# FROM GAS DYNAMICS WITH LARGE FRICTION TO GRADIENT FLOWS DESCRIBING DIFFUSION THEORIES

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Abstract. We study the emergence of gradient flows in Wasserstein distance as high friction limits of an abstract Euler flow generated by an energy functional. We develop a relative energy calculation that connects the Euler flow to the gradient flow in the diffusive limit regime. We apply this approach to prove convergence from the Euler-Poisson system with friction to the Keller-Segel system in the regime that the latter has smooth solutions. The same methodology is used to establish convergence from the Euler-Korteweg theory with monotone pressure laws to the Cahn-Hilliard equation.

## Contents



## 1. Introduction

Following the works of Jordan-Kinderlehrer-Otto [18] and Otto [25] a large interest was generated for diffusive equations induced as gradient flows of functionals in the form:

$$\rho_t - \mathrm{div}_x \left( \rho \nabla_x \frac{\delta \mathcal{E}(\rho)}{\delta \rho} \right) = 0\,. \tag{1.1}$$

A key novelty of the approach introduced in these papers is the use of the Wasserstein space of probability measures as a framework where the gradient flow is considered; for a complete theory we refer to the monograph [1].

2010 *Mathematics Subject Classification.* 35L65, 35B25, 35K55, 35Q31, 76N15.

*Key words and phrases.* relative energy, diffusive relaxation, gradient flows, Euler-Poisson, Keller-Segel system, Cahn-Hilliard equation.

The objective of this work is to explore the induction of such diffusion problems as high friction limits of abstract Euler flows of the form

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}_x(\rho u) = 0 \\ \\ \rho \dfrac{\partial u}{\partial t} + \rho u \cdot \nabla_x u = -\rho \nabla_x \dfrac{\delta \mathcal{E}}{\delta \rho} - \zeta \rho u \,, \end{cases} \tag{1.2}$$

where $\zeta > 0$ is a (large) friction coefficient $\zeta > 0$ and $\mathcal{E}(\rho)$ is a functional on the density that generates the evolution. This problem is introduced in [13] (the Hamiltonian flow case $\zeta = 0$) with the objective to put in a common framework several commonly used systems in applications, like the Euler equations, the Euler-Poisson system and the Euler-Korteweg theory. In this work we study the emergence of the system (1.1) from the system (1.2) in the high friction regime $\zeta \to \infty$.

This type of problem belongs to the general realm of diffusive limits, which has been addressed in various contexts with several techniques; we refer to [11] for a survey. The simplest example of a high friction limit that fits within the present functional framework (from (1.2) to (1.1)) is the limit from the Euler system with friction to the porous media equation. This has been addressed again with various methodologies, see *e.g.* [23, 16, 17] and in particular [21] using the relative energy method adopted here.

We develop a general methodology for treating the diffusive limit from (1.2) to (1.1) and apply it to two examples: First, we consider generalized Keller-Segel type models

$$\begin{cases} \rho_t = \operatorname{div}_x \big( \nabla_x p(\rho) - C_{\mathrm{x}} \rho \nabla_x c \big) \\ -\triangle_x c + \beta c = \rho - <\rho> . \end{cases} \tag{1.3}$$

as high friction imits of the Euler-Poisson system with attractive potentials ($C_{\mathrm{x}} > 0$) and friction:

$$\begin{cases} \rho_t + \operatorname{div}_x m = 0 \\ m_t + \operatorname{div}_x \dfrac{m \otimes m}{\rho} + \nabla_x p(\rho) = -\zeta m + C_{\mathrm{x}} \rho \nabla_x c \\ -\triangle_x c + \beta c = \rho - <\rho> . \end{cases} \tag{1.4}$$

This example corresponds to the choice of the functional

$$\mathcal{E}(\rho) = \int \big( h(\rho) - \tfrac{1}{2} C_{\mathrm{x}} \rho c \big) \, dx \,,$$

where $h$ and $p$ are linked by the thermodynamic consistency relations $\rho h''(\rho) = p'(\rho)$, $\rho h'(\rho) = p(\rho) + h(\rho)$, while $c$ is the solution of the Poisson equation

$$-\triangle_x c + \beta c = \rho - <\rho>, \quad <\rho> = \int \rho dx, \ \beta \geq 0 \,,$$

normalized by requiring $<c> = 0$ for $\beta = 0$. For alternative methodologies on this problem see [9, 22]; related models in the context of semiconductors devices with repulsive potentials $C_{\mathrm{x}} < 0$ are analyzed in [24, 20, 19]. For a study of the limiting Keller-Segel model (1.3) as a gradient flows we refer to [6].

As a second paradigm entering into this framework, we consider the Euler-Korteweg system with friction

$$\begin{cases} \rho_t + \operatorname{div}_x m = 0 \\ m_t + \operatorname{div}_x \dfrac{m \otimes m}{\rho} = -\zeta m - \rho \nabla_x \big( h'(\rho) - C_\kappa \triangle_x \rho \big) \end{cases} \tag{1.5}$$

converging in the high-friction regime to the Cahn-Hilliard equation

$$\rho_t = \operatorname{div}_x \big( \rho \nabla_x \big( h'(\rho) - C_\kappa \triangle_x \rho \big) \big) = \operatorname{div}_x \big( \nabla_x p(\rho) - C_\kappa \rho \nabla_x \triangle_x \rho \big) \,, \tag{1.6}$$

which corresponds to the choice of functional

$$\mathcal{E}(\rho) = \int \big( h(\rho) + \tfrac{1}{2} C_\kappa |\nabla_x \rho|^2 \big) \, dx, \quad C_\kappa > 0 \,.$$

The technical tool consists of a functional form of the relative energy identity introduced in [13], and inspired by [21, 22] and the relative energy calculations of Dafermos [7, 8]. The relative energy monitors the distance between solutions in appropriate norms pertinent to the aforementioned equations (1.2) to (1.1). It provides a very efficient tool to carry out the limiting process, as it is precisely adapted to the functional framework of both problems (1.1) and (1.2).

The outline of this work is as follows. In Section 2 we introduce the relative kinetic energy

$$K(\rho, m|\bar{\rho}, \bar{m}) := \frac{1}{2} \int \rho |u - \bar{u}|^2 \, dx \,,$$

and the relative potential energy

$$\mathcal{E}(\rho|\bar{\rho}) := \mathcal{E}(\rho) - \mathcal{E}(\bar{\rho}) - \big\langle \frac{\delta \mathcal{E}}{\delta \rho}(\bar{\rho}), \rho - \bar{\rho} \big\rangle \,,$$

and use them to derive an identity for the distance between two solutions of (1.2); see (2.17) in Section 2.2. The same tool is used in order to measure the distance between solutions of (1.2) and (1.1) in Section 2.4. It provides a yardstick to measure the distance in the relaxation limit. The identity carries seamlessly to the limit and yields an identity between two solutions of (1.1) in terms of the relative potential energy; see (2.29) in Section 2.5. After this formal calculation, we study the relaxation limits from weak solutions of the hyperbolic relaxing model toward strong solutions of the diffusive equations. This is carried out in two cases: from the Euler-Poisson systems with attractive potentials towards Keller-Segel type models in Section 3, and from the Euler-Korteweg system with friction toward the Cahn-Hilliard equation in Section 4.

## 2. A large friction theory converging toward gradient flows

We start our analysis by presenting a relaxation theory of large friction converging towards gradient flow dynamics. This formalism will unify in a common framework the results on covergence from the compressible Euler system with friction to the porous media equation obtained in [21], with convergence results towards Keller-Segel type systems (see [22] for preliminary results in this direction), or towards the Cahn-Hilliard equation, obtained in the following sections. The specific cases will be obtained as particular examples of the general framework via an appropriate choice of the entropy functional defining the flow of the limiting equation.

To this aim, let us consider the following system of equations consisting of a conservation of mass and a functional momentum equation

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}_x(\rho u) = 0 \\[2ex] \rho \dfrac{\partial u}{\partial t} + \rho u \cdot \nabla_x u = -\rho \nabla_x \dfrac{\delta \mathcal{E}}{\delta \rho} - \zeta \rho u \,, \end{cases} \tag{2.1}$$

where $\rho \geq 0$ is the density and $u$ is the velocity. Moreover, $\dfrac{\delta \mathcal{E}}{\delta \rho}$ stands for the generator of the first variation of the functional $\mathcal{E}(\rho)$ (see the discussion in [13]), and the term $-\zeta \rho u$ accounts for a damping force with frictional coefficient $\zeta > 0$. For large frictions $\zeta = \frac{1}{\varepsilon}$, after a proper scaling of time $\partial_t \mapsto \varepsilon \partial_t$, (2.1) is rewritten as

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{1}{\varepsilon} \operatorname{div}_x(\rho u) = 0 \\[2ex] \rho \dfrac{\partial u}{\partial t} + \dfrac{1}{\varepsilon} \rho u \cdot \nabla_x u = -\dfrac{1}{\varepsilon^2} \rho u - \dfrac{1}{\varepsilon} \rho \nabla_x \dfrac{\delta \mathcal{E}}{\delta \rho} \,, \end{cases} \tag{2.2}$$

or, in terms of $(\rho, m = \rho u)$

$$\begin{cases} \rho_t + \dfrac{1}{\varepsilon} \operatorname{div}_x m = 0 \\[2ex] m_t + \dfrac{1}{\varepsilon} \operatorname{div}_x \dfrac{m \otimes m}{\rho} = -\dfrac{1}{\varepsilon^2} m - \dfrac{1}{\varepsilon} \rho \nabla_x \dfrac{\delta \mathcal{E}(\rho)}{\delta \rho} \,. \end{cases} \tag{2.3}$$

Note that (2.3) is in conservation form except for the term $\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$. Nevertheless, for all examples treated in this paper, we have

$$-\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} = \nabla_x \cdot S \,, \tag{2.4}$$

where $S = S(\rho)$ will be a tensor-valued functional on $\rho$ that plays the role of a stress tensor with components $S_{ij}(\rho)$ with $i, j = 1, ..., d$. We refer to [13] for a discussion of the ramifications of that property.

As $\varepsilon \downarrow 0$ in (2.3), we formally obtain the gradient flow dynamic

$$\rho_t - \mathrm{div}_x \left( \rho \nabla_x \frac{\delta \mathcal{E}(\rho)}{\delta \rho} \right) = 0 \,. \tag{2.5}$$

The objective of this section is to describe this large friction limit using the relative energy identities induced by the functional framework. Particular examples will include various interesting systems (see the examples in [13]) and in particular:

(1) the porous medium equation as limit of the Euler equation with friction [21] corresponds to the choice

$$\mathcal{E}(\rho) = \int h(\rho) dx \,;$$

(2) the Keller-Segel system as limit of the Euler-Poisson system with friction (in the case of attractive potentials), considered in Section 3, is given by the functional

$$\mathcal{E}(\rho) = \int \left( h(\rho) - \tfrac{1}{2} C_{\mathrm{x}} \rho c \right) dx \,,$$

where $C_{\mathrm{x}} > 0$ and $c$ is viewed as a constraint in terms of the relation

$$-\triangle_x c + \beta c = \rho - < \rho >, \quad < \rho >= \int \rho dx, \ \ \beta \geq 0 \,;$$

(3) the Cahn-Hilliard equation as limit of the the Euler-Korteweg system with friction corresponds to the choice

$$\mathcal{E}(\rho) = \int \left( h(\rho) + \tfrac{1}{2} C_\kappa |\nabla_x \rho|^2 \right) dx, \quad C_\kappa > 0$$

and is investigated in Section 4.

