chi^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \chi^k}{\partial x_1} & & \frac{\partial \chi^k}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \chi^n}{\partial x_1} & & \frac{\partial \chi^n}{\partial x_n} \end{pmatrix} = \frac{\partial v^i}{\partial x_i} J$$

$$J_t(x,t) = \sum_i I_i = \sum_i \frac{\partial v^i}{\partial x_i} J = \operatorname{div}_x v(\chi(x,t),t)\, J(x,t)$$

KINEMATICS



$F = \nabla_X \chi$

$V = \frac{\partial \chi}{\partial t}$, v

$V = V_\alpha E_\alpha$ $\quad v = v_i e_i$

$F = F_{i\alpha}\, e_i \otimes E_\alpha$

$a \otimes b$ dyadic product

$(a \otimes b)c = a(b \cdot c)$

Orthogonal tensors

Q is orthogonal if $(Qa)\cdot(Qb) = a\cdot b \quad \forall a,b \in \mathbb{R}^n$

Q orthogonal $\iff QQ^T = Q^TQ = I$ $\quad$ ($\leadsto$ rotation)

Q proper orthogonal if Q ortho, $\det Q = +1$
improper " if Q ortho, $\det Q = -1$

Rigid motion

$\chi : X \mapsto x$ is rigid motion if $|x-y| = |X-Y|$

χ rigid motion $\iff$ $\chi(X,t) = c(t) + Q(t)X$ $\qquad c(t) \in \mathbb{R}^n$
$Q(t)$ orthogonal

Pf. Exercise in linear algebra

POLAR DECOMPOSITION THM

Given F invertible tensor, there exist Q, R orthogonal and U, V pos. definite symmetric tensors such that

$$F = QU = VR$$

Moreover, the right (left) polar decomposition is unique

Pf. F^TF pos. defn, symmetric tensor.

FF^T " " "

$G = F^TF$, $G^T = G$, $a \cdot Ga = a \cdot F^TFa = Fa \cdot Fa > 0$
$a \neq 0$.

Define $U = +\sqrt{G}$ unique pos defn symmetric square root

$U = \sqrt{F^TF}$ $\quad Q = FU^{-1}$

$$Q^TQ = (FU^{-1})^T(FU^{-1}) = U^{-1}F^TFU^{-1} = U^{-1}U^2U^{-1} = I.$$

<u>Uniqueness</u> $\cancel{A = QU = Q'U'}$

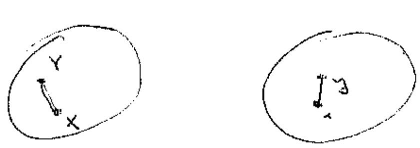
Let $G = QU$, $U = +\sqrt{G}$

Suppose $G = Q'U'$

$G^TG = (Q'U')^T(Q'U') = U'^2 = U^2 \Rightarrow U = U'$ since pos. defn square root is unique

$R' = R$

<u>Significance</u>



$$x(Y) - x(X) = \underbrace{\nabla x(X)}_{F}\,\underbrace{(Y - X)}_{h} + o(|Y - X|)$$

$$\Delta x = Fh + O(h^2).$$

$$= QUh + O(h^2).$$

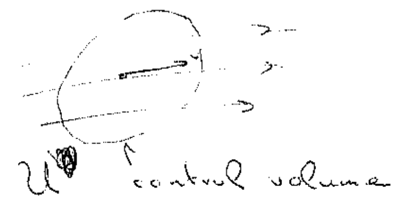
Locally motion is decomposed into stretching (U) and rotation (Q)

Rmk Notion useful for solid deformation

ROTATION - VORTICITY

$\dim = 3$

For fluids the natural setting to do calculations is that of the Eulerian coordinates



measuring $v = v(x,t)$

"deformation" $\nabla_x v$ velocity gradient

$$v(y) - v(x) = \nabla v(x)\underbrace{(y-x)}_{h} + O(|h|^2) \qquad \text{Taylor th.}$$

Introduce $D = \frac{1}{2}(\nabla v + \nabla v^T)$ $\quad D^T = D$ symmetric

$S = \frac{1}{2}(\nabla v - \nabla v^T)$ $\quad S^T = -S$ skew. symmetric

$\boxed{\nabla v = D + S}$ and the decomposition is unique

$$\nabla v = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} \qquad v = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad u_x = \frac{\partial u}{\partial x} \quad \text{etc}$$

$$S = \frac{1}{2}(\nabla v - \nabla v^T) = \begin{pmatrix} 0 & \frac{1}{2}(u_y - v_x) & \frac{1}{2}(u_z - w_x) \\ \frac{1}{2}(v_x - u_y) & 0 & \frac{1}{2}(v_z - w_y) \\ \frac{1}{2}(w_x - u_z) & \frac{1}{2}(w_y - v_z) & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

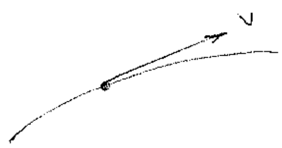
Introduce
$$\xi = \operatorname{curl} v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = (w_y - v_z,\ u_z - w_x,\ v_x - u_y)^T$$
$$= (\xi_1, \xi_2, \xi_3)$$

$$S h = \frac{1}{2}\begin{pmatrix} \xi_2 h_3 - \xi_3 h_2 \\ \xi_3 h_1 - \xi_1 h_3 \\ \xi_1 h_2 - \xi_2 h_1 \end{pmatrix} = \frac{1}{2}\, \xi \times h$$

Thus

$$v(y) - v(x) = D(x)\, h + \frac{1}{2}\, \xi \times h + O(|h|^2), \qquad h = y - x \text{ small}$$

<u>Interpretations</u>



$\frac{dy}{dt} = v(y)$ defines the motion of the fluid.

Locally,

(1) $D(x) h = \frac{1}{2} \nabla_h \left(D(x) h \cdot h \right)$

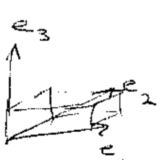
$$\boxed{\frac{dh}{dt} = D h.}$$

Choose a basis $\{e_1, e_2, e_3\}$ so that $D = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & d_3 \end{bmatrix}$ on this basis.
(since D symmetric).

On basis $\{e_1, e_2, e_3\}$ we have $\frac{dh_i}{dt} = d_i h_i$

$$h_i(t) = h_i(0)\, e^{d_i t}.$$

fluid expanding $(d_i > 0)$ or contracting $(d_i < 0)$ on direction e



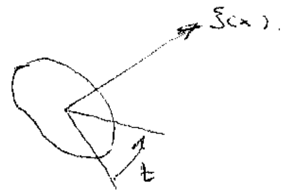
$$\frac{d}{dt}(h_1 h_2 h_3) = \frac{dh_1}{dt} h_2 h_3 + \dots + h_1 h_2 \frac{dh_3}{dt} = (d_1 + d_2 + d_3)(h_1 h_2 h_3)$$

$$\frac{dV}{dt} = (\operatorname{tr} D)\, V = (\operatorname{div} v)\, V$$

Vol change $\sim \operatorname{div} v$

2i $v(x) = \frac{1}{2}\xi(x)\times h$.

$$\boxed{\frac{dh}{dt} = \frac{1}{2}\xi(x)\times h}$$



$h(t) = R(t, \xi(x))\, h(0)$.

