4 The gradient of a scalar field

In this chapter we consider vector fields which are derived from scalar fields. This is a concept which has wide application in many branches of mathematical physics, and its usefulness is almost obvious. A scalar field requires a knowledge only of one scalar function $\phi(x_1, x_2, x_3)$ say, but in the case of a vector field we require information about three components of the field. It is much simpler therefore if we can deduce a vector field, which is physically useful, from a scalar field, as is the case for example in electrostatic theory. In that theory a knowledge of the electrostatic potential ϕ determines completely the vector field of the electric intensity **E**. We describe now the construction of such a vector field from the basic scalar field.

4.1 The construction of grad ϕ

The given scalar field ϕ might be specified as a function of the Cartesian coordinates x_i, or as a function of some other system of coordinates, but for the moment it is convenient to think independently of particular coordinates. Construct the level surfaces of ϕ, and consider the particular level surface $\phi = \phi(P)$ which passes through a generic point P of space. At P draw a unit normal $\mathbf{n}$ to this surface; we remark that we could specify the sense of this normal, but it is not essential to do this. We now define a vector at P, called the gradient of ϕ at P, or $(\text{grad } \phi)_P$, as follows:

$$(\text{grad } \phi)_P = \alpha \mathbf{n},$$

where $\alpha =$ Rate of increase of ϕ with respect to distance measured in the direction of $\mathbf{n}$, evaluated at P. (4.1)

This construction yields a vector at a typical point P, and the same process at all points of space gives the vector field grad ϕ.

4.2 Cartesian components of grad ϕ

Suppose that the scalar field is given in the form $\phi(x_1, x_2, x_3)$. If P is the point with coordinates x_i', then the level surface of ϕ through P is

$$\phi(x_1, x_2, x_3) = \phi(x_1', x_2', x_3'),$$

and a unit normal $\mathbf{n}$ to this surface at P is

$$\mathbf{n} = \left\{\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_i}\right\}^{-\frac{1}{2}}\left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}\right), \tag{1}$$

evaluated at P, using the summation convention in the multiplier.

Now consider a small displacement dx_i along this normal, so that

$$dx_i = \lambda\left(\frac{\partial \phi}{\partial x_i}\right)_P, \tag{2}$$

where λ is small and positive. The change in ϕ, $d\phi$, from its value at P by virtue of this displacement is

$$d\phi = \left(\frac{\partial \phi}{\partial x_i} dx_i\right)_P,$$

using a Taylor series expansion and retaining only the first order terms. Thus, using (2)

$$d\phi = \lambda\left\{\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_i}\right\}_P.$$

The distance moved along the normal

$$= \{dx_i\, dx_i\}^{\frac{1}{2}}$$

$$= \lambda\left\{\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_i}\right\}_P^{\frac{1}{2}},$$

since λ is positive.

Hence the rate of change of ϕ with respect to distance measured along $\mathbf{n}$

$$= \left\{\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_i}\right\}_P^{\frac{1}{2}} = \mu, \quad \text{say}, \tag{3}$$

the limit here being simply a quotient since we have retained first order terms only for $d\phi$. It appears here that μ is positive, so evidently the original choice for the sense of $\mathbf{n}$ has been taken in the direction of increasing ϕ.

In all, we now have

$$(\text{grad } \phi)_P = \mu\mathbf{n} = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}\right).$$

Thus, at any point x_i in space the vector field grad ϕ, in terms of Cartesian components, is

$$\text{grad } \phi = \frac{\partial \phi}{\partial x_i}\mathbf{a}_i. \tag{4.2}$$

4.3 The component of grad ϕ in a given direction

At any point P, coordinates x_i, consider an arbitrary unit vector $\mathbf{b}$. If ds is an element of distance measured along a line from P to P' in the direction of $\mathbf{b}$, then the Cartesian coordinates of P' are

$$x_1 + \alpha\, ds, \quad x_2 + \beta\, ds, \quad x_3 + \gamma\, ds,$$

where (α, β, γ) are the direction cosines of $\mathbf{b}$. Thus

$$\phi(P') - \phi(P) = \frac{\partial \phi}{\partial x_1}\alpha\, ds + \frac{\partial \phi}{\partial x_2}\beta\, ds + \frac{\partial \phi}{\partial x_3}\gamma\, ds + 0(ds^2),$$

using the Taylor expansion for $\phi(P')$, and $0(ds^2)$ to indicate the terms of second order and above in the small element ds.

Hence

$$\lim_{ds \to 0} \frac{\phi(P') - \phi(P)}{ds} = \alpha\frac{\partial \phi}{\partial x_1} + \beta\frac{\partial \phi}{\partial x_2} + \gamma\frac{\partial \phi}{\partial x_3}$$
$$= \mathbf{b} \cdot \text{grad } \phi.$$

This last result, reading backwards, can be stated:

The component of grad ϕ in a given direction = Rate of change of ϕ with respect to distance measured in that direction. (4.3)

This statement provides immediately the method of calculating grad ϕ in any system of orthogonal curvilinear coordinates. For all that is required is the application of the result in each of three directions at right angles, followed by multiplication by the appropriate unit vectors and vector addition of the resulting expressions.

The calculation is easily performed. Consider an orthogonal curvilinear system ξ_i, and let $\mathbf{e}_i$ be the appropriate unit vectors along the local axes at P (see Section 2.4). Then, with reference to Fig. 4.1, if

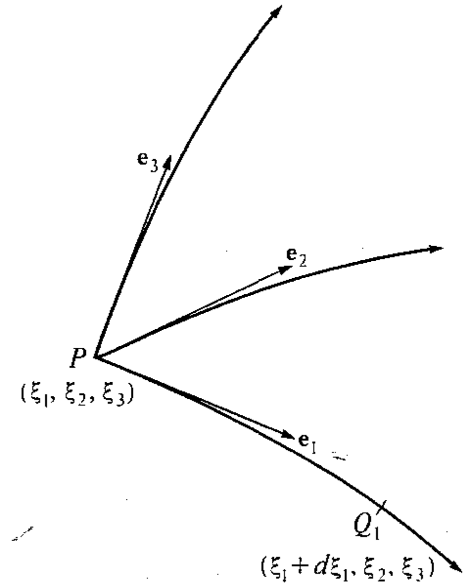

Fig. 4.1 Diagram for the calculation of grad ϕ in orthogonal curvilinear coordinates, Section 4.3.

Q_1 is the point $(\xi_1 + d\xi_1, \xi_2, \xi_3)$, and thus lies on the local axis of $\mathbf{e}_1$ at P, we have:

Change in ϕ in the $\mathbf{e}_1$ direction

$$= \phi(Q_1) - \phi(P) = \frac{\partial \phi}{\partial \xi_1} d\xi_1,$$

correct to the first order of small quantities, since ξ_2 and ξ_3 remain constant.

The distance $PQ_1 = h_1\, d\xi_1$, where h_1 is the appropriate scale factor.

$\therefore$ Rate of change of ϕ with respect to distance measured in the direction $\mathbf{e}_1 = \dfrac{1}{h_1}\dfrac{\partial \phi}{\partial \xi_1}$.

A similar calculation follows for the directions $\mathbf{e}_2$, $\mathbf{e}_3$, and hence, since the system is orthogonal,

$$\text{grad } \phi = \frac{1}{h_1}\frac{\partial \phi}{\partial \xi_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial \xi_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial \xi_3}\mathbf{e}_3. \tag{4.4}$$

grad φ = 1/― ∂φ e₁ + 1/― ∂φ e₂ + 1/― ∂φ e

4.4 Single valued scalar fields

So far we have implied that the scalar fields ϕ which we have discussed are single valued functions of position; that is, for any point P in space, the value of ϕ at P is unique.

It is a simple matter to postulate fields which do not satisfy this requirement. For example, if (r, θ) are plane polar coordinates, and $\phi = k\theta$, then since a point P in the plane is represented equally well by the coordinates (r, θ) or $(r, \theta+2n\pi)$, where n is any integer, then clearly at P, ϕ takes the multiplicity of values $k(\theta+2n\pi)$. Situations of this type do arise in applications, but for the moment we shall restrict attention to scalar fields which are single valued functions of position.

In this context there is a basic theorem which we now prove.

Theorem and converse

(i) *Statement.* If a vector field **F** is the gradient of a single valued scalar field ϕ in a region of space, then the line integral of **F** round any closed path C in the region is zero. (4.5)

(ii) *Statement* (Converse). If a vector field **F** is such that its line integral round any closed path C in a region of space is zero, then **F** is expressible as the gradient of a single valued scalar field ϕ in the region. (4.6)

(i) PROOF First let C be any path in the region, with terminal points A, B. If s is the arc length measured along C, we have:

$$\text{Line integral of } \mathbf{F} \text{ along } C = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \text{grad}\,\phi\cdot d\mathbf{r}$$

$$= \int_A^B \frac{\partial\phi}{ds}\,ds \quad \text{(from result (4.3))}$$

$$= \phi(B)-\phi(A).$$

Now close the path C, so that B coincides with A. Then since ϕ is single valued, $\phi(B) = \phi(A)$, and hence

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0.$$

(ii) PROOF With reference to Figs. 4.2 and 4.3, let O be an origin in the region, and P a generic point. Consider two distinct paths C_1, C_2

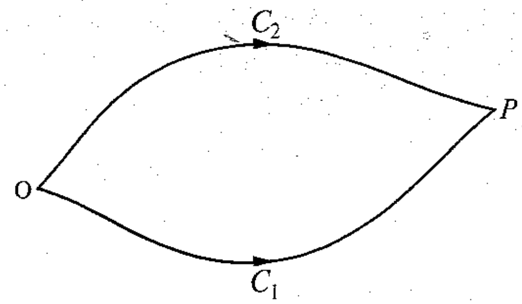

Fig. 4.2 Diagram for theorem (4.6).

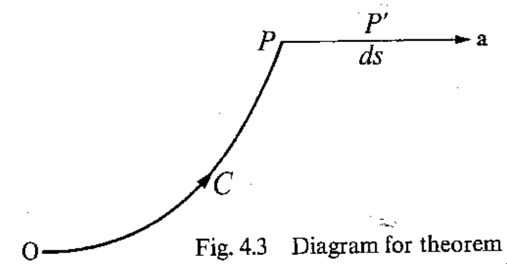

Fig. 4.3 Diagram for theorem (4.6).

joining O to P. Then the path $C_3 = C_1 - C_2$ is a closed path in the region, and hence from the data of (ii),

$$\oint_{C_3} \mathbf{F}\cdot d\mathbf{r} = 0,$$

$$\text{i.e.} \quad \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}.$$

This means that the line integral of **F** along any path joining O and P is independent of the path, and so can only be some single valued function of position $\phi(P)$, regarding O as a fixed origin. Thus, if C is any path joining O and P,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \phi(P).$$

Now consider an arbitrary unit vector **a** at P, and let P' be a point such that $\overline{PP'} = \mathbf{a}\,ds$, where the length element ds is small. Then $\phi(P')$ can be calculated by forming the line integral of **F** along the original path C plus the short section PP'.

Hence

$$\phi(P')-\phi(P) = \int_P^{P'} \mathbf{F}\cdot d\mathbf{r} = \mathbf{F}\cdot\mathbf{a}\,ds$$

to the first order in ds, where $\mathbf{F}$ is the vector field at P.

Now

$$\frac{\phi(P')-\phi(P)}{ds} \to \operatorname{grad}\phi\cdot\mathbf{a} \quad \text{as } ds \to 0,$$

from result (4.3). Hence dividing by ds in the above relation, and proceeding to the limit as $ds \to 0$, we have

$$(\mathbf{F}-\operatorname{grad}\phi)\cdot\mathbf{a} = 0,$$

and therefore, since $\mathbf{a}$ is arbitrary,

$$\mathbf{F} = \operatorname{grad}\phi,$$

which establishes theorem (4.6).

A vector field $\mathbf{F}$ which satisfies the condition of theorem (4.6) is described as a conservative field, associated with the fact that if $\mathbf{F}$ is a force field then no work is done in total by $\mathbf{F}$ in a circuit of a closed path C. Expressed in equivalent form, if $\mathbf{F}$ is a conservative field, the line integral of $\mathbf{F}$ along a path connecting two points is independent of the choice of path between these points. This explains the reason for the results of example (3.2). In this particular case,

$$\mathbf{F} = (2x_1x_3, 3x_2^2, x_1^2),$$

and it is clear that $\mathbf{F}$ is expressible in the form

$$\mathbf{F} = \operatorname{grad}\phi,$$

with

$$\phi = x_1^2x_3+x_2^3+c,$$

where c is an arbitrary constant. This constant is determined explicitly, of course, if the value of ϕ is prescribed at some point.