2.1. **The energy equation.** We start by reviewing and adapting to the relaxation framework certain results from [13]. First, we derive the energy estimate for (2.2) or (2.3) in the functional setting. We assume that the directional derivative (Gateaux derivative) of the functional $\mathcal{E}$ defined by

$$d\mathcal{E}(\rho; \psi) = \lim_{\tau \to 0} \frac{\mathcal{E}(\rho + \tau \psi) - \mathcal{E}(\rho)}{\tau} = \frac{d}{d\tau} \mathcal{E}(\rho + \tau \psi) \Big|_{\tau = 0}$$

is linear in $\psi$ and can be represented via a duality bracket

$$d\mathcal{E}(\rho; \psi) = \frac{d}{d\tau} \mathcal{E}(\rho + \tau \psi) \Big|_{\tau=0} = \langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho), \psi \rangle \,, \tag{2.6}$$

with $\frac{\delta \mathcal{E}}{\delta \rho}(\rho)$ standing for the generator of the bracket. This property is always satisfied for Frechet differentiable functionals. Using (2.4), the potential energy is computed via

$$\frac{d}{dt} \mathcal{E}(\rho) = \langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho), \rho_t \rangle = -\frac{1}{\varepsilon} \langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho), \mathrm{div}_x(\rho u) \rangle = \frac{1}{\varepsilon} \int S : \nabla_x u \, dx \,. \tag{2.7}$$

Now, using again (2.4) and the momentum equation $(2.3)_2$ with the standard multiplier $u$, we obtain the usual kinetic energy relation

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \rho |u|^2 dx &= -\frac{1}{\varepsilon^2} \int \rho |u|^2 dx - \frac{1}{\varepsilon} \int \rho u \cdot \nabla_x \frac{\delta \mathcal{E}(\rho)}{\delta \rho} dx \\ &= -\frac{1}{\varepsilon^2} \int \rho |u|^2 dx - \frac{1}{\varepsilon} \int S : \nabla_x u \, dx \,, \end{aligned}$$

which, added to (2.7), finally leads to the standard energy relation

$$\frac{d}{dt}\left(\mathcal{E}(\rho)+\frac{1}{2}\int\rho|u|^2dx\right)+\frac{1}{\varepsilon^2}\int\rho|u|^2dx=0\,. \tag{2.8}$$

2.2. **Relative energy identity for the relaxation theory.** Next, we compare two different solutions $(\rho, m)$, $(\bar\rho,\bar m)$ of (2.3) using the relative entropy framework. To this end, we define also the second variation of the functional $\mathcal{E}(\rho)$ via

$$d^2\mathcal{E}(\rho;\psi,\varphi)=\lim_{\tau\to 0}\frac{\left\langle\frac{\delta\mathcal{E}}{\delta\rho}(\rho+\tau\varphi),\psi\right\rangle-\left\langle\frac{\delta\mathcal{E}}{\delta\rho}(\rho),\psi\right\rangle}{\tau}$$

(whenever the limit exists), and we assume that this can be represented as a bilinear functional in the form

$$d^2\mathcal{E}(\rho;\psi,\varphi)=\lim_{\tau\to 0}\frac{\left\langle\frac{\delta\mathcal{E}}{\delta\rho}(\rho+\tau\varphi),\psi\right\rangle-\left\langle\frac{\delta\mathcal{E}}{\delta\rho}(\rho),\psi\right\rangle}{\tau}=\left\langle\!\!\left\langle\frac{\delta^2\mathcal{E}}{\delta\rho^2}(\rho),(\psi,\varphi)\right\rangle\!\!\right\rangle\,. \tag{2.9}$$

Moreover, in analogy to (2.6), we assume that the directional derivative of $S(\rho)$ is expressed as a linear functional via a duality bracket,

$$dS(\rho;\psi)=\frac{d}{d\tau}S(\rho+\tau\psi)\Big|_{\tau=0}=\left\langle\frac{\delta S}{\delta\rho}(\rho),\psi\right\rangle, \tag{2.10}$$

in terms of the generator $\frac{\delta S}{\delta\rho}(\rho)$.

2.2.1. *The relative potential energy.* Define the relative potential energy,

$$\mathcal{E}(\rho|\bar\rho):=\mathcal{E}(\rho)-\mathcal{E}(\bar\rho)-\left\langle\frac{\delta\mathcal{E}}{\delta\rho}(\bar\rho),\rho-\bar\rho\right\rangle, \tag{2.11}$$

as the quadratic part of the Taylor series expansion of the functional $\mathcal{E}$ with respect to a reference solution $\bar\rho(x,t)$. If $\mathcal{E}(\rho)$ is convex, this quantity can serve as a measure of distance between the two solutions $\rho$ and $\bar\rho$.

Consider next the weak form of (2.4),

$$\left\langle\frac{\delta\mathcal{E}}{\delta\rho}(\rho),\frac{\partial}{\partial x_j}(\rho\varphi_j)\right\rangle=-\int S_{ij}(\rho)\,\frac{\partial\varphi_i}{\partial x_j}\,dx\,.$$

This relation is viewed as a functional in $\rho$; talking its directional derivative along a direction $\psi$, with $\psi$ a smooth test function, we obtain

$$\begin{aligned}\left\langle\!\!\left\langle\frac{\delta^2\mathcal{E}}{\delta\rho^2}(\rho),\left(\psi,\frac{\partial}{\partial x_j}(\rho\varphi_j)\right)\right\rangle\!\!\right\rangle+\left\langle\frac{\delta\mathcal{E}}{\delta\rho}(\rho),\frac{\partial}{\partial x_j}(\psi\varphi_j)\right\rangle&\\ =-\int\left\langle\frac{\delta S_{ij}}{\delta\rho}(\rho),\psi\right\rangle\frac{\partial\varphi_i}{\partial x_j}\,dx\,.&\end{aligned}$$

The two relations lead to (see [13, Section 2.1] for the details of this computation):

$$\frac{d}{dt}\mathcal{E}(\rho|\bar\rho)=\frac{1}{\varepsilon}\int S_{ij}(\rho|\bar\rho)\frac{\partial\bar u_i}{\partial x_j}\,dx-\frac{1}{\varepsilon}\left\langle\frac{\delta\mathcal{E}}{\delta\rho}(\rho)-\frac{\delta\mathcal{E}}{\delta\rho}(\bar\rho),\operatorname{div}_x\big(\rho(u-\bar u)\big)\right\rangle, \tag{2.12}$$

where $S(\rho|\bar\rho)$ stands for the relative stress tensor:

$$S(\rho|\bar\rho):=S(\rho)-S(\bar\rho)-\left\langle\frac{\delta S}{\delta\rho}(\bar\rho),\rho-\bar\rho\right\rangle. \tag{2.13}$$

2.2.2. *The relative kinetic energy.* Next consider the kinetic energy

$$K(\rho, m) = \int \frac{1}{2}\frac{|m|^2}{\rho} dx \tag{2.14}$$

viewed as a (not strictly) convex functional on the density $\rho$ and the momentum $m = \rho u$. The relative kinetic energy is expressed in the form

$$\begin{aligned} K(\rho, m|\bar\rho, \bar m) &:= \int k(\rho, m) - k(\bar\rho, \bar m) - \nabla k(\bar\rho, \bar m)\cdot(\rho - \bar\rho, m - \bar m)\, dx \\ &= \int \frac{1}{2}\frac{|m|^2}{\rho} - \frac{1}{2}\frac{|\bar m|^2}{\bar\rho} - \big(-\frac{1}{2}\frac{|\bar m|^2}{\bar\rho^2}, \frac{\bar m}{\bar\rho}\big)\cdot(\rho - \bar\rho, m - \bar m)\, dx \\ &= \frac{1}{2}\int \rho|u - \bar u|^2\, dx\,, \end{aligned} \tag{2.15}$$

To compute its evolution, consider the difference of the two equations satisfied by $(\rho, u)$ and $(\bar\rho, \bar u)$, that is

$$\begin{aligned} \partial_t(u - \bar u) &+ \frac{1}{\varepsilon}(u\cdot\nabla_x)(u - \bar u) + \frac{1}{\varepsilon}\big((u - \bar u)\cdot\nabla_x\big)\bar u \\ &= -\frac{1}{\varepsilon^2}(u - \bar u) - \frac{1}{\varepsilon}\nabla_x\left(\frac{\delta\mathcal{E}(\rho)}{\delta\rho} - \frac{\delta\mathcal{E}(\bar\rho)}{\delta\rho}\right)\,. \end{aligned}$$

Multiplying this relation by $u - \bar u$ we end up with

$$\begin{aligned} \frac{1}{2}\partial_t|u - \bar u|^2 &+ \frac{1}{2\varepsilon}(u\cdot\nabla_x)|u - \bar u|^2 + \frac{1}{\varepsilon}\nabla_x\bar u : (u - \bar u)\otimes(u - \bar u) \\ &= -\frac{1}{\varepsilon^2}|u - \bar u|^2 - \frac{1}{\varepsilon}(u - \bar u)\cdot\nabla_x\left(\frac{\delta\mathcal{E}(\rho)}{\delta\rho} - \frac{\delta\mathcal{E}(\bar\rho)}{\delta\rho}\right)\,, \end{aligned}$$

which, using $(2.3)_1$ and integrating over space leads to the balance of the relative kinetic energy

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho|u - \bar u|^2\, dx + \frac{1}{\varepsilon^2}\int \rho|u - \bar u|^2\, dx = \\ &-\frac{1}{\varepsilon}\int \rho\nabla_x\bar u : (u - \bar u)\otimes(u - \bar u)\, dx + \frac{1}{\varepsilon}\big\langle\frac{\delta\mathcal{E}}{\delta\rho}(\rho) - \frac{\delta\mathcal{E}}{\delta\rho}(\bar\rho), \mathrm{div}_x\left(\rho(u - \bar u)\right)\big\rangle\,. \end{aligned} \tag{2.16}$$

2.2.3. *The functional form of the relative energy formula.* Summing (2.12) to (2.16) we obtain the relative energy identity

$$\begin{aligned} \frac{d}{dt}\left(\mathcal{E}(\rho|\bar\rho) + \frac{1}{2}\int \rho|u - \bar u|^2\, dx\right) &+ \frac{1}{\varepsilon^2}\int \rho|u - \bar u|^2\, dx \\ &= \frac{1}{\varepsilon}\int \nabla_x\bar u : S(\rho|\bar\rho)\, dx - \frac{1}{\varepsilon}\int \rho\nabla_x\bar u : (u - \bar u)\otimes(u - \bar u)\, dx\,, \end{aligned} \tag{2.17}$$

where $\mathcal{E}(\rho|\bar\rho)$ and $S(\rho|\bar\rho)$ stand for the relative potential energy and relative stress functionals defined in (2.11) and (2.13), respectively. The main property which leads to the above relation is the fact that the contributions of the term

$$D = \frac{1}{\varepsilon}\big\langle\frac{\delta\mathcal{E}}{\delta\rho}(\rho) - \frac{\delta\mathcal{E}}{\delta\rho}(\bar\rho), \mathrm{div}_x\left(\rho(u - \bar u)\right)\big\rangle$$

in (2.12) and (2.16) offset each other, as for the terms involving the stress tensor $S$ in the derivation of the energy relation (2.8).

