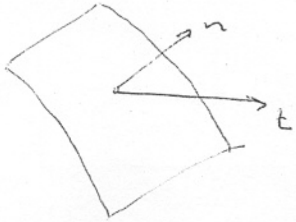
VISCOUS FLUIDS - NAVIER-STOKES EQUATIONS

molecular diffusion
momentum transfer $\rightarrow$ shear forces.



Force on S per unit area

$$= -p(x,t)\,n + \sigma(x,t)\,n$$

on $\partial \Omega$.

Cauchy stress $\qquad S = -pI + \sigma \quad , \quad \sigma = \sigma(\nabla v)$

Restrictions on S (or σ)

- Theory of thermoviscoelastic fluid

$$S = S(F, Z, \theta) \qquad F = \nabla X \;,\; Z = \nabla X_t =$$

$$= \underbrace{\frac{\partial W(F,\theta)}{\partial F}}_{\text{elastic}} + \underbrace{S^v(F,\theta,Z)}_{\text{viscous}}$$

2nd law

Assumptions

- fluid is isotropic
- σ is symmetric $\quad \sigma = \sigma^T$
- $\sigma = \sigma(\nabla v)$

  σ invariant under rigid body rotations and translations

  $$\sigma(Q \nabla v Q^T) = Q\,\sigma(\nabla v)\,Q^T$$

- σ is linear in ∇v

- Compressible Newtonian fluid (thermoviscous)

$$S = -p(\rho,\theta)\, I + \underbrace{\left(\lambda(\rho,\theta)\,(\mathrm{tr}\, D)\, I + 2\mu(\rho,\theta)\, D\right)}_{\sigma}$$

λ, μ Lamé viscosity coefficients, μ first viscosity, λ second "

$$D = \tfrac{1}{2}\left(\nabla v + \nabla v^T\right)$$

- Compressible fluid

$$S = -p(\rho)\, I + \left(\lambda (\mathrm{tr}\, D)\, I + 2\mu D\right)$$

$$= -p(\rho)\, I + \left[\left(\lambda + \tfrac{2}{3}\mu\right)(\mathrm{div}\, v)\, I + 2\mu\left(D - \tfrac{1}{3}(\mathrm{div}\, v)\, I\right)\right]$$

$\mathrm{tr}\, D = \mathrm{div}\, v$

Discussion of the various terms

- $-p(\rho)\, I$ an elastic term due to changes in density

- $$\lambda(\mathrm{tr}\, D)\, I = \begin{pmatrix} \lambda\, \mathrm{tr} D & & \\ & \lambda\, \mathrm{tr} D & \\ & & \lambda\, \mathrm{tr} D \end{pmatrix}$$

bulk viscosity - viscous forces due to volume change

$\mathrm{tr}\, D = \mathrm{div}\, v$ recall $\boxed{J_t = J\, \mathrm{div}\, v}$

- To exemplify consider $D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & d_3 \end{pmatrix}$

~~simple shear~~

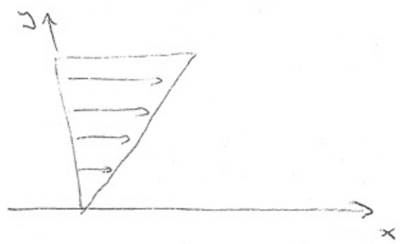
$\cancel{\sigma_{12} = \ldots}$

Recall

$$\frac{dh_1}{dt} = d_1 h_1$$

$$h_1 = e^{d_1 t}\, h_1(0)$$

shear flow



$$u(x,y) = ky$$
$$v(x,y) = 0$$

$$\sigma = 2\mu D = 2\mu \begin{pmatrix} 0 & \frac{1}{2}(u_y + v_x) \\ \frac{1}{2}(u_y+v_x) & 0 \end{pmatrix} = \mu \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

$\mathrm{tr}\, D = 0$ for this flow

$$S = -p(\rho)\, I + \begin{pmatrix} 0 & \mu k \\ \mu k & 0 \end{pmatrix}.$$

Compressible Navier-Stokes equations

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}\, \rho v = 0 \\[2ex] \rho \dfrac{Dv}{Dt} = \rho\left(\dfrac{\partial v}{\partial t} + (v\cdot\nabla) v\right) = -\nabla p + (\lambda+\mu)\nabla(\mathrm{div}\, v) + \mu \Delta v \end{cases}$$

λ, μ constants

Derivation

$$\rho \frac{Dv}{Dt} = \mathrm{div}\, S$$

$$S = -p(\rho)\, I + \left(\lambda (\mathrm{div}\, v) I + 2\mu D\right)$$

$$\partial_{x_j} S_{ij} = \partial_{x_j}\left(-p\,\delta_{ij} + \lambda\left(\frac{\partial v_k}{\partial x_k}\right)\delta_{ij} + \frac{2\mu}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right)$$

$$= -\frac{\partial p}{\partial x_i} + \lambda\, \partial_{x_i}(\mathrm{div}\, v) + \mu \Delta v_i + \mu\, \partial_{x_i}(\partial_{x_j} v_j)$$

$$= -\frac{\partial p}{\partial x_i} + (\lambda+\mu)\, \partial_{x_i}(\mathrm{div}\, v) + \mu \Delta v_i$$