If it is known that $\mathbf{F}$ is a conservative field, it is possible that the associated scalar field ϕ might be constructed by evaluation of the line integral

$$\phi(P)-\phi(O) = \int_C \mathbf{F}\cdot d\mathbf{r},$$

where C is some path connecting a base point O to the current point P in the field. This is illustrated by the following example:

Example 4.1

A system of oblate spheroidal coordinates (ξ, η, α) is defined by

$$x_1 = a\cosh\xi\cos\eta\cos\alpha,$$
$$x_2 = a\cosh\xi\cos\eta\sin\alpha,$$
$$x_3 = a\sinh\xi\sin\eta,$$

where (x_1, x_2, x_3) are Cartesian coordinates. A vector field $\mathbf{F}$ is the gradient of a scalar field ϕ, and has components F_ξ, F_η, F_α appropriate to the above system. On the curve $\xi = 0, \eta = 0$,

$$F_\alpha = 2\cos\alpha;$$

on the curves $\alpha = \text{const.}, \eta = 0$,

$$F_\xi = 3\sinh 2\xi\sin\alpha,$$

and on the curves $\alpha = \text{const.}, \xi = \text{const.}$,

$$F_\eta = 3\sin 2\eta\sin\alpha.$$

Determine ϕ by the evaluation of a line integral, given also that $\phi = 0$ when $\xi = \eta = \alpha = 0$.

The student will easily verify that the system (ξ, η, α) is orthogonal, and that the appropriate scale factors are

$$h_\xi = h_\eta = a\{\sinh^2\xi+\sin^2\eta\}^{\frac{1}{2}},$$
$$h_\alpha = a\cosh\xi\cos\eta.$$

If we denote, for the moment, the curvilinear coordinates of the point P in the field by $(\xi_P, \eta_P, \alpha_P)$, the path of integration between the origin O and P can be constructed from arcs of three coordinate curves along each of which a pair of the coordinates ξ, η, α take constant values in turn. Moreover, since we have an orthogonal system, just one component of $\mathbf{F}$ appears in the line integral at each stage. Thus, recalling that the scale factors appear in the expressions for displacement, we have:

$$\begin{aligned}\phi(\xi_P, \eta_P, \alpha_\alpha) &= \int_O^P \mathbf{F}\cdot d\mathbf{r}\\ &= \int_{\xi=\eta=\alpha=0}^{\alpha=\alpha_P} F_\alpha h_\alpha\, d\alpha + \int_{\substack{\alpha=\alpha_P\\ \xi=\eta=0}}^{\xi=\xi_P} F_\xi h_\xi\, d\xi + \int_{\substack{\alpha=\alpha_P\\ \xi=\xi_P\\ \eta=0}}^{\eta=\eta_P} F_\eta h_\eta\, d\eta\\ &= \int_0^{\alpha_P} 2a\cos\alpha\, d\alpha + \int_0^{\xi_P} a\sinh\xi\, 3\sinh 2\xi\sin\alpha_P\, d\xi\\ &\quad + \int_0^{\eta_P} a\{\sinh^2\xi_P+\sin^2\eta\}^{\frac{1}{2}} 3\sin 2\eta\sin\alpha_P\, d\eta\end{aligned}$$

$$= 2a \sin \alpha_P + 2a \sin \alpha_P \sinh^3 \xi_P$$
$$+2a \sin \alpha_P [\{\sinh^2 \xi_P + \sin^2 \eta\}^{3/2}]_0^{\eta_P}$$
$$= 2a \sin \alpha_P + 2a \sin \alpha_P \{\sinh^2 \xi_P + \sin^2 \eta_P\}^{3/2}.$$

Hence, in terms of current coordinates for P,

$$\phi(\xi, \eta, \alpha) = 2a \sin \alpha \{(\sinh^2 \xi + \sin^2 \eta)^{3/2} + 1\}.$$

Examples, Chapter 4

1. The vector field **F** with Cartesian components

$$F_1 = 2x_1x_2^3x_3 + x_2 \cos(x_1x_2),$$
$$F_2 = 3x_1^2x_2^2x_3 + x_1 \cos(x_1x_2),$$
$$F_3 = x_1^2x_2^3,$$

is such that $\int \mathbf{F} \cdot d\mathbf{r}$ is invariant for any curve joining any two given points in the field. By evaluating the integral along the straight line path

$$x_i = tx_i'$$

joining the origin and the point x_i', find a scalar field ϕ for which $\mathbf{F} = \text{grad}\, \phi$.

Answer: $\phi = x_1^2x_2^3x_3 + \sin(x_1x_2)$.

2. A vector field **F** with components F_i referred to a system of spherical polar coordinates (r, θ, α) is such that $\mathbf{F} = \text{grad}\, \phi$, with $\phi = 0$ when $r = 0$. Calculate ϕ at the point (r, θ, α) by evaluating a line integral taken along a suitable composite path, given that

$$F_2 = r \sin 2\theta\, (a \cos \alpha - r),$$

and, on the axis $\theta = 0$, $F_1 = 3r^2$.

Answer: $\phi = ar^2 \sin^2 \theta \cos \alpha + r^3 \cos^2 \theta$.

3. If (r, θ, z) are cylindrical polar coordinates, and

$$\phi_1 = r^2 (\cos \theta - \tfrac{1}{2}) + z^2,$$
$$\phi_2 = r^2 z g(\theta),$$

determine the function $g(\theta)$ if the level surfaces of ϕ_1 and ϕ_2 intersect orthogonally.

Answer: $g = A \sin^4 \theta$.

4. If **r** is the radius vector of a point with respect to an origin of Cartesian coordinates x_i, unit vectors $\mathbf{a}_i$, and **A** is a constant vector field, obtain the gradients of the following scalar fields:
(i) $\mathbf{A} \cdot \mathbf{r}$, (ii) r^n, (iii) $f(r)$, (iv) $\mathbf{r} \cdot \text{grad}\,(x_1 + x_2 + x_3)$, (v) $\frac{1}{2} \log (\mathbf{A} \times \mathbf{r})^2$.

Answers: (i) **A**, (ii) $nr^{n-2}\mathbf{r}$, (iii) $\dfrac{f'(r)}{r}\mathbf{r}$, (iv) $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$,

(v) $\dfrac{1}{(\mathbf{A} \times \mathbf{r})^2} \mathbf{A} \times (\mathbf{r} \times \mathbf{A})$.

$\mathbf{r} \cdot \text{grad}(x_1 + x_2 + x_3) = x_1 + x_2 + x_3$

5. If ϕ_1 and ϕ_2 are two scalar fields, show that

$$\text{grad}\,(\phi_1\phi_2) = \phi_2 \,\text{grad}\, \phi_1 + \phi_1 \,\text{grad}\, \phi_2.$$

Extend this result for a product of several scalar fields.

$\text{grad}(\varphi_1\varphi_2\ldots\varphi_v) = (\varphi_2\ldots\varphi_v)\,\text{grad}\,\varphi_1 + \ldots + (\varphi_1\ldots\varphi_{v-1})\,\text{grad}\,\varphi_v$

6. If **r** is the position vector of a point in curvilinear coordinates ξ_i, show that

$$\frac{\partial \mathbf{r}}{\partial \xi_i} \cdot \text{grad}\, \xi_j = \delta_{ij} \qquad (i, j = 1, 2, 3),$$

where δ_{ij} is the Krönecker delta.

5 The divergence of a vector field

In contrast to the subject of the previous chapter, we next consider scalar fields which are derived from vector fields. That is, given a vector field **F**, we construct a scalar field described as the *divergence of* **F**, or for brevity, div **F**.

5.1 Definition

The formal definition of this scalar field is a simple matter. Given a vector field **F** throughout a region of space, then with reference to Fig. 5.1, let P be any point of this region. Surround P by a small closed

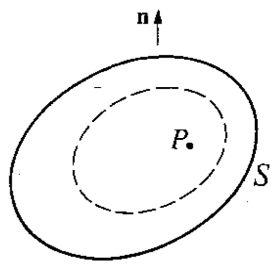

Fig. 5.1 Diagram for the definition of div **F** (Section 5.1). P is the point at which div **F** is to be evaluated, and the closed surface S shrinks to zero, condensing on P.

surface S, enclosing a volume V (in which, of course, P lies), and calculate the flux of **F** across the surface S, namely

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

with the usual convention that the unit normal **n** to S shall be drawn outwards from V. Now form the ratio

$$\frac{\int_S \mathbf{F} \cdot d\mathbf{S}}{V}$$

and consider the limit of this last quantity as S and V both shrink to zero, condensing on P. With this implication for the limiting process, we now define div **F** at P as the limit

$$(\text{div } \mathbf{F})_P = \lim \left\{ \frac{\int_S \mathbf{F} \cdot d\mathbf{S}}{V} \right\}, \tag{5.1}$$

and this is evidently a scalar quantity.

This process can be performed for all points in the region, and so the entire scalar field div **F** can be constructed. Once again we have made no appeal to any special system of coordinates in framing the definition above, but the reader will note that there are mathematical points which require discussion before the definition (5.1) can be accepted as satisfactory. These points can be expressed easily in physical terms; in the first place we note that no particular shape has been specified for the surface S; for example, a rectangular box could be selected, but equally well, some other shape such as a sphere. Secondly, the actual location of P in relation to S is not specified; all that is required is that S should shrink to zero about P. There is no reason to suppose at first sight that the limiting process involved in (5.1) is independent of considerations such as these; thus the result might well depend on the particular choice of the shape of S and the actual location of P in relation to S. Fortunately, there are theorems which enable us to assert the contrary, provided we accept minor restrictions on the original vector field **F**. If F_1, F_2, F_3 are Cartesian components of **F**, then by Gauss's theorem (3.25) we have

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) d\tau.$$

The integrand in the volume integral is supposed to be a continuous function of x_1, x_2 and x_3, and in this case we can apply also a mean value theorem to the integral, which asserts

$$\int_V \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) d\tau = V \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right)_{P'},$$

where P' is some point in V, and the expression in brackets is evaluated at P'. Thus, we have

$$\frac{\int_S \mathbf{F}\cdot d\mathbf{S}}{V} = \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}\right)_{P'}.$$

If we now carry out the limiting process implied in (5.1), then as S shrinks to zero (condensing on P), we must have $P' \to P$, and since the function concerned is continuous, the value of the limit is

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

evaluated at P.

This establishes the validity of the definition, and incidentally gives the value of div **F** in terms of Cartesian coordinates and components of **F**, namely

$$\text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}. \tag{5.2}$$

Finally, Gauss's theorem (3.25) can now be stated more concisely, and reads:

$$\int_V \text{div}\,\mathbf{F}\,d\tau = \int_S \mathbf{F}\cdot d\mathbf{S}. \tag{5.3}$$

5.2 The expression for div F in terms of orthogonal curvilinear coordinates

Although the limiting process described in the previous section constitutes the basic definition of div **F**, it does not follow that it is necessary to carry out this process in detail each time such a calculation is required. We can perform the operation just once for the case of orthogonal curvilinear coordinates, and so obtain a general result which will be available for a wide range of applications.

With reference to Fig. 5.2, P is the point $(\xi_{1P}, \xi_{2P}, \xi_{3P})$ at which div **F** is to be calculated; for the moment we introduce the suffix P to distinguish values of the various quantities at the point P itself from their current values elsewhere in the region. The surface S is the small curvilinear box $A\,D\,H\,G\,C\,K\,B\,E$, formed by the six level surfaces of the coordinates $\xi_{iP} \pm \delta\xi_i$, where the $\delta\xi_i$ are small. Thus, for example, A is the point $(\xi_{1P}+\delta\xi_1, \xi_{2P}-\delta\xi_2, \xi_{3P}-\delta\xi_3)$. In a sense, therefore, P is the 'centre' of the box, the sides of which are of lengths

$$EA = 2h_{1P}\,\delta\xi_1,$$
$$AD = 2h_{2P}\,\delta\xi_2,$$
$$AG = 2h_{3P}\,\delta\xi_3.$$

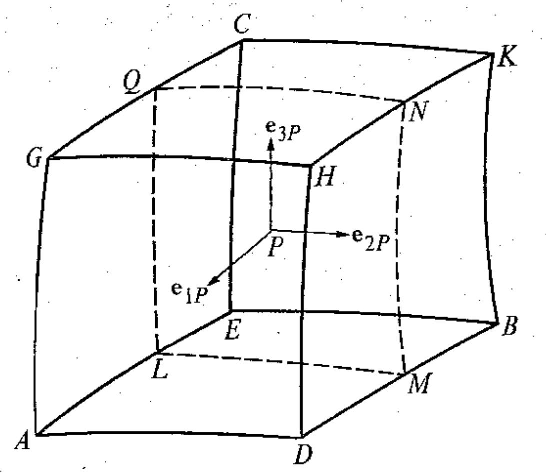

Fig. 5.2 Diagram for the calculation of div **F** in orthogonal curvilinear coordinates.

The unit vectors in the directions of the coordinate axes at P are $\mathbf{e}_{1P}, \mathbf{e}_{2P}, \mathbf{e}_{3P}$, and F_{1P}, F_{2P}, F_{3P} are the appropriate components of the vector field **F** at P.

In the calculation of the flux of **F** across S, some care must be exercised because the areas of the six faces of S are not necessarily equal in pairs. The pair of faces $A\,D\,H\,G$, $E\,B\,K\,C$, for example, are not in general equal in area. However, the flux of **F** across these faces can be deduced by the simple expedient of first calculating this quantity for the level surface $\xi_1 = \xi_{1P}$ (i.e. for $L\,M\,N\,Q$), and then obtaining the fluxes for the two faces mentioned by changing ξ_{1P} to $\xi_{1P}+\delta\xi_1$, and to $\xi_{1P}-\delta\xi_1$ respectively.