2.3. **Confinement potentials.** It is expedient to give an extension of the calculation for systems driven by a confinement potential $V = V(x)$,

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \frac{1}{\varepsilon}\,\mathrm{div}_x(\rho u) = 0 \\ \rho\dfrac{\partial u}{\partial t} + \frac{1}{\varepsilon}\rho u\cdot\nabla_x u = -\frac{1}{\varepsilon^2}\rho u - \frac{1}{\varepsilon}\rho\nabla_x\dfrac{\delta\mathcal{E}}{\delta\rho} - \frac{1}{\varepsilon}\rho\nabla_x V\,. \end{cases} \tag{2.18}$$

The potential energy is now given by the functional

$$\mathcal{F}(\rho) = \mathcal{E}(\rho) + \int \rho V(x)\, dx \tag{2.19}$$

and we require that $\mathcal{E}(\rho)$ satisfies (2.4) for some stress functional $S(\rho)$. Note, that the potential energy functional splits into the potential energy of the contact forces $\mathcal{E}(\rho)$ and the potential energy of the body forces $\int \rho V$. The latter is not expected to be associated to a stress, and also is not invariant under space translations (which is connected to the hypothesis (2.4)). Under the hypothesis (2.4) for $\mathcal{E}$, we may write the weak form of (2.18) for $(\rho, m = \rho u)$

$$\begin{cases} \rho_t + \frac{1}{\varepsilon} \operatorname{div}_x m = 0 \\ m_t + \frac{1}{\varepsilon} \operatorname{div}_x \dfrac{m \otimes m}{\rho} = -\frac{1}{\varepsilon^2} m - \frac{1}{\varepsilon} \nabla_x \cdot S(\rho) - \frac{1}{\varepsilon} \rho \nabla_x V\,. \end{cases} \tag{2.20}$$

Proceeding along the lines of the calculations in Section 2.1 we see that

$$\frac{d}{dt}\mathcal{F}(\rho) = \frac{1}{\varepsilon} \int S : \nabla_x u\, dx + \frac{1}{\varepsilon} \int \rho \nabla_x V \cdot u\, dx\,, \tag{2.21}$$

$$\frac{1}{2}\frac{d}{dt} \int \rho |u|^2 dx = -\frac{1}{\varepsilon^2} \int \rho |u|^2 dx - \frac{1}{\varepsilon} \int S : \nabla_x u\, dx - \frac{1}{\varepsilon} \int \rho \nabla_x V \cdot u\, dx\,, \tag{2.22}$$

and the total energy for (2.18) reads

$$\frac{d}{dt}\left( \mathcal{F}(\rho) + \frac{1}{2} \int \rho |u|^2 dx \right) + \frac{1}{\varepsilon^2} \int \rho |u|^2 dx = 0\,. \tag{2.23}$$

On the other hand, due to the formulas

$$\mathcal{F}(\rho|\bar\rho) = \mathcal{E}(\rho|\bar\rho)\,,$$
$$\frac{\delta \mathcal{F}}{\delta \rho}(\rho) - \frac{\delta \mathcal{F}}{\delta \rho}(\bar\rho) = \frac{\delta \mathcal{E}}{\delta \rho}(\rho) - \frac{\delta \mathcal{E}}{\delta \rho}(\bar\rho)\,,$$

the calculations in Sections 2.2.1 and 2.2.2 remain essentially unaffected, and the final relative energy formula, comparing two solutions $(\rho, u)$ and $(\bar\rho, \bar u)$ of the system (2.18) with confinement potential, takes exactly the same form as (2.17):

$$\begin{aligned} \frac{d}{dt}&\left( \mathcal{E}(\rho|\bar\rho) + \frac{1}{2} \int \rho |u - \bar u|^2\, dx \right) + \frac{1}{\varepsilon^2} \int \rho |u - \bar u|^2\, dx \\ &= \frac{1}{\varepsilon} \int \nabla_x \bar u : S(\rho|\bar\rho)\, dx - \frac{1}{\varepsilon} \int \rho \nabla_x \bar u : (u - \bar u) \otimes (u - \bar u)\, dx\,. \end{aligned} \tag{2.24}$$

2.4. **The analysis of the diffusive limit.** We return now to the system (2.2) without confinement potential. In this section we aim to compare a solution $(\rho, \rho u)$ of (2.3) with a smooth solution $\bar\rho$ of (2.5). To this end, we define

$$\bar m = \bar\rho \bar u = -\varepsilon \bar\rho \nabla_x \frac{\delta \mathcal{E}(\bar\rho)}{\delta \rho} \tag{2.25}$$

and visualize the pair $(\bar\rho, \bar m = \bar\rho \bar u)$ as an approximate solution of (2.3), that is

$$\begin{cases} \bar\rho_t + \dfrac{1}{\varepsilon} \operatorname{div}_x \bar m = 0 \\ \bar m_t + \dfrac{1}{\varepsilon} \operatorname{div}_x \dfrac{\bar m \otimes \bar m}{\bar\rho} = -\dfrac{1}{\varepsilon^2} \bar m - \dfrac{1}{\varepsilon} \bar\rho \nabla_x \dfrac{\delta \mathcal{E}(\bar\rho)}{\delta \rho} + \bar e\,, \end{cases} \tag{2.26}$$

where

$$\bar e = \bar m_t + \frac{1}{\varepsilon} \operatorname{div}_x \frac{\bar m \otimes \bar m}{\bar\rho}\,.$$

Using (2.25) and the smoothness of $\bar\rho$ we see that $\bar e$ is a $O(\varepsilon)$ error term. The only difference from the calculations of section 2.2 lies in the relative kinetic energy (2.16). Presently, $\bar u$ satisfies the approximate equation

$$\bar u_t + \frac{1}{\varepsilon}(\bar u \cdot \nabla_x)\bar u = -\frac{1}{\varepsilon^2}\bar u - \frac{1}{\varepsilon}\nabla_x \frac{\delta \mathcal{E}(\bar\rho)}{\delta\rho} + \frac{\bar e}{\bar \rho}\,,$$

and, following the analysis in section 2.2.2, we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t|u-\bar u|^2 + \frac{1}{2\varepsilon}(u\cdot\nabla_x)|u-\bar u|^2 + \frac{1}{\varepsilon}\nabla_x \bar u : (u-\bar u)\otimes(u-\bar u) \\
&\quad = -\frac{1}{2\varepsilon^2}|u-\bar u|^2 - \frac{1}{\varepsilon}(u-\bar u)\cdot\nabla_x\left(\frac{\delta\mathcal{E}(\rho)}{\delta\rho} - \frac{\delta\mathcal{E}(\bar\rho)}{\delta\rho}\right) - (u-\bar u)\cdot\frac{\bar e}{\bar\rho}\,,
\end{aligned} \tag{2.27}$$

and the relative energy relation

$$\begin{aligned}
&\frac{d}{dt}\left(\mathcal{E}(\rho|\bar\rho) + \frac{1}{2}\int \rho|u-\bar u|^2\,dx\right) + \frac{1}{\varepsilon^2}\int\rho|u-\bar u|^2\,dx \\
&\quad = \frac{1}{\varepsilon}\int \nabla_x\bar u : S(\rho|\bar\rho)\,dx - \frac{1}{\varepsilon}\int \rho\nabla_x \bar u : (u-\bar u)\otimes(u-\bar u)\,dx - \int\rho(u-\bar u)\cdot\frac{\bar e}{\bar\rho}dx\,.
\end{aligned} \tag{2.28}$$

Since $\bar u = O(\varepsilon)$, for smooth solutions the coefficients of the quadratic terms are $O(1)$ in $\varepsilon$. Moreover, the last (error) term at the right hand side of (2.28) is controlled in terms of the distance w.r.t. the equilibrium, i.e. by $\frac{1}{2}\varepsilon^{-2}\rho|u-\bar u|^2$, and an $O(\varepsilon^4)$ term depending on total mass of $\rho$ and the smooth, strictly positive solution $\bar\rho$ of (2.5). This relation is therefore instrumental to control the relaxation limit.

2.5. **Relative energy estimate for the gradient flow.** The above calculations induce a relative energy estimate for comparing two solutions $\rho$ and $\bar\rho$ of the limiting gradient flow (2.5). Indeed, using in (2.17) the expressions (2.25) for the two velocities $u$ and $\bar u$ at equilibrium we obtain

$$\frac{d}{dt}\mathcal{E}(\rho|\bar\rho) + \int\rho\left|\nabla_x\Big(\frac{\delta\mathcal{E}(\rho)}{\delta\rho} - \frac{\delta\mathcal{E}(\bar\rho)}{\delta\rho}\Big)\right|^2 dx = -\int S_{ij}(\rho|\bar\rho)\frac{\partial^2}{\partial x_j\partial x_i}\frac{\delta\mathcal{E}(\bar\rho)}{\delta\rho}dx\,. \tag{2.29}$$

Note that in this calculation the effect of the kinetic energy drops out, and the derivation of (2.29) uses only (2.12) and the transport equations for $\rho$ and $\bar\rho$; the term $D$ becomes dissipative with the velociy choices

$$u = -\varepsilon\nabla_x\frac{\delta\mathcal{E}(\rho)}{\delta\rho} \quad \text{and} \quad \bar u = -\varepsilon\nabla_x\frac{\delta\mathcal{E}(\bar\rho)}{\delta\rho}$$

leading to the gradient flow.