Incompressible Navier-Stokes ($\rho = \rho_0$) and homogeneous

$$\begin{cases} \operatorname{div} v = 0 \\ \rho_0 \dfrac{Dv}{Dt} = -\nabla p + \mu \Delta v \end{cases}$$

written also

$$\frac{Dv}{Dt} = -\nabla\left(\frac{p}{\rho_0}\right) + \frac{\mu}{\rho_0} \Delta v$$

μ viscosity

$\nu = \frac{\mu}{\rho_0}$ kinematic viscosity

units of ν are $\dfrac{L/T^2}{\frac{L}{T L^2}} = \dfrac{L^2}{T}$

units of μ are $\nu \rho_0 = \dfrac{L^2}{T} \dfrac{M}{L^3}$

~~Boundary~~

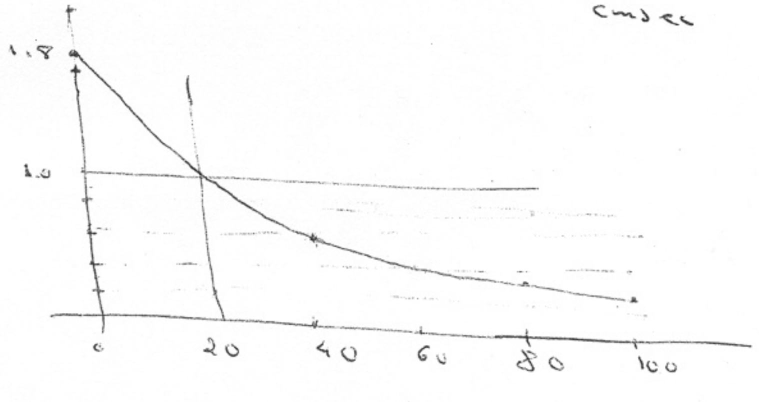
Typical values of viscosity

Water (20°C, atmospheric pressure) : $\mu \approx 10^{-2} \frac{gr}{cm\,sec}$

air $2 \times 10^{-4} \frac{gr}{cm\,sec}$

glycerine $9 \frac{gr}{cm\,sec}$

Dependence of viscosity on temperature (water)



SCALING PROPERTIES OF N-S EQUATIONS

L characteristic length

T " time

U " velocity $\qquad U = \frac{L}{T}$

$u' = \frac{u}{U} \quad x' = \frac{x}{L}, \; t' = \frac{t}{T}$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}\frac{1}{\rho_0} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{U}{T}\frac{\partial u'}{\partial t'} + \frac{U^2}{L}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right) = -\frac{1}{\rho_0}\frac{1}{L}\frac{\partial p}{\partial x'} + \frac{\nu U}{L^2}\Delta_{x'} u'$$

$$\frac{U^2}{L}\left[\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right] = \frac{U^2}{L}\frac{\partial \frac{p}{\rho_0 U^2}}{\partial x'} + \frac{U}{L^2}\nu\,\Delta_{x'} u'$$

scale $\quad p' = \frac{p}{\rho_0 U^2} \qquad \rho_0 U^2$ dimension $\quad \frac{M}{L^3}\left(\frac{L}{T}\right)^2 = \frac{M}{LT^2}$

pressure dim. $\quad M\frac{L}{T^2}\cdot\frac{1}{L^2} = \frac{M}{LT^2}$

$$\frac{\partial u'}{\partial t'} + (u'\cdot\nabla')u' = -\nabla' p' + \frac{\nu}{LU}\Delta u'$$

$$\boxed{R = \frac{LU}{\nu} = \text{Reynolds number}}$$

Selection of char. scales

Role in experiments

$R \to \infty$ slow flow $\qquad$ viscosity dominates

$R \to 0$ fast flow $\qquad$ inertia dominates

Flows with the same geometry and same Reynolds number are called similar

geometry

$L_2 = \lambda L_1$

$R_1 = R_2$

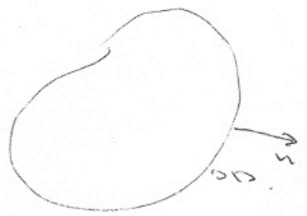
$$\frac{L_1 U_1}{\nu_1} = \frac{L_2 U_2}{\nu_2}$$

The dimensionless fields satisfy same equation in same region.

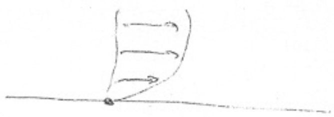
- Experimental models

Boundary conditions

Euler eqn
ideal fluid.



$u \cdot n = 0$ on ∂D



Navier-Stokes $\quad u = 0$ on solid walls

Rmk
- Experiments with dye on flow.
- Experiment with free surface on a wall

Navier b.c

$u(x,t) \sim$ tangential stress

$u(x,t) = -k(x)\,(\,S(x,t)\,\pi\,)$

THE INCOMPRESSIBLE NAVIER-STOKES EQUATION

$$\operatorname{div} u = 0$$

$$\rho_0 \frac{Du}{Dt} = \rho_0\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) = -\nabla p + \mu\Delta u + \rho_0 b = \operatorname{div} S + \rho_0 b$$

$$\boxed{S = -pI + 2\mu D}$$

in dimensionless variables

$$R = \frac{UL}{\nu} = \frac{UL}{\mu/\rho_0}$$

$$\begin{cases} \operatorname{div} u = 0 \\ \dfrac{Du}{Dt} = -\nabla p + \dfrac{1}{Re}\Delta u + b' \end{cases}$$