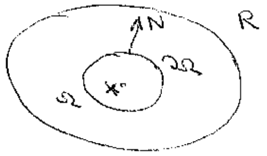
$\downarrow$

matrix representing rotation through angle t about axis $\xi(x)$.

$$v(y) - v(x) = D(x)\,h + \frac{1}{2}\xi\times h + O(|h|^2)$$

In a small nbhd of a point: v is the sum of a (rigid) translation, a deformation, and a (rigid) rotation with rotation vector $\xi = \operatorname{curl} v$.

BALANCE LAWS



monitor a quantity $\mathcal{G}$ over a control volume Ω

$$\underbrace{\mathcal{G}(\Omega)(t_2)-\mathcal{G}(\Omega)(t_1)}_{\text{change of } \mathcal{G}} = \int_{t_1}^{t_2} \underbrace{\mathcal{F}(\partial\Omega)(\tau)}_{\text{flux through } \partial\Omega}\, d\tau + \int_{t_1}^{t_2} \underbrace{\mathcal{H}(\Omega)(\tau)}_{\text{production in } \Omega}\, d\tau$$

Assume All set functions $\mathcal{G}$ can be described through corresponding density functions

(*) $$\int_\Omega G(x,t)\,dx \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\partial\Omega} F(x,t,N)\,dS\,dt + \int_{t_1}^{t_2}\int_\Omega H(x,t)\,dx\,dt.$$

$\forall\ \Omega\subset B,\ t_1<t_2$

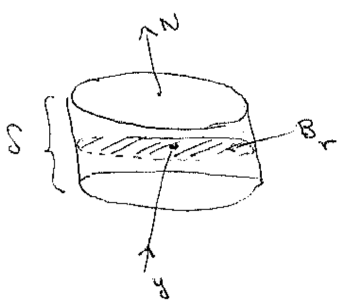
where $G, H : \mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^n$, $N=(N_1,\dots,N_d)$

Cauchy thm If (*) holds for each Ω then

$$F(x,t,N)=\sum_{j=1}^{d} F_j(x,t)\,N_j \qquad F_j:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^n$$

Proof. Say $d=3$,

(2) $$\int_\Omega G_t\,dx = \int_{\partial\Omega} F(x,t,N)\,dS + \int_\Omega H(x,t)\,dx$$

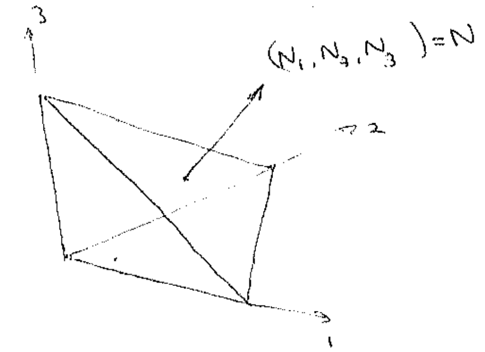


$$O(r^2\delta) = \Big(F(y,t,N)+F(y,t,-N)\Big)|B_r| + O(r\delta) + O(r^2\delta$$

let $\delta\to 0$.

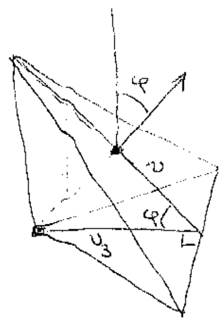
$$F(y,t,N) = -F(y,t,-N).$$

(ii)



Scale by a factor r

$$O(r^3) + \left[F(x,t,N)\,A + F(x,t,-e_3)\,A_3 + F(x,t,-e_2)\,A_2 + F(x,t,-e_1)\,A_1 \right] = 0$$



$$\frac{A_3}{A} = \frac{v_3}{v} = \cos\varphi = N_3 \quad , \quad A = O(r^2)$$

$$F(x,t,N) = \sum_j F(x,t,e_j)\,N_j$$

Assuming G, F_j, H smooth

$$\partial_t G(x,t) = \sum_{j=1}^{d} \partial_{x_j} F_j(x,t) + H(x,t).$$

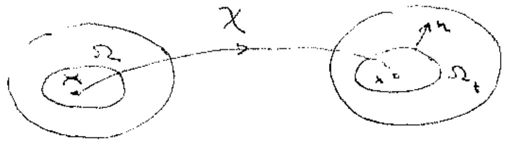
$\Updownarrow$ smooth solns

$$\int_\Omega G(x,t)\,dx \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\partial\Omega} \sum_{j=1}^{d} F_j(x,t)\,N_j\,dS\,dt + \int_{t_1}^{t_2} \int_\Omega H(x,t)\,dx\,dt \qquad \forall\,\Omega,\ t_1 < t_2$$

Rmk ~~The~~ A ~~more~~ general class of fields for which this equivalence can be proved is the class $G, F_j, H \in BV$

Weaker derivation (current topic) see Dafermos.

BALANCE LAWS OF CONTINUUM PHYSICS



X label (material pt) x current pt

Balance of mass

$\dot{\rho}_0 = 0$ $\rho_t + \operatorname{div} \rho v = 0$ $\rho J = \rho_0$

$\rho\, dx = \rho J\, dX = \rho_0\, dX$

Balance of momentum

$$\frac{d}{dt}\int_{\Omega_t} \rho v\, dx = \int_{\partial\Omega_t} t(x,t,n)\, dS + \int_{\Omega_t} \rho b\, dx$$

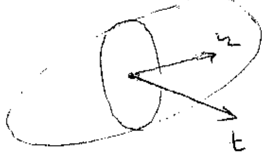
t (contact forces) traction b body force

force/unit area force/unit mass

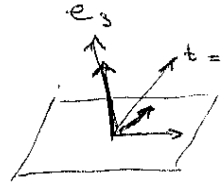
Cauchy thm $t(x,t,n) = S(x,t)\, n$ S Cauchy stress tensor

$S = (S_{ij})$

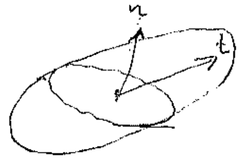
Interpretation



$t = Sn$



$t = Se_3 = \begin{pmatrix} S_{31} \\ S_{32} \\ S_{33} \end{pmatrix}$



S_{ii} called normal stresses

S_{ij} $i \neq j$ shear stresses

$$\frac{Df}{Dt} := \frac{\partial f}{\partial t} + v\cdot\nabla_x f = \frac{d}{dt} f(x(t),t) \qquad \frac{dx}{dt} = v$$

