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RELATIVE ENTROPY METHODS FOR HYPERBOLIC AND DIFFUSIVE LIMITS

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Abstract. We review the relative entropy method in the context of hyperbolic and diffusive relaxation limits of entropy solutions for various hyperbolic models. The main example consists of the convergence from multidimensional compressible Euler equations with friction to the porous medium equation [7]. With small modifications, the arguments used in that case can be adapted to the study of the diffusive limit from the Euler-Poisson system with friction to the Keller-Segel system [8]. In addition, the p–system with friction and the system of viscoelasticity with memory are then reviewed, again in the case of diffusive limits [7]. Finally, the method of relative entropy is described for the multidimensional stress relaxation model converging to elastodynamics [6, Section 3.2], one of the first examples of application of the method to hyperbolic relaxation limits.

1. **Introduction.**

The relative entropy method was introduced in a context of hyperbolic systems by Dafermos and DiPerna [3, 2, 5] and serves as a mathematical tool for studying stability and limiting processes among thermomechanical theories. The method consists of a direct calculation of the relative entropy between a weak, *entropy dissipative* solution and a smooth, *entropy conservative* solution for the underlying thermomechanical processes, and leads to a striking stability formula. The same approach can be used to control hyperbolic relaxation limits [6, 9, 1] as well as diffusive relaxation [7]. The novelty in the latter case lies in the fact that, when dealing with a diffusive limit, the relative entropy method aims to compare weak, *entropy dissipative* solutions of the approximating hyperbolic system with smooth yet *entropy dissipative* solutions of the limit. Therefore, in order to prove that the relative entropy can serve as a Lyapunov–type functional for the model,

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one has in that case to also control the dissipation of the limit diffusive equation in terms of the dissipation of the approximating system.

In the present paper we review certain examples of diffusive relaxation analyzed in [7]: the case of 3–d isentropic gas dynamics with friction in Eulerian coordinates (Section 2), 1–d p–system with friction (Section 4), and 1–d viscoelasticity of the memory type converging to viscoelasticity of the rate type (Section 5.1). In addition to these cases, in Section 3 we shall apply this technique to the Euler-Poisson system with friction in a diffusive scaling, which has been treated in [4] by means of energy methods and compensated compactness tools.

Finally, in Section 5.2, we shall review a result from [6, Sec 3.2] concerning the convergence from viscoelasticity of the memory type to the equations of elastodynamics. This is one of the first examples in the literature where the technique of relative entropy has been utilized in the context of hyperbolic relaxation limits. The general framework of such singular limits has been studied in [9], while the corresponding analysis for general diffusive relaxation limits is still an open problem.

2. **Isentropic gas dynamics in Eulerian coordinates with friction.**

As an example to test the relative entropy method in the context of diffusive relaxation, let us consider the (scaled w.r.t. a diffusive scaling) system of isentropic gas dynamics with friction in three space dimensions:

$$\begin{cases} \rho_t + \dfrac{1}{\varepsilon}\operatorname{div}_x m = 0 \\ m_t + \dfrac{1}{\varepsilon}\operatorname{div}_x \dfrac{m\otimes m}{\rho} + \dfrac{1}{\varepsilon}\nabla_x p(\rho) = -\dfrac{1}{\varepsilon^2} m, \end{cases} \tag{1}$$

where $t\in\mathbb{R}$, $x\in\mathbb{R}^3$, the density $\rho\geq 0$ and the momentum flux $m\in\mathbb{R}^3$. At this level, the pressure $p(\rho)$ is a general function satisfying $p'(\rho)>0$ so that (1) is hyperbolic. The usual example of pressures verifying all needed conditions is given by the γ–laws: $p(\rho)=k\rho^\gamma$ with $\gamma\geq 1$ and $k>0$. The (formal) diffusive relaxation limit $\varepsilon\to 0$ yields the porous media equation

$$\bar\rho_t - \triangle_x p(\bar\rho) = 0\,. \tag{2}$$

In the sequel, we shall establish this limit via the relative entropy method.

An example of an entropy pair for (1) is given by the mechanical energy

$$\eta(\rho,m) = \frac{1}{2}\frac{|m|^2}{\rho} + h(\rho)$$

and the associated flux of mechanical work

$$q(\rho,m) = \frac{1}{2} m\frac{|m|^2}{\rho^2} + m h'(\rho)\,,$$

where $h(\rho)=\rho e(\rho)$ with $e(\rho)$ the internal energy of the gas:

$$e'(\rho) = \frac{p(\rho)}{\rho^2};\quad h''(\rho) = \frac{p'(\rho)}{\rho};\quad \rho h'(\rho) = p(\rho)+h(\rho).$$

For the particular case of γ–law gases, h takes the form

$$h(\rho) = \begin{cases} \dfrac{k}{\gamma-1}\rho^\gamma = \dfrac{1}{\gamma-1}p(\rho) & \text{for } \gamma>1\,, \\ k\rho\log\rho & \text{for } \gamma=1. \end{cases}$$

Smooth solutions of (1) satisfy the energy identity

$$\eta(\rho,m)_t + \frac{1}{\varepsilon}\operatorname{div}_x q(\rho,m) = -\frac{1}{\varepsilon^2}\nabla_m\eta(\rho,m)\cdot m = -\frac{1}{\varepsilon^2}\frac{|m|^2}{\rho} \le 0\,,$$

which reveals the dissipative nature of the mechanical energy $\eta(\rho,m)$ along the process (1).

Let us now consider a weak solution (ρ,m) of (1) that satisfies the weak form of the entropy inequality,

$$\eta(\rho,m)_t + \frac{1}{\varepsilon}\operatorname{div}_x q(\rho,m) + \frac{1}{\varepsilon^2}\frac{|m|^2}{\rho} \le 0\,, \tag{3}$$

and let $\bar\rho \ge 0$ be a smooth solution of the porous media equation (2). Clearly, $\bar\rho$ will also satisfy an energy dissipation identity of the form

$$h(\bar\rho)_t - \operatorname{div}_x\big(h'(\bar\rho)\nabla_x p(\bar\rho)\big) = -\frac{|\nabla_x p(\bar\rho)|^2}{\bar\rho} \le 0\,.$$

Thanks to the relative entropy method, we obtain an identity that monitors the distance between ρ and $\bar\rho$. Such identities have been obtained via the relative entropy method for hyperbolic relaxation in [6, 9, 1], while the first results in a diffusive relaxation framework are derived in [7].