THE ENERGY EQUATION

$$\rho_0 \frac{D}{Dt}\frac{1}{2}|u|^2 = \rho_0 u\cdot\frac{Du}{Dt} = -u\cdot\nabla p + \mu\, u\cdot\Delta u + \rho_0 u\cdot b$$

$$= u\cdot\operatorname{div} S + \rho_0 u\cdot b$$

$$u\cdot\operatorname{div} S = u_i \frac{\partial}{\partial x_j} S_{ij} = u_i\left(-\frac{\partial p}{\partial x_i} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)$$

$$= \frac{\partial}{\partial x_j}\left(u_i S_{ij}\right) - S_{ij}\frac{\partial u_i}{\partial x_j}$$

$$= \operatorname{div}(u\cdot S) - \operatorname{tr}\left(S^T\nabla u\right)$$

$$\boxed{\rho_0 \frac{D}{Dt}\frac{1}{2}|u|^2 = \operatorname{div}(u\cdot S) - \operatorname{tr}\left(S^T\nabla u\right) + \rho_0 u\cdot b}$$

$$\rho_0 \frac{D}{Dt}\frac{1}{2}|u|^2 = -u\cdot\nabla p + \mu\, u\cdot\Delta u + \rho_0\, u\cdot b$$

$$= -u_i\frac{\partial p}{\partial x_i} + \mu\, u_i \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho_0\, u_i b_i$$

$$= +\frac{\partial}{\partial x_i}\left(-u_i p\right) + p\frac{\partial u_i}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\left(u_i \frac{\partial u_i}{\partial x_j}\right) - \mu \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \rho_0 u_i$$

$$= \operatorname{div}\left(-p\,u\cdot I + \mu\, u\cdot(\nabla u)\right) - \mu \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \rho_0\, u\cdot b$$

$$\mu \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} = \mu\, \mathrm{tr}\left(\nabla u^T\cdot\nabla u\right) = \mu|\nabla u|^2 = \mu\left(\nabla u : \nabla u\right)$$

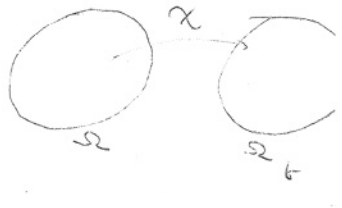
$$\boxed{\rho_0 \frac{D}{Dt}\frac{1}{2}|u|^2 + \mu|\nabla u|^2 = \operatorname{div}\left(-p\,u\cdot I + \mu\, u\cdot(\nabla u)\right) + \rho_0\, u\cdot b}$$

local form.

$$\frac{d}{dt}\int_{\Omega_t}\frac{1}{2}|u|^2\rho_0\,dx \underset{\uparrow}{=} -\int_{\Omega_t}\mu|\nabla u|^2\,dx + \int_{\Omega_t}\rho_0\, u\cdot b\,dx$$

with b.c.

$u=0$ on $\partial\Omega_t$



If there are no external forces, no action on $\partial\Omega_t$

$$\underbrace{\frac{d}{dt}\int_{\Omega_t}\frac{1}{2}|u|^2\rho_0\,dx}_{E_{kinetic}} = -\mu\int_{\Omega_t}|\nabla u|^2\,dx$$

Rmk If we want energy to dissipate, we need to assume $\boxed{\mu \geq 0}$

2nd law of thermodynamics

THE ROLE OF THE PRESSURE

∂D smooth.

<u>Thm</u> Any vector field w on D can be uniquely decomposed in the form

$$w = u + \nabla p.$$

where u has divergence zero, $\operatorname{div} u = 0$ and $u \parallel \partial D$.

$(u \cdot n = 0 \text{ on } \partial D)$

<u>Step 1</u> <u>Orthogonality</u>

Suppose
$$\left.\begin{aligned} &\operatorname{div} u = 0 \quad \text{in } D \\ &u \cdot n = 0 \quad \text{on } \partial D \end{aligned}\right\} \Rightarrow \int_D u \cdot \nabla p = 0$$

<u>Pf</u>.
$$\operatorname{div} pu = (\operatorname{div} u) p + u \cdot \nabla p$$

$$\int_D u \cdot \nabla p = \int_D \operatorname{div}(pu)\, dx = \int_{\partial D} p\, u \cdot n \, d\sigma = 0$$

<u>Step 2</u> Uniqueness

Let
$$w = u_1 + \nabla p_1 \qquad \operatorname{div} u_1 = 0 \quad u_1 \cdot n = 0 \text{ on } \partial D$$
$$w = u_2 + \nabla p_2 \qquad \operatorname{div} u_2 = 0 \quad u_2 \cdot n = 0 \text{ on } \partial D.$$

$$(u_1 - u_2) + \nabla(p_1 - p_2) = 0$$

orthogonal

$$\int |u_1-u_2|^2\,dx + \underbrace{\int \nabla(p_1-p_2)\cdot(u_1-u_2)}_{=0,\ \text{orthogonal}} = 0$$

$$u_1 = u_2$$

$$\nabla(p_1-p_2) = 0 \Rightarrow p_1 - p_2 = \text{const.}$$

<u>Step 3</u> <u>Existence</u>

Analysis. Let $\quad w = u + \nabla p \qquad \operatorname{div} u = 0$

$$\operatorname{div} w = \operatorname{div} u + \operatorname{div} \nabla p = \Delta p$$

$$w\cdot n = u\cdot n + \nabla p\cdot n = \frac{\partial p}{\partial n} \quad \text{on } \partial D$$