Coordinate form

$$\int_{\Omega_t} \rho\left(\frac{\partial v_i}{\partial t} + (v\cdot\nabla_x) v_i\right) dx = \int_{\partial\Omega_t} (Sn)_i \, dS_x + \int_{\Omega_t} \rho b_i \, dx$$

$$= \int_{\Omega_t} (\operatorname{div} S)_i + \rho b_i \, dx$$

$$\int_{\partial\Omega_t} (Sn)_i \, dS = \int_{\partial\Omega_t} \sum_j S_{ij} n_j \, dS_x = \int_{\Omega_t} \sum_j \partial_{x_j} S_{ij} \, dx$$

$$(\operatorname{div} S)_i = \left(\sum_j \partial_{x_j} S_{ij}\right)_i$$

Local form

$$\rho\left(\frac{\partial v_i}{\partial t} + v\cdot\nabla_x v_i\right) = \rho \frac{Dv_i}{Dt} = (\operatorname{div} S)_i + \rho b_i$$

or

$$\rho \frac{Dv}{Dt} = \rho\left(\frac{\partial v}{\partial t} + (v\cdot\nabla) v\right) = \operatorname{div} S + \rho b$$

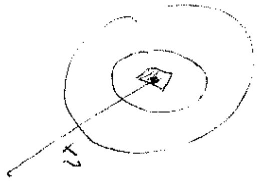
Conservative form

$$\frac{\partial\rho}{\partial t} + \operatorname{div} \rho v = 0$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v\cdot\nabla v_i = (\operatorname{div} S)_i + \rho b_i$$

$$\frac{\partial}{\partial t}(\rho v_i) + \operatorname{div}(\rho v_i v) = (\operatorname{div} S)_i + \rho b_i \quad ,$$
$$\frac{\partial}{\partial t} \rho v + \operatorname{div}(\rho v \otimes v) = \operatorname{div} S + \rho b$$

Balance of angular momentum



$$\frac{d}{dt}\int_{\Omega_t} x \wedge \rho v \, dx = \int_{\partial\Omega_t} \underbrace{x \wedge Sn \, dS_x}_{\text{torque of traction}} + \int_{\Omega_t} \underbrace{x \wedge \rho b \, dx}_{\text{torque of body force}} .$$

$$\int_{\partial\Omega_t} (x \wedge Sn)_i \, dS_x = \int_{\partial\Omega_t} \varepsilon_{ijk} x_j (Sn)_k \, dS_x = \int_{\partial\Omega_t} \varepsilon_{ijk} x_j S_{kl} n_l \, dS_x$$

$(a \wedge b)_i = \varepsilon_{ijk} a_j b_k$

$$= \int_{\Omega_t} \partial_{x_l}\left(\varepsilon_{ijk} x_j S_{kl}\right) dx$$

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{even perm. of } 1,2,3 \\ -1 & \text{odd " " } 1,2,3 \\ 0 & \text{indices coincide} \end{cases}$$

$$= \int_{\Omega_t} \left[\varepsilon_{ijk}\delta_{jl} S_{kl} + \varepsilon_{ijk} x_j \left(\partial_{x_l} S_{kl}\right)\right] dx$$

$$= \int_{\Omega_t} \varepsilon_{ijk}\delta_{jl} S_{kl} + (x \wedge \operatorname{div} S)_i \, dx$$

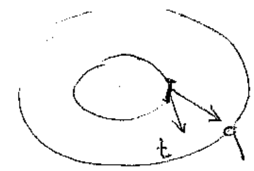
$$\int_{\Omega_t} x \wedge \rho \frac{D}{Dt} v = \int_{\Omega_t} \Big(\underbrace{\varepsilon_{ijk}\delta_{jl} S_{kl}}_{\varepsilon_{ilk} S_{kl}}\Big)_{i=1,2,3} + x \wedge \operatorname{div} S + x \wedge \rho b \, dx$$

Compare with balance of momentum

$$\varepsilon_{ijk}\delta_{jl} S_{kl} = 0 = \varepsilon_{ilk} S_{kl}$$

$$\boxed{S_{ij} = S_{ji}} \quad \text{or} \quad S = S^T.$$

<u>Balance of energy</u>



$$\frac{d}{dt}\int_{\Omega_t} \tfrac{1}{2}\rho|v|^2 + \rho e \, dx = \int_{\partial\Omega_t} Sn\cdot v + q\cdot n \, d\sigma + \int_{\Omega_t} \rho r \, dx + \rho b\cdot v \, dx$$

kinetic energy, internal energy, heat flux, heat supply, work of body force

$$\int_{\partial\Omega_t} S_{ij} n_j v_i + q_j n_j \, d\sigma = \int_{\Omega_t} \partial_j \left(S_{ij} v_i + q_j \right)$$

local

$$\int_{\Omega_t} \rho \frac{D}{Dt}\left(\tfrac{1}{2}|v|^2 + e\right) dx = \int_{\Omega_t} \operatorname{div}\left(v\cdot S + q\right) + \rho r + \rho b\cdot v \, dx$$

$$\rho\left[\underbrace{\frac{\partial}{\partial t}\left(\tfrac{1}{2}|v|^2 + e\right) + v\cdot\nabla\left(\tfrac{1}{2}|v|^2 + e\right)}_{\rho \frac{D}{Dt}\left(\frac{1}{2}|v|^2 + e\right)}\right] = \operatorname{div}\left(v\cdot S + q\right) + \rho r + \rho b\cdot v$$

$$\rho_t + \operatorname{div} \rho v = 0$$

local conservative form

$$\partial_t\left(\rho \tfrac{1}{2}|v|^2 + \rho e\right) + \operatorname{div}\left(\rho v\left(\tfrac{1}{2}|v|^2 + e\right)\right) = \operatorname{div}\left(v\cdot S + q\right) + \rho r + \rho b\cdot v$$

<u>Rmk</u> Constitutive relations are needed to close the system.

SUMMARY

EQUATIONS OF CONTINUUM PHYSICS IN EULERIAN COORDINATES

mass $\quad \frac{\partial \rho}{\partial t} + \operatorname{div} \rho v = 0 \quad$ or $\quad \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho + \rho \operatorname{div} v = 0$

momentum $\quad \rho\left(\frac{\partial v}{\partial t} + (v \cdot \nabla) v\right) = \operatorname{div} S + \rho b$

angular momentum $\quad S^T = S$

energy $\quad \rho \frac{\partial}{\partial t}\left(\frac{1}{2}|v|^2 + e\right) + (v \cdot \nabla)\left(\frac{1}{2}|v|^2 + e\right) = \operatorname{div}(v \cdot S + q) + \rho r$

In $d = 3$, 5 equations, 14 unknowns

ρ, v, e, S (6 due to symmetry), q

+ b, r viewed as externally supplied fields.

