EFFECTIVE EQUATIONS FOR LOCALIZATION AND SHEAR BAND FORMATION*

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Abstract. We develop a quantitative criterion determining the onset of localization and shear band formation at high-strain rate deformations of metals. We introduce an asymptotic procedure motivated by the theory of relaxation and the Chapman-Enskog expansion and derive an effective equation for the evolution of the strain rate, consisting of a second-order nonlinear diffusion regularized by fourth order effects and with parameters determined by the degree of thermal softening, strain hardening and strain-rate sensitivity. The nonlinear diffusion equation changes type across a threshold in the parameter space from forward parabolic to backward parabolic, what highlights the stable and unstable parameter regimes. The fourth order effects play a regularizing role in the unstable region of the parameter range.

Key words. shear band, localization, thermoviscoplasticity, Chapman-Enskog expansion

AMS subject classifications. 74C20, 74H40, 35K65, 35Q72

1. Introduction. One striking instance of material instability is observed in the course of deformations of metals at high strain-rates. It appears as an instability in shear and leads to regions of intensely concentrated shear strain, called shear bands. Since shear bands are often precursors to rupture, their study has attracted attention in the mechanics literature (*e.g.* [1, 7, 8, 13, 14, 17, 22, 24, 25]).

In experimental investigations of high strain-rate deformations of steels, observations of shear bands are typically associated with strain softening response – past a critical strain – of the measured stress-strain curve [8]. It was recognized by Zener and Hollomon [27] that the effect of the deformation speed is twofold: First, an increase in the deformation speed changes the deformation conditions from isothermal to nearly adiabatic. Second, strain rate has an effect *per se*, and needs to be included in the constitutive modeling.

Under isothermal conditions, metals, in general, strain harden and exhibit a stable response. As the deformation speed increases, the heat produced by the plastic work causes an increase in the temperature. For certain metals, the tendency for thermal softening may outweigh the tendency for strain hardening and deliver net softening. A destabilizing feedback mechanism is then induced, which operates as follows ([8]): Nonuniformities in the strain rate result in nonuniform heating. Since the material is softer at the hotter spots and harder at the colder spots, if heat diffusion is too weak to equalize the temperatures, the initial nonuniformities in the strain rate are, in turn, amplified. This mechanism tends to localize the total deformation into narrow regions. On the other hand, there is opposition to this process by "viscous effects" induced by strain-rate sensitivity. The outcome of the competition depends mainly on the relative weights of thermal softening, strain hardening and strain-rate sensitivity, as well as the loading circumstances.

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This qualitative scenario is widely accepted as the mechanism of shear band formation. However, despite several attempts, a quantitative explanation of the phenomenon of shear bands is presently lacking. Moreover, the above picture is somewhat imprecise in terms of what determines (or rules out) the onset of localization. It is this aspect of the problem that we attempt to address in the present work. We use the model

$$
\begin{aligned}
v_t &= \frac{1}{r}\,\sigma_x, \\
\theta_t &= \kappa\theta_{xx} + \sigma\gamma_t, \\
\gamma_t &= v_x,
\end{aligned}
\tag{1.1}
$$

where r, κ are non-dimensional constants, and the stress is given by an empirical power law in the normalized form

$$
\sigma = \theta^{-\alpha}\gamma^m\gamma_t^n\,. \tag{1.2}
$$

appropriate for the flow rule of a viscoplastic material exhibiting thermal softening, strain hardening and strain-rate sensitivity. The model and its relevance to the problem of shear band formation is explained in section 2.

There is an extensive literature on the problem, including experimental [7, 14], mechanics and linearized analysis (e.g. [8, 1, 13, 17, 24, 25] and references therein), numerical [26, 11], as well as nonlinear analysis [9, 18, 19, 20, 4, 3] and asymptotic analysis studies [10, 12, 25]. With regard to the analysis of the shear band formation process, analytical results account for either the case where the forcing is effected by a boundary force [20, 22] causing a shear band at the boundary, or in situations where the initial data involve a localization in shear (or in the temperature) and the subsequent evolution leads to an intensification process to a fully developed band [3, 23]. It is indicated by numerical evidence in [24] and the analysis in [20, 3] that a collapse of the stress-diffusion mechanisms is associated with the development of the bands. There is a class of special solutions to (1.1) describing uniform shearing

$$
\begin{aligned}
v_s &= x, \\
\gamma_s &= t + \gamma_0, \\
\theta_s &= \left[\theta_0^{1+\alpha} + \frac{1+\alpha}{m+1}\Big[(t+\gamma_0)^{m+1} - \gamma_0^{m+1}\Big]\right]^{\frac{1}{1+\alpha}}, \\
\sigma_s &= \theta_s^{-\alpha}(t)(t+\gamma_0)^m\,,
\end{aligned}
\tag{1.3}
$$

and much of the analysis on (1.1) has centered on the issue of their stability. The form of (1.3) suggests the change of variables (3.3) that transforms the stability problem into the study of the asymptotic behavior for the reaction-diffusion type system (3.4), see section 3. In the special case of a fluid with temperature dependent viscosity ($m = 0$) the kinematic equation decouples from the remaining equations and the problem reduces to the study of the simplified model (4.1)-(4.2). This simpler system is indeed the one that has been analyzed in most detail both analytically [3, 9, 18], but also in numerical investigations [26, 11]. Its rescaled variant (4.4) admits invariant rectangles in the parameter range $q = -\alpha + n > 0$ but misses this property in the range $q = -\alpha + n < 0$. It is this dichotomy that provides a quantitative threshold to stability, as shown in section 4: In the parameter range $q > 0$ the invariant rectangles yield asymptotic stability of the uniform shearing solution, *cf.* Theorem 4.1. By contrast, in the complementary region $q < 0$ moderate perturbations of the uniform solutions can lead to instability and formation of shear bands, *cf.* Theorem 4.2.

The analysis on invariant domains of section 4 suggests a connection of the present problem with the theory of relaxation systems (*e.g.* [5]) that turns out to be instrumental for understanding the onset of localization. This connection is studied in detail in section 5, and motivates the derivation of an effective equation for the onset of localization in section 6. We outline the result in the following: Let T be a parameter describing a time-scale, and consider a change of variables of the form

$$\theta(x,t) = (t+1)^{\frac{m+1}{\alpha+1}}\,\Theta^T\big(x, \frac{s(t)}{T}\big), \quad v_x(x,t) = V_x^T\big(x, \frac{s(t)}{T}\big) \tag{1.4}$$

where $s(t)$ is an appropriate rescaling of time (in fact, see (6.1) for the full transformation). The new functions $(U^T, \Theta^T, \Gamma^T, \Sigma^T)$ with $U^T = V_x^T$ satisfy the system (6.3). It is clear that if $(U^T, \Theta^T, \Gamma^T, \Sigma^T)$ stabilizes as $T \to \infty$ then its limiting profile will describe the asymptotic form of (v_x, θ, γ) as $t \to \infty$. This reduces the problem of studying the asymptotic behavior into the problem of identifying the large T behavior of (6.3), which lies within the realm of relaxation theory. Using a technique analogous to the Chapman-Enskog expansion(*e.g.* [5]), we show in section 6 that $U^T = V_x^T$ satisfies for $T >> 1$ and $r = O(T)$ the effective equation

$$\partial_s U = \partial_{xx}\Big(c\,U^p + \frac{\lambda c^2}{T}(\beta s + 1)U^{p-1}\partial_{xx}U^p\Big), \tag{1.5}$$

within order $O(\frac{1}{T^2})$ and with parameters $p = \frac{q}{1+\alpha} = \frac{-\alpha+m+n}{1+\alpha}$, $\beta = \frac{m+1}{1+\alpha}$, $c = \beta^{\frac{\alpha}{1+\alpha}}$ and the coefficient of the fourth order term

$$\lambda = \frac{\alpha(1+m+n) - m(m+1)}{(m+1)(1+\alpha)}\,.$$

We note that (1.5) changes type from forward parabolic when $q = -\alpha + m + n > 0$ to backward parabolic when $q = -\alpha + m + n < 0$ what captures the parameter regime associated with the onset of localization. We also note that in the region of instability $q < 0$ the coefficient $\lambda > 0$ and the fourth order term has a regularizing effect. Numerical comparisons between the effective equation (1.5) and the system (6.3) are performed in section 6 and indicate good agreement between the effective and the actual problem.

2. The Nature of Shear Band Formation.

2.1. Description of the model. As shear bands appear and propagate as one-dimensional structures (up to interaction times) most investigations have focused on one-dimensional, simple shearing deformations. In a Cartesian coordinate system an infinite plate, located between the planes $x = 0$ and $x = d$, is subjected to simple shear. The thermomechanical process is described (upon neglecting the normal stresses) by the list of variables: velocity in the shearing direction $v(x,t)$, shear strain $\gamma(x,t)$, temperature $\theta(x,t)$, heat flux $Q(x,t)$, and shear stress $\sigma(x,t)$. They are connected through the balance of linear momentum

$$\rho v_t = \sigma_x\,, \tag{2.1}$$

the kinematic compatibility relation

$$\gamma_t = v_x\,, \tag{2.2}$$

and the balance of energy equation

$$c\rho\theta_t = Q_x + \beta\sigma\gamma_t \,, \tag{2.3}$$

where ρ is the reference density, c is the specific heat and β the portion of plastic work converted to heat. The upper plate is subjected to a prescribed constant velocity V while the lower plate is held at rest: $v(0,t) = 0$, $v(d,t) = V$. It is further assumed that the plates are thermally insulated: $\theta_x(0,t) = 0$, $\theta_x(d,t) = 0$. Thermal insulation is appropriate for the analysis of shear band formation, since heat transfer at positions distant from the bands via radiation is negligible at sufficiently short loading times.

For the heat flux we will either use the adiabatic assumption $Q = 0$ or a Fourier law $Q = k\theta_x$ with the thermal diffusivity parameter k. Imposing adiabatic conditions projects the belief that, at high strain rates, heat diffusion operates at a slower time scale than the one required for the development of a shear band. It appears a plausible assumption for the shear band initiation process, but not necessarily for the evolution of a developed band, due to the high temperature differences involved.

For the shear stress we set

$$\sigma = f(\theta, \gamma, \gamma_t) \,, \tag{2.4}$$

where f is a smooth function with $f(\theta,\gamma,0) = 0$ and $f_p(\theta,\gamma,p) > 0$, for $p \neq 0$. In terms of classification, the resulting model belongs to the framework of one-dimensional thermoviscoelasticity and is compatible with the requirements imposed by the Clausius-Duhem inequality.

It is instructive to interpret (2.4) in the context of a constitutive theory for thermal elastic-viscoplastic materials. In this context, it is assumed that the shear strain γ is decomposed, additively, into elastic and plastic components: $\gamma = \gamma^e + \gamma^p$. The elastic component γ^e satisfies linear elasticity with shear modulus G_e, that is $\gamma^e = \frac{1}{G_e}\sigma$. The evolution of the plastic component is dictated by a plastic flow rule:

$$\gamma^p_t = g(\theta, \gamma^p, \sigma) \quad \text{or} \quad \sigma = f(\theta, \gamma^p, \gamma^p_t) \,, \tag{2.5}$$

where g is an increasing function in the variable σ, and $f(\theta,\gamma,\cdot)$ is the inverse function of $g(\theta,\gamma,\cdot)$. In summary,

$$\begin{aligned} \frac{1}{G_e}\sigma + \gamma^p &= \gamma, \\ \frac{1}{G_e}\sigma_t + g(\theta,\gamma^p,\sigma) &= v_x \,. \end{aligned} \tag{2.6}$$

Note that (2.4) can be obtained from the constitutive theory (2.6) in the limit as the elastic shear modulus $G_e \to \infty$. Accordingly, γ should then be interpreted as the plastic strain and (2.4) as an inverted plastic flow rule.