Solve
$$\begin{cases} \Delta p = \operatorname{div} w & \text{on } D \\ \dfrac{\partial p}{\partial n} = w\cdot n & \text{on } \partial D \end{cases}$$

this determines p (the pressure)

Then set
$$u = w - \nabla p$$

$$\operatorname{div} u = \operatorname{div} w - \Delta p = 0 \quad \text{in } D$$

$$u\cdot n = w\cdot n - \frac{\partial p}{\partial n} = 0 \quad \text{on } \partial D$$

INTERPRETATION

$$L^2(D) = \left\{ w(x) : D \to \mathbb{R}^3 \;:\; \int_D |w|^2 \, dx < \infty \right\}$$

$L^2(D)$ is equipped with the inner product

$$(w,z) = \int_D w \cdot z \, dx = \int_D w_1 z_1 + w_2 z_2 + w_3 z_3 \, dx.$$

$$\|w\|_{L^2}^2 = \int_D |w|^2 \, dx.$$

$$V = H^1_0(D) = \left\{ w(x) : D \to \mathbb{R}^3, \quad w \in L^2, \; \frac{\partial w}{\partial x_i} \in L^2, \; i = 1,2,3 \right\} \subset H = L^2(D)$$

$$\langle w, z \rangle_{H^1} = \int_D w \cdot z + \underbrace{\nabla w : \nabla z}_{tr(\nabla w^T \nabla z)} \, dx$$

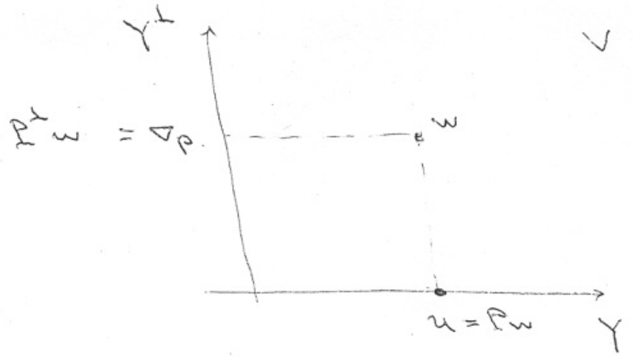
$$\|w\|_{H^1}^2 = \int_D |w|^2 + |\nabla w|^2 \, dx.$$

Define

$$Y = \{ w \in V : \operatorname{div} w = 0 \text{ on } D, \; w \cdot n = 0 \text{ on } \partial D \}$$

$$Y^\perp = \{ w \in V : w = \nabla p \text{ for some } p : D \to \mathbb{R} \}.$$

We show $\quad u \in Y, \; \nabla p \in Y^\perp \Rightarrow (u, \nabla p)_H = 0$



$$V = Y \oplus Y^\perp$$

$\uparrow$

in $(\cdot,\cdot)_{L^2}$

Define projection operators

$\underline{P}$ projection $\qquad \underline{P}u = u$ if $\operatorname{div} u = 0$, $u \cdot n = 0$ on ∂D.

$\qquad \underline{P}(\nabla p) = 0$.

$\underline{P}^{\perp} = \underline{I} - \underline{P}$

Then we have

$$w = \underline{P}w + \nabla p \qquad \text{for any } w \in V$$

<u>Application to incompressible Navier-Stokes</u>

$$V = H_0^1(D) = \{ w(x) : D \to R^3 \quad w = 0 \text{ on } \partial D \}.$$

$$V = Y \oplus Y^{\perp}$$

Consider a soln. of N-S

$$\begin{cases} \operatorname{div} u = 0 \\ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + \frac{1}{R} \Delta u \end{cases}$$

Note $\operatorname{div} u = 0$, $u \cdot n = 0$ on ∂D $\rightarrow$ $u \in Y$.

$$\underline{P}\left(\frac{\partial u}{\partial t} + \nabla p\right) = \underline{P}\left(-(u\cdot\nabla)u + \frac{1}{R}\Delta u\right)$$

$$\underline{P}\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\underline{P}u = \frac{\partial u}{\partial t}$$

$$\boxed{\frac{\partial u}{\partial t} = \underline{P}\left(-(u\cdot\nabla)u + \frac{1}{R}\Delta u\right) \qquad u \in Y} \qquad (*)$$

Conversely, let $u \in Y$ satisfy $(*)$

$\operatorname{div} u = 0$, $u \cdot n = 0$ on ∂D , in fact the stronger $u = 0$ on ∂D

$$\underbrace{-(u\cdot\nabla)u + \frac{1}{R}\Delta u}_{w} = \mathbb{P}\left(-(u\cdot\nabla)u + \frac{1}{R}\Delta u\right) + \nabla p .$$

where p is determined by $\begin{cases} \Delta p = \operatorname{div} w & \text{in } D \\ \frac{\partial p}{\partial n} = w \cdot n & \text{on } \partial D . \end{cases}$

$$\frac{\partial u}{\partial t} = -\nabla p + \left(-(u\cdot\nabla)u + \frac{1}{R}\Delta u\right) .$$

i.e. u sat. N-S

Therefore, we come up with the equivalent formulation

$$\boxed{\frac{\partial u}{\partial t} = \mathbb{P}\left(-(u\cdot\nabla)u + \frac{1}{R}\Delta u\right) \qquad u \in Y}$$

this is a (nonlinear) parabolic equation

<u>Application to Stokes equation</u>

$\operatorname{div} u = 0$

$$\frac{\partial u}{\partial t} = -\nabla p + \frac{1}{R}\Delta u$$

approximation of N-S when viscous terms are dominant over inertial terms

$$\boxed{\partial_t u = \mathbb{P}\left(\frac{1}{R}\Delta u\right) \qquad u \in Y = \{ u \in H_0^1 : \operatorname{div} u = 0 \}}$$

Rmk This idea has ramifications

- Conceptual
- Existence theory for N-S equation
- Development for numerical schemes - finite element spaces
  - divergence free elements

VORTICITY FORMULATION OF INCOMPRESSIBLE N-S.