We recall that the relative entropy is defined as the quadratic part of the Taylor series expansion between two solutions (ρ,m) and $(\bar\rho,\bar m)$:

$$\begin{aligned}\eta(\rho,m\,|\bar\rho,\bar m\,) &:= \eta(\rho,m) - \eta(\bar\rho,\bar m) - \eta_\rho(\bar\rho,\bar m)(\rho-\bar\rho) - \nabla_m\eta(\bar\rho,\bar m)\cdot(m-\bar m)\\ &= \frac12\rho\left|\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right|^2 + h(\rho\,|\bar\rho\,)\,,\end{aligned} \tag{4}$$

while the corresponding relative entropy-flux reads

$$\begin{aligned}q_i(\rho,m\,|\bar\rho,\bar m\,) &:= q_i(\rho,m) - q_i(\bar\rho,\bar m) - \eta_\rho(\bar\rho,\bar m)(m_i-\bar m_i)\\ &\qquad - \nabla_m\eta(\bar\rho,\bar m)\cdot(f_i(\rho,m)-f_i(\bar\rho,\bar m))\\ &= \frac12 m_i\Big|\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\Big|^2 + \rho(h'(\rho)-h'(\bar\rho))\Big(\frac{m_i}{\rho}-\frac{\bar m_i}{\bar\rho}\Big) + \frac{\bar m_i}{\bar\rho}h(\rho\,|\bar\rho\,)\,,\end{aligned} \tag{5}$$

where $i=1,2,3$, f_i stands for the (vector) of the flux in (1),

$$f_i(\rho,m) = m_i\frac{m}{\rho} + p(\rho)I_i\,,$$

and I_i is the i–th column of the 3×3 identity matrix.

As noticed in [7], the novelty in the case of diffusive relaxation lies mainly in the selection of the flux $\bar m$ in (4) and in (5). Indeed $\bar m$ is chosen to adapt itself in the relaxation, what allows to handle a diffusive relaxation process, where both solutions that are compared are energy dissipative. More precisely, we choose $\bar m = -\varepsilon\nabla_x p(\bar\rho)$ and we rewrite (2) in the form of the system of Euler equations with relaxation, plus additional higher–order error terms:

$$\begin{aligned}\bar\rho_t + \frac{1}{\varepsilon}\partial_{x_i}\bar m_i &= 0\\ \bar m_t + \frac{1}{\varepsilon}\partial_{x_i}f_i(\bar\rho,\bar m) &= -\frac{1}{\varepsilon^2}\bar m + e(\bar\rho,\bar m)\,,\end{aligned} \tag{6}$$

where (we use the convention of summation over repeated indices and) $\bar e$ is given by

$$\begin{aligned}\bar e := e(\bar\rho,\bar m) &= \frac{1}{\varepsilon}\,\mathrm{div}_x\left(\frac{\bar m\otimes\bar m}{\bar\rho}\right) - \varepsilon\partial_t\nabla_x p(\bar\rho)\\ &= \varepsilon\,\mathrm{div}_x\left(\frac{\nabla_x p(\bar\rho)\otimes\nabla_x p(\bar\rho)}{\bar\rho}\right) - \varepsilon\nabla_x(p'(\bar\rho)\triangle_x p(\bar\rho))\\ &= O(\varepsilon)\,. \end{aligned}\tag{7}$$

Thanks to the aforementioned rewriting of the limiting equation (2), it is possible to analyze the relative entropy (4) and to prove the following result [7].

Proposition 2.1. *Let (ρ,m) be a weak entropy solution of (1) satisfying (3) and let $(\bar\rho,\bar m)$ be a smooth solution of (6). Then,*

$$\partial_t\eta(\rho,m\,|\bar\rho,\bar m\,) + \frac{1}{\varepsilon}\,\mathrm{div}_x\,q(\rho,m\,|\bar\rho,\bar m\,) \le -\frac{1}{\varepsilon^2}R(\rho,m\,|\bar\rho,\bar m\,) - Q - E\,,\tag{8}$$

where

$$\begin{aligned} R(\rho,m\,|\bar\rho,\bar m\,) &= \rho\left|\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right|^2,\\ Q &= \frac{1}{\varepsilon}\nabla^2_{(\rho,m)}\eta(\bar\rho,\bar m)\begin{pmatrix}\bar\rho_{x_i}\\ \bar m_{x_i}\end{pmatrix}\cdot\begin{pmatrix}0\\ f_i(\rho,m|\bar\rho,\bar m)\end{pmatrix},\\ E &= e(\bar\rho,\bar m)\cdot\frac{\rho}{\bar\rho}\left(\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right),\end{aligned}\tag{9}$$

and $e(\bar\rho,\bar m)$ is defined in (7).

Concerning the relative entropy estimate (8), we point out that the coefficient of the quadratic term Q depends only on $(\bar\rho,\bar m)$ and it is $O(1)$ in ε:

$$\frac{1}{\varepsilon}\Big(\eta_{\rho m_j}(\bar\rho,\bar m)\bar\rho_{x_i} + \eta_{m_k m_j}(\bar\rho,\bar m)\partial_{x_i}\bar m_k\Big) = \frac{1}{\varepsilon}\partial_{x_i}\left(\frac{\bar m_j}{\bar\rho}\right) = -\partial_{x_i x_j}h'(\bar\rho)\,,$$

while the term E is an error term of order $O(\varepsilon)$. The term $R(\rho,m\,|\bar\rho,\bar m\,)$ captures the dissipation of the relaxation system (1) relative to its diffusive scale limit (2). It turns out to be the *quadratic part of the dissipative relaxation term with respect to* $(\bar\rho,\bar m)$, justifying the notation in (9). Clearly, the property $R(\rho,m\,|\bar\rho,\bar m\,)\ge 0$ is crucial in the stability analysis of the relaxation process.

An example of a framework in which to apply the relative entropy identity (8) is that of multidimensional periodic solutions, referred to as $(\mathbf{H_1})$:

(i) $(\rho\,,\,m):(0,T)\times\mathbb{T}^3\to\mathbb{R}^4$ is a (periodic) *dissipative weak solution* of (1) with $\rho\ge 0$, satisfying the weak form of (1) and the integrated form of the entropy inequality (3):

$$\begin{aligned}\iint_{[0,+\infty)\times\mathbb{T}^3}&\left[\left(\frac{1}{2}\frac{|m|^2}{\rho}+h(\rho)\right)\dot\theta(t) - \frac{1}{\varepsilon^2}\frac{|m|^2}{\rho}\theta(t)\right]dxdt\\ &+\int_{\mathbb{T}^3}\left(\frac{1}{2}\frac{|m|^2}{\rho}+h(\rho)\right)\Big|_{t=0}\theta(0)dx \ge 0\,,\end{aligned}$$

for any $\theta(t)$ nonnegative Lipschitz test function compactly supported in $[0,T)$. The family (ρ, m) is assumed to satisfy the (uniform in ε) bounds

$$\sup_{t\in(0,T)} \int_{\mathbb{T}^3} \rho dx \le K_1 < \infty \,,$$
$$\sup_{t\in(0,T)} \int_{\mathbb{T}^3} \left[\frac{1}{2}\frac{|m|^2}{\rho} + h(\rho)\right] dx \le K_2 < \infty \,,$$

which are natural within the given framework, and follow from corresponding uniform bounds on the initial data.

(ii) $\bar\rho$ is a smooth (C^3) periodic solution of the multidimensional porous media equation (2) that avoids vacuum, $\bar\rho \ge \bar\rho^* > 0$; $\bar m$ is defined via $\bar m = -\varepsilon \nabla p(\bar\rho)$.