## 3. From the Euler-Poisson system with friction to the Keller-Segel system

In this section, we shall make precise the functional setting for the Euler–Poisson system with an attractive potential and friction converging to Keller–Segel type models. The Euler-Poisson system reads

$$\begin{cases}
\rho_t + \dfrac{1}{\varepsilon}\operatorname{div}_x m = 0 \\[2mm]
m_t + \dfrac{1}{\varepsilon}\operatorname{div}_x\dfrac{m\otimes m}{\rho} + \dfrac{1}{\varepsilon}\nabla_x p(\rho) = -\dfrac{1}{\varepsilon^2}m + \dfrac{C_{\mathrm{x}}}{\varepsilon}\rho\nabla_x c \\[2mm]
-\triangle_x c + \beta c = \rho - <\rho>,
\end{cases} \tag{3.1}$$

where $t\in\mathbb{R}$, $x\in\mathbb{T}^n$, $n=2,3$ the physically relevant dimensions, $\rho\geq 0$, $c\in\mathbb{R}$, $m=\rho u\in\mathbb{R}^n$. We assume that the internal energy $h(\rho)$ and the pressure $p(\rho)$ are connected through the usual thermodynamic relation

$$\rho h''(\rho) = p'(\rho)\,, \quad \rho h'(\rho) = p(\rho) + h(\rho) \quad \text{with } p'(\rho) > 0. \tag{H}$$

Moreover, we impose the conditions on the pressure that for some constants $k>0$ and $A>0$,

$$h(\rho) = \frac{k}{\gamma-1}\rho^\gamma + o(\rho^\gamma)\,, \quad \text{as } \rho\to+\infty \tag{A$_1$}$$

$$|p''(\rho)| \leq A\frac{p'(\rho)}{\rho} \qquad \forall\rho>0\,. \tag{A$_2$}$$

These conditions are satisfied by the usual $\gamma$–law: $p(\rho) = k\rho^\gamma$ with $\gamma > 1$. For the constants $\gamma > 1$, $\beta \geq 0$ and $C_{\rm x} > 0$ (chemosensitive coefficient) appearing in (3.1) we will impose various smallness/largeness conditions that are precised later. Finally, the elliptic equation in (3.1) is provided with periodic boundary conditions and it shall be intended for zero mean solutions $c$ in the case $\beta = 0$; in that equation, as specified in the previous section, $< \rho >$ stand for the mean of $\rho$.

Formally, after an appropriate scaling of the moment, at the limit $\varepsilon \downarrow 0$ we obtain $m = C_{\rm x}\rho\nabla_x c - \nabla_x p(\rho)$ and therefore the formal limit of (3.1) is given by the Keller-Segel type model:

$$\begin{cases} \rho_t + \mathrm{div}_x \big(C_{\rm x}\rho\nabla_x c - \nabla_x p(\rho)\big) = 0 \\ -\triangle_x c + \beta c = \rho - < \rho > . \end{cases} \tag{3.2}$$

3.1. **Preliminaries.** Standard hyperbolic theory suggests to employ for (3.1) the usual entropy–entropy flux pair

$$\eta(\rho, m) = \frac{1}{2}\frac{|m|^2}{\rho} + h(\rho), \quad , \quad q(\rho, m) = \frac{1}{2} m \frac{|m|^2}{\rho^2} + m h'(\rho)\,, \tag{3.3}$$

depicting the mechanical energy and its flux. We recall that for the particular case of $\gamma$–law gases, $p(\rho) = k\rho^\gamma$, $h$ takes the form

$$h(\rho) = \begin{cases} \dfrac{k}{\gamma - 1}\rho^\gamma = \dfrac{1}{\gamma-1} p(\rho) & \text{for } \gamma > 1; \\ k\rho\log\rho & \text{for } \gamma = 1. \end{cases}$$

An entropy weak solution of (3.1) satisfies in the sense of distribution the entropy inequality

$$\eta(\rho, m)_t + \frac{1}{\varepsilon}\,\mathrm{div}_x\, q(\rho, m) \leq -\frac{1}{\varepsilon^2}\nabla_m \eta(\rho, m)\cdot m = -\frac{1}{\varepsilon^2}\frac{|m|^2}{\rho} + \frac{C_{\rm x}}{\varepsilon} m \cdot \nabla_x c \tag{3.4}$$

On the other hand, smooth solutions of (3.2) satisfy the entropy identity

$$h(\rho)_t + \mathrm{div}_x \big(h'(\rho)(C_{\rm x}\rho\nabla_x c - \nabla_x p(\rho))\big) = -\frac{|\nabla_x p(\rho)|^2}{\rho} + C_{\rm x}\nabla_x p(\rho)\cdot\nabla_x c. \tag{3.5}$$

This form of the energy equations is inadequate to carry out the relative entropy analysis of the forthcoming section.

Next, we present a variant of the energy equation, inspired by the formal analysis of Section 2. To start, the solution of the elliptic equation $(3.1)_3$ can be expressed via convolution with the Green's function,

$$c(x) = (\mathcal{K} * \rho)(x) = \int \mathcal{K}(x-y)\rho(y)\, dy\,,$$

where $\mathcal{K}$ is a symmetric function. The energy of the Euler-Poisson system takes the form

$$\begin{aligned} \mathcal{E}(\rho) &= \int \big(h(\rho) - \tfrac{1}{2}C_{\rm x}\rho c\big)\, dx \\ &= \int h(\rho)\, dx - \tfrac{1}{2}C_{\rm x} \iint \rho(x)\mathcal{K}(x-y)\rho(y) dx dy\,, \end{aligned} \tag{3.6}$$

and the symmetry of $\mathcal{K}$ implies

$$-\rho\nabla_x \frac{\delta\mathcal{E}}{\delta\rho}(\rho) = -\rho\nabla_x(h'(\rho) - C_{\rm x} c)\,. \tag{3.7}$$

Next, we show that for the Euler-Poisson system there is a stress associated with (3.7). Indeed, multiplying $(3.1)_3$ by $\nabla_x c$ we end up with

$$\rho\nabla_x c = \nabla_x\Big(\tfrac{1}{2}|\nabla_x c|^2 + \tfrac{\beta}{2}c^2 + < \rho > c\Big) - \mathrm{div}_x(\nabla_x c \otimes \nabla_x c)\,,$$

so that

$$\begin{aligned} &-\rho\nabla_x(h'(\rho) - C_{\rm x} c) \\ &\quad = \mathrm{div}_x\Big( -\big[p(\rho) - \tfrac{1}{2}C_{\rm x}(\beta c^2 + |\nabla_x c|^2) - C_{\rm x} < \rho > c\big]\mathbb{I} - C_{\rm x}\nabla_x c\otimes\nabla_x c\Big)\,. \end{aligned} \tag{3.8}$$

This determines the stress $S$ in (2.4) as

$$S = -\big[p(\rho) - \tfrac{1}{2}C_{\rm x}(\beta c^2 + |\nabla_x c|^2) - C_{\rm x} < \rho > c\big]\mathbb{I} - C_{\rm x}\nabla_x c\otimes\nabla_x c\,, \tag{3.9}$$

where $\mathbb{I}$ is the identity matrix. Note that the pressure has a contribution coming from the mean-field interaction term. The induction of a pressure from the mean filed interaction can also be see from the energy identity

$$\int \rho c dx = \int \big(\beta c^2 + |\nabla_x c|^2\big) dx \,, \tag{3.10}$$

obtained directly from the elliptic equation $(3.1)_3$.

Following the general framework of Section 2.1, the potential energy satisfies

$$\begin{aligned} \frac{d}{dt}\int \big(h(\rho) - \tfrac{1}{2}C_{\mathrm{x}}\rho c\big)\, dx &= -\frac{1}{\varepsilon}\int \big(h'(\rho) - C_{\mathrm{x}} c\big)\, \mathrm{div}_x(\rho u) dx \\ &= \frac{1}{\varepsilon}\int S : \nabla_x u \, dx \,, \end{aligned}$$

the kinetic energy is

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int \frac{|m|^2}{\rho} dx &= -\frac{1}{\varepsilon^2}\int \frac{|m|^2}{\rho} dx - \frac{1}{\varepsilon}\int \rho u \cdot \nabla_x (h'(\rho) - C_{\mathrm{x}} c) dx \\ &= -\frac{1}{\varepsilon^2}\int \frac{|m|^2}{\rho} dx - \frac{1}{\varepsilon}\int S : \nabla_x u \, dx \,, \end{aligned}$$

and the total energy reads

$$\frac{d}{dt}\int \big(\eta(\rho, m) - \tfrac{1}{2}C_{\mathrm{x}}\rho c\big)\, dx + \frac{1}{\varepsilon^2}\int \frac{|m|^2}{\rho} dx = 0 \,.$$

In accordance to the usual practice in conservation laws, we will define entropy dissipative solutions $(\rho, m, c)$ of (3.1) as weak solutions satifysing the weak form of the energy inequality

$$\frac{d}{dt}\int \big(\eta(\rho, m) - \tfrac{1}{2}C_{\mathrm{x}}\rho c\big)\, dx + \frac{1}{\varepsilon^2}\int \frac{|m|^2}{\rho} dx \le 0 \,, \tag{3.11}$$

in the sense of distributions. Clearly (3.11) and (3.4) are equivalent (for smooth solutions), however the form (3.11) together with (3.10) suggest that to control the potential energy is tantamount to obtaining control of the $H^1$ norm of $c$. The above estimate is the starting point to obtain the stability estimate in terms of relative entropy and the corresponding analysis of the relaxation limit in the next section.

3.2. **Relative energy estimate.** In this section we perform a relative energy computation between a weak solution $(\rho, m, c)$ of (3.1) and a strong solution $(\bar\rho, \bar c)$ of (3.2). The final formula (3.20) turns out to be a special case of formula (2.28) derived in Section 2 for smooth solutions, and of the relative energy computation for the Euler–Poisson system discussed in [13, Section 2.5]. Nevertheless, we shall provide here a direct proof of this identity. The reason is twofold: (i) to justify the relative energy estimate among a weak and a strong solution, (ii) to account for the effect of error terms appropriate for the relaxation limit problem. We recall the framework of weak solutions we shall refer to.

**Definition 3.1.** *A function $(\rho, m, c)$ with $\rho \in C([0,\infty); L^1(\mathbb{T}^n) \cap L^\gamma(\mathbb{T}^n))$, $m \in C\big(([0,\infty); \big(L^1(\mathbb{T}^n)\big)^n\big)$, $c \in C([0,\infty); H^1(\mathbb{T}^n))$ $\rho \ge 0$ and $\frac{m\otimes m}{\rho} \in L^1_{loc}\left(((0,\infty)\times\mathbb{T}^n)\right)^{n\times n}$ is a* dissipative weak periodic solution *of (3.1) with finite total energy if*

- *$(\rho, m, c)$ satisfies the weak form of (3.1);*
- *$(\rho, m, c)$ satisfies the following integrated form of the energy inequality (3.11):*

$$\begin{aligned} -\iint \Big(\eta(\rho, m) - \tfrac{1}{2}C_{\mathrm{x}}\rho c\Big)\dot\theta(t) dx dt + \frac{1}{\varepsilon^2}\iint \frac{|m|^2}{\rho}\theta(t) dx dt \\ \le \int \Big(\eta(\rho, m) - \tfrac{1}{2}C_{\mathrm{x}}\rho c\Big)\Big|_{t=0}\theta(0) dx \,, \end{aligned} \tag{3.12}$$

*for any non-negative $\theta \in W^{1,\infty}[0,\infty)$ compactly supported on $[0,\infty)$,*

- *$(\rho, m)$ satisfies the following bounds, natural within the given framework:*

$$\sup_{t\in(0,T)}\int \rho dx = M < \infty \,,$$

$$\sup_{t\in(0,T)} \int \big(\eta(\rho,m) - \tfrac{1}{2}C_{\rm x}\rho c\big)\, dx < \infty\,. \tag{3.13}$$

*Remark* 3.2. The regularity requested in the above definition is the one needed to rewrite the equation in terms of the divergence of the stress tensor $S$ in (3.9), and it is implied by the finite energy condition. This relies on the $L^\gamma$ integrability of $\rho$ and elliptic regularity estimates for $c$, and it is proved in Section 3.3; see Lemma 3.6. Besides the appropriate smallness condition on the chemosensitive coefficient $C_{\rm x} > 0$ (cfr. $(\mathrm{H}_c)$), we shall require that $\gamma$ lies in the relevant range for which (3.2) has regular solutions, that is $(\mathrm{H}_{exp})$.