3-D $\quad \xi = \text{curl}\, v$

$$\underbrace{\frac{\partial \xi}{\partial t} + (u\cdot\nabla)\xi}_{\text{convected}} - \underbrace{(\xi\cdot\nabla)u}_{\text{stretched}} = \underbrace{\frac{1}{R}\Delta\xi}_{\text{diffused}}$$

Rmk Circulation is no longer constant of the motion

"If $\xi=0$ at $t=0 \implies \xi=0$ at t". Not true because of effect of condition $u=0$ at boundaries.

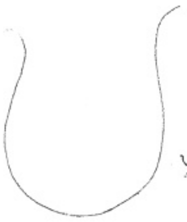
2-D

$$\frac{D\xi}{Dt} = \frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} = \frac{1}{R}\Delta\xi$$

$(u,v)=0$ on ∂D.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Introduce ψ s.t. $\quad u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$



$\underset{\sim}{u}\cdot n = 0$ on ∂D. $\implies \psi = 0$ on ∂D.

$\underset{\sim}{u} = 0$ on $\partial D \implies \frac{\partial \psi}{\partial y} = 0 = -\frac{\partial \psi}{\partial x}$ on ∂D.

Thus we get $\quad \psi = 0$ on ∂D

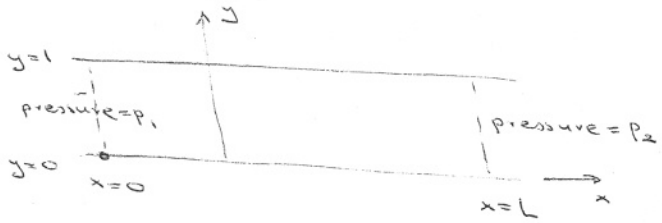
$\frac{\partial \psi}{\partial n} = 0$ on D.

$$\begin{cases} \dfrac{D\xi}{Dt} = \dfrac{1}{R}\Delta\xi \\ \Delta\psi = -\xi, \quad \psi = 0 \text{ on } \partial D. \\ u = \psi_y, \quad v = -\psi_x \end{cases}$$

Come back later to that

SPECIAL SOLUTIONS OF N-S EQUATIONS

Flow through a duct, - Poiseuille flow



Viscous incompressible through two stationary plates $y=0$, $y=1$

Ansatz

$\underset{\sim}{u}(x,y) = \big(u(x,y), 0\big)$ steady flow

p only fn of x. $p(0)=p_1$, $p(L)=p_2$; $p_1 > p_2$

flow caused by pressure gradient

$$\rho \frac{Du}{Dt} = -\nabla p + \mu \Delta u$$

$$\operatorname{div} u = 0$$

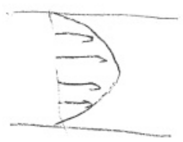
$$\begin{cases} \dfrac{\partial u}{\partial x} = 0 , \quad -\dfrac{\partial p}{\partial y} = 0 \\ \rho u \dfrac{\partial u}{\partial x} = -\dfrac{\partial p}{\partial x} + \mu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ \text{b.c.} \quad u(x,0) = 0 = u(x,1) \end{cases}$$

$p = p(x)$, $u = u(y)$.

$$\frac{dp}{dx}(x) = \mu \frac{d^2u}{dy^2}(y)$$

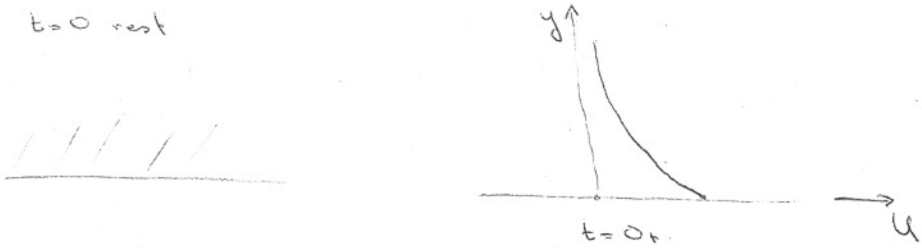
$$\frac{dp}{dx}(x) = \text{const} = -\frac{\Delta p}{L} = \frac{p_1 - p_2}{L} \qquad p(x) = p_1 - \frac{\Delta p}{L} x .$$

$$\left.\begin{array}{l} \dfrac{d^2u}{dy^2} = -\dfrac{\Delta p}{\mu L} \\ u(0) = u(1) = 0 \end{array}\right\} \quad u(y) = y(1-y) \frac{\Delta p}{\mu 2 L}$$