Using the stability property in Proposition 2.1, one controls the distance between the relaxing sequence and the limiting solution by means of the distance function:

$$\varphi(t) = \int_{\mathbb{T}^3} \eta(\rho, m\,|\bar\rho, \bar m)dx \,.$$

The results are valid for pressure laws satisfying quite general conditions (see theorem below), and apply to γ–law pressures $p(\rho) = k\rho^\gamma$, $\gamma \ge 1$. For the proof we refer once again to [7].

Theorem 2.2. *Let $T > 0$ be fixed and assume $p(\rho)$ satisfies*

$$p''(\rho) \le A\frac{p'(\rho)}{\rho} \quad \forall\ \rho > 0$$

and

$$p'(\rho) = k\gamma\rho^{\gamma-1} + o(\rho^{\gamma-1})\,, \quad \text{as } \rho \to +\infty\,.$$

Under hypothesis $(\mathbf{H_1})$*, the stability estimate*

$$\varphi(t) \le C\big(\varphi(0) + \varepsilon^4\big), \quad t \in [0,T]\,,$$

holds, where C is a positive constant depending only on T, K_1, $\bar\rho$ and its derivatives. Moreover, if $\varphi(0) \to 0$ as $\varepsilon \to 0$, then

$$\sup_{t\in[0,T]} \varphi(t) \to 0, \ \text{as } \varepsilon \to 0\,.$$

Remark 2.3. The relative entropy method can also be applied to other frameworks, such as 1–d dissipative weak solutions with different end states at $\pm\infty$ [7, Sec 2.3.2] as well as for comparing entropic measure-valued solutions of the Euler equations with friction to smooth solutions of the porous media equation [7, Sec 2.4].

3. **The diffusive limit from the Euler-Poisson system with friction to the Keller-Segel system.** A variant of the above calculation may be used to establish convergence from the Euler–Poisson system with attractive potentials and friction to the Keller-Segel model. The Euler-Poisson system with friction is

$$\begin{cases} \rho_t + \dfrac{1}{\varepsilon}\operatorname{div}_x m = 0 \\ m_t + \dfrac{1}{\varepsilon}\operatorname{div}_x \dfrac{m\otimes m}{\rho} + \dfrac{1}{\varepsilon}\nabla_x p(\rho) = -\dfrac{1}{\varepsilon^2} m + \dfrac{1}{\varepsilon}\rho\nabla_x c \\ -\triangle_x c + \beta c = \rho, \end{cases} \tag{10}$$

where, as usual, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, $\rho \ge 0$, $c \in \mathbb{R}$, $m \in \mathbb{R}^3$, the pressure $p(\rho)$ satisfies $p'(\rho) > 0$ and β is a positive, sufficiently large constant, as we shall see in the sequel, which captures the effects of screening. In the limit $\varepsilon \to 0$, we obtain

$m = \rho\nabla_x c - \nabla_x p(\rho)$, and therefore the formal limit of (10) is given by the Keller-Segel type model:

$$\begin{cases} \rho_t + \operatorname{div}_x \big(\rho\nabla_x c - \nabla_x p(\rho)\big) = 0 \\ -\triangle_x c + \beta c = \rho. \end{cases} \tag{11}$$

We refer to [4] (and references therein) for convergence results using the compensated compactness method, and discussions of alternate scalings. Here, we focus to the convergence from (10) to (11) as a case study of the relative entropy method.

We again consider the entropy–entropy flux pair

$$\eta(\rho, m) = \frac{1}{2}\frac{|m|^2}{\rho} + h(\rho), \quad q(\rho, m) = \frac{1}{2} m \frac{|m|^2}{\rho^2} + m h'(\rho), \quad h''(\rho) = \frac{p'(\rho)}{\rho},$$

and note that an entropy weak solution of (10) satisfies the entropy inequality

$$\eta(\rho, m)_t + \frac{1}{\varepsilon}\operatorname{div}_x q(\rho, m) \le -\frac{1}{\varepsilon^2}\frac{|m|^2}{\rho} + \frac{1}{\varepsilon} m\cdot\nabla_x c\,. \tag{12}$$

On the other hand, smooth solutions of (11) satisfy the entropy identity

$$h(\rho)_t + \operatorname{div}_x \big(h'(\rho)(\rho\nabla_x c - \nabla_x p(\rho))\big) = -\frac{|\nabla_x p(\rho)|^2}{\rho} + \nabla_x p(\rho)\cdot\nabla_x c\,. \tag{13}$$

Note that (13) is indeed the equilibrium version ($\varepsilon = 0$) of the energy dissipation (12), as can be easily shown via the standard Hilbert expansion analysis.

As it is manifest, neither (12) nor (13) are indeed *dissipative*, due to the extra terms coming from the coupling with the equation for the concentration c. To take into account these extra terms, we consider the following modified entropy–entropy flux pair, again based on the mechanical energy of the system under consideration:

$$\mathcal{H}(\rho, m, c) = \eta(\rho, m) - \rho c\,,$$
$$\mathcal{Q}(\rho, m, c) = q(\rho, m) - mc\,.$$

Then the entropy inequality becomes

$$\mathcal{H}(\rho, m, c)_t + \frac{1}{\varepsilon}\operatorname{div}_x \mathcal{Q}(\rho, m, c) \le -\frac{1}{\varepsilon^2}\frac{|m|^2}{\rho} - \rho c_t\,. \tag{14}$$

Moreover, multiplying $(10)_3$ by c_t we get

$$\rho c_t = \frac{1}{2}\big(\beta c^2 + |\nabla_x c|^2\big)_t - \operatorname{div}_x\big(c_t\nabla_x c\big)\,,$$

which once added to (14) gives

$$\left(\mathcal{H}(\rho, m, c) + \frac{1}{2}\big(\beta c^2 + |\nabla_x c|^2\big)\right)_t + \frac{1}{\varepsilon}\operatorname{div}_x\big(\mathcal{Q}(\rho, m, c) - \varepsilon c_t\nabla_x c\big) \le -\frac{1}{\varepsilon^2}\frac{|m|^2}{\rho}\,. \tag{15}$$

The estimate (15) is the starting point to obtain the stability estimate in terms of relative entropy and the corresponding analysis of the relaxation limit.