Denote the flux of **F** across the surface $L\,M\,N\,Q$ by X. Since this surface is a level surface of ξ_1, the only component of **F** which contributes to X is F_1. Thus

$$X = \int\int F_1 h_2 h_3\,d\xi_2\,d\xi_3,$$

where, in the integrand, ξ_1 has the constant value ξ_{1P}. Now the changes in ξ_2 and ξ_3 over this surface are small, so that we can use a Taylor series expansion for the integrand. Thus, with the substitutions

$$\xi_2 = \xi_{2P}+\alpha,$$
$$\xi_3 = \xi_{3P}+\beta,$$

$$X = \int_{\beta=-\delta\xi_3}^{\delta\xi_3}\int_{\alpha=-\delta\xi_2}^{\delta\xi_2}\left\{(F_1h_2h_3)_P + \left(\frac{\partial}{\partial\xi_2}(F_1h_2h_3)\right)_P\alpha + \left(\frac{\partial}{\partial\xi_3}(F_1h_2h_3)\right)_P\beta + \cdots\right\}d\alpha\, d\beta \quad (1)$$

in which only the first order terms of the Taylor series expansion for $F_1h_2h_3$ are shown. In fact the second and third terms above evidently give no contribution on integration, and the result for X is thus

$$X = 4(F_1h_2h_3)_P\,\delta\xi_2\,\delta\xi_3+0(\delta\xi_2^n\,\delta\xi_3^m), \quad (2)$$

where n, m are integers $\geqslant 2$. That is, the higher terms in (2) are at best of the fourth order of small quantities.

The flux across the faces $A\,D\,H\,G$, $E\,B\,K\,C$ of the box can be deduced now, since the change

from $L\,M\,N\,Q$ to $A\,D\,H\,G$ is due to the increment $+\delta\xi_1$ in ξ_{1P},

and

from $L\,M\,N\,Q$ to $E\,B\,K\,C$ is due to the increment $-\delta\xi_1$ in ξ_{1P}.

Thus the flux of F out of the box across $A\,D\,H\,G$

$$4F_1h_2h_3\,\delta\xi_2\,\delta\xi_3+4\,\delta\xi_1\,\delta\xi_2\,\delta\xi_3\left\{\frac{\partial}{\partial\xi_1}(F_1h_2h_3)+\text{higher order terms}\right\},$$

and the flux of $\mathbf{F}$ out of the box across $E\,B\,K\,C$ (here remember that a negative sign must be introduced to correspond with the correct sense of the normal)

$$= -4F_1h_2h_3\,\delta\xi_2\,\delta\xi_3 + 4\,\delta\xi_1\,\delta\xi_2\,\delta\xi_3\left\{\frac{\partial}{\partial\xi_1}(F_1h_2h_3)+\text{higher order terms}\right\}.$$

In these last results all quantities are evaluated at P, the coordinates of which are now set simply as (ξ_1, ξ_2, ξ_3).

Hence the total flux out of the box due to the above pair of faces is

$$8\,\delta\xi_1\,\delta\xi_2\,\delta\xi_3\,\frac{\partial}{\partial\xi_1}(F_1h_2h_3),$$

correct to terms of the third order of small quantities. Similarly, the contribution from the pair of faces $D\,B\,K\,H$, $A\,E\,C\,G$

$$= 8\,\delta\xi_1\,\delta\xi_2\,\delta\xi_3\,\frac{\partial}{\partial\xi_2}(F_2h_1h_3),$$

and that from the pair of faces $G\,H\,K\,C$, $A\,D\,B\,E$

$$= 8\,\delta\xi_1\,\delta\xi_2\,\delta\xi_3\,\frac{\partial}{\partial\xi_3}(F_3h_1h_2).$$

Also, correct to terms of the third order of small quantities, the volume V of the box is

$$V = 8h_1h_2h_3\,\delta\xi_1\,\delta\xi_2\,\delta\xi_3.$$

Hence applying the limiting process of Section 5.1 (which now becomes a simple ratio, since we have already shed the terms which vanish in the limit) we have, at P,

$$\operatorname{div}\mathbf{F} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial\xi_1}(F_1h_2h_3) + \frac{\partial}{\partial\xi_2}(F_2h_1h_3) + \frac{\partial}{\partial\xi_3}(F_3h_1h_2)\right\}. \quad (5.4)$$

5.3 The Laplacian, and Laplace's equation

If we have a scalar field ϕ, then the vector field grad ϕ can be constructed. Since grad ϕ is a vector field, it follows therefore that the scalar field div (grad ϕ) can be constructed. This last quantity, the *Laplacian of* ϕ, is of considerable importance in applications, as will be seen later. For the moment, it is simply useful to note the expression of this scalar field in terms of orthogonal curvilinear coordinates. Thus, from result (4.4),

$$\operatorname{grad}\phi = \left(\frac{1}{h_1}\frac{\partial\phi}{\partial\xi_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial\xi_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial\xi_3}\right),$$

and hence from result (5.4),

$$\operatorname{div}(\operatorname{grad}\phi) = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial\xi_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial\xi_1}\right) + \frac{\partial}{\partial\xi_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial\xi_2}\right) + \frac{\partial}{\partial\xi_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial\xi_3}\right)\right\}. \quad (5.5)$$

In the case of rectangular Cartesian coordinates (x_1, x_2, x_3), this result is simply

$$\operatorname{div}(\operatorname{grad}\phi) = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}. \quad (5.6)$$

Incidentally, the interpretation of the Laplacian (5.6) in terms of another system of orthogonal curvilinear coordinates is required frequently; result (5.5) is the desired expression.

Further, if

$$\text{div}\,(\text{grad}\,\phi) = 0 \tag{5.7}$$

everywhere throughout a region of space, the original scalar field ϕ is a solution of a certain partial differential equation, namely equation (5.7), and this is *Laplace's equation.* In terms of the coordinates (ξ_1, ξ_2, ξ_3) this equation is clearly

$$\frac{\partial}{\partial \xi_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \phi}{\partial \xi_2}\right) + \frac{\partial}{\partial \xi_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial \xi_3}\right) = 0, \tag{5.8}$$

after removing the factor $(h_1 h_2 h_3)^{-1}$. When ϕ is a solution of Laplace's equation, ϕ is said to be a *harmonic function.*

More generally, there are scalar fields which satisfy the partial differential equation

$$\text{div}\,(\text{grad}\,\phi) = \rho, \tag{5.9}$$

where ρ is a prescribed scalar function of the coordinates. Equation (5.9) is *Poisson's equation,* and Laplace's equation is evidently a special (but important) case of this, namely the case $\rho \equiv 0$. It is clear that the expression of this equation corresponding to equation (5.8) is obtained by replacing the zero on the right hand side of (5.8) by the term $h_1 h_2 h_3 \rho$.

5.4 An expansion formula, and Green's formulae

If $\mathbf{F}$ is a vector field, and ϕ a scalar field, then $\phi\mathbf{F}$ is a vector field, and its divergence can be calculated. There is an expansion formula in this context, which has very wide application, and the formula is

$$\text{div}\,(\phi\mathbf{F}) = \text{grad}\,\phi \cdot \mathbf{F} + \phi\,\text{div}\,\mathbf{F}. \tag{5.10}$$

The proof is simple, and is left as an exercise to the reader (use Cartesian components of $\mathbf{F}$ to establish the result).

This formula, applied in Gauss's theorem, leads directly to Green's formulae. Let ϕ and ψ be two scalar fields defined throughout a region of space V, and on a closed surface S enclosing V. The two formulae are then

$$\int_V \{\phi\,\text{div}\,(\text{grad}\,\psi) + \text{grad}\,\phi \cdot \text{grad}\,\psi\}\,d\tau = \int_S \phi\,\text{grad}\,\psi \cdot d\mathbf{S}, \tag{5.11}$$

and

$$\int_V \{\phi\,\text{div}\,(\text{grad}\,\psi) - \psi\,\text{div}\,(\text{grad}\,\phi)\}\,d\tau = \int_S \{\phi\,\text{grad}\,\psi - \psi\,\text{grad}\,\phi\} \cdot d\mathbf{S}. \tag{5.12}$$

The proof of result (5.11) comes directly from the identity (5.10). For, by forming the volume integral, we have

$$\int_V \{\text{grad}\,\phi \cdot \mathbf{F} + \phi\,\text{div}\,\mathbf{F}\}\,d\tau = \int_V \text{div}\,(\phi\mathbf{F})\,d\tau = \int_S \phi\mathbf{F} \cdot d\mathbf{S},$$

from result (5.3). Now put $\mathbf{F} = \text{grad}\,\psi$ to deduce the formula (5.11). Further, the second formula (5.12) follows by interchanging the roles of ϕ and ψ in the formula (5.11), and then by subtraction from (5.11) itself.

These results are of considerable importance in mathematical physics. They may be applied to establish the uniqueness of solution of certain partial differential equations subject to given boundary conditions. They also lead to the development of Green's functions as a device for deriving the solutions of certain partial differential equations in the form of integrals, but this is essentially outside the range of this book.

5.5 The expression of a scalar field ϕ in terms of a volume and surface integral, when ϕ satisfies Poisson's equation

The intention of this section is to provide a result which is required in the theory of the next chapter. We consider Poisson's equation, namely

$$\text{div}\,(\text{grad}\,\phi) = \rho, \tag{5.13}$$

where ρ is a prescribed scalar field in a region V of space enclosed by a closed surface S, and it will be shown how the scalar field ϕ can be determined within S when certain boundary conditions on S are known.

With reference to Fig. 5.3, let P be a point within S, and let r be distance measured from P of a variable point P' in the region. It is a simple matter to verify that $1/r$ is a solution of Laplace's equation (5.7). This can be seen, for example, by choosing Cartesian coordinates (X_1, X_2, X_3) for P (regarded here as a fixed point), and (x_1, x_2, x_3) for P', so that

$$r = \{(x_1 - X_1)^2 + (x_2 - X_2)^2 + (x_3 - X_3)^2\}^{\frac{1}{2}},$$

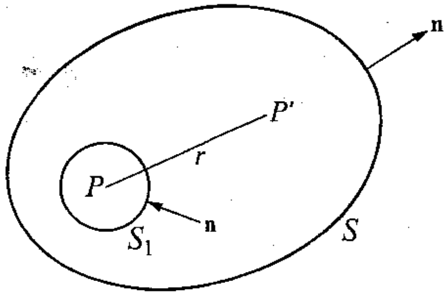

Fig. 5.3 Diagram for Section (5.5).

and the result follows by differentiation and use of expression (5.6), in which ϕ is put equal to $1/r$. Now the function $1/r$ is singular at P itself, so in the first instance we isolate P by constructing a sphere S_1 of small radius ϵ, say, with P as centre, and apply the argument to follow to the region V_1 between S and S_1. In Green's formula (5.12), applied to the region V_1 with its composite boundary S and S_1, put $\psi = 1/r$. Then since

$$\text{div}\,(\text{grad}\,\psi) = 0,$$

and

$$\text{div}\,(\text{grad}\,\phi) = \rho,$$

the formula yields

$$-\int_{V_1} \frac{\rho}{r}\,d\tau = \int_S \{\phi\,\text{grad}\,\psi - \psi\,\text{grad}\,\phi\}\cdot d\mathbf{S}$$

$$+\int_{S_1} \{\phi\,\text{grad}\,\psi - \psi\,\text{grad}\,\phi\}\cdot d\mathbf{S}, \qquad (5.14)$$

with due regard to the convention for the sense of the normal on S and S_1. Now consider the behaviour of the surface integral over S_1 as the radius of S_1 tends to zero. It is assumed that ϕ and grad ϕ are continuous functions of the space variables. Thus, with use of a mean value theorem,

$$\int_{S_1} \phi\,\text{grad}\,\psi\cdot d\mathbf{S} = \phi(P_1)\int_{S_1} \text{grad}\,\psi\cdot d\mathbf{S},$$

where P_1 is some interior point of S_1. Also,

$$\text{grad}\,\psi = -\frac{\mathbf{r}}{r^3} = \frac{\mathbf{n}}{\epsilon^2} \text{ on } S_1;$$

so that

$$\int_{S_1} \text{grad}\,\psi\cdot d\mathbf{S} = 4\pi,$$

and hence,

$$\int_{S_1} \text{grad}\,\psi\cdot d\mathbf{S} = 4\pi\phi(P_1).$$

Further, since grad ϕ is finite, $\int_{S_1} \psi\,\text{grad}\,\phi\cdot d\mathbf{S}$ is of order ϵ, and so tends to zero as $\epsilon \to 0$. In this limiting process, P_1 necessarily tends to P, and V_1 tends to the original volume V. Hence result (5.14) yields

$$4\pi\phi(P) = -\int_V \frac{\rho}{r}\,d\tau + \int_S \left\{\frac{1}{r}\,\text{grad}\,\phi - \phi\,\text{grad}\left(\frac{1}{r}\right)\right\}\cdot d\mathbf{S}. \qquad (5.15)$$

This last result therefore determines ϕ in terms of a volume and surface integral, provided that ϕ and the normal component of grad ϕ are known on the boundary surface S. Now a solution of Poisson's equation is determined uniquely if ϕ alone is known on S, so that the expression obtained does not constitute a solution in this sense. However, there are circumstances in which it is possible to employ the result as a solution, as occurs in the proof of Helmholtz's theorem (Chapter 6).