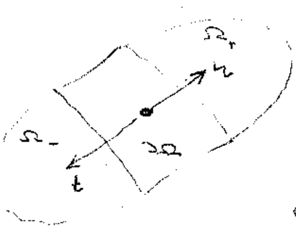
These are supplemented with constitutive relations and are subject to various contraints

- 2^{nd} law of thermodynamics
- objectivity : invariance under Galilean transformation / Lorentz transformation
- material symmetries

IDEAL FLUID.

Constitutive hypothesis

Forces across any surface cut in the fluid are normal to the surface



$$\left(\begin{array}{ll}\text{force across } \partial\Omega & \text{(that } \Omega_+ \\ \text{per unit area} & \text{exercises on } \Omega_-\text{)}\end{array}\right) = \underset{\sim}{t} = -p(x,t)\,\underset{\sim}{n},$$

$$S = -pI$$

For an ideal fluid. If there is no rotation at time $t=0$ there is no rotation afterwards

$$\frac{d}{dt}\int_{\Omega_t} \rho v\, dx = -\int_{\partial\Omega_t} p\, n\, d\sigma + \int_{\Omega} \rho b\, dx$$

$$\rho \frac{Dv}{Dt} = -\nabla p + \rho b$$

Equations

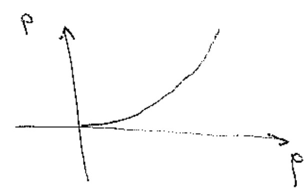
$$\frac{\partial\rho}{\partial t} + \operatorname{div} \rho v = 0$$

$$\rho\left[\frac{\partial v}{\partial t} + (v.\nabla)v\right] = -\nabla p + \rho b$$

5 unknowns ρ, v, p

Isentropic ideal fluid

$p = p(\rho)$, $p'(\rho) > 0$ (for well posedness).



ex $p = A\rho^{\gamma}$

$$\left\{\begin{array}{l}\dfrac{\partial\rho}{\partial t} + \operatorname{div}\rho v = 0 \\ \rho\dfrac{Dv}{Dt} = -\nabla p(\rho) + \rho b\end{array}\right.$$

compressible Euler equations

Incompressible ideal fluid

Incompressible $\quad \dfrac{d}{dt}\displaystyle\int_{\Omega_t} dx = \dfrac{d}{dt} V(\Omega_t) = 0$

$\displaystyle\int_{\Omega_t} dx = \int_{\Omega} J\, dX \qquad$ Incompressible $\quad J = 1$

$\iff \quad J_t = 0$

$J_t = \operatorname{div} v\, J$

$\iff \quad \boxed{\operatorname{div} v = 0}$

$$\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho + \underline{\rho \operatorname{div} v} = 0$$

Obtain $\quad \dfrac{\partial \rho}{\partial t} + v(x,t) \cdot \nabla \rho = 0 \qquad (*)$

Introduce trajectories $\quad \dfrac{dx}{dt} = v(x(t), t)$.

$(*)$ means ρ remains constant on trajectories

$$\rho(x(t), t) = \rho(x(0), 0).$$

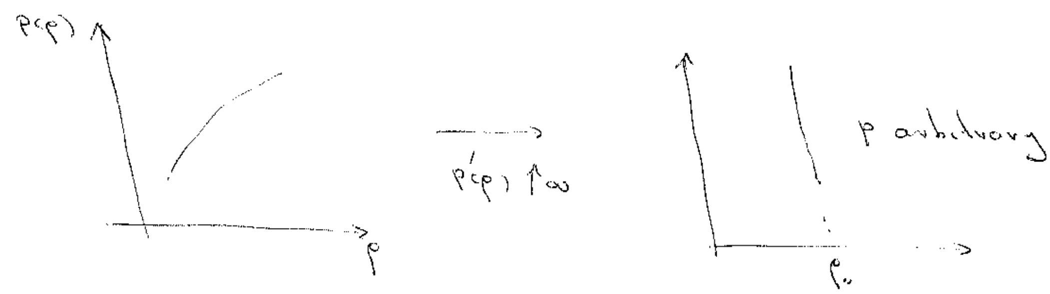
If $\rho = \rho_0$ at time $t = 0$ then $\rho = \rho_0$ thereafter.

Equations become ~~(incom~~

$$\begin{cases} \operatorname{div} v = 0 \\ \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v = -\dfrac{1}{\rho_0} \nabla p + b \end{cases}$$

(incompressible)
Euler eqns.

Compressible



MECHANICAL ENERGY

Isentropic flow

Identities
$$\frac{D}{Dt}\tfrac{1}{2}|v|^2 = \frac{\partial}{\partial t}\frac{v_i v_i}{2} + v_j \partial_j \frac{v_i v_i}{2} = v_i \frac{Dv_i}{Dt} = v\cdot\frac{Dv}{Dt}$$

$$\frac{D}{Dt}G(\rho) = G'(\rho)\frac{D\rho}{Dt}$$

$$\rho\frac{Dv}{Dt} = -\nabla p + \rho b$$

$$\rho\frac{D}{Dt}\tfrac{1}{2}|v|^2 = -v\cdot\nabla p + \rho b\cdot v = \operatorname{div}\big(v\cdot(-pI)\big) + p\operatorname{div}v + \rho b\cdot v$$

$$\frac{D}{Dt}\tfrac{1}{2}|v|^2 = \frac{1}{\rho}\operatorname{div}\big(v\cdot(-pI)\big) + \frac{p}{\rho^2}\frac{D\rho}{Dt} + b\cdot v$$

Let $\epsilon = \int^{\rho}\frac{p(s)}{s^2}\,ds \quad \epsilon'(\rho) = \frac{p}{\rho^2}$

$$\boxed{\rho\frac{D}{Dt}\left(\tfrac{1}{2}|v|^2 + \epsilon\right) = \operatorname{div}\big(v\cdot(-pI)\big) + \rho\, b\cdot v}$$ conservation of mechanical energy

compare with

$$\rho\frac{D}{Dt}\left(\tfrac{1}{2}|v|^2 + e\right) = \operatorname{div}(v\cdot S + q) + \rho\, b\cdot v + \rho r$$

ϵ internal energy $\epsilon' = \frac{p}{\rho^2}$ $\qquad E = E_{kin} + E_{pot}$

$q=0,\ r=0$ no heat flux, heat supply

Incompressible flow ($\rho = \rho_0$)

$$\frac{Dv}{Dt} = -\nabla \frac{p}{\rho_0} + b$$

$$\frac{D}{Dt}\,\frac{1}{2}|v|^2 = -v\cdot\nabla\frac{p}{\rho_0} = \operatorname{div}\left(v\cdot\left(\frac{-pI}{\rho_0}\right)\right) + b\cdot v$$

$$\boxed{\rho_0\frac{D}{Dt}\,\frac{1}{2}|v|^2 = \operatorname{div}\left(v\cdot(-pI)\right) + \rho_0 b\cdot v}$$ conservation of energy

$$E = E_{\text{kinetic}} = \int_{\Omega_t} \rho_0\,\frac{1}{2}|v|^2\,dx$$

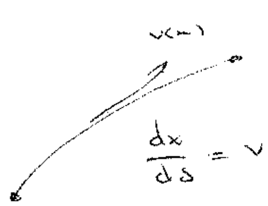
BERNOULLI THM In stationary isentropic or incompressible ($\rho=\rho_0$) flow, the quantity $\frac{1}{2}|v|^2 + w$ is constant on streamlines.