Viewing (2.4) as a plastic flow rule suggests the terminology: The material exhibits thermal softening at state variables (θ,γ,p) where $f_\theta(\theta,\gamma,p) < 0$, strain hardening at state variables where $f_\gamma(\theta,\gamma,p) > 0$, and strain softening when $f_\gamma(\theta,\gamma,p) < 0$. The amounts of the slopes of f in the directions θ, γ and p measure the degree of thermal softening, strain hardening (or softening) and strain-rate sensitivity, respectively. The difficulty of performing high strain-rate experiments causes uncertainty as to the specific form of the constitutive relation (2.4). Examples that have been extensively used are the power law or the Arrhenius law outlined later.

2.2. Non-dimensionalization. To turn the system into a dimensionless form we introduce the following non-dimensional variables:

$$\hat{x} = \frac{x}{d}, \qquad \hat{t} = t\dot{\gamma}_0, \qquad \hat{v} = \frac{v}{V}, \qquad \hat{\theta} = \frac{\theta}{\tau_0/\rho c}, \qquad \hat{\sigma} = \frac{\sigma}{\tau_0}, \tag{2.7}$$

where we introduce a nominal stress τ_0 to be appropriately selected later, a nominal temperature $\theta_0 = \frac{\tau_0}{\rho c}$ and a nominal strain rate $\dot{\gamma}_0 = \frac{V}{d}$.

With these choices we obtain the nondimensional system

$$\begin{aligned} \hat{v}_{\hat{t}} &= \frac{1}{r}\ \hat{\sigma}_{\hat{x}}, \\ \hat{\theta}_{\hat{t}} &= \kappa \hat{\theta}_{\hat{x}\hat{x}} + \beta\hat{\sigma}\ \hat{v}_{\hat{x}}, \\ \hat{\gamma}_{\hat{t}} &= \hat{v}_{\hat{x}}, \end{aligned} \tag{2.8}$$

where the nondimensional numbers are

$$r = \frac{\rho V^2}{\tau_0}, \quad \kappa = \frac{k}{\rho c V d}, \tag{2.9}$$

while β is non-dimensional by its very nature. The number r is a ratio of inertial to viscoplastic stresses and depends on the choice of the normalizing stress τ_0. The constitutive law (2.4) turns to the non-dimensional form

$$\hat{\sigma} = \hat{f}(\hat{\theta}, \hat{\gamma}, \hat{\gamma}_{\hat{t}}) = \frac{1}{\tau_0} f(\frac{\tau_0}{\rho c}\hat{\theta}, \hat{\gamma}, \dot{\gamma}_0 \hat{\gamma}_{\hat{t}}).$$

The freedom in the choice of τ_0 is useful in normalizing the form of $\hat{f}$.

2.3. Power laws. In the experimental literature on shear bands at high strain-rates there is extensive use of constitutive laws in the form of power laws (*e.g* [15], [14]),

$$\sigma = G\ \left(\frac{\theta}{\theta_r}\right)^{-\alpha}\ \left(\frac{\gamma}{\gamma_r}\right)^{m}\ \left(\frac{\gamma_t}{\dot{\gamma}_r}\right)^{n} = G_0\ \theta^{-\alpha}\gamma^m\gamma_t^n. \tag{2.10}$$

Here, α, m, n denote the thermal softening, strain hardening and strain rate sensitivity parameters respectively, G is a material constant, and $\theta_r,\ \gamma_r, \dot{\gamma}_r$ are some reference values for temperature, strain and strain rate respectively. Specifically, $\gamma_r \simeq 0.01$ is the strain at yield in a quasi-static simple shear test at a nominal strain rate $\dot{\gamma} = 10^{-4}/s$ for most steels. There is no unique choice for the other reference values, but the simplest choices are $\dot{\gamma}_r = 10^3/s,\ \theta_r = 300K$. This corresponds to the nominal strain rate and ambient temperature of the usual torsional experiment. In (2.10) θ and θ_r are measured in Kelvin. This power law model has been used extensively to model steels that exhibit shear bands, [15], [14], and is entirely empirical, but it allows considerable flexibility in fitting experimental data over an extended range. According to experimental data for most steels we have $\alpha = O(10^{-1}),\ m = O(10^{-2})$ and $n = O(10^{-2})$.

The freedom in the choice of the nominal stress τ_0 is useful to simplify the form of $\hat{f}$. For (2.10), if we select τ_0 such that

$$\frac{1}{\tau_0} G \left(\frac{\frac{\tau_0}{\rho c}}{\theta_r}\right)^{-\alpha}\ \left(\frac{1}{\gamma_r}\right)^{m}\ \left(\frac{\dot{\gamma}_0}{\dot{\gamma}_r}\right)^{n} = 1,$$

it yields the non-dimensional form $\hat{\sigma} = \hat{\theta}^{-\alpha}\hat{\gamma}^m\hat{\gamma}_t^n$.

2.4. The mathematical model. We collect the non-dimensional form of the equations, dropping the hats, in the form

$$\begin{aligned} v_t &= \frac{1}{r}\,\sigma_x, \\ \theta_t &= \kappa\theta_{xx} + \sigma\gamma_t, \\ \gamma_t &= v_x, \\ \sigma &= f(\theta, \gamma, \gamma_t), \end{aligned} \tag{2.11}$$

where r, κ are given by (2.9) and we have taken the (not so important) constant $\beta = 1$. For the stress we use the empirical power law, which for the appropriate choice of τ_0 takes the normalized form

$$\sigma = \theta^{-\alpha}\gamma^m\gamma_t^n\,. \tag{2.12}$$

The parameters $\alpha > 0$, m and $n > 0$ serve as measures of the degree of thermal softening, strain hardening (or softening) and strain-rate sensitivity. Another commonly used constitutive relation is the Arrhenius law

$$\sigma = e^{-\alpha\theta}\gamma_t^n\,. \tag{2.13}$$

In the form (2.13) the Arrhenius law does not exhibit any strain hardening and the parameters α and n measure the degree of thermal softening and strain-rate sensitivity, respectively. The boundary conditions are prescribed velocities at the ends of the plates, in the non-dimensional form

$$v(0,t) = 0, \quad v(1,t) = 1, \quad t \geq 0\,, \tag{2.14}$$

and thermal insulation at the two ends

$$\theta_x(0,t) = 0, \quad \theta_x(1,t) = 0, \quad t \geq 0. \tag{2.15}$$

We impose initial conditions

$$v(x,0) = v_0(x)\,,\ \theta(x,0) = \theta_0(x) > 0,\ \gamma(x,0) = \gamma_0(x) > 0, \quad x \in [0,1]\,. \tag{2.16}$$

For the initial data we take $v_{0x} > 0$, in which case a maximum principle shows that $\gamma_t = v_x > 0$ at all times and thus all powers are well defined.

2.5. Isothermal vs. adiabatic deformations. To illustrate the effect of thermal softening on *spatially uniform* deformations, the isothermal and adiabatic cases are contrasted. Consider a deformation where the plate is subjected to steady shearing, with boundary velocities: $v = 0$ at $x = 0$ and $v = 1$ at $x = 1$.
(i) In an *isothermal* deformation the temperature is kept constant, say θ_0, by appropriately removing the produced heat due to the plastic work. The "measured" stress-strain response in this idealized situation coincides with the $\sigma - \gamma$ graph of the function $\sigma = f(\theta_0, \gamma, 1)$. The slope of the graph is measured by $f_\gamma(\theta_0, \gamma, 1)$, and, for a strain-hardening material the graph $\sigma - \gamma$ is monotonically increasing.
(ii) The situation in an *adiabatic* deformation is understood by studying a special class of solutions describing *uniform shearing.* These are:

$$\begin{aligned} v_s(x,t) &= x, \\ \gamma_s(x,t) &= \gamma_s(t) = t + \gamma_0, \\ \theta_s(x,t) &= \theta_s(t)\,, \end{aligned}$$

where

$$\frac{d\theta_s}{dt} = f(\theta_s, t + \gamma_0, 1),$$
$$\theta_s(0) = \theta_0$$

and γ_0, θ_0 are positive constants, standing for the initial values of the strain and temperature. The resulting stress is given by the graph of the function

$$\sigma_s(t) = f(\theta_s(t), t + \gamma_0, \dot{\gamma}_0)\,,$$

which may be interpreted as stress vs. time but also as stress vs. (average) strain. This effective stress-strain curve coincides with the $\sigma_s - t$ graph, and the material exhibits effective hardening in the increasing parts of the graph and effective softening in the decreasing parts. The slope is determined by the sign of the quantity $f_\theta\, f + f_\gamma$, when this sign is negative the combined effect of strain hardening and thermal softening delivers net softening. For instance, for a strain-hardening ($m > 0$) power law (2.12) the uniform shearing solution reads

$$\begin{aligned}
\gamma_s(t) &= t + \gamma_0, \\
\theta_s(t) &= \left[\theta_0^{1+\alpha} + \frac{1+\alpha}{m+1}\left[(t+\gamma_0)^{m+1} - \gamma_0^{m+1}\right]\right]^{\frac{1}{1+\alpha}}, \\
\sigma_s(t) &= \theta_s^{-\alpha}(t)(t+\gamma_0)^m\,,
\end{aligned} \tag{2.17}$$

and a simple computation yields

$$\frac{d\sigma_s}{dt} = \theta_s(t)^{-2\alpha-1}\gamma_s(t)^{2m}\left[\frac{-\alpha+m}{m+1} + \frac{m}{(t+\gamma_0)^{m+1}}\left[\theta_0^{1+\alpha} - \frac{1+\alpha}{m+1}\gamma_0^{m+1}\right]\right].$$

For parameter values ranging in the region $m > \alpha$ the graph $\sigma_s(t)$-t is increasing and the material exhibits net hardening. By contrast, for parameter values ranging in the region $m < \alpha$, $\sigma_s(t)$ may initially increase but eventually decreases with t. In this range, the combined effect of thermal softening and strain-hardening results to net softening, and it is precisely this effect that is considered as a necessary (though not sufficient) cause of the shear band formation process.

2.6. Strain softening vs. strain-rate sensitivity. It is generally maintained that strain softening has a destabilizing influence, tending to amplify small nonuniformities. To illustrate the nature of the instability consider the model

$$\begin{aligned}
v_t &= \tau(\gamma)_x, \\
\gamma_t &= v_x\,,
\end{aligned} \tag{2.18}$$

with $\tau'(u) < 0$. This model describes isothermal motions of a strain-softening, inelastic material. The system (2.18) is elliptic in the t-direction, and the initial value problem is ill-posed. The uniform shearing solution

$$\hat{v} = x\,, \quad \hat{\gamma} = t + \gamma_0\,,$$

γ_0 constant, is still a special class of solutions to this problem.