We rewrite the equilibrium system (11) in the variables $\bar\rho$, $\bar c$ and

$$\bar m = -\varepsilon\big(\nabla_x p(\bar\rho) - \bar\rho\nabla_x\bar c\big) = -\varepsilon\bar\rho\nabla_x\big(h'(\bar\rho) - \bar c\big)$$

in the form:

$$\begin{cases} \bar\rho_t + \frac{1}{\varepsilon}\operatorname{div}_x \bar m = 0 \\ \bar m_t + \frac{1}{\varepsilon}\operatorname{div}_x \frac{\bar m\otimes\bar m}{\bar\rho} + \frac{1}{\varepsilon}\nabla_x p(\bar\rho) = -\frac{1}{\varepsilon^2}\bar m + \frac{1}{\varepsilon}\bar\rho\nabla_x\bar c + e(\bar\rho, \bar m) \\ -\triangle_x\bar c + \beta\bar c = \bar\rho, \end{cases} \tag{16}$$

where the error term $e(\bar\rho,\bar m)$ is

$$\begin{aligned}\bar e := e(\bar\rho,\bar m) &= \frac{1}{\varepsilon}\operatorname{div}_x\left(\frac{\bar m\otimes\bar m}{\bar\rho}\right) - \varepsilon\partial_t\big(\nabla_x p(\bar\rho) - \bar\rho\nabla_x\bar c\big) \qquad (17)\\ &= \varepsilon\operatorname{div}_x\Big(\bar\rho\nabla_x\big(h'(\bar\rho)-\bar c\big)\otimes\nabla_x\big(h'(\bar\rho)-\bar c\big)\Big) - \varepsilon\partial_t\big(\bar\rho\nabla_x\big(h'(\bar\rho)-\bar c\big)\big) = O(\varepsilon)\,.\end{aligned}$$

In turn, (13) is rewritten as

$$\eta(\bar\rho,\bar m)_t + \frac{1}{\varepsilon}\operatorname{div}_x q(\bar\rho,\bar m) = -\frac{1}{\varepsilon^2}\frac{|\bar m|^2}{\bar\rho} + \frac{1}{\varepsilon}\bar m\cdot\nabla_x\bar c + \nabla_m\eta(\bar\rho,\bar m)\cdot\bar e\,,$$

or, equivalently,

$$\begin{aligned}&\left(\mathcal{H}(\bar\rho,\bar m,\bar c) + \frac12\big(\beta\bar c^2 + |\nabla_x\bar c|^2\big)\right)_t + \frac{1}{\varepsilon}\operatorname{div}_x\big(\mathcal{Q}(\bar\rho,\bar m,\bar c) - \varepsilon\bar c_t\nabla_x\bar c\big)\\ &= -\frac{1}{\varepsilon^2}\frac{|\bar m|^2}{\bar\rho} + \nabla_m\eta(\bar\rho,\bar m)\cdot\bar e\,.\end{aligned}$$

We now define the relative entropy

$$\mathcal{H}(\rho,m,c\,|\,\bar\rho,\bar m,\bar c\,) = \eta(\rho,m\,|\,\bar\rho,\bar m\,) - (\rho-\bar\rho)(c-\bar c)\,,$$

with corresponding relative entropy flux

$$\mathcal{Q}(\rho,m,c\,|\,\bar\rho,\bar m,\bar c\,) = q(\rho,m\,|\,\bar\rho,\bar m\,) - (m-\bar m)(c-\bar c)\,,$$

and show that:

Proposition 3.1. *For any weak, entropy solution (ρ,m,c) of (10) and any smooth solution $(\bar\rho,\bar m,\bar c)$ of (16) it holds*

$$\begin{aligned}&\partial_t\left(\mathcal{H}(\rho,m,c\,|\,\bar\rho,\bar m,\bar c\,) + \frac12\big(\beta(c-\bar c)^2 + |\nabla_x(c-\bar c)|^2\big)\right)\\ &\quad + \frac{1}{\varepsilon}\operatorname{div}_x\Big(\mathcal{Q}(\rho,m,c\,|\,\bar\rho,\bar m,\bar c\,) - \varepsilon(c-\bar c)_t\nabla_x(c-\bar c)\Big)\\ &\le -\frac{1}{\varepsilon^2}R(\rho,m\,|\,\bar\rho,\bar m\,) - Q - P - E\,, \qquad (18)\end{aligned}$$

where R, Q and E are defined in Proposition 2.1, but with $\bar e$ defined in (17), and

$$P = \frac{1}{\varepsilon}\frac{\bar m}{\bar\rho}(\rho-\bar\rho)\cdot\nabla_x(c-\bar c)\,.$$

Proof. We sketch the proof of (18) starting from Proposition 2.1 and analyzing the extra terms coming from the coupling with the elliptic equation involving the variable c. The estimate for $\eta(\rho,m\,|\,\bar\rho,\bar m\,)$ becomes in this case

$$\begin{aligned}&\eta(\rho,m\,|\,\bar\rho,\bar m\,)_t + \frac{1}{\varepsilon}\operatorname{div}_x q(\rho,m\,|\,\bar\rho,\bar m\,)\\ &\quad\le -\frac{1}{\varepsilon^2}R(\rho,m\,|\,\bar\rho,\bar m\,) - Q - E + \frac{1}{\varepsilon}\rho\left(\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right)\cdot\nabla_x(c-\bar c)\,.\end{aligned}$$

Then, we multiply

$$-\triangle_x(c-\bar c) + \beta(c-\bar c) = \rho-\bar\rho$$

by $(c-\bar c)_t$ to conclude

$$(\rho-\bar\rho)(c-\bar c)_t = \frac12\big(\beta(c-\bar c)^2 + |\nabla_x(c-\bar c)|^2\big)_t - \operatorname{div}_x\big((c-\bar c)_t\nabla_x(c-\bar c)\big)\,.$$

Putting all relations together, we end up with

$$\begin{aligned}
&\partial_t \left(\mathcal{H}(\rho, m, c \,|\, \bar\rho, \bar m, \bar c\,) + \frac{1}{2}\big(\beta(c-\bar c)^2 + |\nabla_x (c-\bar c)|^2\big)\right)\\
&\quad + \frac{1}{\varepsilon}\operatorname{div}_x \big(\mathcal{Q}(\rho, m, c \,|\, \bar\rho, \bar m, \bar c\,) - \varepsilon (c-\bar c)_t \nabla_x (c-\bar c)\big)\\
&\le -\frac{1}{\varepsilon^2} R(\rho, m \,|\, \bar\rho, \bar m\,) - Q - E\\
&\quad + \frac{1}{\varepsilon}\rho\left(\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right)\cdot \nabla_x (c-\bar c) - \frac{1}{\varepsilon}(m-\bar m)\cdot\nabla_x(c-\bar c)\\
&= -\frac{1}{\varepsilon^2} R(\rho, m \,|\, \bar\rho, \bar m\,) - Q - E - \frac{1}{\varepsilon}\frac{\bar m}{\bar\rho}(\rho-\bar\rho)\cdot\nabla_x(c-\bar c)\,,
\end{aligned}$$

which is exactly (18). □

We conclude our analysis by using the inequality (18) in the particular case $p(\rho) = h(\rho) = k\rho^2$ for which the limit system becomes

$$\begin{cases} \rho_t + \operatorname{div}_x\big(\rho\nabla_x(c-2\rho)\big) = 0\\ -\triangle_x c + \beta c = \rho. \end{cases} \tag{19}$$

Indeed, in that case, if $\beta > \frac{1}{2k}$, the relative entropy gives directly the L^2 control of the relaxation process, and in particular of the difference $\rho - \bar\rho$, and this is exactly what is needed in the estimate of the extra term P obtained above. Clearly, other frameworks of applications can be considered, for instance γ–laws for $\gamma \ge 2$ and ρ, $\bar\rho \ge \rho_* > 0$, for which we have in particular $h(\rho\,|\,\bar\rho\,) \ge C(\rho-\bar\rho)^2$.