Impulsively moving plane

$t=0$ rest



ansatz

$$\left\{\begin{array}{l} \underset{\sim}{u} = \left(u(y,t), 0, 0\right) \\ \dfrac{\partial u}{\partial t} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu \dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial p}{\partial y} = \dfrac{\partial p}{\partial z} = 0 \qquad p = p(x,t). \end{array}\right.$$

~~constr rectilin~~ shear flow

satisfies continuity eqn

$$p = p(x) \quad \left\{\begin{array}{l} \left(\dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial y^2}\right)(y,t) = -\dfrac{1}{\rho}\dfrac{dp}{dx}(x,t). \\ \dfrac{dp}{dx} = \text{function of } t \text{ only} \end{array}\right.$$

Impulsively moving plane

$$\frac{dp}{dx} = 0 \qquad p = \text{const}.$$

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2} \\ u(y,0) = 0 \\ u(0,t) = U \end{array}\right.$$

problem invariant under scaling transformation

$$y \to \alpha y \qquad t \to \alpha^2 t$$

$$u(\alpha y, \alpha^2 t) = u(y,t) \qquad \forall \alpha$$

$$u(y,t) = f\left(\frac{y}{\sqrt{t}}\right)$$

n fact try $\quad u = f\left(\frac{y}{\sqrt{\nu t}}\right)$

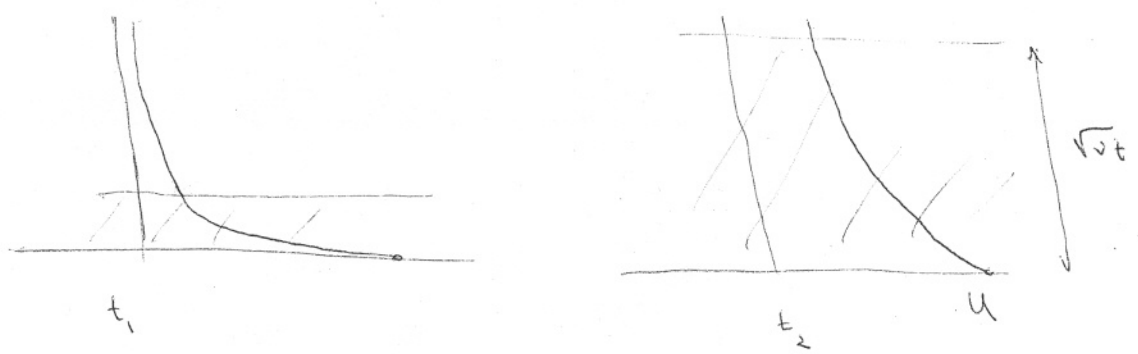
obtain $\begin{cases} f'' + \frac{1}{2}\xi f' = 0 \\ f(\infty) = 0, \quad f(0) = U \end{cases}$

$$f'(\xi) = B\, e^{-\frac{\xi^2}{4}}$$

$$f(\xi) = A + B\int_0^{\xi} e^{-\frac{s^2}{4}}\, ds$$

$$u = U\left[1 - \frac{1}{\sqrt{\pi}}\int_0^{\xi} e^{-\frac{s^2}{4}}\, ds\right] \qquad \xi = \frac{y}{\sqrt{\nu t}}$$

$$\omega = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi \nu t}}\, e^{-\frac{y^2}{4\nu t}}$$



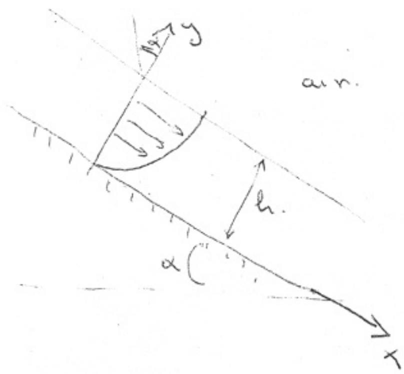
Region of significant vorticity $\sim \sqrt{\nu t}$

* Smoothing of an initial vortex sheet near boundary

Vorticity diffuses a distance of order $\sqrt{\nu t}$ in time t
Time taking vorticity to diffuse a dist. of order L
viscous diff. time $= O\left(\frac{L^2}{\nu}\right)$

adj :

Flow under gravity down an inclined plane



Ansatz $\quad u = (u(y), 0, 0)$.

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{d^2 u}{dy^2} + g \sin\alpha$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad - g \cos\alpha.$$

$$p = -\rho g y \cos\alpha + f(x).$$

$$\boxed{p = p_0 . \quad \mu \frac{du}{dy} = 0 \quad \text{at } y = h.}$$

p_0 atmospheric pressure.