The existence theory for the Euler-Poisson system (3.1) with attractive potential $C_{\rm x} > 0$ (treated here) is largely an open problem. Repulsive potentials, $C_{\rm x} < 0$, offer a subtle stabilizing mechanism leading to global smooth solutions for potential flows with small velocities [14], as well as existence results for global weak entropy solutions in one-space dimension (*e.g.* [15]).

Let $(\bar\rho, \bar c) : (0,T)\times\mathbb{T}^n \to \mathbb{R}^{n+1}$ be a strong (conservative) periodic solution of (3.2) with $\bar\rho \geq \delta > 0$ for some $\delta > 0$, where the regularity "strong" refers to the boundedness of all the derivatives which will appear lately in the relative energy relation.

As in [21, 22] and in Section 2.4, we rewrite the equilibrium system (3.2) in the variables $\bar\rho$, $\bar c$ and

$$\bar m = -\varepsilon\big(\nabla_x p(\bar\rho) - C_{\rm x}\bar\rho\nabla_x\bar c\big) = -\varepsilon\bar\rho\nabla_x\big(h'(\bar\rho) - C_{\rm x}\bar c\big) \tag{3.14}$$

as follows:

$$\begin{cases} \bar\rho_t + \dfrac{1}{\varepsilon}\operatorname{div}_x \bar m = 0 \\ \bar m_t + \dfrac{1}{\varepsilon}\operatorname{div}_x \dfrac{\bar m\otimes\bar m}{\bar\rho} + \dfrac{1}{\varepsilon}\nabla_x p(\bar\rho) = -\dfrac{1}{\varepsilon^2}\bar m + \dfrac{C_{\rm x}}{\varepsilon}\bar\rho\nabla_x\bar c + e(\bar\rho,\bar m) \\ -\triangle_x\bar c + \beta\bar c = \bar\rho - <\bar\rho>, \end{cases} \tag{3.15}$$

where the term $e(\bar\rho,\bar m)$ is given by

$$\begin{aligned} \bar e := e(\bar\rho,\bar m) &= \frac{1}{\varepsilon}\operatorname{div}_x\left(\frac{\bar m\otimes\bar m}{\bar\rho}\right) + \bar m_t \\ &= \varepsilon\operatorname{div}_x\Big(\bar\rho\nabla_x\big(h'(\bar\rho) - C_{\rm x}\bar c\big)\otimes\nabla_x\big(h'(\bar\rho) - C_{\rm x}\bar c\big)\Big) - \varepsilon\partial_t\big(\bar\rho\nabla_x\big(h'(\bar\rho) - C_{\rm x}\bar c\big)\big) \\ &= O(\varepsilon)\,. \end{aligned} \tag{3.16}$$

Then, the equations satisfied by the differences $\rho-\bar\rho$, $m-\bar m$, $c-\bar c$ are given by

$$\begin{cases} (\rho-\bar\rho)_t + \dfrac{1}{\varepsilon}\partial_{x_i}(m_i - \bar m_i) = 0 \\ (m-\bar m)_t + \dfrac{1}{\varepsilon}\partial_{x_i}(f_i(\rho,m) - f_i(\bar\rho,\bar m)) \\ \qquad\qquad = -\dfrac{1}{\varepsilon^2}(m-\bar m) + \dfrac{C_{\rm x}}{\varepsilon}\big(\rho\nabla_x c - \bar\rho\nabla_x\bar c\big) - \bar e \\ -\triangle_x(c-\bar c) + \beta(c-\bar c) = (\rho-\bar\rho) - <\rho-\bar\rho>, \end{cases} \tag{3.17}$$

where $i = 1,\ldots,n$, $f_i$ stands for the (vector) of the flux in (3.1),

$$f_i(\rho,m) = m_i\frac{m}{\rho} + p(\rho)\mathbb{I}_i \tag{3.18}$$

and $\mathbb{I}_i$ is the $i$–th column of the identity matrix. With this notations, the integrated version of the entropy relation at the limit (3.5) becomes

$$\frac{d}{dt}\int_{\mathbb{T}^n}\Big(\eta(\bar\rho,\bar m) - \tfrac{1}{2}C_{\rm x}\bar\rho\bar c\Big)dx + \frac{1}{\varepsilon^2}\int_{\mathbb{T}^n}\frac{|\bar m|^2}{\bar\rho}dx = \int_{\mathbb{T}^n}\bar e\cdot\frac{\bar m}{\rho}dx\,. \tag{3.19}$$

**Theorem 3.3.** *Let $(\rho,m,c)$ be as in Definition 3.1 and let $(\bar\rho,\bar c)$ be a smooth solution of* (3.2)*. Then*

$$\begin{aligned} &\int_{\mathbb{T}^n}\Big[\eta(\rho,m\,|\bar\rho,\bar m) - \tfrac{1}{2}C_{\rm x}(\rho-\bar\rho)(c-\bar c)\Big]dx\Big|_t \\ &\leq \int_{\mathbb{T}^n}\Big[\eta(\rho,m\,|\bar\rho,\bar m) - \tfrac{1}{2}C_{\rm x}(\rho-\bar\rho)(c-\bar c)\Big]dx\Big|_0 \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\varepsilon^2} \iint_{[0,t]\times\mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar m}{\bar\rho} \right|^2 dxd\tau - \iint_{[0,t]\times\mathbb{T}^n} e(\bar\rho,\bar m)\cdot \frac{\rho}{\bar\rho}\left(\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right) dxd\tau \\
&- \frac{1}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} \mathrm{div}_x \big(\frac{\bar m}{\bar\rho}\big) p(\rho\,|\bar\rho) dxd\tau \\
&+ \frac{C_{\mathrm{x}}}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} \mathrm{div}_x \big(\frac{\bar m}{\bar\rho}\big)\big(\tfrac{\beta}{2}(c-\bar c)^2 + \tfrac12 |\nabla_x(c-\bar c)|^2 + (c-\bar c) < \rho - \bar\rho > \big) dxd\tau \\
&- \frac{C_{\mathrm{x}}}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} \nabla_x\big(\frac{\bar m}{\bar\rho}\big) : \nabla_x(c-\bar c)\otimes\nabla_x(c-\bar c) dxd\tau \\
&- \frac{1}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} \rho\nabla_x\left(\frac{\bar m}{\bar\rho}\right) : \left(\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right)\otimes\left(\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right) dxd\tau\,, \qquad (3.20)
\end{aligned}
$$

*where*

$$
\eta(\rho, m\,|\bar\rho,\bar m) = h(\rho|\bar\rho) + \tfrac12 \rho \Big|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\Big|^2 . \qquad (3.21)
$$

*Proof.* Let $(\rho, m, c)$ be a weak solution according to Definition 3.1 and $(\bar\rho,\bar m,\bar c)$ a strong solution of (3.15). We introduce in (3.12) the standard choice of test function

$$
\theta(\tau) := \begin{cases} 1, & \text{for } 0 \le \tau < t, \\ \frac{t-\tau}{\mu} + 1, & \text{for } t \le \tau < t+\mu, \\ 0, & \text{for } \tau \ge t+\mu, \end{cases} \qquad (3.22)
$$

and let $\mu \downarrow 0$; we then obtain

$$
\int_{\mathbb{T}^n} \big(\eta(\rho,m) - \tfrac12 C_{\mathrm{x}}\rho c\big) dx\Big|_{\tau=0}^{t} \le -\frac{1}{\varepsilon^2}\iint_{[0,t]\times\mathbb{T}^n} \frac{|m|^2}{\rho} dxd\tau\,. \qquad (3.23)
$$

Moreover, time integration of (3.19) gives

$$
\begin{aligned}
&\int_{\mathbb{T}^n} \Big(\eta(\bar\rho,\bar m) - \tfrac12 C_{\mathrm{x}}\bar\rho\bar c\Big) dx\Big|_{\tau=0}^{t} \\
&\quad = -\frac{1}{\varepsilon^2}\iint_{[0,t]\times\mathbb{T}^n} \frac{|\bar m|^2}{\bar\rho} dxd\tau + \iint_{[0,t]\times\mathbb{T}^n} \frac{\bar m}{\bar\rho}\cdot\bar e dxd\tau\,. \qquad (3.24)
\end{aligned}
$$

Next, for the linear part of the relative energy, we consider the weak formulation for the equations satisfied by the differences $(\rho-\bar\rho, m-\bar m)$ in (3.17):

$$
- \iint_{[0,+\infty)\times\mathbb{T}^n} \psi_t(\rho-\bar\rho) + \frac{1}{\varepsilon}\psi_{x_i}(m_i - \bar m_i) dxdt - \int_{\mathbb{T}^n} \psi(\rho-\bar\rho)\Big|_{t=0} dx = 0\,, \qquad (3.25)
$$

$$
\begin{aligned}
&- \iint_{[0,+\infty)\times\mathbb{T}^n} \left(\varphi_t\cdot(m-\bar m) + \frac{1}{\varepsilon}\partial_{x_i}\varphi_j\left(\frac{m_i m_j}{\rho} - \frac{\bar m_i\bar m_j}{\bar\rho}\right)\right. \\
&\quad \left. + \frac{1}{\varepsilon}\,\mathrm{div}_x\,\varphi\big(p(\rho) - p(\bar\rho)\big)\right) dxdt - \int_{\mathbb{T}^n} \varphi\cdot(m-\bar m)\Big|_{t=0} dx \\
&= -\frac{1}{\varepsilon^2}\iint_{[0,+\infty)\times\mathbb{T}^n} \varphi\cdot(m-\bar m) dxdt \\
&\quad + \frac{1}{\varepsilon}\iint_{[0,+\infty)\times\mathbb{T}^n} C_{\mathrm{x}}\varphi\cdot\big(\rho\nabla_x c - \bar\rho\nabla_x\bar c\big) dxdt - \iint_{[0,+\infty)\times\mathbb{T}^n} \varphi\cdot\bar e dxdt\,, \qquad (3.26)
\end{aligned}
$$

where $\varphi$, $\psi$ are Lipschitz test functions compactly supported in $[0,+\infty)$ in time and periodic in space. In the above relations we introduce the test functions

$$
\psi = \theta(\tau)\left(h'(\bar\rho) - C_{\mathrm{x}}\bar c - \frac12\frac{|\bar m|^2}{\bar\rho^2}\right), \quad \varphi = \theta(\tau)\frac{\bar m}{\bar\rho}\,,
$$

with $\theta(\tau)$ as in (3.22). This choice in (3.25), sending $\mu \downarrow 0$, leads to

$$\int_{\mathbb{T}^n} \left( h'(\bar\rho) - C_{\mathrm{x}} \bar c - \frac{1}{2} \frac{|\bar m|^2}{\bar\rho^2} \right) (\rho - \bar\rho) \Big|_{\tau=0}^{t} dx - \iint_{[0,t]\times\mathbb{T}^n} \partial_\tau \left( h'(\bar\rho) - C_{\mathrm{x}} \bar c - \frac{1}{2} \frac{|\bar m|^2}{\bar\rho^2} \right) (\rho - \bar\rho) dx d\tau - \frac{1}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} \nabla_x \left( h'(\bar\rho) - C_{\mathrm{x}} \bar c - \frac{1}{2} \frac{|\bar m|^2}{\bar\rho^2} \right) \cdot (m - \bar m) dx d\tau = 0 \,. \tag{3.27}$$