Therefore, we denote

$$\psi(t) = \int_{\mathbb{T}^3} \left(\frac{1}{2}\rho\left|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right|^2 + \frac{1}{2}\big(\beta(c-\bar c)^2 + |\nabla_x(c-\bar c)|^2\big) + k(\rho-\bar\rho)^2 - (\rho-\bar\rho)(c-\bar c)\right) dx$$

and we observe

$$\psi(t) \ge C\left(\int_{\mathbb{T}^3} \frac{1}{2}\rho\left|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right|^2 dx + \|c-\bar c\|^2_{L^2} + \|\nabla_x(c-\bar c)\|^2_{L^2} + \|\rho-\bar\rho\|^2_{L^2}\right)$$

for β as above.

We again consider dissipative weak solutions of (10), for which in particular an integrated version of the relative entropy estimate (18) can be rigorously derived; more specific results in this direction are under investigation in [8]. We place the following hypotheses, referred to as $(\mathbf{H_2})$:

(i) $(\rho\,,\, m\,, c) : (0,T)\times\mathbb{T}^3 \to \mathbb{R}^5$ is a (periodic) *dissipative weak solution* of (10) with $p(\rho) = k\rho^2$, $\beta > \frac{1}{2k}$, with $\rho \ge 0$, satisfying the weak form of (10) and the integrated form of the relative entropy inequality (18):

$$\begin{aligned}
&\iint_{[0,+\infty)\times\mathbb{T}^3} \Bigg[\left(\mathcal{H}(\rho, m, c \,|\, \bar\rho, \bar m, \bar c\,) + \frac{1}{2}\big(\beta(c-\bar c)^2 + |\nabla_x(c-\bar c)|^2\big)\right)\dot\theta(t)\\
&\qquad - \left(\frac{1}{\varepsilon^2} R(\rho, m \,|\, \bar\rho, \bar m) - Q - P - E\right)\theta(t)\Bigg]\, dxdt\\
&\qquad + \int_{\mathbb{T}^3} \left(\mathcal{H}(\rho, m, c \,|\, \bar\rho, \bar m, \bar c\,) + \frac{1}{2}\big(\beta(c-\bar c)^2 + |\nabla_x(c-\bar c)|^2\big)\right)\Bigg|_{t=0}\theta(0)dx \ge 0\,,
\end{aligned}$$

for any $\theta(t)$ nonnegative Lipschitz test function compactly supported in $[0,T)$. The family (ρ,m,c) is assumed to satisfy the (uniform in ε) bounds

$$\sup_{t\in(0,T)}\int_{\mathbb{T}^3}\rho dx\le K_1<\infty\,,$$
$$\int_{\mathbb{T}^3}\left(\mathcal{H}(\rho,m,c\,|\bar\rho,\bar m,\bar c)+\frac12\big(\beta(c-\bar c)^2+|\nabla_x(c-\bar c)|^2\big)\right)\Big|_{t=0}dx\le K_2<\infty\,,$$

which are natural within the given framework.

(ii) $(\bar\rho,\bar c)$ is a smooth (C^3) periodic solution of (19) such that $\bar\rho\ge\bar\rho^*>0$; $\bar m$ is defined via $\bar m=-\varepsilon\bar\rho\nabla_x\big(2\rho-c\big)$.

Then the following theorem holds.

Theorem 3.2. *Let $T>0$ be fixed and assume that hypothesis $(\mathbf{H_2})$ holds. Then, the following stability estimate holds:*

$$\psi(t)\le C(\psi(0)+\varepsilon^4)\,,$$

for any $t\in[0,T]$, with C a positive constant depending only on T, K_1, $\bar\rho$, $\bar c$ and their derivatives. Moreover, if $\psi(0)\to 0$ as $\varepsilon\to 0$, then

$$\sup_{t\in[0,T]}\psi(t)\to 0,\ \text{ as }\varepsilon\to 0\,.$$

Proof. As usual in this context, in $(\mathbf{H_2})$ we choose the test function

$$\theta(\tau):=\begin{cases}1, & \text{for } 0\le\tau<t,\\ \frac{t-\tau}{\kappa}+1, & \text{for } t\le\tau<t+\kappa,\\ 0, & \text{for } \tau\ge t+\kappa,\end{cases}$$

to get, as $\kappa\to 0$,

$$\psi(t)+\frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{T}^3}R(\rho,m\,|\bar\rho,\bar m)dxd\tau\le\psi(0)+\int_0^t\int_{\mathbb{T}^3}(|Q|+|E|+|P|)dxd\tau\,.$$

Now, the hypotheses of Theorem 2.2 are satisfied for the particular case $p(\rho)=k\rho^2$. Therefore, we can carry out here the same estimates for the terms $|Q|$ and $|E|$ as follows:

$$\int_0^t\int_{\mathbb{T}^3}|Q|dxd\tau\le C_1\int_0^t\psi(\tau)d\tau\,,$$

where C_1 depends on $\|\partial_{x_ix_j}\bar\rho\|_{L^\infty}$. The error term E in (9) is estimated by

$$\begin{aligned}\int_0^t\int_{\mathbb{T}^3}|E|dxd\tau&\le\frac{\varepsilon^2}{2}\int_0^t\int_{\mathbb{T}^3}\left|\frac{\bar e}{\bar\rho}\right|^2\rho dxd\tau+\frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^3}\rho\left|\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right|^2dxd\tau\\&\le C_2\varepsilon^4\,t+\frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^3}R(\rho,m\,|\bar\rho,\bar m)dxd\tau\,,\end{aligned}$$

where C_2 depends on K_1, T and $\bar\rho$ through the following norms of derivatives up to third order:

$$\left\|\frac{1}{\bar\rho}\,\mathrm{div}_x\left(\bar\rho\nabla_x\big(\bar\rho-\bar c\big)\otimes\nabla_x\big(\bar\rho-\bar c\big)\right)\right\|_{L^\infty}+\left\|\frac{1}{\bar\rho}\partial_t\big(\bar\rho\nabla_x\big(\bar\rho-\bar c\big)\big)\right\|_{L^\infty}.$$

Finally, Young's inequality implies

$$\int_0^t\int_{\mathbb{R}^3}|P|dxd\tau\le C\|\nabla_x(2\bar\rho-\bar c)\|_\infty\int_0^t\psi(\tau)d\tau$$

and the proof follows from the Gronwall Lemma. □

4. **The p–system with friction.** Another, actually easier case in which one can apply the above technique is given by the p–system with friction in one space dimension:

$$\begin{aligned} u_t - \frac{1}{\varepsilon} v_x &= 0 \\ v_t - \frac{1}{\varepsilon}\tau(u)_x &= -\frac{1}{\varepsilon^2} v\,, \end{aligned} \tag{20}$$

where τ satisfies $\tau'(u) > 0$ to guarantee strict hyperbolicity. The system (20) is a model either for elasticity with friction or for isentropic gas dynamics in Lagrangian coordinates. Then u stands for the strain (or the specific volume for gases), v for the velocity and τ for the stress.