By contrast, strain-rate dependence tends to diffuse nonuniformities in the strain-rate and/or the stress, and it may hinder or even altogether suppress instability. That is confirmed, for example, by considering the system

$$\begin{aligned} v_t &= (\tau(\gamma) v_x^n)_x, \\ \gamma_t &= v_x \, , \end{aligned} \tag{2.19}$$

with $\tau(u) > 0$ and $\tau'(u) < 0$. This system is a parabolic regularization of the elliptic problem (2.18) and it is precisely the competition between an ill-posed equation and a regularizing effect that is hidden behind the shear band formation problem. If one considers the linearization of the uniform shear solution

$$v = x + \hat{V} \, , \quad \gamma = t + \gamma_0 + \hat{\Gamma}, \tag{2.20}$$

we see that the linearized problem from $(\hat{V}, \hat{\Gamma})$ reads

$$\begin{aligned} \hat{V}_t &= n\tau(t+\gamma_0)\hat{V}_{xx} + \tau'(t+\gamma_0)\hat{\Gamma}_x, \\ \hat{\Gamma}_t &= \hat{V}_x. \end{aligned} \tag{2.21}$$

The form of the linearized problem is a parabolic regularization of an elliptic initial value problem and indicates that strain-rate sensitivity provides a stabilizing effect to the destabilizing mechanism of strain softening.

To quantify the role of the various effects – thermal softening, strain-hardening, strain-rate sensitivity and heat diffusion – at the level of the linearized problem, Molinari and Clifton [17] suggest the notion that the uniform shearing solution is stable if the perturbation of the uniform shearing solution grows slower than the basic solution (2.17), and is unstable if the perturbation grows faster than the solution (2.17). It has been conjectured in [17], based on linearized analysis of such "relative perturbations" and some additional plausibility arguments, that for power laws the uniform shearing solution is stable in the parameter range $q = m + n - \alpha > 0$ and unstable in the complementary region $q = m + n - \alpha < 0$. The relative perturbation analysis is not straightforward to rigorously justify, as it requires stability analysis for non-autonomous systems. Nevertheless, the linearized analysis was carried out for (2.21) using maximum principles, see [22], and in this case the conjecture was verified.

Nonlinear analysis has been more efficient for providing stability results and validating the above criterion in various special cases [9, 18, 19, 21]. A complete understanding for the full model only exists in the case of stress boundary conditions which is energetically fairly demanding: it is shown in [20] that unstable response and formation of shear bands occurs in certain parameter regimes, and that the process of shear band formation is concurrent with a collapse of the stress diffusion mechanisms of the material. The case of velocity boundary conditions is energetically more benign and closer to the experimental setup. For this case, the only available instability results concern a temperature dependent Newtonian fluid [3] (or a strain softening rate-sensitive solid [22]) and indicate that a large perturbation of the temperature (or the strain) can lead to localization and formation of bands at large times. However, a precise quantification of the onset of localization is at present unavailable and will be pursued in later sections of this work.

2.7. Effect of thermal diffusion. Although the early deformation can with no considerable error be regarded as adiabatic, when localization sets in and temperature gradients across a band become very large, thermal diffusion effects can no longer be regarded as negligible. The morphology of a fully formed band in the late stages of deformation is thus influenced by heat conduction balancing the heat production from plastic work. In [26], an extensive numerical treatment of fully developed shear bands, Walter noticed that, due to heat conduction, the strain rate essentially becomes independent of time in late stages of deformation, even though the temperature and stress continue to evolve. We refer to [16, 10, 12] for studies of the effect of heat conduction in a variety of models.

3. Adiabatic shear. We consider now the adiabatic form ($\kappa = 0$) of the non-dimensional system (2.11) with a power law stress

$$\begin{aligned} v_t &= \frac{1}{r}\,\sigma_x, \\ \theta_t &= \sigma\gamma_t, \\ \gamma_t &= v_x, \\ \sigma &= \theta^{-\alpha}\gamma^m\gamma_t^n. \end{aligned} \tag{3.1}$$

In this section we outline various formulations of the problem that are useful in the sequel.

Stress formulation. There is a reformulation of the problem (3.1) in the form of a reaction-diffusion system, that has been quite instructive in the development of shear-band theory (see [20]). Suppose that σ, θ, γ are considered the independent variables. A simple but lengthy computation shows that they satisfy the reaction-diffusion system

$$\begin{aligned} \sigma_t &= \frac{n}{r}\,\theta^{-\frac{\alpha}{n}}\,\gamma^{\frac{m}{n}}\,\sigma^{\frac{n-1}{n}}\sigma_{xx} + \left(-\alpha\frac{\sigma}{\theta} + \frac{m}{\gamma}\right)\theta^{\frac{\alpha}{n}}\,\gamma^{-\frac{m}{n}}\,\sigma^{\frac{n+1}{n}}, \\ \gamma_t &= \theta^{\frac{\alpha}{n}}\,\gamma^{-\frac{m}{n}}\,\sigma^{\frac{1}{n}}, \\ \theta_t &= \theta^{\frac{\alpha}{n}}\,\gamma^{-\frac{m}{n}}\,\sigma^{\frac{n+1}{n}}. \end{aligned} \tag{3.2}$$

Conversely, given a solution (σ, θ, γ) of (3.2), if we define v_x by

$$v_x = \theta^{\frac{\alpha}{n}}\,\gamma^{-\frac{m}{n}}\,\sigma^{\frac{1}{n}}$$

then (v, θ, γ) satisfies (3.1).

Time rescaling. Motivated by the form of the uniform shearing solutions (2.17), one may introduce a rescaling of the dependent variables and time in the form:

$$\begin{aligned} &\theta(x,t) = (t+1)^{\frac{m+1}{\alpha+1}}\Theta(x,\tau(t)), \quad \gamma(x,t) = (t+1)\Gamma(x,\tau(t)), \\ &\sigma(x,t) = (t+1)^{\frac{m-\alpha}{\alpha+1}}\Sigma(x,\tau(t)), \quad v(x,t) = V(x,\tau(t)), \quad \tau = \ln(1+t). \end{aligned} \tag{3.3}$$

In the new variables $(V, \Theta, \Gamma, \Sigma)$ the system (3.1) becomes :

$$\begin{aligned} V_\tau &= \frac{1}{r}e^{\frac{m+1}{1+\alpha}\tau}\,\Sigma_x, \\ \Gamma_\tau &= V_x - \Gamma, \\ \Theta_\tau &= \Sigma V_x - \frac{m+1}{1+\alpha}\Theta, \\ \Sigma &= \Theta^{-\alpha}\Gamma^m V_x^n. \end{aligned} \tag{3.4}$$

Accordingly, the system (3.2) takes the form,

$$
\begin{aligned}
\Sigma_\tau &= \frac{n}{r}\, e^{\frac{m+1}{1+\alpha}\tau}\Theta^{-\frac{\alpha}{n}}\ \Gamma^{\frac{m}{n}}\ \Sigma^{\frac{n-1}{n}}\Sigma_{xx} \\
&\qquad + \left(-\alpha\frac{\Sigma}{\Theta}+\frac{m}{\Gamma}\right)\Theta^{\frac{\alpha}{n}}\ \Gamma^{-\frac{m}{n}}\ \Sigma^{\frac{n+1}{n}} + \Sigma\frac{\alpha-m}{1+\alpha}, \\
\Gamma_\tau &= \Theta^{\frac{\alpha}{n}}\ \Gamma^{-\frac{m}{n}}\ \Sigma^{\frac{1}{n}} - \Gamma, \\
\Theta_\tau &= \Theta^{\frac{\alpha}{n}}\ \Gamma^{-\frac{m}{n}}\ \Sigma^{\frac{n+1}{n}} - \frac{m+1}{1+\alpha}\Theta.
\end{aligned}
\tag{3.5}
$$

Various properties of (3.5) will be noted in forthcoming sections. To understand its usefulness, note that the rescaled variants of the uniform shearing solutions (2.17), given by

$$
\theta_s(t) = (t+1)^{\frac{m+1}{\alpha+1}}\Theta_s(\tau(t)), \quad \gamma_s(t) = (t+1)\Gamma_s(\tau(t)), \quad \sigma_s(t) = (t+1)^{\frac{m-\alpha}{\alpha+1}}\Sigma_s(\tau(t)),
$$

have the long time behavior

$$
\begin{aligned}
&\Theta_s(\tau) \to \left(\frac{1+\alpha}{1+m}\right)^{\frac{1}{1+\alpha}}, \qquad && \Gamma_s(\tau) \to 1, \\
&\Sigma_s(\tau) \to \left(\frac{1+\alpha}{1+m}\right)^{-\frac{\alpha}{1+\alpha}}, \qquad && \Sigma_s^{\frac{1}{n}}\Theta_s^{\frac{\alpha}{n}}\Gamma_s^{-\frac{m}{n}} \to 1,
\end{aligned}
\tag{3.6}
$$

as $\tau \to \infty$ independently of the values of the initial constants θ_0, γ_0.

4. Non-Newtonian fluids with temperature dependent viscosity. Various simplified models have been used in the mathematical and mechanics literature of shear band formation. One example is models that neglect thermal softening like (2.19). Another class neglects the effect of strain hardening ($m = 0$) in which case the kinematic compatibility equation decouples and the system consists of two equations. In the adiabatic case ($\kappa = 0$), the resulting model reads

$$
\begin{aligned}
v_t &= \frac{1}{r}\sigma_x, \\
\theta_t &= \sigma\, v_x\,,
\end{aligned}
\tag{4.1}
$$

with the power law

$$
\sigma = \theta^{-\alpha} v_x^n\,, \tag{4.2}
$$

and may be viewed as describing a non-Newtonian fluid with temperature dependent viscosity. The problem is set in $[0,1]$ with velocity boundary conditions (2.14) and the objective is to examine the stability of uniform shearing flows (2.17). In this context, the question becomes whether the destabilizing effect of the decreasing and spatially nonhomogeneous viscosity is sufficiently powerful to overcome the stabilizing influence of momentum diffusion and induce localization of shear. We introduce the change of variables

$$
\begin{aligned}
&\theta(x,t) = (t+1)^{\frac{1}{\alpha+1}}\Theta(x,\tau(t)), \quad \sigma(x,t) = (t+1)^{-\frac{\alpha}{\alpha+1}}\Sigma(x,\tau(t))\,, \\
&v(x,t) = V(x,\tau(t)), \quad \tau = \ln(1+t).
\end{aligned}
\tag{4.3}
$$

(analogous to (3.3)) and obtain the rescaled system

$$
\begin{aligned}
\Sigma_\tau &= \frac{n}{r}\, e^{\frac{1}{1+\alpha}\tau}\Theta^{-\frac{\alpha}{n}}\ \Sigma^{\frac{n-1}{n}}\Sigma_{xx} + \left(-\alpha\Theta^{\frac{\alpha}{n}-1}\Sigma^{\frac{n+1}{n}} + \frac{\alpha}{1+\alpha}\right)\Sigma, \\
\Theta_\tau &= \left(\Theta^{\frac{\alpha}{n}-1}\ \Sigma^{\frac{n+1}{n}} - \frac{1}{1+\alpha}\right)\Theta\,.
\end{aligned}
\tag{4.4}
$$

(V, Θ, Σ) also satisfy the equations

$$
\Sigma = \Theta^{-\alpha} V_x^n, \tag{4.5}
$$

$$
V_\tau = \frac{1}{r} e^{\frac{1}{1+\alpha}\tau}\Sigma_x. \tag{4.6}
$$

The rescaled variants of the uniform shearing solution (2.17) enjoy the asymptotic behavior

$$
\begin{aligned}
&\Theta_s(\tau) \to (1+\alpha)^{\frac{1}{1+\alpha}}, \\
&\Sigma_s(\tau) \to (1+\alpha)^{-\frac{\alpha}{1+\alpha}}, \\
&\Sigma_s^{\frac{1}{n}}\Theta_s^{\frac{\alpha}{n}} \to 1\,, \qquad\qquad \text{as} \quad \tau \to \infty\,.
\end{aligned}
$$

In this section we discuss the stability properties of the uniform shearing solutions in the model (4.4).