Examples, Chapter 5

1. Obtain the divergence of the following vector fields:
(i) $x_2\mathbf{a}_1 + x_3\mathbf{a}_2 + x_1\mathbf{a}_3$, (ii) $x_1x_2x_3(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)$, (iii) $\mathbf{r}/r^3$,
where $\mathbf{r}$ is the radius vector of a point with respect to an origin of Cartesian coordinates x_i, unit vectors $\mathbf{a}_i$. Show that

$$\text{div}\,(\text{grad}\,r^n) = n(n+1)r^{n-2}.$$

Answers: (i) 0, (ii) $x_2x_3 + x_1x_3 + x_1x_2$, (iii) 0.

2. In terms of the stated coordinate system, establish the following results for div (grad ϕ):
(i) Cylindrical polar coordinates (r, θ, z);

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2},$$

(ii) Spherical polar coordinates (r, θ, α);

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\alpha^2},$$

(iii) Oblate spheroidal coordinates (ξ, η, θ)
$x_1 = a\cosh\xi\cos\eta\cos\theta,\quad x_2 = a\cosh\xi\cos\eta\sin\theta,$
$x_3 = a\sinh\xi\sin\eta;$

$$\frac{1}{a^2(\sinh^2\xi+\sin^2\eta)}\left\{\frac{1}{\cosh\xi}\frac{\partial}{\partial\xi}\left(\cosh\xi\frac{\partial\phi}{\partial\xi}\right)+\frac{1}{\cos\eta}\frac{\partial}{\partial\eta}\left(\cos\eta\frac{\partial\phi}{\partial\eta}\right)\right\}$$
$$+\frac{1}{a^2\cosh^2\xi\cos^2\eta}\frac{\partial^2\phi}{\partial\theta^2}.$$

3. If the scalar fields u, v, w are harmonic, and the level surfaces of u, v, w intersect everywhere at right angles, show that the scalar field $u\,v\,w$ is harmonic.

4. Prove Kelvin's generalisation of Green's formula, namely

$$\int_V \xi \operatorname{grad}\phi\cdot\operatorname{grad}\psi\, d\tau = \int_S \phi\xi\operatorname{grad}\psi\cdot d\mathbf{S} - \int_V \phi\operatorname{div}(\xi\operatorname{grad}\psi)\,d\tau$$
$$= \int_S \psi\xi\operatorname{grad}\phi\cdot d\mathbf{S} - \int_V \psi\operatorname{div}(\xi\operatorname{grad}\phi)\,d\tau,$$

in a usual notation, where ϕ, ψ, ξ are scalar fields.

[N.B. Consider expansions for $\operatorname{div}(\phi\mathbf{F})$, $\operatorname{div}(\psi\mathbf{G})$, where $\mathbf{F}=\xi\operatorname{grad}\psi$, $\mathbf{G}=\xi\operatorname{grad}\phi$.]

5. If the scalar field ϕ satisfies the equation

$$\frac{\partial\phi}{\partial t} = \operatorname{div}(k\operatorname{grad}\phi)$$

everywhere, where k is a prescribed scalar field, and ϕ is zero over a closed surface S, show that

$$\int_V\left\{\phi\frac{\partial\phi}{\partial t} + k(\operatorname{grad}\phi)^2\right\}d\tau = 0,$$

where the volume integration is taken over the region V enclosed by S.

6 The curl of a vector field

In previous chapters we have considered the derivation of a vector field from a scalar field (the gradient), and that of a scalar field from a vector field (the divergence). The last construction which we need to consider is the derivation of a new vector field from a given vector field. The reason why the particular construction given below exhausts the situations which it is necessary to consider will be clear from a fundamental theorem to be proved later in this chapter.

Let $\mathbf{F}$ be a given vector field; a new vector field $\mathbf{A}$, called the *curl of* $\mathbf{F}$ ($\mathbf{A} = \operatorname{curl}\mathbf{F}$), is constructed from $\mathbf{F}$ in the following manner:

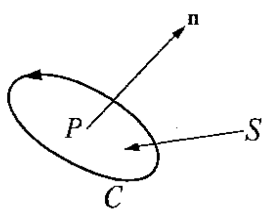

Fig. 6.1 Diagram for the calculation of the component of curl **F** in the direction **n** at a point *P*.

6.1 Definition

With reference to Fig. 6.1, let P be any point in space, and let $\mathbf{n}$ be a unit vector at P. Take any smooth surface through P, to which $\mathbf{n}$ is normal at P, and construct a small closed curve C in this surface, enclosing P and an area S of the surface. Form the line integral of F and round C, namely

$$\oint_C \mathbf{F}\cdot d\mathbf{r},$$

where the sense of integration round C is related to $\mathbf{n}$ by the right-hand screw rule. We now define *the component of curl* $\mathbf{F}$ *at* P *in the direction* $\mathbf{n}$, $(\text{curl }\mathbf{F})_n$, by the limit

$$(\text{curl }\mathbf{F})_n = \lim\left\{\frac{\oint_C \mathbf{F}\cdot d\mathbf{r}}{S}\right\}, \tag{6.1}$$

where the limiting process implies that the contour C is to shrink to zero, so that S also shrinks to zero, condensing ultimately on P.

If this definition is accepted for the moment, note that it is necessary to perform three calculations such as (6.1), corresponding to three mutually perpendicular directions at P, with unit vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ say, in order to calculate the complete field of curl $\mathbf{F}$ at P, since (6.1) provides only one component of curl $\mathbf{F}$. There are, however, two important requirements which must be guaranteed before this construction can be accepted as valid. In the first place, the limiting process (6.1) must be shown to be independent of the particular choice of the surface and its embedded contour C. Secondly, no restriction has been placed on the choice of the orthogonal triad $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, and naturally the corresponding components of curl $\mathbf{F}$ will be different for different triads; the question arises whether, on combination of components, different triads yield the same composite vector field. We can satisfy ourselves on these points by an appeal to Stokes's theorem.

In a fixed Cartesian frame of reference $Ox_1x_2x_3$, let the components of the given vector field $\mathbf{F}$ be F_1, F_2, F_3. Then, with reference to the definition (6.1), if α, β, γ are the direction cosines of the normal to the surface at any point, Stokes's theorem (3.27) gives

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_S\left\{\left(\frac{\partial F_3}{\partial x_2}-\frac{\partial F_2}{\partial x_3}\right)\alpha+\left(\frac{\partial F_1}{\partial x_3}-\frac{\partial F_3}{\partial x_1}\right)\beta+\left(\frac{\partial F_2}{\partial x_1}-\frac{\partial F_1}{\partial x_2}\right)\gamma\right\}dS.$$

If we now assume that the integrand in the surface integral is a continuous function of the space variables, a mean value theorem can be applied to the integral to give

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \left\{\left(\frac{\partial F_3}{\partial x_2}-\frac{\partial F_2}{\partial x_3}\right)\alpha+\left(\frac{\partial F_1}{\partial x_3}-\frac{\partial F_3}{\partial x_1}\right)\beta+\left(\frac{\partial F_2}{\partial x_1}-\frac{\partial F_1}{\partial x_2}\right)\gamma\right\}_{P'} S,$$

where P' is some point of the surface interior to C, and this suffix indicates that the quantity in brackets above is to be evaluated at P'.

Divide now by S, and form the limiting process implied in (6.1); then P' tends to P in the limit, and we have

$$\lim \frac{\oint_C \mathbf{F}\cdot d\mathbf{r}}{S} = \left(\frac{\partial F_3}{\partial x_2}-\frac{\partial F_2}{\partial x_3}\right)\alpha+\left(\frac{\partial F_1}{\partial x_3}-\frac{\partial F_3}{\partial x_1}\right)\beta+\left(\frac{\partial F_2}{\partial x_1}-\frac{\partial F_1}{\partial x_2}\right)\gamma$$

where the right-hand side above is evaluated at P, so that α, β, γ are now the components of $\mathbf{n}$ at P.

This last result ensures both the requirements which were mentioned previously. Firstly the result is entirely independent of the surface and of the shape of C, and secondly it states that the component claimed for the vector field $\mathbf{A}$ in the direction $\mathbf{n}$ is exactly

$$A_1\alpha + A_2\beta + A_3\gamma,$$

where

$$\left.\begin{aligned} A_1 &= \frac{\partial F_3}{\partial x_2}-\frac{\partial F_2}{\partial x_3},\\ A_2 &= \frac{\partial F_1}{\partial x_3}-\frac{\partial F_3}{\partial x_1},\\ A_3 &= \frac{\partial F_2}{\partial x_1}-\frac{\partial F_1}{\partial x_2}, \end{aligned}\right\} \tag{6.2}$$

which are independent of $\mathbf{n}$. This is precisely the form which is required if indeed the construction is to yield a definite vector field. Thus the construction is justified, and incidentally we have obtained the Cartesian components of curl $\mathbf{F}$, namely A_1, A_2, A_3.

Finally, we recall the statement (3.28) of Stokes's theorem, which now becomes

$$\int_S (\text{curl }\mathbf{F})\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r}, \tag{6.3}$$

where S is any open surface bounded by a closed contour C.

6.2 The components of curl F in orthogonal curvilinear coordinates

The limiting process of (6.1) will be applied for three directions at right angles. Naturally in this case these directions are those of the local axes of the system at a point P. One component only of curl $\mathbf{F}$ will be calculated in detail; the remaining components are inferred by arguments of symmetry.

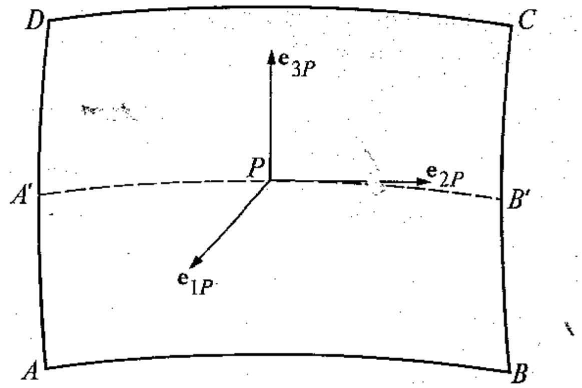

Fig. 6.2 Diagram for the calculation of a component of curl **F** in orthogonal curvilinear coordinates.

With reference to Fig. (6.2), the curvilinear coordinates of the point P are $(\xi_{1P}, \xi_{2P}, \xi_{3P})$, and $\mathbf{e}_{1P}, \mathbf{e}_{2P}, \mathbf{e}_{3P}$ are the unit vectors of the system at P. To deduce the component of curl **F** in the direction $\mathbf{e}_{1P}$, consider the small circuit $A\,B\,C\,D$. This lies in the level surface of ξ_1 through P, and the coordinates of A, B, C, D are

$$(\xi_{1P}, \xi_{2P}\pm\delta\xi_2, \xi_{3P}\pm\delta\xi_3),$$

where the appropriate selection of signs is clear from the diagram. If the scale factors of the system are h_1, h_2, h_3, with those at P indicated by the additional suffix P, the lengths of the edges AB, BC correct to the first order of small quantities are

$$AB = 2h_{2P}\,\delta\xi_2,$$
$$BC = 2h_{3P}\,\delta\xi_3,$$

and the area S enclosed by the circuit is, correct to the second order of small quantities,

$$S = 4h_{2P}h_{3P}\,\delta\xi_2\,\delta\xi_3.$$

The line integral of **F** round $A\,B\,C\,D$

$$= \int_A^B \mathbf{F}\cdot d\mathbf{r} + \int_B^C \mathbf{F}\cdot d\mathbf{r} + \int_C^D \mathbf{F}\cdot d\mathbf{r} + \int_D^A \mathbf{F}\cdot d\mathbf{r}.$$

In order to simplify the calculation of the various elements in this expression, consider first the line integral of **F** along $A'B'$, where $A'B'$ is the coordinate curve formed by the intersection of the level surfaces of ξ_1, ξ_3 through P. Denote this integral by T, so that

$$T = \int_{A'}^{B'} \mathbf{F}\cdot d\mathbf{r} = \int_{A'}^{B'} F_2h_2\,d\xi_2.$$

Put $F_2h_2 = H$, $\xi_2 = \xi_{2P}+\alpha$. Then, since α is small on $A'B'$, use of a Taylor expansion for H gives

$$T = \int_{-\delta\xi_2}^{\delta\xi_2} \left\{(H)_P+\alpha\left(\frac{\partial H}{\partial\xi_2}\right)_P+\frac{1}{2!}\alpha^2\left(\frac{\partial^2 H}{\partial\xi_2^2}\right)_P+\cdots\right\}d\alpha,$$

since ξ_2 alone varies on $A'B'$.

$$\therefore\quad T = 2\,\delta\xi_2(H)_P+0(\delta\xi_2^3).$$

Now the line integral of **F** along AB is obtained from T by a change $-\delta\xi_3$ in ξ_3, and that of **F** along DC by a change $+\delta\xi_3$ in ξ_3. Hence, correct to terms of the second order of small quantities,

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = 2F_2h_2\,\delta\xi_2-\frac{\partial}{\partial\xi_3}(2F_2h_2)\,\delta\xi_2\,\delta\xi_3,$$

where the suffix P is now suppressed, on the understanding that quantities are evaluated at P.