In the limit $\varepsilon \to 0$, solutions of (20) converge towards solutions of the parabolic equation

$$u_t - \tau(u)_{xx} = 0\,. \tag{21}$$

This limit may be obtained via the relative entropy estimate as we describe below; we refer to [7] for the details.

To this aim, let us consider the mechanical energy

$$\mathcal{E}(u,v) = \frac{1}{2}v^2 + W(u)\,,$$

where

$$W(u) = \int_0^u \tau(s)ds$$

stands for the stored energy, and its associated flux

$$\mathcal{F}(u,v) = -v\tau(u)\,.$$

The corresponding entropy inequality is

$$\mathcal{E}(u,v)_t + \frac{1}{\varepsilon}\mathcal{F}(u,v)_x \le -\frac{1}{\varepsilon^2}v^2 \le 0\,, \tag{22}$$

and captures the dissipation of the mechanical energy for weak solutions of (20). Smooth solutions of (21) satisfy the energy dissipation identity

$$\mathcal{E}(u,0)_t + \mathcal{F}(u,\tau(u)_x)_x = -\big(\tau(u)_x\big)^2 \le 0\,.$$

The latter is the equilibrium ($\varepsilon = 0$) limit of (22).

The relative entropy is again defined as the quadratic part of the Taylor expansion of $\mathcal{E}(u,v)$ relative to the "algebraic–differential equilibrium" $(\bar u, \bar v)$, where $\bar u$ is a smooth solution of (21) and $\bar v = \varepsilon\tau(\bar u)_x$. Namely,

$$\begin{aligned} \mathcal{E}(u,v\,|\bar u,\bar v\,) &= \mathcal{E}(u,v) - \mathcal{E}(\bar u,\bar v) - \mathcal{E}_u(\bar u,\bar v)(u-\bar u) - \mathcal{E}_v(\bar u,\bar v)(v-\bar v) \\ &= \frac{1}{2}(v-\bar v)^2 + W(u\,|\bar u\,)\,. \end{aligned}$$

As corresponding flux we shall consider

$$\begin{aligned} \mathcal{F}(u,v\,|\bar u,\bar v\,) &= \mathcal{F}(u,v) - \mathcal{F}(\bar u,\bar v) + \mathcal{E}_u(\bar u,\bar v)(v-\bar v) + \mathcal{E}_v(\bar u,\bar v)(\tau(u)-\tau(\bar u)) \\ &= -(v-\bar v)(\tau(u)-\tau(\bar u))\,. \end{aligned}$$

As in the previous sections, we rewrite the equilibrium equation (21) as a damped p–system

$$\begin{cases} \bar u_t - \dfrac{1}{\varepsilon}\bar v_x = 0 \\ \bar v_t - \dfrac{1}{\varepsilon}\tau(\bar u)_x = -\dfrac{1}{\varepsilon^2}\bar v + \bar v_t\,, \end{cases} \tag{23}$$

where the term $\bar{v}_t = \varepsilon\tau(\bar{u})_{xt}$ is an error of order ε. Then a direct computation gives the following proposition.

Proposition 4.1. *For any weak, entropy solution (u, v) of (20) and any smooth solution $(\bar{u}, \bar{v})$ of (23) it holds:*

$$\mathcal{E}(u, v\,|\bar{u}, \bar{v})_t + \frac{1}{\varepsilon}\mathcal{F}(u, v\,|\bar{u}, \bar{v})_x \leq -\frac{1}{\varepsilon^2}(v-\bar{v})^2 + \tau(\bar{u})_{xx}\tau(u\,|\bar{u}) - \varepsilon\tau(\bar{u})_{xt}(v-\bar{v})\,. \quad (24)$$

The terms in the right hand side of (24) are analogous to the terms in (8) of Proposition 2.1 for the Eulerian case: the first term is dissipative and is due to the friction in the relaxation system, the second is quadratic in the flux, while the last term is a linear error term.

Finally, using this result, one can obtain stability and convergence of the relaxation limit in terms of the quantity

$$\int_{\mathbb{R}} \mathcal{E}(u, v\,|\bar{u}, \bar{v})dx\,,$$

provided $\tau(u)$ satisfies appropriate growth conditions at infinity; see [7] for details.

5. **Viscoelasticity with memory.** It is well known that the system of viscoelasticity of memory type can yield in different scaling limits both the equations of viscoelasticity of the rate type as well as the equations of dynamic elasticity. We consider such scaling limits from the perspective of the relative entropy method, hoping to indicate the remarkably wide applicability of the methodology. We start by considering a quasilinear model (1–d for simplicity) with a diffusive scaling, thus entering in the framework of diffusive relaxations [7], and then we shall review the multidimensional model of stress relaxation approximating the equations of elastodynamics considered in [6, Sec 3.2].

5.1. **From viscoelasticity of the memory type to viscoelasticity of the rate type.** First, we consider a diffusive scaling limit leading to a hyperbolic – parabolic system describing the dynamics of a 1–d viscoelastic material of the rate type. To this end, consider the following 3×3, one dimensional, quasilinear system of viscoelasticity with memory effects:

$$\begin{aligned} u_t - v_x &= 0 \\ v_t - \sigma(u)_x - \frac{1}{\varepsilon}z_x &= 0 \\ z_t - \frac{\mu}{\varepsilon}v_x &= -\frac{1}{\varepsilon^2}z\,, \end{aligned} \quad (25)$$

where $\mu > 0$ and the elastic stress function σ satisfies the usual condition $\sigma'(u) > 0$ for hyperbolicity. In (25), the stress $S = \sigma(u) + \frac{1}{\varepsilon}z$ is decomposed in a purely elastic part and a viscoelastic part of the memory type (see $(25)_3$ for z), scaled so that it relaxes as $\varepsilon \to 0$ to the equations of viscoelasticity of the rate type:

$$\begin{aligned} u_t - v_x &= 0 \\ v_t - \sigma(u)_x &= \mu v_{xx}\,. \end{aligned} \quad (26)$$

Indeed, in (26) the stress is given by $\sigma(u) + \mu v_x$, that is again the same elastic part plus a Newtonian viscous stress.