4.1. Equilibria and orbits. Consider the reaction part of the system (4.4), i.e. the associated system of ordinary differential equations ,

$$
\begin{aligned}
\Sigma_\tau &= -\alpha\Sigma\left(\Theta^{\frac{\alpha}{n}-1}\Sigma^{\frac{n+1}{n}} - \frac{1}{1+\alpha}\right), \\
\Theta_\tau &= \Theta\left(\Theta^{\frac{\alpha}{n}-1}\ \Sigma^{\frac{n+1}{n}} - \frac{1}{1+\alpha}\right).
\end{aligned}
\tag{4.7}
$$

We observe that

$$
\frac{\Sigma_\tau}{\Sigma} + \alpha\frac{\Theta_\tau}{\Theta} = 0 \Longleftrightarrow \partial_\tau\left(\frac{\Sigma}{\Theta^{-\alpha}}\right) = 0,
$$

which implies that $\Sigma\ \Theta^\alpha$ is constant along the orbits of (4.7). The equilibria of (4.7) are located on the curve

$$
\Sigma = \left(\frac{1}{1+\alpha}\right)^{\frac{n}{n+1}}\ \Theta^{\frac{n-\alpha}{n+1}}. \tag{4.8}
$$

The curve of equilibria changes monotonicity depending on the sign of

$$
q = -\alpha + n\,, \tag{4.9}
$$

it is increasing for $q > 0$ and decreasing for $q < 0$. In Figure 4.1 the orbits of the system (4.7) along with the curve (4.8) are presented in the case $q > 0$ as well as for the case $q < 0$. The point

$$
(\Theta_m, \Sigma_m) = ((1+\alpha)^{\frac{1}{1+\alpha}}, (1+\alpha)^{-\frac{\alpha}{1+\alpha}}), \tag{4.10}
$$

see Figure 4.1, corresponds to the asymptotic state of uniform shear in the rescaled variables. When the uniform shearing solution (3.6) is asymptotically stable then trajectories of the system (4.4) should approach the point (Θ_m, Σ_m) as $\tau \to \infty$.

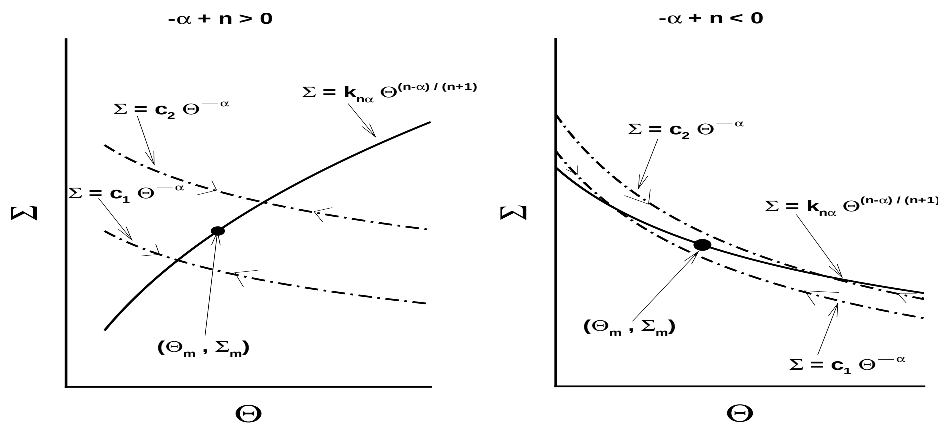


FIG. 4.1. *Orbits of o.d.e system* (4.7), $k_{n\alpha} = \left(\frac{1}{\alpha+1}\right)^{\frac{n}{n+1}}$.

4.2. Invariant regions, the stable regime. In the case $q = -\alpha + n > 0$ any rectangle

$$[\Theta_-, \Theta_+] \times [\Sigma_-, \Sigma_+] \quad \text{with } (\Theta_-, \Sigma_-)\,,\ (\Theta_+, \Sigma_+) \text{ equilibria,}$$

is invariant under the flow of (4.7). The theory of invariant regions for parabolic systems [6] then implies that such rectangles are also invariant under the flow of the reaction-diffusion system (4.4). For initial data taking values in one of the invariant rectangles

$$\begin{aligned} \Theta_- &< \Theta_0(x) < \Theta_+, \\ \Sigma_- &< \Sigma_0(x) < \Sigma_+, \end{aligned}$$

the solutions of (4.4) satisfy

$$\begin{aligned} \Theta_- &< \Theta(x,\tau) < \Theta_+, \\ \Sigma_- &< \Sigma(x,\tau) < \Sigma_+, \\ \Sigma_-^{\frac{1}{n}} \Theta_-^{\frac{\alpha}{n}} &< V_x(x,\tau) = \Sigma^{\frac{1}{n}}(x,\tau)\Theta^{\frac{\alpha}{n}}(x,\tau) < \Sigma_+^{\frac{1}{n}} \Theta_+^{\frac{\alpha}{n}}. \end{aligned} \tag{4.11}$$

The reader should note that the invariant regions property is lost in the complementary region $q < 0$, and no rectangle of the form $[\Theta_-, \Theta_+] \times [\Sigma_-, \Sigma_+]$ is invariant for the reaction system (4.7). The bounds (4.11) yield time-dependent estimates for solutions of (4.1), (4.2),

$$\begin{aligned} \Theta_-(t+1)^{\frac{1}{1+\alpha}} &< \theta(x,t) < \Theta_+ t+1)^{\frac{1}{1+\alpha}}, \\ \Sigma_-(t+1)^{\frac{-\alpha}{1+\alpha}} &< \sigma(x,t) < \Sigma_+(t+1)^{\frac{-\alpha}{1+\alpha}}, \\ \Sigma_-^{\frac{1}{n}} \Theta_-^{\frac{\alpha}{n}} &< v_x(x,t) < \Sigma_+^{\frac{1}{n}} \Theta_+^{\frac{\alpha}{n}}. \end{aligned} \tag{4.12}$$

These are used to show that the uniform shear solution is asymptotically stable:

THEOREM 4.1. *Let $q = -\alpha + n > 0$ and consider a solution (v, θ) of* (4.1), (4.2) *with initial data $\theta_0(x) > 0$ and $\sigma_0(x) > 0$. Then (v, θ) is defined for all times and has* *the asymptotic behavior*

$$v_x(x,t) = 1 + O\left((t+1)^{-\frac{n-\alpha}{n(1+\alpha)}}\right), \tag{4.13}$$

$$\theta(x,t) = \Theta_m (t+1)^{\frac{1}{1+\alpha}} \Big(1 + O\big((t+1)^{-\frac{n-\alpha}{n(1+\alpha)}}\big)\Big), \tag{4.14}$$

$$\sigma(x,t) = \Sigma_m (t+1)^{\frac{-\alpha}{1+\alpha}} \Big(1 + O\big((t+1)^{-\frac{n-\alpha}{n(1+\alpha)}}\big)\Big), \tag{4.15}$$

as $t \to \infty$, *where* (Θ_m, Σ_m) *are given in* (4.10).

Theorem 4.1 was proved in [9, 18] using detailed energy estimates to derive the time-dependent bounds (4.12). These estimations are considerably simplified using the invariant regions presented above. The remainder of the asymptotic stability proof is presented in the appendix.

4.3. The unstable regime. The stability of the uniform shearing solution in the complementary region $q = -\alpha + n < 0$ is at present unknown. In fact, it is even unknown whether solutions exist globally in time or, in contrast, blow up in finite time. Numerical investigations indicate development of shear bands in this regime. In addition, there are two theoretical results that are also backing this direction. First, consider initial data $\theta_0(x)$, $v_0(x)$ such that

$$v_0(x) = x, \tag{4.16}$$

$$\theta_0(x) = \begin{cases} \bar\theta & x \notin I_\delta, \\ U(x) & x \in I_\delta, \end{cases} \tag{4.17}$$

where $\bar\theta$ is a constant, $I_\delta = (y-\delta, y+\delta)$ is a (small) interval centered around a given point $y \in (0,1)$, and $U(x)$ is the initial temperature profile in I_δ and is selected so that θ_0 is smooth. Then:

Theorem 4.2. *Let* $q < 0$ *and* (v,θ) *be a solution of* (4.1), (4.2) *with initial data* (4.16), (4.17). *If* $U(y)$ *is selected sufficiently large, then either the solution blows up in finite time,*

$$\limsup_{t\to T^*} \sup_{x\in[0,1]} \theta(x,t) \to \infty \quad \text{for some } T^* < \infty,$$

or else $T^* = \infty$ *and* (v,θ) *has the asymptotic behavior*

$$v(x,t) = \begin{cases} 0 + O\left((t+1)^{-\frac{1}{n+1}}\right) & x < y - \delta, \\ 1 + O\left((t+1)^{-\frac{1}{n+1}}\right) & y + \delta < x, \end{cases} \tag{4.18}$$

and $\theta(x,t)$ *approaches a limiting temperature profile for* $x \notin I_\delta$ *as* $t \to \infty$.

Theorem 4.2 was proved in [3] for the case of a Newtonian fluid ($n = 1$). We extend this result for non-Newtonian temperature dependent flows. The proof is provided in the appendix and it yields a quantitative criterion for a size of an initial temperature perturbation $U(x)$ that suffices to induce instability. The reader can check that even a moderate perturbation will suffice, but an arbitrarily small perturbation of the uniform temperature is excluded. The question remains what is the basic mechanism that induces this instability. This question is answered at the level of the full system (3.1) in the following two sections, of course including the special case (4.1), (4.2).

5. The long-time response of adiabatic shear as a relaxation limit. Our next goal is to derive an effective equation describing the long-time behavior of the system (3.1) by using convenient scaling limits. This is done in three steps: First, we consider the system (3.5) or the equivalent form (3.4) and point out certain analogies with the structure of relaxation systems (like equilibrium manifolds, moment equations). Then we consider a modified version of the system (3.4) and show how to introduce a scaled limit that describes the long-time response, and how to compute the effective response by a process analogous to the Chapman-Enskog expansion. The analysis applies to the modified system (see (5.9)) which shares the same general structure as (3.5) but also has an important difference. Then in section 6, we consider the original system (3.5) and modify the change of variables (3.3) keeping in mind that we calculate perturbed profiles of time-dependent solutions. An analogous procedure then leads to the effective equation describing the long-time response of (3.1).