Similarly,

$$\int_D^C \mathbf{F}\cdot d\mathbf{r} = 2F_2h_2\,\delta\xi_2+\frac{\partial}{\partial\xi_3}(2F_2h_2)\,\delta\xi_2\,\delta\xi_3.$$

Thus
$$\int_A^B \mathbf{F}\cdot d\mathbf{r}+\int_C^D \mathbf{F}\cdot d\mathbf{r} = -4\frac{\partial}{\partial\xi_3}(F_2h_2)\,\delta\xi_2\,\delta\xi_3.$$

In a similar manner, we deduce

$$\int_B^C \mathbf{F}\cdot d\mathbf{r}+\int_D^A \mathbf{F}\cdot d\mathbf{r} = +4\frac{\partial}{\partial\xi_2}(F_3h_3)\,\delta\xi_2\,\delta\xi_3.$$

Hence the limiting process (6.1), which now amounts to a simple ratio since we have retained only terms of the same order in both numerator and denominator, gives

$$(\text{curl }\mathbf{F})_{\mathbf{e}_1} = \frac{1}{h_3h_2}\left\{\frac{\partial}{\partial\xi_2}(F_3h_3)-\frac{\partial}{\partial\xi_3}(F_2h_2)\right\}.$$

The other components of curl **F** (i.e. in the directions $\mathbf{e}_2$, $\mathbf{e}_3$) are calculated in the same way, but we can infer the result by an argument of symmetry from the last expression. Thus, we find

$$\text{curl }\mathbf{F} = \frac{\mathbf{e}_1}{h_2h_3}\left\{\frac{\partial}{\partial\xi_2}(F_3h_3)-\frac{\partial}{\partial\xi_3}(F_2h_2)\right\}+\cdots$$

$$+\frac{\mathbf{e}_2}{h_1h_3}\left\{\frac{\partial}{\partial\xi_3}(F_1h_1)-\frac{\partial}{\partial\xi_1}(F_3h_3)\right\}$$
$$+\frac{\mathbf{e}_3}{h_1h_2}\left\{\frac{\partial}{\partial\xi_1}(F_2h_2)-\frac{\partial}{\partial\xi_2}(F_1h_1)\right\}. \qquad (6.4)$$

This can be expressed more compactly in the form of a determinant involving differential operators, provided it is clearly understood that there is no question of rearrangement of rows in this case. The determinant is:

$$\text{curl}\,\mathbf{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\ \frac{\partial}{\partial\xi_1} & \frac{\partial}{\partial\xi_2} & \frac{\partial}{\partial\xi_3}\\ F_1h_1 & F_2h_2 & F_3h_3\end{vmatrix} \qquad (6.5)$$

Explicitly, the determinant is to be evaluated in terms of elements of its first row, to yield expression (6.4).

6.3 The vector operator ∇

At this stage it is convenient to introduce the operator ∇ (pronounced 'del'), for it occurs widely in the literature. It is defined in terms of a Cartesian system x_i, with unit vectors $\mathbf{a}_i$, by the combination

$$\nabla \equiv \mathbf{a}_i\frac{\partial}{\partial x_i}, \qquad (6.6)$$

in which, as usual, the summation convention is implied with i ranging over values from 1 to 3.

Thus, the gradient of a scalar field ϕ can be written

$$\text{grad}\,\phi = \nabla\phi,$$

since

$$\nabla\phi = \mathbf{a}_i\frac{\partial\phi}{\partial x_i}.$$

The divergence of a vector field **F** can be displayed as the dot product of ∇ with **F** (in that order, since the operator must precede the quantity on which it operates). Thus

$$\nabla\cdot\mathbf{F}=\left(\mathbf{a}_1\frac{\partial}{\partial x_1}+\mathbf{a}_2\frac{\partial}{\partial x_2}+\mathbf{a}_3\frac{\partial}{\partial x_3}\right)\cdot(\mathbf{a}_1F_1+\mathbf{a}_2F_2+\mathbf{a}_3F_3)$$

$$=\frac{\partial F_1}{\partial x_1}+\frac{\partial F_2}{\partial x_2}+\frac{\partial F_3}{\partial x_3}=\text{div}\,\mathbf{F},$$

because $\mathbf{a}_i\cdot\mathbf{a}_j=\delta_{ij}$, and also the unit vectors $\mathbf{a}_i$ are unaffected by differentiation since they are Cartesian unit vectors and hence are invariant throughout space. This last remark is important, because difficulties can arise in the indiscriminate use of the operator in other systems of coordinates. For example, in a system of orthogonal curvilinear coordinates, the operator is

$$\nabla\equiv\frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial\xi_1}+\frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial\xi_2}+\frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial\xi_3},$$

so that if a vector field **F** is expressed in terms of its components in this system, namely

$$\mathbf{F}=F_i\mathbf{e}_i,$$

then

$$\text{div}\,\mathbf{F}=\nabla\cdot\mathbf{F}=\left(\frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial\xi_1}+\frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial\xi_2}+\frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial\xi_3}\right)\cdot(\mathbf{e}_1F_1+\mathbf{e}_2F_2+\mathbf{e}_3F_3).$$

However, on performing the differentiations here, it must be remembered that the unit vectors $\mathbf{e}_i$ are now also spacially dependent, in general, and so are subject to differentiation in the same way as the components of **F**. Thus the operation becomes much more complicated, and there is no advantage here in pursuing the use of the operator ∇. In general, therefore, the use of the operator is primarily confined to operations in a Cartesian frame of reference.

Corresponding to the case of the divergence of a vector field, the curl of a vector field, curl **F**, is expressed by the cross product ∇ × **F**, as can be verified by a similar calculation. Hence, in all we have

$$\left.\begin{aligned}\text{grad}\,\phi&=\nabla\phi,\\ \text{div}\,\mathbf{F}&=\nabla\cdot\mathbf{F},\\ \text{curl}\,\mathbf{F}&=\nabla\times\mathbf{F}.\end{aligned}\right\} \qquad (6.7)$$

Further, the Laplacian of ϕ, div (grad ϕ) is evidently

$$\text{div}\,(\text{grad}\,\phi)=\nabla\cdot(\nabla\phi),\ \text{which is written}\ \nabla^2\phi. \qquad (6.8)$$

6.4 Further expansion formulae

Two results which are needed frequently are the following:

$$\text{curl}\,(\phi\mathbf{F})=\text{grad}\,\phi\times\mathbf{F}+\phi\,\text{curl}\,F, \qquad (6.9)$$

$$\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F}-\mathbf{F}\cdot\operatorname{curl}\mathbf{G}, \qquad (6.10)$$

in which ϕ is a scalar field, and **F**, **G** are vector fields. These results are established easily by use of Cartesian coordinates and components, but clearly they are universal (that is, they are applicable in any system). The first result is proved as an illustration:

$$\begin{aligned}\operatorname{curl}(\phi\mathbf{F}) &= \mathbf{a}_1\left\{\frac{\partial}{\partial x_2}(\phi F_3)-\frac{\partial}{\partial x_3}(\phi F_2)\right\}+\mathbf{a}_2\left\{\frac{\partial}{\partial x_3}(\phi F_1)-\frac{\partial}{\partial x_1}(\phi F_3)\right\}\\ &\quad+\mathbf{a}_3\left\{\frac{\partial}{\partial x_1}(\phi F_2)-\frac{\partial}{\partial x_2}(\phi F_1)\right\}\\ &= \phi\left\{\mathbf{a}_1\left(\frac{\partial F_3}{\partial x_2}-\frac{\partial F_2}{\partial x_3}\right)+\mathbf{a}_2\left(\frac{\partial F_1}{\partial x_3}-\frac{\partial F_3}{\partial x_1}\right)+\mathbf{a}_3\left(\frac{\partial F_2}{\partial x_1}-\frac{\partial F_1}{\partial x_2}\right)\right\}\\ &\quad+\mathbf{a}_1\left\{F_3\frac{\partial\phi}{\partial x_2}-F_2\frac{\partial\phi}{\partial x_3}\right\}+\mathbf{a}_2\left\{F_1\frac{\partial\phi}{\partial x_3}-F_3\frac{\partial\phi}{\partial x_1}\right\}\\ &\quad+\mathbf{a}_3\left\{F_2\frac{\partial\phi}{\partial x_1}-F_1\frac{\partial\phi}{\partial x_2}\right\}\\ &= \phi\operatorname{curl}\mathbf{F}+\begin{vmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\\ \dfrac{\partial\phi}{\partial x_1} & \dfrac{\partial\phi}{\partial x_2} & \dfrac{\partial\phi}{\partial x_3}\\ F_1 & F_2 & F_3\end{vmatrix}\\ &= \phi\operatorname{curl}\mathbf{F}+\operatorname{grad}\phi\times\mathbf{F}.\end{aligned}$$

There are various other expansion formulae involving the 'curl', but their usefulness tends to be restricted to the use of a Cartesian system of reference. One such result which is widely used is an expression for curl (curl **F**). Clearly, if **F** is a vector field, curl **F** is also a vector field, and thus the curl of this second field also can be calculated. The expansion, which can be established readily by use of a Cartesian system, is

$$\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F})-\nabla^2\mathbf{F}. \qquad (6.11)$$

In this formula it must be remembered that Cartesian components of **F** are implied, and the meaning of the last term is

$$\nabla^2\mathbf{F} \equiv \mathbf{a}_1\,\nabla^2F_1+\mathbf{a}_2\,\nabla^2F_2+\mathbf{a}_3\,\nabla^2F_3,$$

where

$$\nabla^2F_i = \frac{\partial^2F_i}{\partial x_1^2}+\frac{\partial^2F_i}{\partial x_2^2}+\frac{\partial^2F_i}{\partial x_3^2}.$$

Thus no attempt should be made to employ such a result for other systems of coordinates. In such cases the only certain method of deduc-

ing a correct result is to evaluate curl (curl **F**) directly, by a repeated application of result (6.4). This vector field occurs so frequently, both in fluid mechanics and in electro-magnetic theory, that it is worth performing this calculation as an example for one system of coordinates.

Example 6.1

Evaluate curl (curl **F**) in terms of its components in the system of paraboloidal coordinates (ξ, η, α), where

$$x_1 = \xi\eta\cos\alpha,\quad x_2 = \xi\eta\sin\alpha,\quad x_3 = \tfrac{1}{2}(\xi^2-\eta^2).$$

The usual tests establish that (ξ, η, α) is an orthogonal system of curvilinear coordinates. The scale factors h_1, h_2, h_3 are

$$h_1 = h_2 = \{\xi^2+\eta^2\}^{\frac{1}{2}} = h, \quad\text{say,}\quad h_3 = \xi\eta.$$

With $\mathbf{F} = F_1\mathbf{e}_1+F_2\mathbf{e}_2+F_3\mathbf{e}_3$, where the $\mathbf{e}_i$ are the appropriate unit vectors for the system, result (6.4) yields

$$\operatorname{curl}\mathbf{F} = \mathbf{e}_1G_1+\mathbf{e}_2G_2+\mathbf{e}_3G_3 = \mathbf{G},\ \text{say,}$$

where

$$\begin{aligned}G_1 &= \frac{1}{hh_3}\left\{\frac{\partial}{\partial\eta}(F_3h_3)-\frac{\partial}{\partial\alpha}(F_2h)\right\},\\ G_2 &= \frac{1}{hh_3}\left\{\frac{\partial}{\partial\alpha}(F_1h)-\frac{\partial}{\partial\xi}(F_3h_3)\right\},\\ G_3 &= \frac{1}{h^2}\left\{\frac{\partial}{\partial\xi}(F_2h)-\frac{\partial}{\partial\eta}(F_1h)\right\}.\end{aligned}$$

Thus curl (curl **F**) = curl **G**, and the components of curl **G** are obtained by replacing the F_i by G_i in the right-hand sides of the expressions above. These components are thus very complicated; let us select simply the $\mathbf{e}_3$ component for illustration:

$$\begin{aligned}\{\operatorname{curl}(\operatorname{curl}\mathbf{F})\}_3 &= \frac{1}{h^2}\left\{\frac{\partial}{\partial\xi}(G_2h)-\frac{\partial}{\partial\eta}(G_1h)\right\}\\ &= \frac{1}{h^2}\left[\frac{\partial}{\partial\xi}\left\{\frac{1}{h_3}\frac{\partial}{\partial\alpha}(F_1h)-\frac{1}{h_3}\frac{\partial}{\partial\xi}(F_3h_3)\right\}-\cdots\right.\end{aligned}$$

$$-\frac{\partial}{\partial\eta}\left\{\frac{1}{h_3}\frac{\partial}{\partial\eta}(F_3h_3)-\frac{1}{h_3}\frac{\partial}{\partial\alpha}(F_2h)\right\}\Bigg]$$

$$=\frac{1}{h^2}\left[\frac{\partial}{\partial\xi}\left\{\frac{h}{h_3}\frac{\partial F_1}{\partial\alpha}\right\}-\frac{\partial}{\partial\xi}\left\{\frac{\partial F_3}{\partial\xi}+\frac{1}{\xi}F_3\right\}\right.$$
$$\left.-\frac{\partial}{\partial\eta}\left\{\frac{\partial F_3}{\partial\eta}+\frac{1}{\eta}F_3\right\}+\frac{\partial}{\partial\eta}\left(\frac{h}{h_3}\frac{\partial F_2}{\partial\alpha}\right)\right],$$

(on substituting partially for the h_i)

$$=-\frac{1}{(\xi^2+\eta^2)}\left[\frac{\partial^2F_3}{\partial\xi^2}+\frac{\partial^2F_3}{\partial\eta^2}+\frac{\partial}{\partial\xi}\left(\frac{F_3}{\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{F_3}{\eta}\right)\right.$$
$$-\frac{\partial}{\partial\xi}\left\{\frac{(\xi^2+\eta^2)^{\frac{1}{2}}}{\xi\eta}\frac{\partial F_1}{\partial\alpha}\right\}$$
$$\left.-\frac{\partial}{\partial\eta}\left\{\frac{(\xi^2+\eta^2)^{\frac{1}{2}}}{\xi\eta}\frac{\partial F_2}{\partial\alpha}\right\}\right].$$

6.5 Special types of vector fields

It is appropriate at this stage to consider certain types of vector fields which occur commonly in applications.