Pf.

$$(v\cdot\nabla)v = -\nabla w \qquad \begin{cases} w = \dfrac{p}{\rho_0} & \text{incompressible, hom flow} \\ w' = \dfrac{p'(\rho)}{\rho} & \text{isentropic} \end{cases}$$

Identity $\quad \frac{1}{2}\nabla|v|^2 = (v\cdot\nabla)v + v\times(\nabla\times v)$

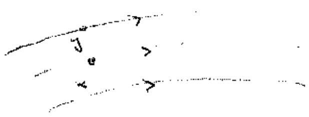
Using this $\quad \nabla\left(\frac{1}{2}|v|^2 + w\right) = v\times(\nabla\times v)$



$$\frac{dx}{ds}\cdot\nabla\left(\frac{1}{2}|v|^2+w\right) = \frac{dx}{ds}\cdot\left(v\times\nabla\times v\right) \overset{v\,\perp\,v\times(\nabla\times v)}{=} 0$$

$$0 = \int_{s_1}^{s_2}\frac{dx}{ds}\cdot\nabla_x\left(\frac{1}{2}|v|^2+w\right)ds = \frac{1}{2}|v|^2+w\,\Big|_{x(s_1)}^{x(s_2)}$$

VORTICITY

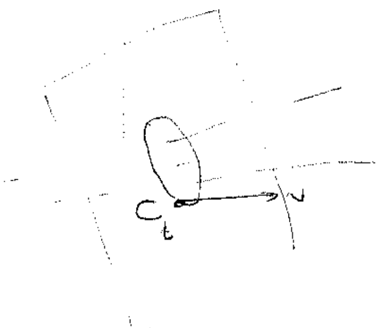


$$v(y) = v(x) + D(x)h + \frac{1}{2}\xi(x)\times h + O(|h|^2)$$

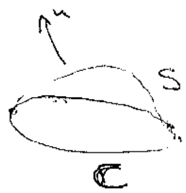
$y - x = h$

$$D(x) = \frac{1}{2}(\nabla v + \nabla v^T)$$

$$\xi(x) = \text{curl}\, v(x)$$

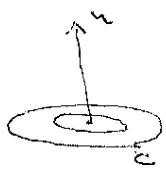


Circulation $\Gamma_{C_t} = \oint_{C_t} v\cdot dx$



Stokes thm

$$\int_C F\cdot dx = \int_S \text{curl}\, F\cdot n\, d\sigma$$



$$\Gamma_C = \int_C v\cdot dx = \int_S \text{curl}\, v\cdot n\, d\sigma = \int_S \xi\cdot n\, d\sigma$$

$$\text{curl}\, v\cdot n = \lim_{|S_\epsilon|\to 0}\frac{1}{|S|}\int_{S_\epsilon}\xi\cdot n\, d\sigma = \lim_{|S|\to 0}\frac{1}{|S|}\int_C v\cdot dx.$$

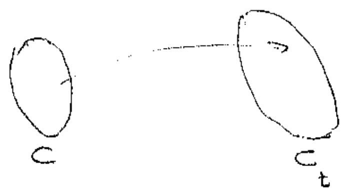
$= \text{circulation}/\text{unit area}$

$\frac{Dv}{Dt} = -\nabla w$ isentropic flow

incompress. ($\rho = \rho_0$)

KELVIN'S CIRCULATION THEOREM

For isentropic flow Γ_{C_t} is constant in time



$\int_{C_t} v \cdot dx$ is const. in time

Lemma (Transport thm for curves)

v velocity field

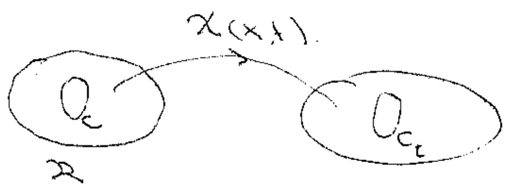
C closed loop.

$C_t = \chi_t(C)$ image of loop through the flow

Then

$$\frac{d}{dt}\int_{C_t} v \cdot dx = \int_{C_t} \frac{Dv}{Dt} \cdot dx.$$

Pf.



$V = \frac{\partial \chi}{\partial t}$

$v(\chi(x,t),t) = V(x,t)$

$C: \quad z = x(s) \qquad 0 \le s \le 1$

$C_t: \quad z_t = \chi(x(s),t) \qquad 0 \le s \le 1, \quad t \in \mathbb{R}^+$

$$\frac{d}{dt}\int_{C_t} v \cdot dx = \frac{d}{dt}\int_0^1 v(\chi(x(s),t),t) \cdot \frac{\partial}{\partial s}\chi(x(s),t)\, ds$$

$$= \int_0^1 \frac{\partial v}{\partial t}(\;) + \frac{\partial \chi}{\partial t} \cdot \nabla_x v(\;) \cdot \frac{\partial}{\partial s}\chi(x(s),t)\, ds$$
$$+ \int_0^1 v(\;) \cdot \frac{\partial}{\partial t}\frac{\partial}{\partial s}\chi(\;)\, ds$$

$$= \int_0^1 \underbrace{\left(\frac{\partial v}{\partial t} + v\cdot\nabla_x v\right)}_{\frac{Dv}{Dt}}(\chi(x(s),t),t)\cdot\frac{\partial}{\partial s}\chi(x(s),t)\,ds$$

$$+ \int_0^1 v(\chi(x(s),t),t)\cdot\frac{\partial}{\partial s}\underbrace{\frac{\partial\chi}{\partial t}(x(s),t)}_{V(x(s),t)\,=\,v(\chi(x(s),t),t)}\,ds.$$

$$= \int_0^1 \frac{Dv}{Dt}\cdot\frac{\partial}{\partial s}(\chi(x(s),t))\,ds + \underbrace{\int_0^1 \frac{\partial}{\partial s}\frac{1}{2}V\cdot V\,ds}_{=\,0}$$