The mechanical energy – energy flux couple for (25) is

$$\mathbb{E}(u,v,z) = \int_0^u \sigma(s)ds + \frac{1}{2}v^2 + \frac{1}{2\mu}z^2 = \Sigma(u) + \frac{1}{2}v^2 + \frac{1}{2\mu}z^2,$$
$$\mathbb{F}_\varepsilon(u,v,z) = -(\varepsilon\sigma(u)v + vz)\,.$$

Hence, the dissipation of mechanical energy for weak solutions of (25) reads

$$\mathbb{E}(u,v,z)_t + \frac{1}{\varepsilon}\mathbb{F}_\varepsilon(u,v,z)_x \leq -\frac{1}{\mu\varepsilon^2}z^2$$

and the corresponding relation for smooth solutions of (26) is given by

$$\mathbb{E}(u,v,0)_t + \mathbb{F}_1(u,v,\sigma(u)_x)_x = -\mu(v_x)^2 \leq 0\,,$$

for

$$\mathbb{E}(u,v,0) = \Sigma(u) + \frac{1}{2}v^2, \quad \mathbb{F}_1(u,v,\sigma(u)_x) = -(\sigma(u)v + \mu v v_x)\,.$$

We rewrite the equilibrium system (26) and the corresponding stress–strain response for the variables $(\bar{u},\bar{v},\bar{z})$ with $\bar{z} = \varepsilon\mu\bar{v}_x$ as follows:

$$\begin{cases} \bar{u}_t - \bar{v}_x = 0 \\ \bar{v}_t - \sigma(\bar{u})_x - \frac{1}{\varepsilon}\bar{z}_x = 0 \\ \bar{z}_t - \frac{\mu}{\varepsilon}\bar{v}_x = -\frac{1}{\varepsilon^2}\bar{z} + \bar{z}_t\,, \end{cases} \tag{27}$$

where we shall treat the term $\bar{z}_t$ as an $O(\varepsilon)$ error:

$$\bar{z}_t = \varepsilon\mu\bar{v}_{xt} = \varepsilon\mu\big(\sigma(\bar{u})_x + \mu\bar{v}_{xx}\big)_x\,.$$

Finally, the relative entropy and relative entropy flux, respectively,

$$\begin{aligned} \mathbb{E}(u,v,z\,|\bar{u},\bar{v},\bar{z}\,) &= \mathbb{E}(u,v,z) - \mathbb{E}(\bar{u},\bar{v},\bar{z}) \\ &\quad - \mathbb{E}_u(\bar{u},\bar{v},\bar{z})(u-\bar{u}) - \mathbb{E}_v(\bar{u},\bar{v},\bar{z})(v-\bar{v}) - \mathbb{E}_z(\bar{u},\bar{v},\bar{z})(z-\bar{z})\,, \\ \mathbb{F}_\varepsilon(u,v,z\,|\bar{u},\bar{v},\bar{z}\,) &= \mathbb{F}_\varepsilon(u,v,z) - \mathbb{F}_\varepsilon(\bar{u},\bar{v},\bar{z}) - \mathbb{E}_u(\bar{u},\bar{v},\bar{z})\big(-\varepsilon(v-\bar{v})\big) \\ &\quad - \mathbb{E}_v(\bar{u},\bar{v},\bar{z})\big(-\varepsilon(\sigma(u)-\sigma(\bar{u})) - (z-\bar{z})\big) - \mathbb{E}_z(\bar{u},\bar{v},\bar{z})\big(v-\bar{v}\big)\,, \end{aligned}$$

verify the following identity:

Proposition 5.1. *Let (u,v,z) be a weak entropy solution of (25) and let $(\bar{u},\bar{v},\bar{z})$ be a smooth solution of (27). Then*

$$\begin{aligned} \partial_t\mathbb{E}(u,v,z\,|\bar{u},\bar{v},\bar{z}\,) &+ \frac{1}{\varepsilon}\partial_x\mathbb{F}_\varepsilon(u,v,z\,|\bar{u},\bar{v},\bar{z}\,) \\ &\leq -\frac{1}{\mu\varepsilon^2}(z-\bar{z})^2 + \bar{v}_x\sigma(u\,|\bar{u}\,) - \varepsilon\bar{v}_{xt}(z-\bar{z})\,. \end{aligned}$$

As in the previous cases, Proposition 5.1 suggests to measure the distance between systems (25) and (26) by means of

$$\int_{\mathbb{R}} \mathbb{E}(u,v,z\,|\bar{u},\bar{v},\bar{z}\,)dx\,,$$

and this can be done under appropriate structural condition on the stress function $\sigma(u)$ [7].

5.2. **A model of stress relaxation approximating the equations of elastodynamics.** As a final application of the relative entropy method, we shall review the case of the hyperbolic–hyperbolic relaxation limit $\varepsilon \to 0$ for the model of stress relaxation

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha S_{i\alpha} \\ \partial_t (S_{i\alpha} - f_{i\alpha}(F)) &= -\frac{1}{\varepsilon}(S_{i\alpha} - T_{i\alpha}(F)) \,. \end{aligned} \tag{28}$$

In (28), i, $\alpha = 1,2,3$, F stands for the deformation gradient, v for the velocity and the stress S is again decomposed in an elastic part and a viscoelastic part with memory effects:

$$S = f(F) + \int_{-\infty}^{t} \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}(t-\tau)} h(F(\cdot,\tau))\, d\tau \,.$$

In turn, the equilibrium stress is accordingly decomposed as $T(F) = f(F) + h(F)$. Following [6], we shall derive a relative entropy relation for smooth solutions (v, F, S) of (28) and smooth solutions $(\bar{v}, \bar{F})$ of its limit, that is the elasticity system

$$\begin{aligned} \partial_t \bar{F}_{i\alpha} &= \partial_\alpha \bar{v}_i \\ \partial_t \bar{v}_i &= \partial_\alpha T_{i\alpha}(\bar{F}) \,, \end{aligned} \tag{29}$$

even if with nowadays technologies the same relation can be rigorously justified for dissipative weak solutions of (28). To this end, let us consider the framework

$$\begin{aligned} T(F) = \nabla_F W(F) &= f(F) + h(F) \,, \\ f(F) = \nabla_F W_I(F), &\quad h(F) = -\nabla_F W_v(F) \end{aligned} \tag{a}$$

and $W_v = W_I - W$ is convex. Under these structural hypotheses, the dissipation of the mechanical energy reads:

$$\begin{aligned} &\partial_t \left(\frac{1}{2}|v|^2 + \Psi(F, S - f(F)) \right) - \partial_\alpha (v_i S_{i\alpha}) \\ &\qquad + \frac{1}{\varepsilon}(F_{i\alpha} - h^{-1}_{i\alpha}(S - f(F)))(S_{i\alpha} - T_{i\alpha}(F)) = 0 \,. \end{aligned} \tag{30}$$

In (30), the free energy function Ψ is of the form

$$\Psi(F, A) = W_I(F) + A \cdot F + G(A) \,,$$

where G is a convex function such that $\nabla_A G = -h^{-1}$. Indeed, the condition that the inverse of h is a gradient is equivalent to the existence of a free energy function for (28) compatible with the Clausius-Duhem inequality. In (30) this is expressed by the positivity of the last term, revealing the dissipation arising from of the viscoelastic stresses [6].