Irrotational fields

If $\mathbf{F}$ is a vector field such that curl $\mathbf{F} = \mathbf{O}$ everywhere in a given region, then $\mathbf{F}$ is described as an irrotational field. This nomenclature arises in the context of hydrodynamics, in which the curl of the velocity field measures the local state of rotation of the fluid, and this is zero if the velocity field is irrotational. Alternative descriptions are non-curl and lamellar.

Theorem

If curl $\mathbf{F} = \mathbf{O}$ everywhere in a simply connected region of space, then $\mathbf{F}$ is expressible as the gradient of a single valued scalar field ϕ. (6.12)

This theorem is easily proved. For if C is any closed contour drawn in the region, it can be spanned by an open surface S which also lies in the region, and thus by Stokes's theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_S \text{curl } \mathbf{F}\cdot d\mathbf{S} = 0, \quad \text{since curl } \mathbf{F} = \mathbf{O}.$$

Hence, by theorem (4.6), $\mathbf{F} = \text{grad } \phi$, where ϕ is a single valued scalar field.

Note carefully the requirement that the region should be simply connected. If this is not so, while curl $\mathbf{F} = \mathbf{O}$ everywhere in the region, it may still be possible to construct a scalar potential ϕ, with $\mathbf{F} = \text{grad } \phi$, but this potential may not be single valued. This situation occurs frequently in applications to hydrodynamics.

The above result explains the description of the irrotational field as *lamellar*, which refers to shell-like structure, namely the family of level surfaces of the scalar field ϕ.

Solenoidal fields

If a vector field $\mathbf{F}$ is such that div $\mathbf{F} = 0$ everywhere in a given region of space, then $\mathbf{F}$ is described as a solenoidal field. An alternative, but less direct, definition is the statement that a vector field $\mathbf{F}$ is solenoidal if it can be expressed as curl $\mathbf{A}$, where $\mathbf{A}$ is another vector field. This last statement automatically implies that div $\mathbf{F} = 0$, since it is a simple matter to verify that

$$\text{div (curl } \mathbf{A}) \equiv 0. \tag{6.13}$$

This can be seen either by direct differentiation, with use of the Cartesian forms for div and curl, or alternatively as follows:

Consider an arbitrary region V, bounded by a closed surface S, and let S be divided into two open surfaces S_1 and S_2 by a closed contour C drawn on S (Fig. 6.3).

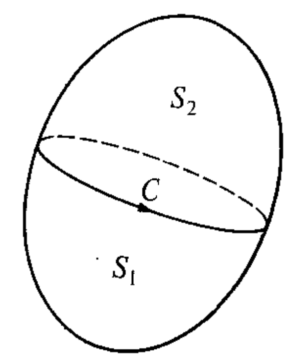

Fig. 6.3 Diagram for the proof that div(curl $\mathbf{A}$) $\equiv$ 0.

curl (grad φ) = 0
div (curl $\underset{\sim}{A}$) = 0

Now consider the volume integral of div (curl **A**) over V. By Gauss's theorem we have

$$\int_V \operatorname{div}(\operatorname{curl}\mathbf{A})\,d\tau = \int_S \operatorname{curl}\mathbf{A}\cdot d\mathbf{S}$$

$$= \int_{S_1} \operatorname{curl}\mathbf{A}\cdot d\mathbf{S} + \int_{S_2} \operatorname{curl}\mathbf{A}\cdot d\mathbf{S}.$$

Now apply Stokes's theorem to each of the last surface integrals. They are, respectively,

$$\pm \oint_C \mathbf{A}\cdot d\mathbf{r},$$

and thus the sum is identically zero. Hence

$$\int_V \operatorname{div}(\operatorname{curl}\mathbf{A})\,d\tau = 0,$$

and therefore, since V is arbitrary, div (curl **A**) $\equiv 0$.

Theorem

If a vector field **F** is solenoidal everywhere in a given region of space, it can be expressed in the form **F** = curl **A**. (6.14)

This can be established by use of the Cartesian form for div **F**, namely

$$\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = 0, \qquad (1)$$

since **F** is solenoidal. Now write

$$F_2 = -\frac{\partial B_3}{\partial x_1},$$

$$F_3 = \frac{\partial B_2}{\partial x_1},$$

since it is certainly possible to find functions B_2, B_3 (by integration with respect to x_1), which satisfy the above relations. Substitution in (1), followed by integration with respect to x_1, then gives

$$F_1 = \frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} + \frac{\partial}{\partial x_2} g(x_2, x_3),$$

$$= \frac{\partial}{\partial x_2}\{B_3+g\} - \frac{\partial B_2}{\partial x_3},$$

where g is some arbitrary differentiable function of x_2 and x_3. Hence the components of **F** can be expressed in the form

$$F_1 = \frac{\partial}{\partial x_2}\{B_3+g\} - \frac{\partial B_2}{\partial x_3},$$

$$F_2 = -\frac{\partial}{\partial x_1}\{B_3+g\}, \quad \text{(since } g \text{ is independent of } x_1\text{)},$$

$$F_3 = \frac{\partial B_2}{\partial x_1}.$$

That is, **F** = curl **A**, where **A** is the vector field with components $(0, B_2, B_3+g)$.

The vector field **A** is described as a *vector potential* for **F**, when **F** is solenoidal. It is clear from the above analysis that **A** is by no means unique for a given solenoidal field. Indeed, if **A** is a vector potential for **F**, so also is

$$\mathbf{A} + \operatorname{grad}\phi,$$

where ϕ is any single valued scalar field. For it is easily established that

$$\operatorname{curl}(\operatorname{grad}\phi) \equiv 0, \qquad (6.15)$$

by use of Cartesian representations, for example. Thus

$$\operatorname{curl}(\mathbf{A} + \operatorname{grad}\phi) = \operatorname{curl}\mathbf{A}.$$

It will be seen later that there are important applications of theorem (6.14). That is, it will be convenient to represent a solenoidal field **F** by a vector potential **A**, in which the degree of arbitrariness usually is removed by imposing some restriction on **A**. It is true that the simplicity of the irrotational field, with its scalar potential, appears to be lost, but fortunately in practice the structure of **A** is frequently much simpler than that of **F** itself.

6.6 Construction of a vector potential by a line integral

If **A** is a vector potential for a solenoidal field **F**, then by Stokes's theorem

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \int_S \operatorname{curl}\mathbf{A}\cdot d\mathbf{S} = \int_S \mathbf{F}\cdot d\mathbf{S},$$

where C is a closed contour bounding an open surface S. Now let S be the surface of a cone, with its vertex O as the origin of coordinates

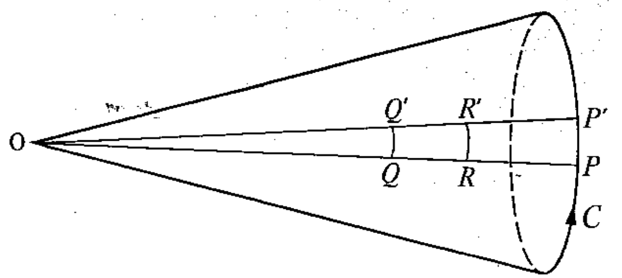

Fig. 6.4 Diagram for the construction of a vector potential (Section 6.6).

(Fig. 6.4). Then the above result suggests that if $\overline{OP} = \mathbf{r}$, and $\overline{PP'} = d\mathbf{r}$, where PP' is a small element of C, then

$$\mathbf{A}\cdot d\mathbf{r} = \int_{S_1} \mathbf{F}\cdot d\mathbf{S},$$

where S_1 is the triangular element OPP' of S. Now consider the subdivision of S_1 into small elements of which a typical member is $QRR'Q'$, and where $\overline{QQ'}$, $\overline{RR'}$ are parallel to $\overline{PP'}$.

The vector $\overline{OQ} = t\mathbf{r}$, where t is a scalar parameter such that $0 \leqslant t \leqslant 1$. Thus, with dt as the change in t on passing from Q to R,

$$\overline{QR} = \mathbf{r}\,dt, \quad \text{and} \quad \overline{QQ'} = t\,\overline{PP'} = t\,d\mathbf{r}.$$

Hence the vector element of area $QRR'Q'$ is $\overline{QR}\times\overline{QQ'}$

$$= \mathbf{r}\times d\mathbf{r}\;t\;dt.$$

The given vector field $\mathbf{F}$ is $\mathbf{F}(\mathbf{r})$, and so has the value $\mathbf{F}(t\mathbf{r})$ at Q. Hence the contribution to

$$\int_{S_1} \mathbf{F}\cdot d\mathbf{S}$$

from the above element is $\mathbf{F}(t\mathbf{r})\cdot\{\mathbf{r}\times d\mathbf{r}\;t\;dt\}$, and therefore

$$\int_{S_1} \mathbf{F}\cdot d\mathbf{S} = \int_{t=0}^{1} \mathbf{F}(t\mathbf{r})\cdot\{\mathbf{r}\times d\mathbf{r}\;t\;dt\}$$

$$= \left\{\int_0^1 \mathbf{F}(t\mathbf{r})\,t\,dt\right\}\cdot(\mathbf{r}\times d\mathbf{r}),$$

since $\mathbf{r}$ and $d\mathbf{r}$ are fixed in this integration. Hence

$$\mathbf{A}\cdot d\mathbf{r} = (\mathbf{r}\times d\mathbf{r})\cdot\int_0^1 \mathbf{F}(t\mathbf{r})\,t\,dt$$

$$= d\mathbf{r}\cdot\left\{\int_0^1 \mathbf{F}(t\mathbf{r})\,t\,dt\times\mathbf{r}\right\},$$

on application of the rules for a scalar triple product. Now the original contour C is arbitrary, so $d\mathbf{r}$ is arbitrary. It follows therefore that

$$\mathbf{A} = \int_0^1 \mathbf{F}(t\mathbf{r})\,t\,dt\times\mathbf{r} \tag{6.16}$$

is a vector potential for the solenoidal field $\mathbf{F}$.*

Example 6.2

Find a vector potential for the vector field $\mathbf{F}$, where $\mathbf{F}$ has Cartesian components $\mathbf{F} = (x_1^2, x_2, -2x_3x_1 - x_3)$. This vector field is solenoidal (div $\mathbf{F} = 0$), and since

$$\mathbf{F}(\mathbf{r}) = x_1^2\mathbf{a}_1 + x_2\mathbf{a}_2 - (2x_3x_1 + x_3)\mathbf{a}_3,$$

then

$$\mathbf{F}(t\mathbf{r}) = x_1^2t^2\mathbf{a}_1 + x_2t\mathbf{a}_2 - (2x_3x_1t^2 + x_3t)\mathbf{a}_3,$$

so that from result (6.16) a vector potential $\mathbf{A}$ for $\mathbf{F}$ is

$$\mathbf{A} = \int_0^1 \{x_1^2t^3\mathbf{a}_1 + x_2t^2\mathbf{a}_2 - (2x_3x_1t^3 + x_3t^2)\mathbf{a}_3\}\,dt\times(x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3)$$

$$= (\tfrac{2}{3}x_2x_3 + \tfrac{1}{2}x_1x_2x_3)\mathbf{a}_1 - (\tfrac{3}{4}x_1^2x_3 + \tfrac{1}{3}x_1x_3)\mathbf{a}_2 + (\tfrac{1}{4}x_1^2x_2 - \tfrac{1}{3}x_1x_2)\mathbf{a}_3,$$

on evaluation.