no tangential force at boundary with air

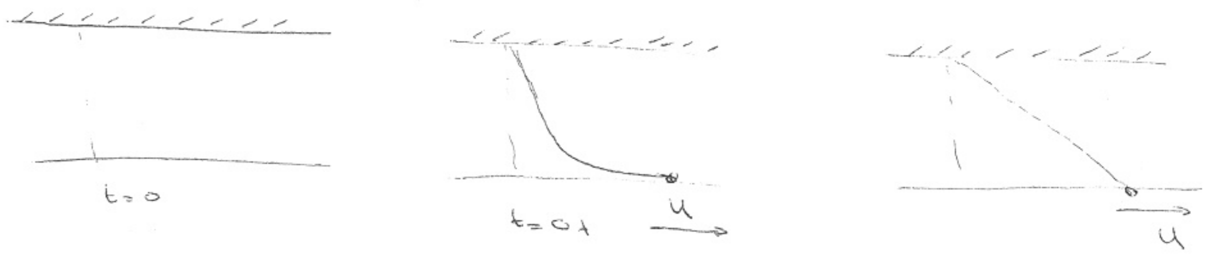
$$p - p_0 = \rho g (h - y) \cos\alpha.$$

$$\frac{\partial p}{\partial x} = 0.$$

$$\left.\begin{array}{l} \nu \dfrac{d^2 u}{dy^2} = -g \sin\alpha \\[2ex] u = 0 \quad \text{at } y = 0 \\[2ex] \mu \dfrac{du}{dy} = 0 \quad \text{at } y = h \end{array}\right\} \quad u = \frac{g}{2\nu}\, y\,(2h - y) \sin\alpha$$

Flux $\quad Q = \int_0^h u\,dy = \dfrac{g h^3}{3\nu} \sin\alpha.$

Impulse flow between two plates



$\underset{\sim}{u} = (u(y,t), 0, 0)$.

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2} \\ u(y,0) = 0 \\ u(0,t) = U \quad t>0 \\ u(h,t) = 0 \quad t>0 \end{array}\right.$$

$$u = U\left(1 - \frac{y}{h}\right) + u_1 \qquad \left\{\begin{array}{l} \dfrac{\partial u_1}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2} \\ u_1(y,0) = -U\left(1 - \dfrac{y}{h}\right). \\ u_1(0,t) = 0 \ , \ u_1(h,t) = 0. \end{array}\right.$$

$$u_1 = \sum_n A_n e^{-\left(\frac{n\pi}{h}\right)^2 \nu t} \sin\left(\frac{n\pi y}{h}\right)$$

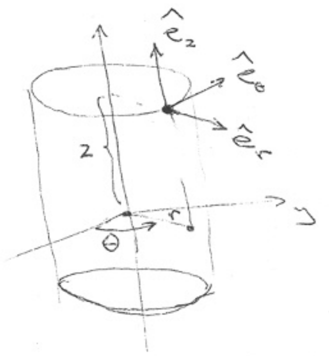
$$u(y,t) = U\left(1 - \frac{y}{h}\right) - \frac{2U}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\left(\frac{n\pi}{h}\right)^2 \nu t} \sin\left(\frac{n\pi y}{h}\right).$$

$u(y,t) \sim U\left(1 - \frac{y}{h}\right)$ for times $t \gtrsim \frac{h^2}{\nu}$. (first eigenvalue)

flow reaches a steady state

vorticity unif. distributed along flow

NAVIER-STOKES IN CYLINDRICAL COORDINATES



$$\underset{\sim}{u} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \qquad \frac{\partial \hat{e}_z}{\partial \theta} = 0$$

how that the N-S take the form.

$$\frac{\partial u_r}{\partial t} + (\underset{\sim}{u}\cdot\nabla) u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\frac{\partial u_\theta}{\partial t} + (\underset{\sim}{u}\cdot\nabla) u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\Delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

$$\frac{\partial u_z}{\partial t} + (\underset{\sim}{u}\cdot\nabla) u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\, \Delta u_z$$

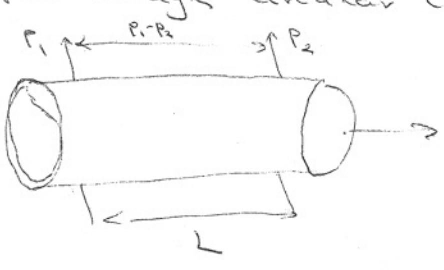
$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

where $(\underset{\sim}{u}\cdot\nabla) = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

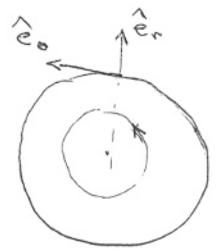
xercise 1 (a) Derive N-S equations in cylindrical coordinates

(b) Apply this to obtain the solution for ~~circular~~ Poiseuille flow through circular cross section



Flow through blood vessels

EQUATIONS OF CIRCULAR FLOWS



$\underset{\sim}{u} = u_\theta(r,t)\,\hat{e}_\theta$ in cylindrical coordinates

Circular flow – streamlines are circles in 2-D. (or 3-D)

-S equations become

continuity is automatically satisfied

irection $\qquad -\dfrac{u_\theta^2}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r}$

irection $\qquad \dfrac{\partial u_\theta}{\partial t} = -\dfrac{1}{\rho r}\dfrac{\partial p}{\partial \theta} + \nu\left(\dfrac{\partial^2 u_\theta}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r^2}\right)$ $\qquad$ 2-D or 3-D

irection $\qquad 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}$

$$u_\theta = u_\theta(r,t) \xRightarrow{2^{nd}\text{ eqn}} \frac{\partial p}{\partial \theta} = P(r,t)$$

$$\xRightarrow{3^{rd}\text{ eqn}} p = P(r,t)\,\theta + f(r,t)$$

$p(\theta=0) = p(\theta=2\pi)$ because of periodicity

We conclude $\quad p = f(r,t)$

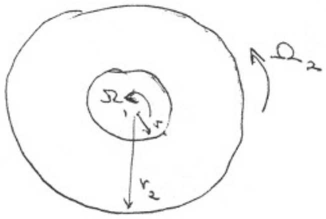
$$P(r,t) = 0$$

Thus

$$\boxed{\frac{\partial u_\theta}{\partial t} = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right)}$$

$$p = f(r,t).$$

.1 Steady flow between rotating cylinders



$$u_\theta = u_\theta(r)$$

$$r^2 \frac{d^2 u_\theta}{dr^2} + r \frac{du_\theta}{dr} - u_\theta = 0 \qquad \text{Euler-eqn}$$