5.1. Some analogies to the theory of relaxation systems. We first point out certain analogies between (3.4) and the theory of relaxation processes. Consider a solution (Σ, Θ, Γ) of (3.5), and note that it satisfies the identity

$$\frac{\Sigma_\tau}{\Sigma} - m\frac{\Gamma_\tau}{\Gamma} + \alpha\frac{\Theta_\tau}{\Theta} = \frac{n}{r}\; e^{\frac{m+1}{1+\alpha}\tau}\Theta^{-\frac{\alpha}{n}}\;\Gamma^{\frac{m}{n}}\;\Sigma^{-\frac{1}{n}}\Sigma_{xx}\,, \tag{5.1}$$

that is $U = (\Sigma\;\Theta^\alpha\;\Gamma^{-m})^{\frac{1}{n}}$ satisfies a conservation law

$$\partial_\tau U = \frac{1}{r}\; e^{\frac{m+1}{1+\alpha}\tau}\;\Sigma_{xx}\,. \tag{5.2}$$

Equation (5.2) is precisely the first equation in (3.4), and it may be interpreted as a conservation law for the quantity $U = V_x$ that arises as a moment equation for the reaction-diffusion system (3.5). The reaction system associated to (3.5) is

$$\begin{aligned}
\Sigma_\tau &= -\alpha\Sigma\left(\frac{\Sigma}{\Theta}\;\Theta^{\frac{\alpha}{n}}\;\Gamma^{-\frac{m}{n}}\;\Sigma^{\frac{1}{n}} - \frac{m+1}{1+\alpha}\right) + m\Sigma\left(\frac{1}{\Gamma}\;\Theta^{\frac{\alpha}{n}}\;\Gamma^{-\frac{m}{n}}\;\Sigma^{\frac{1}{n}} - 1\right),\\
\Gamma_\tau &= \Gamma\left(\frac{1}{\Gamma}\;\Theta^{\frac{\alpha}{n}}\;\Gamma^{-\frac{m}{n}}\;\Sigma^{\frac{1}{n}} - 1\right),\\
\Theta_\tau &= \Theta\left(\frac{\Sigma}{\Theta}\;\Theta^{\frac{\alpha}{n}}\;\Gamma^{-\frac{m}{n}}\;\Sigma^{\frac{1}{n}} - \frac{m+1}{1+\alpha}\right).
\end{aligned} \tag{5.3}$$

For the initial data, we assume that $\Gamma_0(x) > 0$, $\Theta_0(x) > 0$, $\Sigma_0(x) > 0$, and they depend parametrically on x.

Equilibria. The equilibria of (5.3) are the solutions of the algebraic system

$$\begin{aligned}
&\frac{1}{\Gamma}\;\Theta^{\frac{\alpha}{n}}\;\Gamma^{-\frac{m}{n}}\;\Sigma^{\frac{1}{n}} - 1 = 0,\\
&\frac{\Sigma}{\Theta}\;\Theta^{\frac{\alpha}{n}}\;\Gamma^{-\frac{m}{n}}\;\Sigma^{\frac{1}{n}} - \frac{m+1}{1+\alpha} = 0,
\end{aligned}$$

or equivalently

$$\frac{\Sigma\Gamma}{\Theta} = \frac{m+1}{1+\alpha}, \quad \Sigma = \Theta^{-\alpha}\;\Gamma^{m+n}. \tag{5.4}$$

The equilibria form a one parameter family, determined in terms of a parameter $U \in \mathbb{R}^+$, by the equations

$$\begin{aligned} \Gamma &= U, \\ \Theta &= \left(\frac{1+\alpha}{m+1}\right)^{\frac{1}{1+\alpha}} U^{\frac{m+n+1}{1+\alpha}}, \\ \Sigma &= \left(\frac{1+\alpha}{m+1}\right)^{-\frac{\alpha}{1+\alpha}} U^{\frac{-\alpha+m+n}{1+\alpha}}. \end{aligned} \tag{5.5}$$

Orbits. Although the system (5.3) is complex in appearance, its orbits are easily computed due to the property that solutions of (5.3) satisfy the conservation law

$$\frac{\Sigma_\tau}{\Sigma} - m\frac{\Gamma_\tau}{\Gamma} + \alpha\frac{\Theta_\tau}{\Theta} = 0 \Longleftrightarrow \partial_\tau\left(\Sigma\ \Theta^\alpha\ \Gamma^{-m}\right) = 0\,.$$

The quantity $\Sigma\ \Theta^\alpha\ \Gamma^{-m}$ thus remains constant along an orbit,

$$\Sigma\ \Theta^\alpha\ \Gamma^{-m} = U^n = \text{constant in time}\,, \tag{5.6}$$

with the value of $U = V_x$ determined by the values of the initial data. As a result of (5.6), Γ satisfies the differential equation

$$\frac{\Gamma_\tau}{\Gamma} = \frac{U}{\Gamma} - 1$$

Since U is constant in time and the initial data $\Gamma_0(x) > 0$ are positive, $\Gamma(x,\tau)$ remains bounded above and below by positive bounds depending on $U(x)$ and $\Gamma_0(x)$ and that

$$\Gamma \to U \qquad \text{as } \tau \to \infty.$$

Furthermore, concerning the dynamics of the system of differential equations (5.3), we have

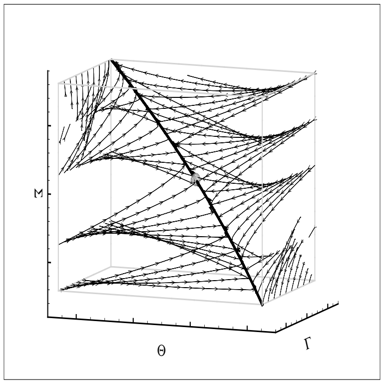
$$\begin{aligned} \partial_\tau\left(\frac{\Sigma\ \Gamma}{\Theta}\right) &= \frac{\Sigma\ \Gamma}{\Theta}\left(\frac{\Sigma_\tau}{\Sigma} + \frac{\Gamma_\tau}{\Gamma} - \frac{\Theta_\tau}{\Theta}\right) = \frac{\Sigma\ \Gamma}{\Theta}\Theta^{\frac{\alpha}{n}}\ \Gamma^{-\frac{m}{n}}\ \Sigma^{\frac{1}{n}}\left(\frac{m+1}{\Gamma} - (\alpha+1)\frac{\Sigma}{\Theta}\right) \\ &= -(\alpha+1)\left(\frac{\Sigma\ \Gamma}{\Theta}\right)\frac{U}{\Gamma}\left(\frac{\Sigma\Gamma}{\Theta} - \frac{m+1}{1+\alpha}\right) \end{aligned}$$

This implies $\left|\frac{\Sigma\ \Gamma}{\Theta} - \frac{m+1}{1+\alpha}\right|$ is a decreasing function of τ and

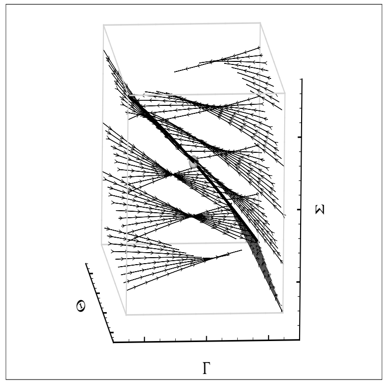
$$\frac{\Sigma\ \Gamma}{\Theta} - \frac{m+1}{1+\alpha} \to 0 \ \text{ as } \tau \to \infty\,. \tag{5.7}$$

Finally, the identity

$$\begin{aligned} \frac{\partial_\tau\left(\Sigma^{\frac{1}{n}}\ \Theta^{\frac{\alpha}{n}}\ \Gamma^{-\frac{m+n}{n}}\right)}{\left(\Sigma^{\frac{1}{n}}\ \Theta^{\frac{\alpha}{n}}\ \Gamma^{-\frac{m+n}{n}}\right)} &= \frac{1}{n}\ \frac{\Sigma_\tau}{\Sigma} + \frac{\alpha}{n}\ \frac{\Theta_\tau}{\Theta} - \frac{m+n}{n}\ \frac{\Gamma_\tau}{\Gamma} \\ &= -\left(\Sigma^{\frac{1}{n}}\ \Theta^{\frac{\alpha}{n}}\ \Gamma^{-\frac{m+n}{n}} - 1\right) \end{aligned}$$



(a)



(b)

FIG. 5.1. *Two views of the flow of the o.d.e. system* (5.3) *in the stable case :* $-\alpha+m+n>0$

implies that $\Phi(\tau)=\Sigma^{\frac{1}{n}}\ \Theta^{\frac{\alpha}{n}}\ \Gamma^{-\frac{m+n}{n}}$ satisfies the ordinary differential equation

$$\partial_\tau \Phi = -\Phi\,(\Phi-1)\,,$$

and thus $\Phi(\tau)\to 1$ as $\tau\to\infty$, that is

$$\Sigma\ \Theta^{\alpha}\ \Gamma^{-(m+n)} \to 1 \quad \text{as } \tau\to\infty. \tag{5.8}$$

We conclude that the orbits of the differential system (5.3) approach the line of equilibria (5.5). Each orbit lies entirely on the surface (5.6) and the specific value of the parameter U is selected by the initial data. Unlike the system with two equations, strong numerical evidence (Figure 5.1), indicates that the system of three equations does not have invariant regions. In particular, Figure 5.1 shows the flow of the vector field generated by (5.3). Notice in the upper left corner and/or in the lower right corner of Figure 5.1(a) there are orbits always exiting the box. One can notice the same type of behavior in the bottom and/or the top part of Figure 5.1(b), there are orbits that are no longer confined in a box around the equilibrium manifold.

5.2. Effective equation for a simplified system. Our ultimate goal is to give an effective equation describing the long-time response of the system (3.1). We will present an argument that leads to such an effective equation in the following section. In preparation, we consider a simplified problem that accounts for the main structure of (3.4), by studying a variant where the time dependence of the diffusion term is frozen,

$$\begin{aligned} V_\tau &= \frac{1}{r}\Sigma_x, \\ \Gamma_\tau &= V_x - \Gamma, \\ \Theta_\tau &= \Sigma V_x - \frac{m+1}{1+\alpha}\Theta, \\ \Sigma &= \Theta^{-\alpha}\ \Gamma^{m} V_x^{n}. \end{aligned} \tag{5.9}$$

For this modified system we calculate an effective equation in an asymptotic limit motivated by the theory of relaxation approximations. To this end, set $U=V_x$ in

(5.9), and consider a rescaling of time in the form

$$
\begin{aligned}
&U(x,\tau) = \bar{U}^T(x, \frac{\tau}{T})\,, \quad \Theta(x,\tau) = \bar{\Theta}^T(x, \frac{\tau}{T}), \\
&\Gamma(x,\tau) = \bar{\Gamma}^T(x, \frac{\tau}{T})\,, \quad \Sigma(x,\tau) = \bar{\Sigma}^T(x, \frac{\tau}{T})\,, \qquad s = \frac{\tau}{T}.
\end{aligned}
\tag{5.10}
$$