6.7 The trajectories of a vector field

A family of curves is associated with a given vector field $\mathbf{F}$, and the construction of this family may be visualised in the following way. In a Cartesian system of coordinates, let P be a point x_i in the field, and let F_i be the components of $\mathbf{F}$. Consider now a small displacement dx_i from P in the direction of the field $\mathbf{F}$ at P. Thus

$$\frac{dx_1}{F_1(x_1, x_2, x_3, t)} = \frac{dx_2}{F_2(x_1, x_2, x_3, t)} = \frac{dx_3}{F_3(x_1, x_2, x_3, t)}, \tag{6.17}$$

in which the F_i have been written explicitly to indicate their dependence on x_i, and also generally on the time t. Equations (6.17) may be regarded as a pair of differential equations, the solution of which defines a curve which has the property that at any point on it the corresponding field $\mathbf{F}$ is tangential to it. Equations (6.17), of course, define a whole family of

* I am indebted to Dr T. A. S. Jackson for pointing out this result.

curves, since there are two arbitrary constants in the solution corresponding to different choices of the initial field point through which the solution of equations (6.17) is to pass. This family constitutes the system of trajectories of the given vector field **F** (Fig. 6.5).

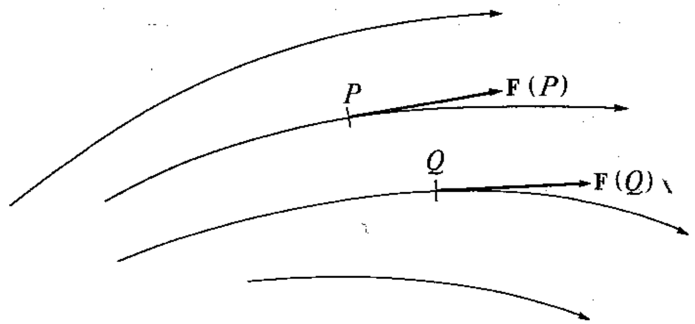

Fig. 6.5 Trajectories of a vector field **F**.

A familiar example of such a system is that of the family of streamlines in hydrodynamics, in which the appropriate vector field is the velocity field **V**. If the motion is steady, namely if **V** is a function only of the space variables x_i and not of the time t, then the streamlines of the flow constitute the actual paths of the particles of fluid.

Example 6.3

Find the trajectories of the vector field **F**, where **F** has Cartesian components

$$\left(-\frac{ax_2}{(x_1^2+x_2^2)}, \frac{ax_1}{(x_1^2+x_2^2)}, 1\right),$$

and a is constant.

From (6.17) the trajectories are the solutions of the differential equations

$$-\frac{(x_1^2+x_2^2)\,dx_1}{ax_2} = \frac{(x_1^2+x_2^2)\,dx_2}{ax_1} = dx_3.$$

The relation between dx_1 and dx_2 gives

$$x_1\,dx_1+x_2\,dx_2 = 0,$$

which integrates immediately to yield

$$x_1^2+x_2^2 = c_1^2, \tag{1}$$

where c_1 is an arbitrary constant. Use of this result in the relation between dx_2 and dx_3 gives

$$dx_3 = \frac{c_1^2\,dx_2}{a\{c_1^2-x_2^2\}^{\frac{1}{2}}},$$

and hence, integrating,

$$x_3 = \frac{c_1^2}{a}\sin^{-1}(x_2/c_1)-c_2,$$

or

$$x_2 = c_1 \sin\left\{\frac{a}{c_1^2}(x_3+c_2)\right\}, \tag{2}$$

where c_2 is a second arbitrary constant. For given values of c_1 and c_2, relations (1) and (2) specify a curve, and as c_1 and c_2 are varied we obtain the whole family of trajectories of the field **F**.

6.8 Vector tubes

The concept of the trajectories of a vector field leads naturally to that of vector tubes. Consider a closed circuit C drawn in the field; then the set of trajectories of the field **F** which pass through points of C constitute a tube, as illustrated in Fig. 6.6.

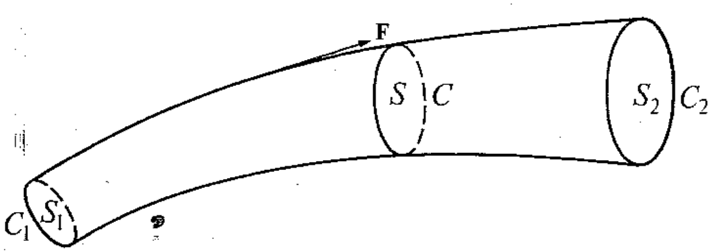

Fig. 6.6 A vector tube, formed from a set of trajectories of the vector field **F**.

If **F** is a solenoidal vector field, we can also define an intensity for a given vector tube, namely the flux of **F** across a cross section of the tube. This scalar quantity is a constant for the tube. For consider a finite volume V of a tube, terminated by open surfaces S_1 and S_2 which span closed contours C_1 and C_2, respectively, drawn on the surface of the tube. Then by Gauss's theorem,

$$\int_V \text{div } \mathbf{F} \, d\tau = \int_{S_1} \mathbf{F} \cdot d\mathbf{S} + \int_{S_2} \mathbf{F} \cdot d\mathbf{S},$$

since on the surface of the tube itself **F** is tangential to the surface, so there is no contribution to the surface integral. Also div **F** = 0 everywhere, since **F** is solenoidal, and hence

$$\int_{S_1} \mathbf{F} \cdot d\mathbf{S} + \int_{S_2} \mathbf{F} \cdot d\mathbf{S} = 0.$$

In this result the normals to S_1 and S_2 are in a direction out of the volume V. If we now arrange for these normals to be drawn in the same sense, the result then reads

$$\int_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

The intensity I of the tube is

$$I = \int_S \mathbf{F} \cdot d\mathbf{S}, \tag{6.18}$$

where S is any cross section of the tube, and the above result establishes that I is constant. This property, incidentally, explains the description of the field as solenoidal, referring to the tube-like structure with constant intensity for a given tube.

Certain consequences are implicit in the theory above. The solenoidal vector field **F** is necessarily continuous, so that a vector tube cannot terminate abruptly in a cross section of finite area inside the field. Thus a vector tube must be closed, or originate from a point and terminate in like manner. It is not possible for this latter case to occur within the field, for the fact that the intensity of the tube is constant implies that the field **F** itself must become infinite as the cross section of the tube shrinks to zero. Thus if a vector tube is not closed it must terminate on the boundary of the space occupied by the field, or move off to infinity in the case where the field is unbounded.

6.9 Expression of a given vector field as a sum of an irrotational field and a solenoidal field

At this stage we are in a position to establish a fundamental theorem in the subject of vector field theory, but we precede this by a theorem which is essentially a lemma for its successor.

Theorem

If a vector field **F**(**r**) is such that

$$\text{div } \mathbf{F} = \mathbf{O} \text{ and } \quad \text{curl } \mathbf{F} = \mathbf{O}$$

everywhere in the whole of space, and **F** decays at infinity like $r^{-\epsilon}$, where $\epsilon > 0$, then $\mathbf{F} \equiv \mathbf{O}$ everywhere. (6.19)

With use of Cartesian forms for the vector fields, we have

$$\text{div } \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = 0, \tag{1}$$

and (since curl **F** = **O**),

$$\frac{\partial F_2}{\partial x_1} = \frac{\partial F_1}{\partial x_2}, \quad \frac{\partial F_3}{\partial x_1} = \frac{\partial F_1}{\partial x_3}, \quad \frac{\partial F_2}{\partial x_3} = \frac{\partial F_3}{\partial x_2}. \tag{2}$$

Differentiate relation (1) with respect to x_1, and the first two of relations (2) by x_2, x_3 respectively. Thus eliminate F_2 and F_3 to yield

$$\nabla^2 F_1 = 0.$$

Similar equations hold for F_2 and F_3, so that each of the Cartesian components of F is a solution of Laplace's equation. Now use result (5.15), in which for this case $\rho \equiv 0$, and the surface integral is taken over a sphere of large radius R with P as centre. Thus

$$4\pi F_1(P) = \int_S \left\{ \frac{1}{r} \text{grad } F_1 - F_1 \text{ grad} \left(\frac{1}{r}\right) \right\} \cdot d\mathbf{S}.$$

Now $F_1 = 0(R^{-\epsilon})$, $|\text{grad } F_1| = 0(R^{-1-\epsilon})$, $|\text{grad } (1/r)| = 0(R^{-2})$, and $|d\mathbf{S}| = 0(R^2)$, so the surface integral is of order $R^{-\epsilon}$ and therefore vanishes as $R \to \infty$. Hence $F_1 = 0$, and similarly $F_2 = F_3 = 0$, which establishes the theorem.

The theorem of Helmholtz

The following theorem has been described as the fundamental theorem of vector analysis. It asserts that any vector field, subject only to modest restrictions, can be expressed as the sum of an irrotational field and a solenoidal field.

STATEMENT Let **F**(**r**) be a vector field, with div **F** and curl **F** both continuous functions, defined throughout the whole of space. Further, suppose that $|\mathbf{F}|$ tends to zero at infinity like $r^{-1-\epsilon}$ (where $\epsilon > 0$), and $|\text{div } \mathbf{F}|$, $|\text{curl } \mathbf{F}|$ do likewise with behaviour like $r^{-2-\epsilon}$. Then **F** can be expressed in the form

$$\mathbf{F} = \mathbf{H} + \mathbf{G},$$

where **H** is an irrotational, and **G** a solenoidal, vector field. (6.20)

In the above statement, of course, $|\text{div }\mathbf{F}|$ is simply the modulus of a scalar quantity, while $|\mathbf{F}|$, $|\text{curl }\mathbf{F}|$ are the (positive) magnitudes of vector fields. The restriction on the tendency to zero of $|\mathbf{F}|$ is sharper than the requirement of theorem (6.19), owing to the fact that in this theorem it is necessary to introduce volume integrals, the convergence of which must be guaranteed.

Assume tentatively that the resolution is possible, namely that

$$\mathbf{F} = \mathbf{H}+\mathbf{G}, \tag{1}$$

where

$$\text{curl }\mathbf{H} = \mathbf{O}, \quad \text{and} \quad \text{div }\mathbf{G} = 0. \tag{2}$$

The space is assumed to be simply connected, so the first of conditions (2) implies that

$$\mathbf{H} = \text{grad }\phi, \tag{6.21}$$

where ϕ is a single valued scalar field. Now form the divergence of relation (1), which yields, since $\text{div }\mathbf{G} = 0$,

$$\nabla^2\phi = \text{div }\mathbf{F}. \tag{3}$$

Thus ϕ is a solution of Poisson's equation, in which the right hand side, div **F**, is a prescribed scalar field. We use result (5.15) to obtain a solution of equation (3), and we select this solution to be explicitly

$$\phi(P) = -\frac{1}{4\pi}\int_V \frac{\text{div }\mathbf{F}}{r}\,d\tau. \tag{6.22}$$

In this result, V is the whole of space and r is the distance from the field point P to a general point in space. Notice that the surface integral in result (5.15), which in this case is an integral over the surface of an infinite sphere S with centre P, is taken to be zero. This is justified *a posteriori* from result (6.22). That is, we can establish that ϕ and $|\text{grad }\phi|$ are vanishingly small on S to such an order that the surface integral is zero.

Consider the behaviour of ϕ, when P is at large distance R, say, from the origin O of coordinates (Fig. 6.7). This behaviour can be estimated by separating the volume integral (6.22) into a volume integral I_1 over a sphere V_1, centre P and radius $\frac{1}{2}R$, plus a volume integral I_2 over the whole of the remainder of space V_2. With use of spherical polar coordinates (r, θ, α) centred on P, the volume element is

$$d\tau = r^2\,dr\,d\theta\,d\alpha\,\sin\theta,$$

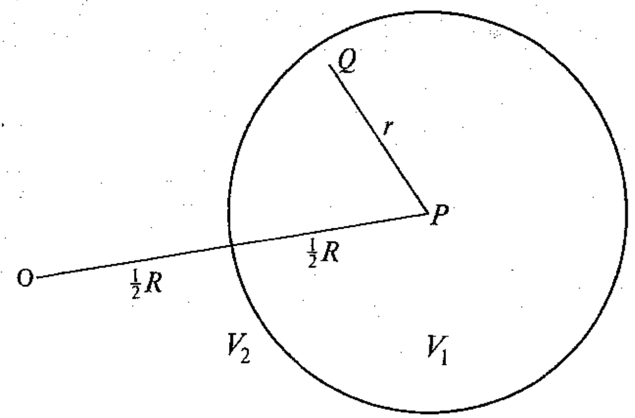

Fig. 6.7 Division of the infinite volume of integration in result (6.22) into two regions V_1, V_2, for the purpose of estimating the behaviour of $\phi(P)$ when P is at large distance R from the origin of coordinates O.

and therefore

$$I_1 = -\frac{1}{4\pi}\int_{V_1}\text{div }\mathbf{F}\,r\sin\theta\,dr\,d\theta\,d\alpha$$

$$= -\frac{1}{4\pi}\int_{r=0}^{\frac{1}{2}R}\int_{\theta=0}^{\pi}\int_{\alpha=0}^{2\pi}\text{div }\mathbf{F}\,r\sin\theta\,dr\,d\theta\,d\alpha.$$

We may apply a mean value theorem to this integral, to give

$$I_1 = -\frac{1}{4\pi}(\text{div }\mathbf{F})_M\int_{r=0}^{\frac{1}{2}R}\int_{\theta=0}^{\pi}\int_{\alpha=0}^{2\pi}\sin\theta r\,dr\,d\theta\,d\alpha,$$

where $(\text{div }\mathbf{F})_M$ is the value of div **F** at some point within V_1. Hence

$$I_1 = -\tfrac{1}{8}R^2(\text{div }\mathbf{F})_M.$$

Now $(\text{div }\mathbf{F})_M$ is a value of div **F** calculated at some point whose least distance from O is $\frac{1}{2}R$, and R is large. Thus $(\text{div }\mathbf{F})_M \doteqdot 0(R^{-2-\epsilon})$, and hence

$$I_1 = 0(R^{-\epsilon}).$$

For I_2 we have

$$\left|\int_{V_2}\frac{\text{div }\mathbf{F}}{r}\,d\tau\right| \leqslant \int_{V_2}\left|\frac{\text{div }\mathbf{F}}{r}\right|d\tau = \int_{V_2}\frac{|\text{div }\mathbf{F}|}{r^{\epsilon_1}r^{1-\epsilon_1}}\,d\tau,$$

where ϵ_1 is a positive number less than ϵ (i.e. $0 < \epsilon_1 < \epsilon$).