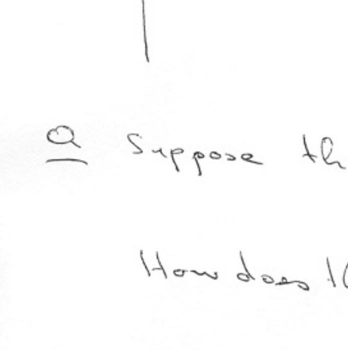
solns r^α.

$$u_\theta(r) = Ar + \frac{B}{r}$$

No slip condition
$$\begin{cases} u_\theta(r_1) = \Omega_1 r_1 \\ u_\theta(r_2) = \Omega_2 r_2 \end{cases}$$

$$\left.\begin{aligned} Ar_1 + \frac{B}{r_1} &= \Omega_1 r_1 \\ Ar_2 + \frac{B}{r_2} &= \Omega_2 r_2 \end{aligned}\right\} \longrightarrow \quad A = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2}, \quad B = \frac{\Omega_1 - \Omega_2}{\frac{1}{r_1^2} - \frac{1}{r_2^2}}$$

.2 Viscous decay of vortex line



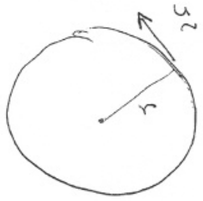
$$\underset{\sim}{u} = \frac{\Gamma_0}{2\pi r} \hat{e}_\theta \qquad \Gamma_0 \text{ constant}$$

$$\underset{\sim}{\xi} = \operatorname{curl} \underset{\sim}{u} = \begin{cases} 0 & r > 0 \\ \infty & r = 0 \end{cases}$$

<u>Q</u> Suppose that $\underset{\sim}{u}(x, 0) = \frac{\Gamma_0}{2\pi r} \hat{e}_\theta$

How does the vorticity diffuse?

Cross section



$\underset{\sim}{u} = u_\theta(r,t)\,\hat{e}_\theta$

Circulation of this velocity field on a circle of radius r

$$\Gamma(r,t) = 2\pi r\, u_\theta(r,t)$$

$$\frac{1}{\nu}\frac{\partial u_\theta}{\partial t} = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}$$

$$\frac{1}{\nu}\frac{\partial}{\partial t}\left(r u_\theta\right) = r\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

$$= \frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r} = \frac{\partial}{\partial r}\frac{\partial}{\partial r}\left(r u_\theta\right) - \frac{\partial}{\partial r}u_\theta - \frac{u_\theta}{r}$$

$$= \frac{\partial^2}{\partial r^2}\left(r u_\theta\right) - \frac{1}{r}\left(r\frac{\partial u_\theta}{\partial r} + u_\theta\right) = \frac{\partial^2}{\partial r^2}\left(u_\theta r\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r u_\theta\right)$$

Can

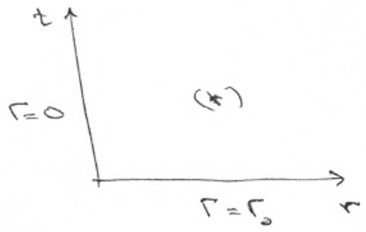
$$\frac{\partial \Gamma}{\partial t} = \nu\left(\frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r}\frac{\partial \Gamma}{\partial r}\right)$$

Γ becomes the dep. variable

tial condition $\Gamma(r,0) = \Gamma_0$, $r > 0$

$|u_\theta|$ finite at any $t > 0$ $\Longrightarrow$ $\Gamma(0,t) = 0$, $t > 0$

olve



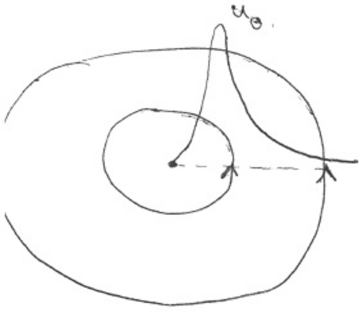
Use similarity methods.

$$\Gamma(r,t) = f\left(\frac{r}{\sqrt{\nu t}}\right)$$

Obtain

$$\Gamma(r,t) = \Gamma_0 \left(1 - e^{-\frac{r^2}{4\nu t}}\right)$$

$$u_\theta(r,t) = \frac{\Gamma_0}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right)$$



$r \ll \sqrt{4\nu t}$ $\qquad u_\theta(r,t) \sim \frac{\Gamma_0}{2\pi r}\left(+\frac{r^2}{4\nu t}\right) = \frac{\Gamma_0}{8\pi\nu t}\, r$

Note $\quad u_\theta(r,t) \sim \omega_0 r \qquad$ not irrotational flow

$$\omega_0 = \frac{\Gamma_0}{8\pi\nu t}$$

$r \gg \sqrt{4\nu t}$ $\qquad u_\theta(r,t) \sim \frac{\Gamma_0}{2\pi r} \qquad$ irrotational flow