At this point, we define the relative energy $\mathcal{E}_r(v, F, S \,|\, \bar{v}, \bar{F}, h(\bar{F}))$ generated by the mechanical energy relative to an equilibrium as follows:

$$\begin{aligned} \mathcal{E}_r &:= \frac{1}{2}|v - \bar{v}|^2 + \Psi(F, S - f(F)) - \Psi(\bar{F}, h(\bar{F})) \\ &\qquad - \nabla_F \Psi(\bar{F}, h(\bar{F})) \cdot (F - \bar{F}) - \nabla_A \Psi(\bar{F}, h(\bar{F})) \cdot (S - f(F) - h(\bar{F})) \\ &= \frac{1}{2}|v - \bar{v}|^2 + \Psi(F, S - f(F)) - W(\bar{F}) - \nabla_F W(\bar{F}) \cdot (F - \bar{F}) \,, \end{aligned}$$

by selecting an appropriate normalization so that $\Psi(F, h(F)) = W(F)$. The associated relative fluxes are then given by

$$\mathcal{F}_r^\alpha = (v_i - \bar v_i)(S_{i\alpha} - T_{i\alpha}(\bar F)) \,.$$

The relative entropy computation is performed as follows: observe that (v, F, S) satisfies (30) and that the smooth solution $(\bar v, \bar F)$ satisfies the energy identity

$$\partial_t \frac{1}{2}\Big(|\bar v|^2 + W(\bar F) \Big) - \partial_\alpha \Big(\bar v_i T_{i\alpha}(\bar F) \Big) = 0 \,. \tag{31}$$

From

$$\begin{aligned} \partial_t (F_{i\alpha} - \bar F_{i\alpha}) &= \partial_\alpha (v_i - \bar v_i) \\ \partial_t (v_i - \bar v_i) &= \partial_\alpha (S_{i\alpha} - T_{i\alpha}(\bar F)) \end{aligned}$$

and (29) we conclude

$$\begin{aligned} \partial_t \Big(& \frac{\partial W}{\partial F_{i\alpha}}(\bar F)(F_{i\alpha} - \bar F_{i\alpha}) + \bar v_i (v_i - \bar v_i) \Big) \\ & \quad - \partial_\alpha \Big(T_{i\alpha}(\bar F)(v_i - \bar v_i) + \bar v_i (S_{i\alpha} - T_{i\alpha}(\bar F)) \Big) \\ &= \partial_t \Big(\frac{\partial W}{\partial F_{i\alpha}}(\bar F) \Big)(F_{i\alpha} - \bar F_{i\alpha}) + (\partial_t \bar v_i)(v_i - \bar v_i) \\ & \quad - (\partial_\alpha T_{i\alpha}(\bar F))(v_i - \bar v_i) - (\partial_\alpha \bar v_i)(S_{i\alpha} - T_{i\alpha}(\bar F)) \\ &= -(\partial_\alpha \bar v_i) \Big(S_{i\alpha} - T_{i\alpha}(\bar F) - \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}(\bar F)(F_{j\beta} - \bar F_{j\beta}) \Big) \,. \end{aligned} \tag{32}$$

Then, (30), (31) and (32) imply

$$\begin{aligned} \partial_t \mathcal{E}_r & - \partial_\alpha \Big((v_i - \bar v_i)(S_{i\alpha} - T_{i\alpha}(\bar F)) \Big) \\ & + \frac{1}{\varepsilon}(F_{i\alpha} - h_{i\alpha}^{-1}(S - f(F)))(S_{i\alpha} - T_{i\alpha}(F)) \\ = (\partial_\alpha \bar v_i) & \Big(T_{i\alpha}(F) - T_{i\alpha}(\bar F) - \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}(\bar F)(F_{j\beta} - \bar F_{j\beta}) \Big) + (\partial_\alpha \bar v_i)(S_{i\alpha} - T_{i\alpha}(F)) \,. \end{aligned} \tag{33}$$

This relative entropy identity can be used to obtain stability and convergence of the relaxation system (28) as long as the solution of (29) remains smooth. Indeed, under appropriate conditions for the potentials W and W_I, namely that there exist positive constants $\gamma_I > \gamma_v > 0$ and $M > 0$ such that

$$\nabla_F^2 W_I(F) \geq \gamma_I I > \gamma_v I \geq \nabla_F^2 (W_I - W)(F) > 0 \,, \tag{b}$$

$$|\nabla_F^2 W_I(F)| \leq M \,. \quad |\nabla^3 W(F)| \leq M \,, \quad \forall F \,, \tag{c}$$

we get that $\Psi(F, A)$ is uniformly convex and therefore

$$\mathcal{E}_r \geq c \big(|v - \bar v|^2 + |F - \bar F|^2 + |A - h(\bar F)|^2 \big)$$

for a positive $c > 0$. Condition (b) is roughly equivalent to what is called subcharacteristic condition in the theory of relaxation. In addition, uniform convexity of $G(A)$ leads to

$$\nabla_A^2 G(A) = \big(- \nabla_F h \big)^{-1} = \big(\nabla_F^2 (W_I - W) \big)^{-1} \geq \frac{1}{\gamma_v} I$$

so that

$$(F - h^{-1}(S - f(F))) \cdot (S - T(F)) = (\nabla_A G(A) - \nabla_A G(h(F))) \cdot (A - h(F))$$
$$\geq \frac{1}{\gamma_v}|A - h(F)|^2 = \frac{1}{\gamma_v}|S - T(F)|^2\,,$$

giving the dissipation property of the relaxation term. Moreover, the first term on the right hand side of (33) is quadratic in $F - \bar{F}$, while the last term can be controlled by the dissipative relaxation term plus an $O(\varepsilon)$ error term. Hence, the following result holds (we refer to [6] for the technical details and the proof).

Theorem 5.2. *Let $(v^\varepsilon, F^\varepsilon, S^\varepsilon)$ be smooth solutions of (28) and $(\bar{v}, \bar{F})$ be a smooth solution of* (29) *defined on $\mathbb{R}^3 \times [0,T]$ and emanating from smooth data $(v_0^\varepsilon, F_0^\varepsilon, S_0^\varepsilon)$ and $(\bar{v}_0, \bar{F}_0)$. Then, under hypotheses* (a), (b), (c), *the relative energy $\mathcal{E}_r$ satisfies (33), and, for $R > 0$, there exist constants s and $C > 0$ independent of ε such that*

$$\int_{|x|<R} \mathcal{E}_r(x,t)dx \leq C\left(\int_{|x|<R+st} \mathcal{E}_r(x,0)dx + \varepsilon\right).$$

In particular, if the data satisfy

$$\int_{|x|<R+sT} \mathcal{E}_r(x,0)dx \longrightarrow 0\,, \quad \text{as } \varepsilon \to 0\,,$$

then

$$\sup_{t\in[0,T]} \int_{|x|<R} \left(|v^\varepsilon - \widehat{v}|^2 + |F^\varepsilon - \widehat{F}|^2 + |A^\varepsilon - h(\widehat{F})|^2\right) dx \longrightarrow 0\,.$$

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