Now since the least value of r in V_2 is $\frac{1}{2}R$, this last integral is less than

$$\frac{1}{(\frac{1}{2}R)^{\epsilon_1}} \int_{V_2} \frac{|\text{div } \mathbf{F}|}{r^{1-\epsilon_1}} \, d\tau,$$

and the integral here, which involves the infinite domain $\frac{1}{2}R \leqslant r < \infty$, is a convergent integral, by virtue of the conditions on $|\text{div } \mathbf{F}|$. Thus $I_2 = 0(R^{-\epsilon_1})$. Hence in all $\phi(P)$ is at best $0(R^{-\epsilon_1})$ for large values of R, and therefore $|\text{grad } \phi|$ is at best $0(R^{-1-\epsilon_1})$. This establishes that the surface integral contribution to result (6.22) is indeed zero.

Consider now the vector field $\mathbf{G}$; since div $\mathbf{G} = 0$, a vector potential $\mathbf{A}$ can be introduced such that

$$\mathbf{G} = \text{curl } \mathbf{A}. \tag{6.23}$$

From relation (1), on formation of the curl of each side, we obtain (since curl $\mathbf{H} = 0$),

$$\text{curl (curl } \mathbf{A}) = \text{curl } \mathbf{F}. \tag{4}$$

The expansion formula (6.11) for curl (curl $\mathbf{A}$), in which Cartesian components of the vector fields are implied, gives an alternative form for equation (4), namely

$$\text{grad (div } \mathbf{A}) - \nabla^2 \mathbf{A} = \text{curl } \mathbf{F}.$$

The vector potential $\mathbf{A}$ is not unique, and we may impose the restriction

$$\text{div } \mathbf{A} = 0, \tag{5}$$

so that $\mathbf{A}$, considered in terms of its Cartesian components, now satisfies the equation

$$\nabla^2 \mathbf{A} = -\text{curl } \mathbf{F}. \tag{6}$$

This equation can now be separated into three scalar equations, so that the three components of $\mathbf{A}$ each satisfy an equation of Poisson's form, in which the prescribed right hand sides are respectively the Cartesian components of curl $\mathbf{F}$. Thus these equations can be solved in the same manner as equation (3), and give, on assembling the separate Cartesian components, the vector result

$$\mathbf{A}(P) = \frac{1}{4\pi} \int_V \frac{\text{curl } \mathbf{F}}{r} \, d\tau. \tag{6.24}$$

Once again, for the reasons previously stated, there are no surface integral contributions in result (6.24), and the vector field $\mathbf{G}$ (=curl $\mathbf{A}$) has similar behaviour to $\mathbf{F}$ at infinity. There remains, however, condi-

tion (5) to be satisfied by $\mathbf{A}$, so we must verify that expression (6.24) does ensure this.

Take (X_1, X_2, X_3) as the Cartesian coordinates of P, in order to distinguish P from the coordinates (x_1, x_2, x_3) of a general field point, which coordinates are to serve as parameters for integration.

In result (6.24),

$$r = \{(X_1 - x_1)^2 + (X_2 - x_2)^2 + (X_3 - x_3)^2\}^{\frac{1}{2}}, \tag{7}$$

and this quantity alone involves the coordinates of P. Thus, if (B_1, B_2, B_3) are the Cartesian components of curl $\mathbf{F}$,

$$4\pi \text{ div } \mathbf{A} = \iiint_V \left\{ B_1 \frac{\partial}{\partial X_1}\left(\frac{1}{r}\right) + B_2 \frac{\partial}{\partial X_2}\left(\frac{1}{r}\right) + \right.$$

$$\left. + B_3 \frac{\partial}{\partial X_3}\left(\frac{1}{r}\right) \right\} dx_1 \, dx_2 \, dx_3.$$

Now from (7),

$$\frac{\partial}{\partial X_1}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right), \quad \text{etc.}$$

Thus

$$4\pi \text{ div } \mathbf{A} = -\int_V \text{curl } \mathbf{F} \cdot \text{grad}\left(\frac{1}{r}\right) d\tau,$$

in which grad $(1/r)$ as well as curl $\mathbf{F}$ now appears as calculated with respect to the general field coordinates (x_1, x_2, x_3). Hence, with use of the expansion formula (5.10) and the identity div (curl $\mathbf{F}$) $\equiv 0$,

$$4\pi \text{ div } \mathbf{A} = -\int_V \text{div}\left(\frac{\text{curl } \mathbf{F}}{r}\right) d\tau$$

$$= -\int_S \frac{\text{curl } \mathbf{F}}{r} \cdot d\mathbf{S}, \quad \text{by Gauss's theorem.}$$

In this last expression, the closed surface S is an infinite sphere, with centre at the point P. If we consider first a sphere of large radius R, then the conditions of the theorem ensure that the surface integral is of order $R^{-1-\epsilon}$, and so tends to zero as $R \to \infty$. Hence div $\mathbf{A} = 0$, as required.

All that has been established so far is that if $\mathbf{F}$ can be expressed in the form $\mathbf{H} + \mathbf{G}$, with the given conditions on $\mathbf{H}$ and $\mathbf{G}$, then these two vector fields can be calculated uniquely. It remains to show that the sum of the two fields as calculated is actually equal to **F**.

Consider the vector field

$$\mathbf{K} = \mathbf{F} - \mathbf{H} - \mathbf{G}.$$

From the definitions of **H** and **G**, we have

$$\operatorname{div} \mathbf{K} = 0,$$
$$\text{and} \quad \operatorname{curl} \mathbf{K} = 0.$$

Also $|\mathbf{K}|$ is of order $r^{-1-\epsilon}$ as $r \to \infty$, where ϵ is now the least of the positive numbers introduced in the analysis, because $|\mathbf{F}|, |\mathbf{G}|, |\mathbf{H}|$ are all of this order at best. Hence theorem (6.19) is applicable, and we conclude therefore that $\mathbf{K} \equiv \mathbf{0}$.

Hence

$$\mathbf{F} = \mathbf{H} + \mathbf{G}.$$

Thus, finally, we have

$$\mathbf{F} = \operatorname{grad} \phi + \operatorname{curl} \mathbf{A},$$

where

$$\phi = -\frac{1}{4\pi} \int_V \frac{\operatorname{div} \mathbf{F}}{r}\, d\tau,$$

$$\mathbf{A} = \frac{1}{4\pi} \int_V \frac{\operatorname{curl} \mathbf{F}}{r}\, d\tau, \qquad (6.25)$$

where the volume integrals extend over the whole of space.

Examples, Chapter 6

1. Use Stokes's theorem applied to the vector field **F** with Cartesian components $\left(\frac{x_2^2}{a^2+x_1^2}, \frac{x_1^2}{a^2+x_1^2}, 0\right)$ to evaluate the integral

$$\int_S \left\{\frac{a^2(x_1-x_2)-x_1^2 x_2}{(a^2+x_1^2)^2}\right\} dS,$$

where the surface integral is taken over the quadrant $x_1, x_2 \geqslant 0$ of the ellipse

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \quad x_3 = 0.$$

Answer: $\frac{b}{2} + \frac{b^2}{2a} - \frac{\pi b^2}{4a} - \frac{b}{4\sqrt{2}} \log\left\{\frac{\sqrt{2}+1}{\sqrt{2}-1}\right\}.$

2. If the vector fields **A** and **B** satisfy the relations

$$\operatorname{curl} \mathbf{A} = \mathbf{B},$$
$$\operatorname{curl} \mathbf{B} = \mathbf{A},$$

show that, in a system of Cartesian components,

$$\nabla^2 \mathbf{A} + \mathbf{A} = 0.$$

Show also that $\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B}^2 - \mathbf{A}^2$.

3. Use Gauss's divergence theorem to show that if **F** is any vector field,

$$\int_V \operatorname{curl} \mathbf{F}\, d\tau = -\int_S \mathbf{F} \times d\mathbf{S},$$

where S is a closed surface enclosing the region V. (Use an expansion for $\operatorname{div}(\mathbf{F} \times \mathbf{G})$, and take **G** to be an arbitrary constant vector field.)

4. Verify that in a system of cylindrical polar coordinates (r, θ, z), with unit vectors $\mathbf{e}_i$, the vector field

$$\mathbf{F} = \frac{1}{r}\mathbf{e}_1$$

is solenoidal. Use the method of Section 6.6 to show that a vector potential for **F** is $-z\mathbf{e}_2/r$.

5. Show that if F_i are the components appropriate to cylindrical polar coordinates of a vector field **F** which satisfies the equation

$$\operatorname{curl}(\operatorname{curl} \mathbf{F}) = -c^2 \frac{\partial^2 \mathbf{F}}{\partial t^2},$$

where c is constant, and t is the time, then the three corresponding scalar equations are:

$$\frac{1}{r^2}\frac{\partial^2 F_1}{\partial \theta^2} + \frac{\partial^2 F_1}{\partial z^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial F_2}{\partial \theta}\right) - \frac{\partial^2 F_3}{\partial z\, \partial r} = c^2 \frac{\partial^2 F_1}{\partial t^2},$$

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rF_2) - \frac{1}{r}\frac{\partial F_1}{\partial \theta}\right\} + \frac{\partial^2 F_2}{\partial z^2} - \frac{1}{r}\frac{\partial^2 F_3}{\partial z\, \partial \theta} = c^2 \frac{\partial^2 F_2}{\partial t^2},$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial F_3}{\partial r} - r\frac{\partial F_1}{\partial z}\right\} + \frac{1}{r^2}\frac{\partial^2 F_3}{\partial \theta^2} - \frac{1}{r}\frac{\partial^2 F_2}{\partial \theta\, \partial z} = c^2 \frac{\partial^2 F_3}{\partial t^2}.$$

6. If u and v are scalar fields, show that grad $u \times$ grad v is a solenoidal field, and that

$$\tfrac{1}{2}(u \text{ grad } v - v \text{ grad } u)$$

is a vector potential for it.

7. A closed surface S encloses a region V in which the vector field $\mathbf{F}$ is solenoidal. If ϕ is a scalar field which assumes a constant value on S, show that

$$\int \text{grad } \phi \cdot \mathbf{F} \, d\tau = 0.$$

8. Use the expansion formula (6.9), and Stokes's theorem, to show that if ϕ, $\mathbf{F}$ are scalar and vector fields respectively,

$$\int_S \phi \text{ curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \phi \mathbf{F} \cdot d\mathbf{r} - \int_S (\text{grad } \phi \times \mathbf{F}) \cdot d\mathbf{S},$$

where the line integral is taken round the closed curve C bounding an open surface S. Deduce the result

$$\int_S (\text{grad } \phi \times \text{grad } \psi) \cdot d\mathbf{S} = \oint_C \phi \text{ grad } \psi \cdot d\mathbf{r}.$$

7 Electro-magnetic theory

The intention in this chapter, and in the next, is to provide a brief account of the subject of the title with the object of illustrating practical applications of the theory of previous chapters. The subjects of electro-magnetic theory and of hydrodynamics are extensive, and are served by many excellent text-books, but the concepts of vector field theory occur so naturally in both these subjects that it is desirable to give some account of them in this book.

We shall use the SI system of units, in which mass, length, time and electric current form the four primary quantities of the subject, and thus avoid the confusion of the Gaussian system with its separate development of units for electrostatic and magnetic phenomena, and consequent embarrassment when the two are related. Admittedly the modern system appears to be more complicated initially, but has the merit of producing ultimately the fundamental equations of the subject in very simple form.

7.1 The electrostatic field

If two point charges q_1 and q_2 are situated *in vacuo* at O and P respectively, each charge exerts a force on the other in the direction of the line joining the charges. Explicitly, the force $\mathbf{F}$ exerted on q_2 by q_1 is

$$\mathbf{F} = \frac{q_1 q_2 \mathbf{r}}{4\pi\epsilon_0 r^3}, \tag{7.1}$$

where r is the vector $\overline{OP}$. In this expression ϵ_0 is a constant, the dielectric constant of free space, and its dimensions are defined by relation (7.1)