Then, $(\bar{U}^T, \bar{\Theta}^T, \bar{\Gamma}^T, \bar{\Sigma}^T)$ satisfies the system of equations

$$
\begin{aligned}
U_s &= \frac{T}{r}\Sigma_{xx}, \\
\Gamma_s &= T(U-\Gamma), \\
\Theta_s &= T(\Sigma U - \frac{m+1}{1+\alpha}\Theta), \\
\Sigma \ &= \Theta^{-\alpha}\ \Gamma^m U^n,
\end{aligned}
\tag{5.11}
$$

where in (5.11) we dropped the bars to simplify notations. Given a family of solutions of (5.11) we may use the relations

$$
\lim_{T\to\infty} U(x, Ts) = \lim_{T\to\infty} \bar{U}^T(x,s)\,,
$$

in order to calculate the long-time behavior of solutions of (5.9). Therefore, it suffices to calculate the effective response in the limit $T\to\infty$ of solutions to (5.11). This goal can be achieved by using the procedure of the Chapman-Enskog expansion (*e.g.* [5]), familiar from the kinetic theory of gases. In order for the conservation law $(5.11)_1$ to provide a nontrivial effective response we will consider the limit $T\to\infty$, $r\to\infty$ such that $\frac{T}{r}<\infty$ (and for simplicity will take $\frac{T}{r}=1$). Consider the ansatz for solutions of (5.11):

$$
\begin{aligned}
&\bar{U}^T = U_0 + \frac{1}{T}U_1 + O\big(\frac{1}{T^2}\big)\,, \quad \bar{\Gamma}^T = \Gamma_0 + \frac{1}{T}\Gamma_1 + O\big(\frac{1}{T^2}\big), \\
&\bar{\Theta}^T = \Theta_0 + \frac{1}{T}\Theta_1 + O\big(\frac{1}{T^2}\big)\,, \quad \bar{\Sigma}^T = \Sigma_0 + \frac{1}{T}\Sigma_1 + O\big(\frac{1}{T^2}\big).
\end{aligned}
\tag{5.12}
$$

Upon introducing the expansions into (5.11) and expanding in orders of T keeping in mind that $r=T\to\infty$, we obtain consecutively

$$
\partial_s\big(U_0 + \frac{1}{T}U_1 + ...\big) = \partial_{xx}\big(\Sigma_0 + \frac{1}{T}\Sigma_1 + ...\big) \tag{5.13}
$$

$$
\partial_s\big(\Gamma_0 + \frac{1}{T}\Gamma_1 + ...\big) = T\big(U_0 - \Gamma_0\big) + \big(U_1 - \Gamma_1\big) + ... \tag{5.14}
$$

$$
\partial_s\big(\Theta_0 + \frac{1}{T}\Theta_1 + ...\big) = T\big(\Sigma_0 U_0 - \frac{m+1}{1+\alpha}\Theta_0\big) \tag{5.15}
$$

$$
+ \big(\Sigma_1 U_0 + \Sigma_0 U_1 - \frac{m+1}{1+\alpha}\Theta_1\big) + ... \tag{5.16}
$$

and finally

$$
\begin{aligned}
&\Sigma_0 + \frac{1}{T}\Sigma_1 + ... \\
&\quad = \big(\Theta_0 + \frac{1}{T}\Theta_1 + ...\big)^{-\alpha}\big(\Gamma_0 + \frac{1}{T}\Gamma_1 + ...\big)^m\big(U_0 + \frac{1}{T}U_1 + ...\big)^n \\
&\quad = \Theta_0^{-\alpha}\Gamma_0^m U_0^n \left(1 + \frac{1}{T}\big(-\alpha\frac{\Theta_1}{\Theta_0} + m\frac{\Gamma_1}{\Gamma_0} + n\frac{U_1}{U_0}\big)\right).
\end{aligned}
\tag{5.17}
$$

Collecting terms of the same order together, we obtain for the 0-th order perturbations the equations

$$
\begin{aligned}
U_0 &= \Gamma_0, \\
\Sigma_0 U_0 &= \frac{m+1}{1+\alpha}\Theta_0, \\
\Sigma_0 &= \Theta_0^{-\alpha}\Gamma_0^m U_0^n,
\end{aligned}
\tag{5.18}
$$

while for the 1-st order perturbations we deduce the equations

$$
\begin{aligned}
U_1 - \Gamma_1 &= \partial_s \Gamma_0, \\
\Sigma_1 U_0 + \Sigma_0 U_1 - \frac{m+1}{1+\alpha}\Theta_1 &= \partial_s \Theta_0, \\
\big(-\alpha\frac{\Theta_1}{\Theta_0} + m\frac{\Gamma_1}{\Gamma_0} + n\frac{U_1}{U_0}\big)\Sigma_0 &= \Sigma_1, \\
\partial_s U_1 &= \partial_{xx}\Sigma_1.
\end{aligned}
\tag{5.19}
$$

Solving (5.18) we obtain that all 0-th order terms can be expressed in terms of the conserved quantity U_0

$$
\begin{aligned}
\Gamma_0 &= U_0, \\
\Sigma_0 &= \Big(\frac{m+1}{1+\alpha}\Big)^{\frac{\alpha}{1+\alpha}} U_0^{\frac{-\alpha+m+n}{1+\alpha}}, \\
\Theta_0 &= \left(\frac{m+1}{1+\alpha}\right)^{-\frac{1}{1+\alpha}} U_0^{\frac{1+m+n}{1+\alpha}}, \\
\partial_s U_0 &= \partial_{xx}\Sigma_0.
\end{aligned}
\tag{5.20}
$$

Note that U_0 is the conserved quantity and that such structure is typical in the theory of relaxation. In addition, we may obtain an evolution equation governing the behavior of the 0-th order approximation in closed form as

$$
\partial_s U_0 = \partial_{xx}\left(\left(\frac{m+1}{1+\alpha}\right)^{\frac{\alpha}{1+\alpha}} U_0^{\frac{-\alpha+m+n}{1+\alpha}}\right). \tag{5.21}
$$

The nature of the equation (5.21) changes, depending on the sign of

$$
q = -\alpha + m + n
$$

from forward parabolic when $q > 0$ to backward parabolic when $q < 0$. For the parameters ranging in the region $q < 0$ equation (5.21) is ill-posed and one needs to derive the next order of the asymptotics. This is accomplished by solving the equations (5.19) for the 1-st order approximants $(\Gamma_1, \Theta_1, \Sigma_1)$ and using the expressions (5.20). We then obtain

$$
\begin{aligned}
\Gamma_1 &= U_1 - \partial_s U_0, \\
\frac{\Sigma_1}{\Sigma_0} &= \frac{-\alpha+m+n}{1+\alpha}\frac{U_1}{U_0} - \frac{m}{1+\alpha}\frac{\partial_s U_0}{U_0} + \frac{\alpha}{1+\alpha}\frac{\partial_s\Theta_0}{\Sigma_0 U_0}, \\
\frac{\Theta_1}{\Theta_0} &= \frac{1+m+n}{1+\alpha}\frac{U_1}{U_0} - \frac{m}{1+\alpha}\frac{\partial_s U_0}{U_0} - \frac{1}{1+\alpha}\frac{\partial_s\Theta_0}{\Sigma_0 U_0}.
\end{aligned}
\tag{5.22}
$$

The corrected form – up to order $O(\frac{1}{T^2})$ – of the effective equation is now easily calculated, using (5.20) and (5.22), leads to the equation

$$\begin{aligned}
\partial_s\Big(U_0 + \frac{1}{T}U_1 + O(\frac{1}{T^2})\Big) &= \partial_{xx}\Big(\Sigma_0 + \frac{1}{T}\Sigma_1 + O(\frac{1}{T^2})\Big) \\
&= \partial_{xx}\Bigg[\Big(\frac{m+1}{1+\alpha}\Big)^{\frac{\alpha}{1+\alpha}} U_0^{\frac{-\alpha+m+n}{1+\alpha}} \\
&\quad + \frac{1}{T}\Sigma_0\Big(\frac{-\alpha+m+n}{1+\alpha}\frac{U_1}{U_0} - \frac{m}{1+\alpha}\frac{\partial_s U_0}{U_0} + \frac{\alpha}{1+\alpha}\frac{\partial_s\Theta_0}{\Sigma_0 U_0}\Big) + O(\frac{1}{T^2})\Bigg] \\
&= \partial_{xx}\Bigg[\Big(\frac{m+1}{1+\alpha}\Big)^{\frac{\alpha}{1+\alpha}} U_0^{\frac{-\alpha+m+n}{1+\alpha}} \Big(1 + \frac{-\alpha+m+n}{1+\alpha}\frac{U_1}{U_0}\frac{1}{T} + O(\frac{1}{T^2})\Big) \\
&\quad + \frac{1}{T}\Sigma_0\Big(-\frac{m}{1+\alpha}\frac{\partial_s U_0}{U_0} + \frac{\alpha}{1+\alpha}\frac{\partial_s\Theta_0}{\Sigma_0 U_0}\Big) + O(\frac{1}{T^2})\Bigg]. \qquad (5.23)
\end{aligned}$$

The last objective is to obtain an equation for U that up to 2-nd order will agree with the equation (5.23). To this end, observe that the first term in the right side of (5.23) satisfies

$$I_1 = \Big(\frac{m+1}{1+\alpha}\Big)^{\frac{\alpha}{1+\alpha}} U^{\frac{-\alpha+m+n}{1+\alpha}} + O(\frac{1}{T^2}),$$

while the second term may be re-expressed using (5.20) and (5.21) as

$$\begin{aligned}
I_2 &= \Sigma_0\Big(-\frac{m}{1+\alpha}\frac{\partial_s U_0}{U_0} + \frac{\alpha}{1+\alpha}\frac{\partial_s\Theta_0}{\Sigma_0 U_0}\Big) = \Sigma_0 \frac{\alpha(1+m+n) - m(m+1)}{(m+1)(1+\alpha)}\frac{\partial_s U_0}{U_0} \\
&= \frac{\alpha(1+m+n) - m(m+1)}{(m+1)(1+\alpha)}\Big(\frac{m+1}{1+\alpha}\Big)^{\frac{2\alpha}{1+\alpha}} U_0^{\frac{-\alpha+m+n}{1+\alpha}-1}\partial_{xx}\big(U_0^{\frac{-\alpha+m+n}{1+\alpha}}\big).
\end{aligned}$$

The effective equation is thus, up to order $O(\frac{1}{T^2})$,

$$\partial_s U = \partial_{xx}\Big(c\,U^p + \frac{\lambda c^2}{T}U^{p-1}\partial_{xx}U^p\Big), \qquad (5.24)$$

where

$$p = \frac{-\alpha+m+n}{1+\alpha}, \quad c = \Big(\frac{m+1}{1+\alpha}\Big)^{\frac{\alpha}{1+\alpha}}, \quad \lambda = \frac{\alpha(1+m+n) - m(m+1)}{(m+1)(1+\alpha)}.$$

We thus see that the 2-nd order approximation acquires an additional effect comprising of a nonlinear fourth order term. When $q = -\alpha+m+n > 0$ this term is a perturbation of a forward parabolic equation. As the forward parabolic term has a stabilizing response the fourth order term is a small perturbation and the sign of λ has no effect on the stability properties. By contrast, in the region $q < 0$, the first term provides backward parabolic response and any stabilization is only due to the fourth order term. Note that

$$\lambda = \frac{(\alpha - m - n) + n(1+\alpha) + (\alpha - m)m}{(m+1)(1+\alpha)}$$

and thus when $q = -\alpha+m+n < 0$ it is $\lambda > 0$. The uniform shear solution corresponds to $U = 1$ and for this reason we write $U = 1 + u$ and compute the linearized equation for the perturbation u. This reads

$$\partial_s u = c\, p\, \partial_{xx} u + \frac{\lambda c^2}{T}\, p\, \partial_{xxxx} u\,. \tag{5.25}$$

The Fourier transform of equation (5.25) satisfies

$$\partial_s \hat{u} = \left(-c\, p\, \xi^2 + \frac{\lambda c^2}{T}\, p\, \xi^4 \right) \hat{u}$$

and thus when $q < 0$ the low frequencies will grow but the high frequency modes still decay. Hence, for $\lambda > 0$, the linearized equation (5.25) is well posed.