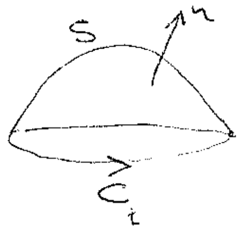
C closed

$$= \int_{C_t}\frac{Dv}{Dt}\cdot dx.$$

Pf. of Kelvin circ. thm

$$\frac{Dv}{Dt} = -\nabla w \qquad \text{no body force}$$

$$\frac{Dv}{Dt} = -\nabla(w+\mathcal{U}) \qquad \text{or potential body force } b = -\nabla\mathcal{U}$$



$$\frac{d}{dt}\Gamma_{C_t} = \frac{d}{dt}\int_{C_t} v\cdot dx$$

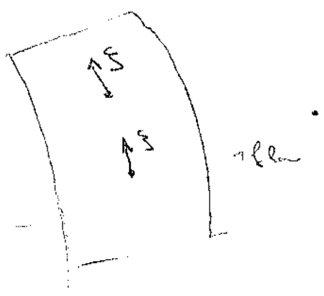
$$= \int_{C_t}\frac{Dv}{Dt}\cdot dx$$

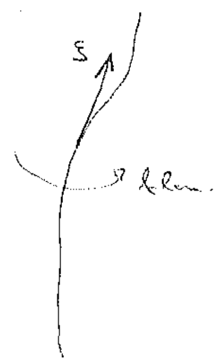
$$= -\int_{C_t}\nabla w\cdot dx = -\int_S \operatorname{curl}(\nabla w)\cdot n\,d\sigma$$

$$= 0$$

<u>Vortex sheet</u> is a surface S that is tangent to vorticity vector ξ at every point

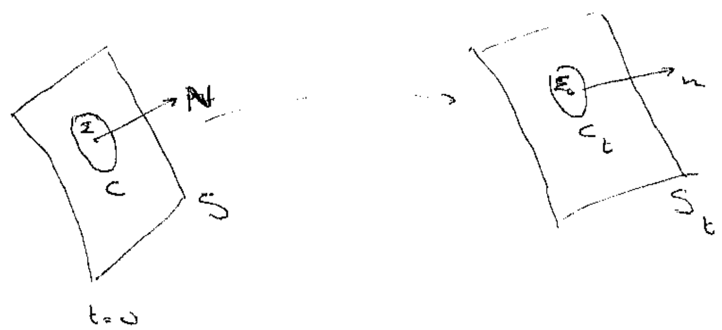
<u>Vortex line</u> is a line L that is "





<u>Prop</u> If a surface (curve) moves with the flow of a isentropic fluid and is a vortex sheet (line) at $t=0$ then it remains so at all times.

<u>Pf.</u>



$$\Gamma_C = \int_\Sigma \operatorname{curl} v \cdot N \, d\sigma = 0$$

$$\Gamma_{C_t} = \Gamma_C = 0$$

$$\int_{\Sigma_t} \operatorname{curl} v \cdot n \, d\sigma = 0 \qquad \forall \Sigma_t$$

$$\operatorname{curl} v \cdot n = 0 \quad \text{on } S_t$$

Vorticity equation in 3-D

$$\frac{Dv}{Dt} = -\nabla w$$ isentropic + $\frac{D\rho}{Dt} = -\rho \operatorname{div} v$

incompressible $w = \frac{p}{\rho_0}$, $\operatorname{div} v = 0$

Identities
$$\frac{1}{2}\nabla(F.F) = F \times \operatorname{curl} F + (F.\nabla)F$$
$$\operatorname{curl}(F \times G) = F \operatorname{div} G - G \operatorname{div} F + (G.\nabla)F - (F.\nabla)G.$$

$$\frac{\partial v}{\partial t} + (v.\nabla)v = -\nabla w$$

$$\frac{\partial v}{\partial t} - v \times \operatorname{curl} v = -\frac{1}{2}\nabla |v|^2 - \nabla w$$

$$\frac{\partial}{\partial t}\operatorname{curl} v - \operatorname{curl}(v \times \operatorname{curl} v) = 0$$

$\xi = \operatorname{curl} v$

$$\boxed{\frac{\partial}{\partial t}\xi - \operatorname{curl}(v \times \xi) = 0}$$

$$\frac{\partial \xi}{\partial t} - \Big[v \underbrace{\operatorname{div} \xi}_{\substack{\operatorname{div}\operatorname{curl} v \\ \shortparallel \\ 0}} - \xi \operatorname{div} v + (\xi.\nabla)v - (v.\nabla)\xi \Big] = 0$$

$$\underbrace{\frac{\partial \xi}{\partial t} + (v.\nabla)\xi}_{\frac{D\xi}{Dt}} - (\xi.\nabla)v + \xi \operatorname{div} v = 0$$

Incompressible flow
$$\frac{D\xi}{Dt} = (\xi\cdot\nabla)v$$

Isentropic flow

Let $\quad \omega = \frac{\xi}{\rho}$

$$\frac{D\omega}{Dt} = \frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{1}{\rho}\frac{D\xi}{Dt} - \frac{1}{\rho^2}\frac{D\rho}{Dt}\xi = \frac{1}{\rho}\frac{D\xi}{Dt} + \omega \operatorname{div} v$$

$$\frac{D\omega}{Dt} = \left[(\omega\cdot\nabla)v - \omega \operatorname{div} v\right] + \omega\operatorname{div} v = (\omega\cdot\nabla)v$$

In both cases

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)v \qquad \omega = \frac{\xi}{\rho} \quad \text{isentropic}$$
$$\omega = \frac{\xi}{\rho_0} \quad \text{incompressible}$$

Pf of identities

$$(F\times \operatorname{curl} F)_i = \varepsilon_{ijk}F_j(\operatorname{curl}F)_k = \varepsilon_{ijk}\varepsilon_{k\alpha\beta}F_j\frac{\partial F_\beta}{\partial x_\alpha} \qquad i,j,k,\alpha,\beta = 1,2,3$$

$$= \varepsilon_{ijk}\varepsilon_{kij}F_j\frac{\partial F_j}{\partial x_i} + \varepsilon_{ijk}\varepsilon_{kji}F_j\frac{\partial F_i}{\partial x_j} \qquad i \text{ not summed}$$

i is not summed

$$= \frac{\partial}{\partial x_i}\frac{1}{2}F_jF_j - F_j\frac{\partial F_i}{\partial x_j}$$

$$F\times\operatorname{curl}F = \nabla\frac{1}{2}|F|^2 - (F\cdot\nabla)F$$

$$\left(\operatorname{curl} F\times G\right)_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(F\times G\right)_k = \varepsilon_{ijk}\varepsilon_{k\alpha\beta}\frac{\partial}{\partial x_j}\left(F_\alpha G_\beta\right)$$

$$= \varepsilon_{ijk}\varepsilon_{k\alpha\beta}\left[\frac{\partial F_\alpha}{\partial x_j}G_\beta + F_\alpha\frac{\partial G_\beta}{\partial x_j}\right]$$

$$= \varepsilon_{ijk}\varepsilon_{kij}\left(\frac{\partial F_i}{\partial x_j}G_j + F_i\frac{\partial G_j}{\partial x_j}\right) + \varepsilon_{ijk}\varepsilon_{kji}\left(\frac{\partial F_j}{\partial x_j}G_i + F_j\frac{\partial G_i}{\partial x_j}\right)$$