Similarly, from (3.26) we get

$$\int_{\mathbb{T}^n} \frac{\bar m}{\bar\rho} \cdot (m - \bar m) \Big|_{\tau=0}^{t} dx - \iint_{[0,t]\times\mathbb{T}^n} \partial_\tau \left( \frac{\bar m}{\bar\rho} \right) \cdot (m - \bar m) dx d\tau - \frac{1}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} \left( \partial_{x_i} \big(\frac{\bar m_j}{\bar\rho}\big) \Big( \frac{m_i m_j}{\rho} - \frac{\bar m_i \bar m_j}{\bar\rho} \Big) + \mathrm{div}_x \left( \frac{\bar m}{\bar\rho} \right) \big( p(\rho) - p(\bar\rho) \big) \right) dx d\tau = -\frac{1}{\varepsilon^2} \iint_{[0,t]\times\mathbb{T}^n} \frac{\bar m}{\bar\rho} \cdot (m - \bar m) dx d\tau + \frac{1}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} C_{\mathrm{x}} \frac{\bar m}{\bar\rho} \cdot \big( \rho \nabla_x c - \bar\rho \nabla_x \bar c \big) dx d\tau - \iint_{[0,t]\times\mathbb{T}^n} \frac{\bar m}{\bar\rho} \cdot \bar e \, dx d\tau \,. \tag{3.28}$$

Since $c = \mathcal{K} * \rho$, $\bar c = \mathcal{K} * \bar\rho$ and $\mathcal{K}$ is symmetric,

$$\int_{\mathbb{T}^n} c \bar\rho dx = \int_{\mathbb{T}^n} \bar c \rho dx$$

and therefore

$$\int_{\mathbb{T}^n} \Big( -\tfrac{1}{2} C_{\mathrm{x}} \rho c + \tfrac{1}{2} C_{\mathrm{x}} \bar\rho \bar c + C_{\mathrm{x}} \bar c (\rho - \bar\rho) \Big) dx = -\int_{\mathbb{T}^n} \tfrac{1}{2} C_{\mathrm{x}} (\rho - \bar\rho)(c - \bar c) dx \,.$$

Hence, we compute (3.23) - (3.24) - $\big($(3.27) + (3.28)$\big)$ to obtain

$$\int_{\mathbb{T}^n} \Big[ \eta(\rho, m \,|\, \bar\rho, \bar m\,) - \tfrac{1}{2} C_{\mathrm{x}} (\rho - \bar\rho)(c - \bar c) \Big] \Big|_{\tau=0}^{t} dx \le -\frac{1}{\varepsilon^2} \iint_{[0,t]\times\mathbb{T}^n} \Big[ \rho |u|^2 - \bar\rho |\bar u|^2 - \bar u \cdot (\rho u - \bar\rho \bar u) \Big] dx d\tau - \iint_{[0,t]\times\mathbb{T}^n} \Big[ \partial_\tau \big( h'(\bar\rho) - C_{\mathrm{x}} \bar c - \tfrac{1}{2} |\bar u|^2 \big) (\rho - \bar\rho) + \partial_\tau(\bar u) \cdot (\rho u - \bar\rho \bar u) \Big] dx d\tau - \frac{1}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} \nabla_x \big( h'(\bar\rho) - C_{\mathrm{x}} \bar c - \tfrac{1}{2} |\bar u|^2 \big) \cdot (\rho u - \bar\rho \bar u) dx d\tau - \frac{1}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} \partial_{x_i}(\bar u_j) \Big( (\rho u_i u_j - \bar\rho \bar u_i \bar u_j) + (p(\rho) - p(\bar\rho)) \delta_{ij} \Big) dx d\tau - \frac{1}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} C_{\mathrm{x}} \bar u \cdot \big( \rho \nabla_x c - \bar\rho \nabla_x \bar c \big) dx d\tau \,, \tag{3.29}$$

for $m = \rho u$ and $\bar m = -\varepsilon \bar\rho \nabla_x \big( h'(\bar\rho) - C_{\mathrm{x}} \bar c \big) = \bar\rho \bar u$. Since $\bar u$ verifies the equation

$$\partial_t \bar u + \frac{1}{\varepsilon} (\bar u \cdot \nabla_x) \bar u = -\frac{1}{\varepsilon^2} \bar u - \frac{1}{\varepsilon} \nabla_x \big( h'(\bar\rho) - C_{\mathrm{x}} \bar c \big) + \frac{\bar e}{\bar\rho} \,,$$

then

$$\partial_\tau \big( -\tfrac{1}{2} |\bar u|^2 \big) (\rho - \bar\rho) + \partial_\tau(\bar u) \cdot (\rho u - \bar\rho \bar u) + \frac{1}{\varepsilon} \nabla_x \big( -\tfrac{1}{2} |\bar u|^2 \big) \cdot (\rho u - \bar\rho \bar u) + \frac{1}{\varepsilon} \partial_{x_i}(\bar u_j)(\rho u_i u_j - \bar\rho \bar u_i \bar u_j) = -\frac{1}{\varepsilon^2} \rho \bar u \cdot (u - \bar u) - \frac{1}{\varepsilon} \rho \nabla_x \big( h'(\bar\rho) - C_{\mathrm{x}} \bar c \big) \cdot (u - \bar u)$$

$$+\bar{e}\frac{\rho}{\bar{\rho}}\cdot(u-\bar{u})+\frac{1}{\varepsilon}\rho\nabla_x(\bar{u}):(u-\bar{u})\otimes(u-\bar{u})\,.$$

Then, using the above relation and the continuity equation for $\bar{\rho}$, after some straightforward calculations (3.29) becomes

$$\begin{aligned}
&\int_{\mathbb{T}^n}\Big[\eta(\rho,m\,|\bar{\rho},\bar{m})-\tfrac{1}{2}C_{\rm x}(\rho-\bar{\rho})(c-\bar{c})\Big]\Bigg|_{\tau=0}^{t}dx\\
&\leq-\frac{1}{\varepsilon^2}\iint_{[0,t]\times\mathbb{T}^n}\rho|u-\bar{u}|^2dxd\tau-\iint_{[0,t]\times\mathbb{T}^n}\bar{e}\frac{\rho}{\bar{\rho}}\cdot(u-\bar{u})dxd\tau\\
&\quad-\frac{1}{\varepsilon}\iint_{[0,t]\times\mathbb{T}^n}\rho\nabla_x(\bar{u}):(u-\bar{u})\otimes(u-\bar{u})dxd\tau\\
&\quad-\frac{1}{\varepsilon}\iint_{[0,t]\times\mathbb{T}^n}{\rm div}_x(\bar{u})p(\rho\,|\bar{\rho})dxd\tau\\
&\quad-\iint_{[0,t]\times\mathbb{T}^n}\Big[\frac{1}{\varepsilon}C_{\rm x}\rho\bar{u}\cdot\nabla_x(c-\bar{c})-C_{\rm x}(\rho-\bar{\rho})\partial_\tau\bar{c}\Big]dxd\tau
\end{aligned}\tag{3.30}$$

and we are left with the control of the last term in (3.30). Now we use again the symmetry of $\mathcal{K}$ and the equation for $\bar{\rho}$ to conclude

$$\begin{aligned}
\int_{\mathbb{T}^n}(\rho-\bar{\rho})\partial_\tau\bar{c}dx&=\int_{\mathbb{T}^n}(\rho-\bar{\rho})\mathcal{K}*\bar{\rho}_\tau dx=\int_{\mathbb{T}^n}(\mathcal{K}*\rho-\mathcal{K}*\bar{\rho})\bar{\rho}_\tau dx\\
&=\int_{\mathbb{T}^n}(c-\bar{c})\bar{\rho}_\tau dx\\
&=-\frac{1}{\varepsilon}\int_{\mathbb{T}^n}(c-\bar{c})\,{\rm div}_x(\rho\bar{u})dx+\frac{1}{\varepsilon}\int_{\mathbb{T}^n}(c-\bar{c})\,{\rm div}_x\big((\rho-\bar{\rho})\bar{u}\big)dx
\end{aligned}$$

and the last term of (3.30) is rewritten as

$$I=-\frac{C_{\rm x}}{\varepsilon}\int_{\mathbb{T}^n}\nabla_x(c-\bar{c})\cdot\bar{u}(\rho-\bar{\rho})dx\,.$$

Finally, $(3.17)_3$ and integration by parts allow to reexpress

$$\begin{aligned}
I=&-\frac{C_{\rm x}}{\varepsilon}\int_{\mathbb{T}^n}\nabla\bar{u}:\nabla(c-\bar{c})\otimes\nabla(c-\bar{c})dx\\
&+\frac{C_{\rm x}}{\varepsilon}\int_{\mathbb{T}^n}{\rm div}_x\,\bar{u}\big(\tfrac{1}{2}|\nabla(c-\bar{c})|^2+\tfrac{\beta}{2}(c-\bar{c})^2+(c-\bar{c})<\rho-\bar{\rho}>\big)dx
\end{aligned}$$

and complete the proof of (3.20). □

3.3. **Stability estimate and convergence of the relaxation limit.** In this section we obtain a stability estimate in terms of the relative energy inequality, and establish convergence in the diffusive relaxation limit from (3.1) to (3.2). We restrict to pressure functions satisfying (H), $(\mathrm{A}_1)$, $(\mathrm{A}_2)$ with

$$\gamma\geq2-\frac{2}{n},\ \text{if } n\geq3;\quad \gamma>2-\frac{2}{n}=1,\ \text{if } n=2\,.\tag{H$_{exp}$}$$

We note that the $\gamma$ threshold obtained here is the same obtained for the Keller–Segel equilibrium system in the study of the existence of global in time weak solutions (no blow-up for finite time) for general initial data [26, 27], both being related with Hardy-Littlewood-Sobolev inequalities.