6. Effective response at the onset of localization. We now consider the system (3.1) and will calculate an effective equation for its time response. We consider a modified time-rescaling of the form

$$\begin{aligned} \theta(x,t) &= (t+1)^{\frac{m+1}{\alpha+1}}\, \Theta(x, \frac{s(t)}{T})\,, \quad \gamma(x,t) = (t+1)\Gamma(x, \frac{s(t)}{T})\,, \\ \sigma(x,t) &= (t+1)^{\frac{m-\alpha}{\alpha+1}}\, \Sigma(x, \frac{s(t)}{T})\,, \quad v_x(x,t) = V_x(x, \frac{s(t)}{T})\,, \end{aligned} \tag{6.1}$$

where T is a parameter representing a change of time-unit and $s(t) : [0,\infty) \to [0,\infty)$ is be selected as a monotone increasing, surjective map that represents a change of time-scale. This should be compared to the equations (3.3) that has been used before. Introducing this transformation to (3.1), we obtain the equations

$$\begin{aligned} \partial_s V_x &= \frac{T}{r}\frac{1}{\dot{s}}(t+1)^{\frac{m-\alpha}{1+\alpha}}\, \Sigma_{xx}, \\ (t+1)\frac{\dot{s}}{T}\Theta_s &= \Sigma V_x - \frac{m+1}{1+\alpha}\Theta, \\ (t+1)\frac{\dot{s}}{T}\Gamma_s &= V_x - \Gamma, \\ \Sigma &= \Theta^{-\alpha}\Gamma^m V_x^n. \end{aligned}$$

We select $s(t)$ so that

$$\dot{s} = (t+1)^{\frac{m-\alpha}{1+\alpha}} \tag{6.2}$$

that is

$$s(t) = \frac{1}{\beta}\big[(t+1)^\beta - 1\big] \iff t(s) = \big(1+\beta s\big)^{\frac{1}{\beta}} - 1,$$

where $\beta = \frac{m+1}{1+\alpha}$. We are interested in the limit $T \to \infty$, $r \to \infty$ so that $\frac{T}{r} = O(1)$, and for simplicity we will take $T = r$. The choice $r = O(T)$ is done for the following reasons: (i) it is expected that the inertial terms play an important role in the shear band formation process, (ii) we wish to retain both terms of the momentum equation at $O(1)$ as $T \to \infty$ (see also Remark 6.1). With these identifications we deduce that the scaled functions Θ^T, Σ^T, $U^T = V_x^T$ and Γ^T satisfy

$$\begin{aligned} \partial_s U &= \Sigma_{xx}, \\ \frac{1}{T}(\beta s+1)\Theta_s &= \Sigma U - \frac{m+1}{1+\alpha}\Theta, \\ \frac{1}{T}(\beta s+1)\Gamma_s &= U - \Gamma, \\ \Sigma &= \Theta^{-\alpha}\Gamma^m U^n. \end{aligned} \tag{6.3}$$

If in the limit $T \to \infty$ the functions $(U^T, \Gamma^T, \Theta^T, \Sigma^T) \to (U_0, \Gamma_0, \Theta_0, \Sigma_0)$ then the limiting $(U_0, \Gamma_0, \Theta_0, \Sigma_0)$ lies in the "equilibrium" manifold

$$\begin{aligned} \Sigma_0 U_0 &= \beta \Theta_0, \\ U_0 &= \Gamma_0, \\ \Sigma_0 &= \Theta_0^{-\alpha} \Gamma_0^m U_0^n \end{aligned} \tag{6.4}$$

and satisfies the conservation law

$$\partial_s U_0 = \partial_{xx} \Sigma_0. \tag{6.5}$$

The latter can be expressed into closed form and leads to an effective equation for the long-time response, in the form

$$\begin{aligned} \Sigma_0 &= \beta^{\frac{\alpha}{1+\alpha}} U_0^{\frac{-\alpha+m+n}{1+\alpha}}, \\ \partial_s U_0 &= \partial_{xx} \left(\beta^{\frac{\alpha}{1+\alpha}} U_0^{\frac{-\alpha+m+n}{1+\alpha}} \right). \end{aligned} \tag{6.6}$$

Observe that U_0 describes the limiting dynamics of v_x as can be seen by taking the change of variable formula

$$v_x(x, t(\tau T)) = U^T(x, \tau), \tag{6.7}$$

to the limit as the unit-scale $T \to \infty$. Equation (6.6) changes type from forward parabolic for $q > 0$ to backward parabolic for $q < 0$, where $q = -\alpha + m + n$.

Remark 6.1. In order to preserve the moment equation at $O(1)$ in the limit $T \to \infty$ we have used the parameter r, by assuming $r = O(T)$. The same effect can be achieved by using a different scaling: One could alternatively scale the x-variable in the parabolic scaling $x \to \frac{x}{\sqrt{T}}$ and retain $r = O(1)$. This scaling would again preserve the equation $\partial_s U = \partial_{xx} \Sigma$ and lead to the same system (6.3) with a coefficient $r = O(1)$. In return, the equation (6.7) would be replaced by

$$v_x \left(y\sqrt{T}, t(\tau T) \right) = U^T(y, \tau)$$

which indicates that the limit $T \to \infty$ would proceed along parabolic rays.

Finally, we perform the Chapman-Enskog procedure to calculate the next term of the correction. This is entirely analogous to the procedure outlined in Section 5.2 and the details are omitted. To this end, the ansatz (5.12) is introduced into (6.3); collecting terms of the same order together, we obtain (6.4), (6.5) at the 0-th order (leading to (6.6)) and obtain at the 1-st order the equations

$$\begin{aligned} (\beta s + 1)\partial_s \Theta_0 &= \Sigma_0 U_1 + \Sigma_1 U_0 - \beta \Theta_1, \\ (\beta s + 1)\partial_s \Gamma_0 &= U_1 - \Gamma_1, \\ \Sigma_1 &= \Sigma_0 \left(-\alpha \frac{\Theta_1}{\Theta_0} + m \frac{\Gamma_1}{\Gamma_0} + n \frac{U_1}{U_0} \right), \end{aligned}$$

together with

$$\partial_s U_1 = \partial_{xx} \Sigma_1 \,. \tag{6.8}$$

Solving the former equations, yields

$$\begin{aligned}
\frac{\Gamma_1}{U_0} &= \frac{U_1}{U_0} - (\beta s+1)\frac{\partial_s U_0}{U_0}, \\
\frac{\Sigma_1}{\Sigma_0} &= \frac{-\alpha+m+n}{1+\alpha}\frac{U_1}{U_0} - \frac{m}{1+\alpha}(\beta s+1)\frac{\partial_s U_0}{U_0} + \frac{\alpha}{1+\alpha}(\beta s+1)\frac{\partial_s \Theta_0}{\Sigma_0 U_0}, \\
\frac{\Theta_1}{\Theta_0} &= \frac{1+m+n}{1+\alpha}\frac{U_1}{U_0} - \frac{m}{1+\alpha}(\beta s+1)\frac{\partial_s U_0}{U_0} - \frac{1}{1+\alpha}(\beta s+1)\frac{\partial_s \Theta_0}{\Sigma_0 U_0},
\end{aligned}$$

We next introduce the values of U_0, U_1, Σ_0, Σ_1 into the equation

$$\partial_s\big(U_0 + \frac{1}{T}U_1 + O\big(\frac{1}{T^2}\big)\big) = \partial_{xx}\big(\Sigma_0 + \frac{1}{T}\Sigma_1 + O\big(\frac{1}{T^2}\big)\big), \tag{6.9}$$

and regroup the terms following section 5.2 to deduce that U^T satisfies up to order $O\big(\frac{1}{T^2}\big)$ the equation

$$\partial_s U = \partial_{xx}\left(c\,U^p + \frac{\lambda c^2}{T}(\beta s+1)U^{p-1}\partial_{xx}U^p\right), \tag{6.10}$$

where

$$p = \frac{-\alpha+m+n}{1+\alpha}, \quad \beta = \frac{m+1}{1+\alpha}, \quad c = \beta^{\frac{\alpha}{1+\alpha}}, \quad \lambda = \frac{\alpha(1+m+n) - m(m+1)}{(m+1)(1+\alpha)}.$$

When $q < 0$ the second order term is backward parabolic and has a destabilizing role, at the same time $\lambda > 0$ and the fourth order term offers a stabilizing influence.

System vs Effective equation : Numerical Comparison Next, we compare numerically the solution of system (6.3) with (6.10). The effective equation (6.10) is a highly nonlinear fourth order equation whose behavior depends drastically on the sign of p. In the stable case, $p > 0$, (6.10) is a forward parabolic equation and a fourth order correction term whose sign depends on the parameter λ. In the stable case the fourth order term does not have a definite sign since the parameter λ can be either positive or negative, but nevertheless without any essential effect on the qualitative behavior of the effective equation. We compare numerically the solution of the system (6.3) and (6.10) for $T = 1000$ at various time instances. We use standard finite element method for the spatial descritization, coupled with Newton's method for linearization, while the Crank-Nicolson method is used for time stepping. In this case we expect the solution $U = V_x$ to converge as $t \to \infty$ to the uniform shearing solution (2.17). In Figure 6.1 we see an excellent agreement of the two solutions, especially as t grows.

In the unstable case $p < 0$ the behavior of the effective equation (6.10) changes drastically. The leading second order term has a negative sign, thus the effective equation changes to an unstable backward parabolic type. On the other hand the parameter λ in the unstable case is positive thus the fourth order term provides a stabilizing effect. The balance of these two mechanisms is a delicate issue theoretically as well as numerically. In general the numerical solution of linear or nonlinear fourth order equations is not a trivial task. Special numerical techniques have to be applied to capture correctly the behavior of the underlying phenomena. These issues go beyond the scope of the present article and will be the subject of a future work. Detailed numerical results on the behavior of systems (2.11) and the companion system with Fourier heat conduction can be found in [2].