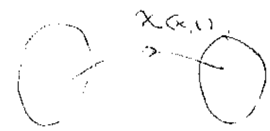
i not summed

$$= G_j\frac{\partial F_i}{\partial x_j} + F_i\frac{\partial G_j}{\partial x_j} - G_i\frac{\partial F_j}{\partial x_j} - F_j\frac{\partial G_i}{\partial x_j}$$

$$\operatorname{curl} F\times G = (G\cdot\nabla)F - (F\cdot\nabla)G + F\operatorname{div}G - G\operatorname{div}F$$

Thm For isentropic flow

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)v$$



where $\xi = \operatorname{curl} v$, $\omega = \frac{\xi}{\rho}$

and $$\omega(X(x,t),t) = \nabla X(x,t)\cdot\omega(X(x,0),0)$$

Pf.

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)v$$

Let $$G(x,t) = g(X(x,t),t) = \nabla X(x,t)\cdot\omega(X(x,0),0).$$

$$\frac{Dg}{Dt} = \frac{\partial G}{\partial t} = \frac{\partial}{\partial t}\nabla X(x,t)\cdot\omega(X(x,0),0)$$

$$= \nabla_x \underbrace{\frac{\partial X}{\partial t}(x,t)}_{\underbrace{V(x,t)}_{v(X(x,t),t)}}\cdot\omega(X(x,0),0)$$

$$= (\nabla_x v)\,\nabla_x X(x,t)\,\underbrace{\omega(X(x,0),0)}_{G(x,t)=g(\,\text{—}\,)}$$

$$= \frac{\partial v_i}{\partial x_j}g_j = (g\cdot\nabla)v$$

Since at $t=0$ $$\omega(X(x,0),0) = \omega(x,0) = \underbrace{\nabla X(x,0)}_{I}\cdot\omega(X(x,0),0).$$

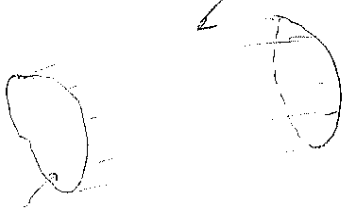
$$= g(X(x,0),0)$$

we have

$$\omega = \nabla X(x,t)\cdot\omega(X(x,0),0)$$

Vortex tube

vortex line



S transversal to ξ

vortex lines through S
integral curves of ξ.

S transversal to ξ

Helmholtz' thm

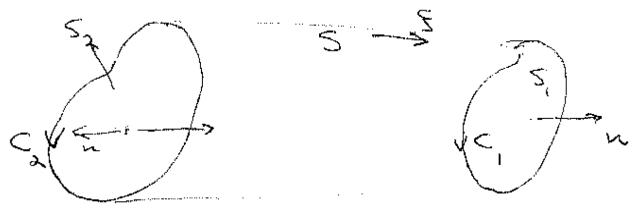
Isentropic fluid

(2) C_1, C_2 two curves encircling the vortex tube, then

$$\int_{C_1} v \cdot dx = \int_{C_2} v \cdot dx \qquad \left(\overset{\text{called}}{=} \text{strength of vortex tube} \right)$$

(3) The strength of the vortex tube is constant in time, as the tube moves with the flow.

Pf.



V volume of tube

S lateral surface, ξ tangent to S.

$$0 = \int_V \underbrace{\operatorname{div} \xi}_{\operatorname{div} \operatorname{curl} v} \, dx = \int_{S_1 \cup S_2 \cup S} \xi \cdot n \, d\sigma$$

$$= \int_{S_1} \xi \cdot n \, d\sigma + \int_{S_2} \xi \cdot n \, d\sigma + \underbrace{\int_S \xi \cdot n \, d\sigma}_{\overset{\parallel}{0} \quad \xi \parallel S}$$

By Stokes thm

$$\int_{C_1} v \cdot dx = \int_{S_1} \operatorname{curl} v \cdot n \, d\sigma$$

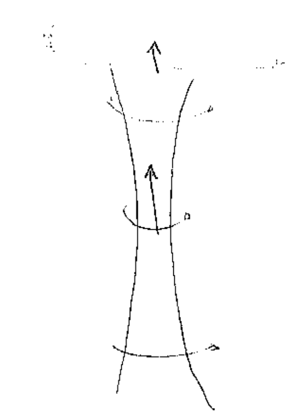
$$\int_{C_2} v \cdot dx = -\int_{S_2} \operatorname{curl} v \cdot n \, d\sigma$$

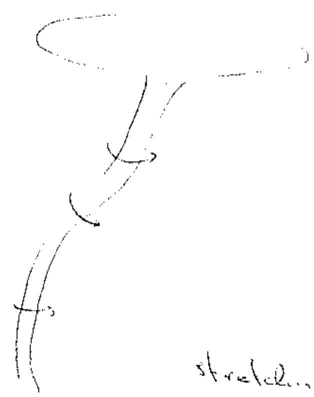
Conclude

$$\int_{C_1} v \cdot dx = \int_{C_2} v \cdot dx .$$

(ii) Follows because vortex sheet S preserved by the flow

Rmk

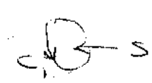




stretching of vortex tube

A vortex tube cannot end in the interior of fluid

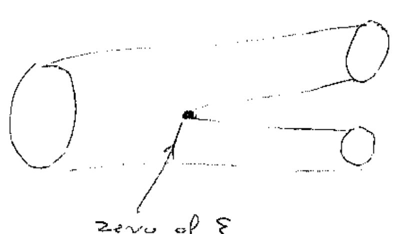
Suppose tube ends on S.



Since it cannot be extended $\xi = 0$ on C_1

Strength on $C_1 = 0$ contradiction.

Not a theorem!



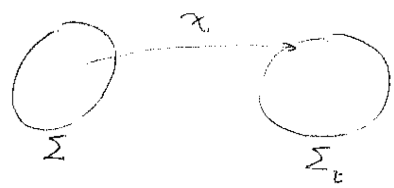
Implicitly assumed "nice geometry"

VORTICITY IN 2-D FLOWS

$\underset{\sim}{v} = (u, v, 0)$

$\underset{\sim}{\xi} = (0, 0, \xi) \qquad \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

Circulation thm



$\Sigma_t = \chi(\Sigma, t)$. region.