We start with some preliminary results.

**Lemma 3.4.** *(a) Let $h\in C^0[0,+\infty)\cap C^2(0,+\infty)$ satisfy $h''(\rho)>0$ and $(\mathrm{A}_1)$ for $\gamma>1$. If $\bar{\rho}\in K=[\delta,\bar{R}]$ with $\delta>0$ and $\bar{R}<+\infty$, then there exist positive constants $R_0$ (depending on $K$) and $C_1$, $C_2$ (depending on $K$ and $R_0$) such that*

$$h(\rho\,|\bar{\rho})\geq\begin{cases}C_1|\rho-\bar{\rho}|^2, & \text{for } 0\leq\rho\leq R_0,\ \bar{\rho}\in K,\\ C_2|\rho-\bar{\rho}|^\gamma, & \text{for } \rho>R_0,\ \bar{\rho}\in K.\end{cases}\tag{3.31}$$

*(b) If $p(\rho)$ and $h(\rho)$ satisfy* (H) *and* ($A_2$) *then*

$$|p(\rho|\bar{\rho})| \le A\, h(\rho|\bar{\rho}) \quad \forall \rho, \bar{\rho} > 0\,. \tag{3.32}$$

*Proof.* Part (a) is proved in [21, Lemma 2.4]. To show (b), one first easily checks the identity

$$\begin{aligned} p(\rho|\bar{\rho}) &= p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})(\rho - \bar{\rho}) \\ &= (\rho - \bar{\rho})^2 \int_0^1 \int_0^\tau p''(s\rho + (1-s)\bar{\rho}) ds d\tau\,, \end{aligned}$$

and a similar identity holds for $h(\rho|\bar{\rho})$. Recall now that $h'' = \frac{p'}{\rho}$. Then hypothesis ($A_2$) implies $|p''| \le Ah''$ and thus

$$\begin{aligned} |p(\rho|\bar{\rho})| &\le (\rho - \bar{\rho})^2 \int_0^1 \int_0^\tau |p''(s\rho + (1-s)\bar{\rho})| ds d\tau \\ &\le A(\rho - \bar{\rho})^2 \int_0^1 \int_0^\tau h''(s\rho + (1-s)\bar{\rho}) ds d\tau \\ &= A\, h(\rho|\bar{\rho}) \end{aligned}$$

and (3.31) follows. □

*Remark* 3.5. For exponents $\gamma \ge 2$, by enlarging if necessary $R_0$ so that $|\rho - \bar{\rho}| \ge 1$ for $\rho > R_0$ and $\bar{\rho} \in K = [\delta, \bar{R}]$, we obtain

$$h(\rho\,|\bar{\rho}) \ge c_0 |\rho - \bar{\rho}|^2, \quad \text{for } \gamma \ge 2,\ \rho \ge 0,\ \bar{\rho} \in K\,, \tag{3.33}$$

where $c_0 > 0$ depends solely on $K$.

Application of standard elliptic theory to the equation

$$-\triangle_x (c - \bar{c}) + \beta(c - \bar{c}) = (\rho - \bar{\rho}) - <\rho - \bar{\rho}> \ . \tag{3.34}$$

provides another preliminary estimate.

**Lemma 3.6.** *Let $(\rho, m, c)$ be as in Definition 3.1 and let $(\bar{\rho}, \bar{c})$ be a smooth solution of* (3.2) *such that $\bar{\rho} \ge \rho^* > 0$ Then, for any $q \in \left[\frac{2n}{n+2}, +\infty\right)$ for $n \ge 3$, or $q \in (1, \infty)$ for $n = 2$, one has*

$$\int_{\mathbb{T}^n} \left(\beta |c - \bar{c}|^2 dx + |\nabla_x (c - \bar{c})|^2\right)\, dx = \left| \int_{\mathbb{T}^n} (\rho - \bar{\rho})(c - \bar{c}) dx \right| \le C \|\rho - \bar{\rho}\|^2_{L^q(\mathbb{T}^n)}\,, \tag{3.35}$$

*where $C$ depends only on the space dimension $n$ and on $\mathbb{T}^n$.*

*Proof.* First of all, we note that $c - \bar{c}$ is of zero mean in $\mathbb{T}^n$ for $t \ge 0$ and for any $\beta > 0$, and also by definition for $\beta = 0$. We apply elliptic estimates for (3.34) for any non negative screening coefficient $\beta$, including $\beta = 0$. Indeed, we multiply this equation by $c - \bar{c}$ and integrate over $\mathbb{T}^n$ to get

$$\begin{aligned} \left| \int_{\mathbb{T}^n} (\rho - \bar{\rho})(c - \bar{c}) dx \right| &\le \beta \int_{\mathbb{T}^n} |c - \bar{c}|^2 dx + \int_{\mathbb{T}^n} |\nabla_x (c - \bar{c})|^2 dx \\ &\le C \int_{\mathbb{T}^n} |\nabla_x (c - \bar{c})|^2 dx, \end{aligned}$$

using Poincaré's inequality if $\beta > 0$, for a constant $C$ depending only on $\mathbb{T}^n$. We use the Sobolev embedding theorem in the form

$$W^{1,q}(\mathbb{T}^n) \subset L^2(\mathbb{T}^n) \quad \text{ for } q \ge \tfrac{2n}{n+2}$$

and then $L^q$ elliptic regularity estimates ($q \ne 1$) to conclude

$$\begin{aligned} \|\nabla_x (c - \bar{c})\|_{L^2(\mathbb{T}^n)} &\le C\, \|\nabla_x (c - \bar{c})\|_{W^{1,q}(\mathbb{T}^n)} \\ &\le C\, \|c - \bar{c}\|_{W^{2,q}(\mathbb{T}^n)} \\ &\le C\, \|\rho - \bar{\rho}\|_{L^q(\mathbb{T}^n)} \end{aligned}$$

for any $\frac{2n}{n+2} \le q < \infty$ for $n \ge 3$, restricted to $\frac{2n}{n+2} = 1 < q < \infty$ in the case $n = 2$. □

**Lemma 3.7.** *Let $\gamma$ satisfy* ($\mathrm{H}_{exp}$)*, let $(\rho, m, c)$ and $(\bar\rho, \bar c)$ be as in Lemma 3.6. There exists a constant $K > 0$ such that*

$$\left|\int_{\mathbb{T}^n} (\rho-\bar\rho)(c-\bar c)dx\right| \le K\int_{\mathbb{T}^n} h(\rho\,|\bar\rho)dx\,. \tag{3.36}$$

*Proof.* We work in the range $2-\frac{2}{n}\le\gamma$ for $n=3$ and $2-\frac{2}{n}<\gamma$ for $n=2$ and split the proof in two cases: $\gamma<2$ and $\gamma\ge 2$.

*Case $\gamma<2$:* Note that, for any $n\ge 2$, $2-\frac{2}{n}\ge\frac{2n}{n+2}$. Using (3.35) with $\frac{2n}{n+2}\le q<\gamma<2$ for $n=3$, and $\frac{2n}{n+2}<q<\gamma<2$ for $n=2$, we obtain

$$\begin{aligned}\left|\int_{\mathbb{T}^n} (\rho-\bar\rho)(c-\bar c)dx\right| &\le C\left(\int_{\mathbb{T}^n\cap\{\rho\le R_0\}}|\rho-\bar\rho|^q dx\right)^{\frac{2}{q}} + C\left(\int_{\mathbb{T}^n\cap\{\rho> R_0\}}|\rho-\bar\rho|^q dx\right)^{\frac{2}{q}}\\ &\le \bar C_1(\mathbb{T}^n)\int_{\mathbb{T}^n\cap\{\rho\le R_0\}}|\rho-\bar\rho|^2 dx + C\left(\int_{\mathbb{T}^n\cap\{\rho> R_0\}}|\rho-\bar\rho|^q dx\right)^{\frac{2}{q}}\,,\end{aligned} \tag{3.37}$$

For $1<q<\gamma$, the second term on the right–hand–side of (3.37) is treated via interpolation between the $L^1$ and $L^\gamma$ spaces: For $\theta$ satisfying $\frac{1}{q}=\frac{\theta}{\gamma}+1-\theta$, it is

$$\|\rho-\bar\rho\|^2_{L^q(\mathbb{T}^n\cap\{\rho>R_0\})} \le \|\rho-\bar\rho\|^{2-2\theta}_{L^1(\mathbb{T}^n\cap\{\rho>R_0\})}\|\rho-\bar\rho\|^{2\theta}_{L^\gamma(\mathbb{T}^n\cap\{\rho>R_0\})}, \tag{3.38}$$

We select $2\theta=\gamma$ and the corresponding $q$ becomes $q=\frac{2}{3-\gamma}$; this $q$ is indeed admissible (i.e. $1<\frac{2n}{n+2}\le q<\gamma$ if $n=3$; $1=\frac{2n}{n+2}<q<\gamma$ if $n=2$), in view of ($\mathrm{H}_{exp}$). Hence, (3.38), the conservation of mass, and (3.31) give

$$\begin{aligned}\left(\int_{\mathbb{T}^n\cap\{\rho>R_0\}}|\rho-\bar\rho|^q dx\right)^{\frac{2}{q}} &\le C(M+\bar M)^{2-\gamma}\int_{\mathbb{T}^n\cap\{\rho>R_0\}}|\rho-\bar\rho|^\gamma dx\\ &\le \frac{C}{C_2}(M+\bar M)^{2-\gamma}\int_{\mathbb{T}^n\cap\{\rho>R_0\}}h(\rho\,|\bar\rho)dx\,.\end{aligned} \tag{3.39}$$

Combining (3.37), (3.39) and (3.31) we derive (3.36) for $\gamma<2$.

