Άσκηση 3η. Παράδοση: Πέμπτη 5/4 στην τάξη

1. **An one dimensional eigenvalue problem**
   We consider the following eigenvalue problem,

$$\text{Find } \lambda \in \mathbb{R} \text{ and } u \neq 0 \text{ such that:}$$

$$\begin{cases} -u''(x) = \lambda u(x), \quad x \in]0,1[\\ u(0) = 0 \\ u(1) = -u'(1) \end{cases} \tag{1}$$

   (α′) Write the variational formulation of problem (1).

   (β′) Conclude that there is a sequence of strictly positive eigenvalues $\lambda_n \to +\infty$ and that the associated eigenfunctions u_n form a Hilbert basis of $L^2(]0,1[)$.

   (γ′) Compute explicitly λ_n and u_n. Compute the limit $\lim_{n\to+\infty} \frac{\lambda_n}{n^2}$.

   (δ′) Let $u \in L^2(]0,1[)$ and $a_n = \int_0^1 u(x)u_n(x)$. Show that $\sum_{n\geq 1} a_n^2$ converges. To what?

   (ε′) Discuss as a function of α and f the existence and uniqueness of the solution of the following problem:

$$\begin{cases} -u''(x) + \alpha u(x) = f(x), \quad x \in]0,1[\\ u(0) = 0 \\ u(1) = -u'(1) \end{cases}$$

2. **Characterization of the first eigenvalue** Let Ω be a bounded sub-domain of $\mathbb{R}^n$. We consider the following eigenvalue problem :

$$\text{Find } u \in H, u \neq 0, \text{ s.t. } a(u,v) = \lambda(u,v), \ \forall v \in H$$

   We suppose that H is a closed subspace of $H^1(\Omega)$ dense in $L^2(\Omega)$ and $a(u,v)$ is a symmetric, coercive, bi-linear form on H. The L^2 inner product is denoted by (u,v) and we set $|u| = \sqrt{(u,u)}$. Let $(\lambda_m)_{m\geq 1}$ be the sequence of eigenvalues and $(u_m)_{m\geq 1}$ the corresponding eigenfunctions that form a Hilbert basis of $L^2(\Omega)$. For all $u \in H, \ u \neq 0$ we define :

$$\mathcal{R}(u) = \frac{a(u,u)}{|u|^2}.$$

(α′) Show that for all $u \in H, \ u \neq 0$ we have,

$$\mathcal{R}(u) = \frac{\sum_{n=1}^{+\infty} \lambda_n (u, u_n)^2}{\sum_{n=1}^{+\infty} (u, u_n)^2}$$

(β′) Show that :

$$\lambda_1 = \min_{u \in H, u \neq 0} \mathcal{R}(u)$$

3. Ω being an open bounded set of $\mathbb{R}^n$ and $k(x)$ a continuous, strictly positive function on $\overline{\Omega}$, we denote $(\lambda_m(k,\Omega))_{m \geq 1}$ the eigenvalues of the following Dirichlet problem:

$$\begin{aligned} &-\operatorname{div}(k(x)\nabla u) = \lambda u, && x \in \Omega, \\ &u = 0, && x \in \partial\Omega \end{aligned}$$

(α′) Show that, if $\Omega_1 \subset \Omega_2$ then $\lambda_1(k, \Omega_1) \geq \lambda_1(k, \Omega_2)$.

(β′) Show that, if $k_1(x) \leq k_2(x)$ then $\lambda_1(k_1, \Omega) \leq \lambda_1(k_2, \Omega)$.