$\int_{\Sigma_t} \xi \, dA = \text{constant in time}$

2-D incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} \qquad p = \frac{P}{\rho_0}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}$$

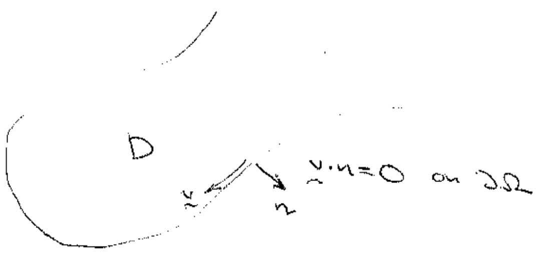
$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

$$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial \xi}{\partial t} + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \underbrace{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}}_{\frac{\partial v}{\partial x}(\operatorname{div}\underset{\sim}{v}) - \frac{\partial u}{\partial y}(\operatorname{div}\underset{\sim}{v})} = 0$$

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} = 0 = \frac{D\xi}{Dt}$$

Take a flow in a domain D.



Assume D is <u>simply connected</u> (i.e. no holes).

$$u_x = -v_y$$

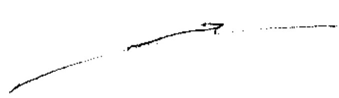
$\exists$ unique scalar fn $\psi(x,y,t)$ in D s.t.

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

<u>Rmk</u> For t fixed, streamlines lie on level curves of ψ.

$(x(s), y(s))$ streamline

t frozen



$$\frac{dx}{ds}(s) = u(x(s), y(s), t)$$

$$\frac{dy}{ds}(s) = v(x(s), y(s), t).$$

$$\frac{d}{ds}\psi(x(s), y(s), t) = \frac{\partial \psi}{\partial x}\dot{x} + \frac{\partial \psi}{\partial y}\dot{y} = -vu + uv = 0$$

<u>Rmk</u> ∂D is a streamline

∂D lies on a level curve of ψ.

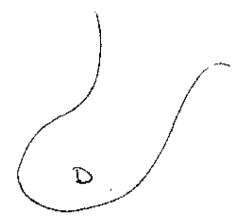
Vorticity then becomes

$$\xi = v_x - u_y = -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} = -\Delta\psi.$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ 2-D Laplacian.

Equations of 2-D incompressible flow

$$\boxed{\begin{aligned} &\partial_t\xi + u\frac{\partial\xi}{\partial x} + v\frac{\partial\xi}{\partial y} = 0 \qquad (*). \\ &\Delta\psi = -\xi \quad , \quad \psi = 0 \text{ on } \partial D \\ &\text{with } \quad u = \frac{\partial\psi}{\partial y} \quad , \quad v = -\frac{\partial\psi}{\partial x} \end{aligned}}$$



Rmk Maximum principle

Apply the method of characteristics on (*)

Define characteristics

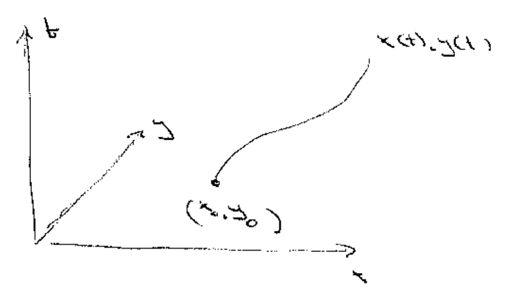
$$\frac{dx}{dt} = u(x(t), y(t), t).$$

$$\frac{dy}{dt} = v(x(t), y(t), t).$$

$x(0) = x_0, \ y(0) = y_0$

trajectories of flow.

These define curves

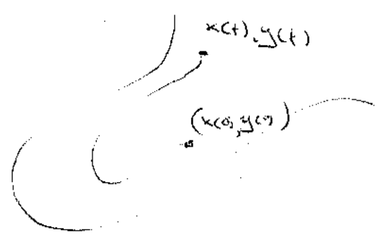


Along these curves

$$\frac{d}{dt}\xi(x(t),y(t),t) = \frac{D\xi}{Dt} = 0$$

$$\xi(x(t),y(t),t) = \xi(x(0),y(0),0).$$

<u>In particular</u>



$$|\xi(x,y,t)| \le \max|\xi(x,y,0)|$$

This is the basic observation that makes work.

<u>Thm</u> For equation of 2D incompressible flow.
If initial data are smooth, $\exists$ unique globally defined smooth solution.

3-D EULER - VORTICITY FORMULATION

$\operatorname{div} u = 0$

$\frac{Du}{Dt} = -\nabla p$

$\xi = \nabla \times u$ satisfies $\frac{D\xi}{Dt} = (\xi \cdot \nabla) u$

$\operatorname{div} u = 0 \Longrightarrow u = \nabla \times A$ with $\operatorname{div} A = 0$ (*) Pf 2.
D has no solid holes

$\xi = \nabla \times u = \nabla \times (\nabla \times A) = -\Delta A + \nabla(\operatorname{div} A) = -\Delta A$

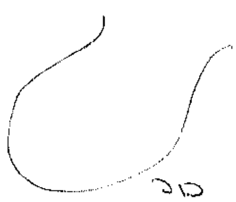
Obtain the formulation

$$\frac{D\xi}{Dt} = (\xi \cdot \nabla) u$$

$$\Delta A = -\xi, \quad \operatorname{div} A = 0$$

$$u = \nabla \times A$$

Rmk



∂D

On ∂D we do not know $A = 0$
or adequate b.c. for A. $(\nabla \times A) \cdot n = 0$ on ∂D

This does not suffice to determine A uniquely

<u>Proposition</u> Suppose D has no solid holes.

$\operatorname{div} u = 0$ on D.

Then $\exists\, A$ so that $u = \operatorname{curl} A$

<u>Pf.</u> $$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$$

Write $$u_2 = -\frac{\partial A_3}{\partial x_1}$$
$$u_3 = \frac{\partial A_2}{\partial x_1}$$ $\longrightarrow$ A_2, A_3 given by integration.

$$\frac{\partial u_1}{\partial x_1} - \frac{\partial^2 A_3}{\partial x_1 \partial x_2} + \frac{\partial^2 A_2}{\partial x_1 \partial x_3} = 0$$

$$u_1 = \frac{\partial}{\partial x_2} A_3 - \frac{\partial A_2}{\partial x_3} + \frac{\partial}{\partial x_2} g(x_2, x_3)$$

So $$\begin{cases} u_1 = \dfrac{\partial}{\partial x_2}(A_3 + g) - \dfrac{\partial A_2}{\partial x_3} \\ u_2 = \qquad\qquad -\dfrac{\partial}{\partial x_1}(A_3 + g) \\ u_3 = \dfrac{\partial A_2}{\partial x_1} \end{cases}$$ $$u = \operatorname{curl}(0, A_2, A_3 + g)$$

<u>Rmk</u> A is by no means unique

$u = \operatorname{curl}(A + \nabla\phi)$ for any ϕ (called gauge).

Given A, $\operatorname{div}(A + \nabla\phi) = 0$ means $\Delta\phi = -\operatorname{div} A$

determines ϕ.