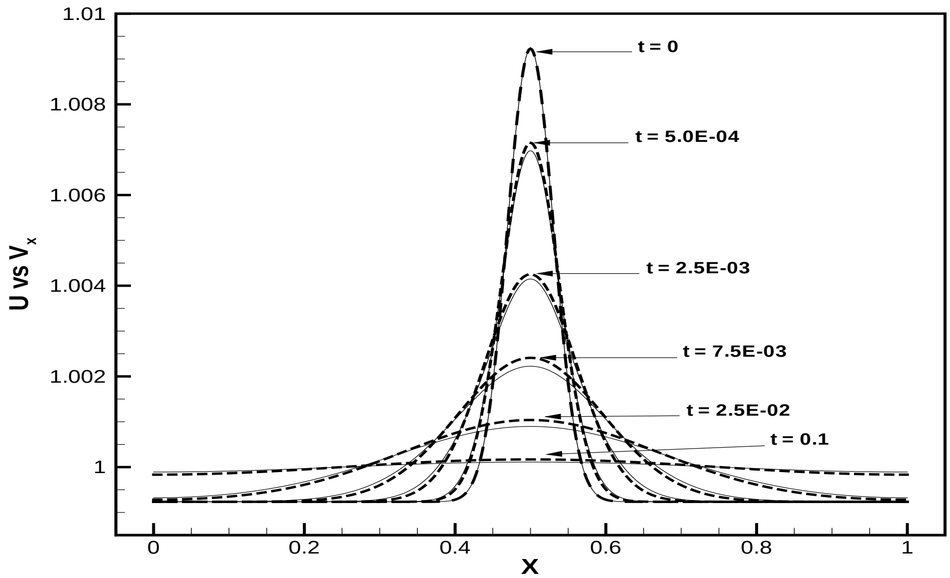


FIG. 6.1. *Comparison in the parameter range* $p > 0$ *of system* (6.3) *(solid line) vs effective equation* (6.9)*(dashed line) for* $T = 1000$.

7. Appendix. We present in this appendix the proofs of Theorems 4.1 and 4.2. Let (v, θ, σ) be a smooth solution of (4.1), (4.2), and (V, Θ, Σ) the rescaled functions defined in (4.3) and satisfying the equations (4.4), (4.5) and (4.6).

Proof of Theorem 4.1 In the range $q = -\alpha + n > 0$, the system (4.4) is endowed with invariant regions, and (Θ, Σ) and V_x satisfy the bounds (4.11). In the sequel, C stands for a generic constant that is independent of time.

Set $g(\tau) = \frac{1}{r} e^{\frac{1}{1+\alpha}\tau}$. Using (4.6), we obtain the identity $\partial_\tau \Big(\frac{1}{2g^2} V_\tau^2 \Big) = \frac{1}{g} \Sigma_{x\tau} V_\tau$. Integrating by parts over $[0, 1]$ and using $(4.4)_1$, (4.6) and (2.14), we obtain

$$
\begin{aligned}
&\frac{d}{d\tau} \frac{1}{2} \int_0^1 \frac{1}{g^2} V_\tau^2 dx + \int_0^1 n \Theta^{-\frac{\alpha}{n}} \Sigma^{\frac{n-1}{n}} \frac{1}{g} V_{x\tau}^2 \, dx \\
&\qquad\qquad = \int_0^1 \alpha \frac{1}{g} \Big(\Theta^{\frac{\alpha}{n}-1} \Sigma^{\frac{n+1}{n}} - \frac{1}{1+\alpha} \Big) \Sigma V_{x\tau} \, dx
\end{aligned}
$$

Next use of the bounds (4.11) and the Poincarè inequality to obtain

$$
\frac{d}{d\tau} \int_0^1 \frac{1}{g^2} V_\tau^2 dx + g(\tau) \frac{1}{C} \int_0^1 \frac{1}{g^2} V_\tau^2 dx \le \frac{C}{g(\tau)}
$$

In turn, Gronwall's inequality implies

$$
\int_0^1 V_\tau^2(x, \tau) dx \le C \,, \tag{7.1}
$$

and, by (4.6),

$$
\int_0^1 |\Sigma_x| dx \le \Big(\frac{1}{g^2} \int_0^1 V_\tau^2 dx \Big)^{\frac{1}{2}} \le C e^{-\frac{1}{1+\alpha}\tau} \,. \tag{7.2}
$$

From (4.4) we obtain the identity

$$\partial_\tau \left(e^{\frac{n-\alpha}{n(1+\alpha)}\tau} \big(\Theta^{1-\frac{\alpha}{n}}\big)_x \right) = \frac{n-\alpha}{n} e^{\frac{n-\alpha}{n(1+\alpha)}\tau} \big(\Sigma^{\frac{n+1}{n}}\big)_x$$

which using (4.11) and (7.2) leads to using (4.11) and (7.2),

$$\int_0^1 |\Theta_x| dx \le C e^{-\frac{n-\alpha}{n(1+\alpha)}\tau} \,. \tag{7.3}$$

Moreover, starting from (4.5) we obtain $V_{xx} = \frac{1}{n}\Sigma^{\frac{1}{n}-1}\Theta^{\frac{\alpha}{n}}\Sigma_x + \frac{\alpha}{n}\Theta^{\frac{\alpha}{n}-1}\Theta_x\Sigma^{\frac{1}{n}}$ which is used, in conjunction with (7.2) and (7.3), to deduce

$$|V_x(x,\tau) - 1| \le \int_0^1 |V_{xx}|\, dx = O\left(e^{-\frac{n-\alpha}{n(1+\alpha)}\tau} \right) \,. \tag{7.4}$$

Finally, from $(4.4)_2$, (4.5) and (7.4), we obtain

$$\partial_\tau \left(e^\tau \frac{1}{1+\alpha} \Theta^{1+\alpha} \right) = e^\tau V_x^{n+1} = e^\tau \left(1 + O\big(e^{-\frac{n-\alpha}{n(1+\alpha)}\tau}\big) \right)^{n+1}$$

which implies

$$\Theta = (1+\alpha)^{\frac{1}{1+\alpha}} + O\left(e^{-\frac{n-\alpha}{n(1+\alpha)}\tau} \right) \tag{7.5}$$

$$\Sigma = \Theta^{-\alpha} V_x^n = (1+\alpha)^{-\frac{\alpha}{1+\alpha}} + O\left(e^{-\frac{n-\alpha}{n(1+\alpha)}\tau} \right) \,. \tag{7.6}$$

The estimates (7.4), (7.5) and (7.6) together with (4.3) yield the asymptotic behaviors stated in Theorem 4.1.

Proof of Theorem 4.2 Let $q = -\alpha + n < 0$, consider initial data satisfying (4.16) and (4.17), and let (v, θ, σ) be a smooth solution of (4.1), (4.2) defined on a maximal interval of existence $[0, T_\star)$. The stress σ satisfies the boundary value problem for the parabolic equation

$$\begin{aligned} \sigma_t &= \frac{n}{r}\, \theta^{-\frac{\alpha}{n}}\, \sigma^{\frac{n-1}{n}} \sigma_{xx} - \alpha \theta^{\frac{\alpha}{n}-1}\, \sigma^{2+\frac{1}{n}} \,, \\ \sigma_x(0,t) &= \sigma_x(1,t) = 0 \,. \end{aligned} \tag{7.7}$$

By the maximum principle $\sigma(x,t) > 0$. Let $S(t)$ be the solution of the initial value problem

$$\begin{cases} \frac{dS}{dt} = -\alpha \underline{\theta_0}^{\frac{\alpha}{n}-1} S^{2+\frac{1}{n}} \\ S(0) = S_0 = \sup_{x\in[0,1]} \sigma_0(x) \end{cases} \tag{7.8}$$

where $\underline{\theta_0} = \inf_{x\in[0,1]} \theta_0(x)$. S is given by the formula

$$S(t) = \left(S_0^{-\frac{n+1}{n}} + \alpha \frac{n+1}{n} \underline{\theta_0}^{\frac{\alpha}{n}-1}\, t \right)^{-\frac{n}{n+1}} \,.$$

For $\frac{\alpha}{n} > 1$, S is a supersolution of (7.7) and thus

$$0 < \sigma(x,t) \le O\left((t+1)^{-\frac{n}{n+1}} \right) \,, \quad x \in [0,1] \,,\ 0 < t < T_* \,. \tag{7.9}$$

We multiply $(4.1)_1$ by $\sigma^{\frac{1}{n}-1}v_t$, integrate over $[0,1]$ by parts, and use $(4.1)_2$ to obtain

$$\begin{aligned}\frac{n}{2}\int_0^1 \theta^{-\frac{\alpha}{n}} v_x^2 dx + \int_0^t\int_0^1 \sigma^{\frac{1-n}{n}} v_t^2 dx dt + \frac{\alpha}{2}\int_0^t\int_0^1 \theta^{-\frac{\alpha}{n}-1-\alpha} v_x^{n+3} dx \\ = \frac{n}{2}\int_0^1 \theta_0(x)^{-\frac{\alpha}{n}} v_{0x}(x)^2 dx = I_0\end{aligned}$$

Next, we employ the calculus inequality

$$f^2(x) - \big(1+\frac{1}{r}\big)f^2(y) \le (1+r)\big(f(x)-f(y)\big)^2 \le (1+r)\int_0^1 f_x^2 d\xi$$

and put $f = \sigma^{\frac{n+1}{2n}}$ in order to obtain (for $r=2$ say)

$$\sigma^{\frac{n+1}{n}}(x,t) - \frac{3}{2}\sigma^{\frac{n+1}{n}}(y,t) \le 3\Big(\frac{n+1}{2n}\Big)^2 \int_0^1 \sigma^{\frac{1-n}{n}} v_t^2 dx$$

Let now the data be as in (4.16), (4.17). In the region $\frac{\alpha}{n} > 1$, the identity

$$\partial_t \theta^{-\left(\frac{\alpha}{n}-1\right)} = -\big(\frac{\alpha}{n}-1\big)\sigma^{\frac{n+1}{n}}\,,$$

taken at two distinct points x and y, gives

$$\begin{aligned}\theta^{-\left(\frac{\alpha}{n}-1\right)}(x,t) - \frac{3}{2}\theta^{-(\frac{\alpha}{n}-1)}(y,t) &= \big(\theta_0^{-\left(\frac{\alpha}{n}-1\right)}(x) - \frac{3}{2}\theta_0^{-\left(\frac{\alpha}{n}-1\right)}(y)\big) \\ &\quad - \big(\frac{\alpha}{n}-1\big)\int_0^t \big(\sigma^{\frac{n+1}{n}}(x,\tau) - \frac{3}{2}\sigma^{\frac{n+1}{n}}(y,\tau)\big)d\tau \\ &\ge \theta_0^{-\left(\frac{\alpha}{n}-1\right)}(x) - \frac{3}{2}\theta_0^{-\left(\frac{\alpha}{n}-1\right)}(y) - C(\alpha,n)I_0 \\ &= m(x,y)\end{aligned} \tag{7.10}$$

where $C(\alpha,n)$ is an explicit positive constant. Suppose now that the solution θ does not blow up. Then $\theta(x,t)$ can be estimated outside the band I_δ by the bound (7.10). The latter is of course meaningful only when $m(x,y) > 0$. This can be achieved provided the value $\theta_0(y) = U(y)$ in (4.17) is sufficiently large, and the base state $\bar{\theta}$ suitably chosen.

Once x, y are selected so that $m(x,y) > 0$, equation (7.10) provides a bound for the temperature

$$\theta(x,t) \le M \quad x \notin I_\delta,\ 0 < t < \infty\,.$$

Since θ is increasing it converges to a limiting profile for $x \notin I_\delta$. Inside the band θ might increase indefinitely. In addition, (7.9) and (4.2) imply

$$v_x(x,t) = O\left((t+1)^{-\frac{1}{n+1}}\right)\,, \quad x \notin I_\delta,\ 0 < t < \infty\,,$$

in turn giving (4.18).

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