*Case $\gamma\ge 2$:* In that case, we select $q=2>\frac{2n}{n+2}$, and use (3.33) and (3.35) to immediately obtain

$$\left|\int_{\mathbb{T}^n} (\rho-\bar\rho)(c-\bar c)dx\right| \le \frac{C}{c_0}\int_{\mathbb{T}^n} h(\rho\,|\bar\rho)dx\,, \tag{3.40}$$

where $C$, $c_0>0$ depend solely on the dimension, $\mathbb{T}^n$ and on the bounds of the smooth limit solution $\bar\rho$. □

Our objective is to use the relative energy of the Euler-Poisson system

$$\Phi(t)=\int_{\mathbb{T}^n}\left(\tfrac12\rho\left|\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right|^2 + h(\rho|\bar\rho) - \tfrac12 C_{\mathrm{x}}(\rho-\bar\rho)(c-\bar c)\right)dx \tag{3.41}$$

as a yardstick for estimating the error terms in the relative entropy identity (3.20). To ensure that (3.41) will measure distance of solutions, we restrict the values of the chemosensitive coefficient $C_{\mathrm{x}}$:

**Proposition 3.8.** *Let $(\rho, m, c)$ be as in Definition 3.1 and let $(\bar\rho,\bar c)$ be a smooth solution of* (3.2) *such that $\bar\rho\ge\rho^*>0$. If the parameter*

$$C_{\mathrm{x}} < \frac{2}{K}\,, \tag{$\mathrm{H}_c$}$$

*where $K$ is defined in* (3.36)*, then for $\lambda := 1-\frac12 KC_{\mathrm{x}}>0$ the relative energy $\Phi(t)$ satisfies*

$$\Phi(t)\ge\int_{\mathbb{T}^n}\left(\tfrac12\rho\left|\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right|^2 + \lambda h(\rho|\bar\rho)\right)dx>0\,. \tag{3.42}$$

We prove:

**Theorem 3.9.** *Let $T > 0$ be fixed, let $(\rho, m, c)$ be as in Definition 3.1 and $(\bar\rho, \bar c)$ be a smooth solution of (3.2) such that $\bar\rho \geq \delta > 0$. Assume that the pressure $p(\rho)$ satisfies* (H), (A$_1$), (A$_2$)*, with exponent $\gamma$ restricted by* (H$_{exp}$)*, the coefficient $C_{\rm x}$ is restricted by* (H$_c$)*, and $\beta \geq 0$. Then, the stability estimate*

$$\Phi(t) \leq C\big(\Phi(0) + \varepsilon^4\big), \quad t \in [0, T]\,,$$

*holds true, where $C$ is a positive constant depending on $T$, $C_{\rm x}$, $M$, $\bar\rho$ and its derivatives. Moreover, if $\Phi(0) \to 0$ as $\varepsilon \downarrow 0$, then*

$$\sup_{t\in[0,T]} \Phi(t) \to 0, \ \text{ as } \varepsilon \downarrow 0\,.$$

*Proof.* We employ (3.20) for the relative energy (3.41) to conclude

$$\Phi(t) + \frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{T}^n} \rho\left|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right|^2 dxd\tau \leq \Phi(0) + \int_0^t\int_{\mathbb{T}^n} \big(|Q| + |E|\big)dxd\tau\,, \tag{3.43}$$

for $t \in [0, T]$. The quadratic $Q$ and error $E$ terms in (3.43) are defined by

$$\begin{aligned}
E &:= \bar e \cdot \frac{\rho}{\bar\rho}\left(\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right)\\
Q &:= \frac{1}{\varepsilon}\,\mathrm{div}_x\left(\frac{\bar m}{\bar\rho}\right)\Big[-p(\rho\,|\bar\rho) + C_{\rm x}\big(\tfrac12\beta(c-\bar c)^2 + \tfrac12|\nabla_x(c-\bar c)|^2 + (c-\bar c) < \rho - \bar\rho >\big)\Big]\\
&\quad - \frac{C_{\rm x}}{\varepsilon}\nabla_x\left(\frac{\bar m}{\bar\rho}\right) : \nabla_x(c-\bar c)\otimes\nabla_x(c-\bar c) - \frac{1}{\varepsilon}\rho\nabla_x\left(\frac{\bar m}{\bar\rho}\right) : \left(\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right)\otimes\left(\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right).
\end{aligned}$$

Using (3.34) and integration by parts, we compute

$$\begin{aligned}
&\frac{C_{\rm x}}{\varepsilon}\int_{\mathbb{T}^n} \mathrm{div}_x\left(\frac{\bar m}{\bar\rho}\right)(c-\bar c) < \rho - \bar\rho > dx\\
&\quad = \frac{C_{\rm x}}{\varepsilon}\int_{\mathbb{T}^n} \mathrm{div}_x\left(\frac{\bar m}{\bar\rho}\right)(c-\bar c)(\rho-\bar\rho)dx - \frac{C_{\rm x}}{\varepsilon}\int_{\mathbb{T}^n} \mathrm{div}_x\left(\frac{\bar m}{\bar\rho}\right)\beta(c-\bar c)^2dx\\
&\qquad - \frac{C_{\rm x}}{\varepsilon}\int_{\mathbb{T}^n} \mathrm{div}_x\left(\frac{\bar m}{\bar\rho}\right)|\nabla_x(c-\bar c)|^2dx - \frac{C_{\rm x}}{\varepsilon}\int_{\mathbb{T}^n}(c-\bar c)\nabla_x\left(\mathrm{div}_x\left(\frac{\bar m}{\bar\rho}\right)\right)\cdot\nabla_x(c-\bar c)dx\,.
\end{aligned}$$

From (3.14) and the smoothness of $\bar\rho$, we have that $\bar u = \frac{\bar m}{\bar\rho} = O(\varepsilon)$ and $\frac{1}{\varepsilon}|\nabla_x\bar u| = O(1)$ as well as $\frac{1}{\varepsilon}|\nabla_x^2\bar u| = O(1)$. Thus, we use Young's inequality to bound the last term of the above relation as follows

$$\frac{C_{\rm x}}{\varepsilon}\int_{\mathbb{T}^n}\left|(c-\bar c)\nabla_x\left(\mathrm{div}_x\left(\frac{\bar m}{\bar\rho}\right)\right)\cdot\nabla_x(c-\bar c)\right|dx \leq C\int_{\mathbb{T}^n}\big((c-\bar c)^2 + |\nabla_x(c-\bar c)|^2\big)\,dx \leq C\Phi(t)$$

using (3.35), (3.36), (3.42) and the Poincaré inequality for the zero mean function $c - \bar c$ in $\mathbb{T}^n$ if $\beta = 0$, and where $C > 0$ (here and below) denotes a generic positive constant. The estimates for the remaining terms in $Q$ are straightforward, again by virtue of (3.35), (3.36), and in view of (3.32) and (3.42) we end up with

$$\int_0^t\int_{\mathbb{T}^n}|Q|dxd\tau \leq C\int_0^t\Phi(\tau)d\tau.$$

Moreover, we recall $\bar e = O(\varepsilon)$ so that

$$\begin{aligned}
\int_0^t\int_{\mathbb{T}^n}|E|dxd\tau &\leq \frac{\varepsilon^2}{2}\int_0^t\int_{\mathbb{T}^n}\left|\frac{\bar e}{\bar\rho}\right|^2\rho dxd\tau + \frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^n}\rho\left|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right|^2dxd\tau\\
&\leq CMT\varepsilon^4 + \frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^n}\rho\left|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right|^2dxd\tau\,.
\end{aligned}$$

Hence (3.43) gives

$$\Phi(t) + \frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^n}\rho\left|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right|^2dxd\tau \leq \Phi(0) + C\varepsilon^4 + C\int_0^t\Phi(\tau)d\tau$$

and Gronwall's Lemma gives the desired result. □

## 4. From the Euler-Korteweg system with friction to the Cahn–Hilliard equation

In this section we discuss the relaxation limit from the Euler–Korteweg system for the motion for capillary fluids towards the Cahn–Hilliard equation. In our discussion we restrict to monotone pressure laws $p(\rho)$ and thus the model does not cover the case of phase transitions. This model fits under the context of the general theory described in Section 2 for a potential energy

$$\mathcal{E}(\rho) = \int \big(h(\rho) + \tfrac{1}{2}C_\kappa|\nabla_x\rho|^2\big)\,dx, \quad C_\kappa > 0\,,$$

for which the generator $\frac{\delta\mathcal{E}}{\delta\rho}(\rho)$ takes the form

$$\frac{\delta\mathcal{E}}{\delta\rho}(\rho) = \big(\partial_\rho - \operatorname{div}_x\partial_q\big)\big(h(\rho) + \tfrac{1}{2}C_\kappa|q|^2\big)\big|_{q=\nabla_x\rho} = h'(\rho) - C_\kappa\triangle_x\rho\,,$$

the stress functional $S$ in (2.4) is given by

$$S = \left(-p(\rho) + \frac{1}{2}C_\kappa|\nabla_x\rho|^2 + C_\kappa\rho\triangle_x\rho\right)\mathbb{I} - C_\kappa\nabla_x\rho\otimes\nabla_x\rho\,, \tag{4.1}$$

and $\mathbb{I}$ stands for the identity matrix. Hence, (2.3) may be expressed as

$$\begin{cases} \rho_t + \dfrac{1}{\varepsilon}\operatorname{div}_x m = 0 \\ m_t + \dfrac{1}{\varepsilon}\operatorname{div}_x\dfrac{m\otimes m}{\rho} = -\dfrac{1}{\varepsilon^2}m - \dfrac{1}{\varepsilon}\rho\nabla_x\big(h'(\rho) - C_\kappa\triangle_x\rho\big) \\ \qquad\qquad\qquad\qquad = -\dfrac{1}{\varepsilon^2}m + \dfrac{1}{\varepsilon}\operatorname{div}_x S\,, \end{cases} \tag{4.2}$$

where $t\in\mathbb{R}$, $x\in\mathbb{T}^n$, $\rho\ge 0$, $C_\kappa$ is a non negative constant, $m\in\mathbb{R}^3$, and the pressure $p(\rho)$ is defined by $p'(\rho) = \rho h''(\rho)$ and satisfies the monotonicity condition $p'(\rho) > 0$ and the stress tensor $S$ is given by (4.1). As $\varepsilon\downarrow 0$, system (4.2), after an appropriate scaling of the momentum $m$, formally reduces to the following Cahn–Hilliard equation

$$\rho_t = \operatorname{div}_x\big(\rho\nabla_x\big(h'(\rho) - C_\kappa\triangle_x\rho\big)\big) = \operatorname{div}_x\big(\nabla_x p(\rho) - C_\kappa\rho\nabla_x\triangle_x\rho\big)\,. \tag{4.3}$$

Our goal is to rigorously validate this limit using the relative entropy identity (2.28) of Section 2.

We recall that the balance of total energy (2.8), when expressed in the context of the system (4.2) takes the form:

$$\begin{aligned} &\partial_t\left(\eta(\rho,m) + \frac{C_\kappa}{2}|\nabla_x\rho|^2\right) + \frac{1}{\varepsilon}\operatorname{div}_x\big(q(\rho,m) - C_\kappa m\triangle_x\rho + C_\kappa\nabla_x\rho\operatorname{div}_x m\big) \\ &= -\frac{1}{\varepsilon^2}\frac{|m|^2}{\rho} \le 0, \end{aligned} \tag{4.4}$$

where we used the notation

$$\eta(\rho,m) = \frac{1}{2}\frac{|m|^2}{\rho} + h(\rho)\,, \qquad q(\rho,m) = \frac{1}{2}m\frac{|m|^2}{\rho^2} + mh'(\rho)\,,$$

for the mechanical energy of the compressible Euler equations and its associated flux. Moreover, smooth solutions $\rho$ of (4.3) satisfy the energy dissipation identity

$$\begin{aligned} &\partial_t\left(h(\rho) + \frac{C_\kappa}{2}|\nabla_x\rho|^2\right) - \operatorname{div}_x\Big[\rho h'(\rho)\nabla_x\big(h'(\rho) - C_\kappa\triangle_x\rho\big) \\ &+ C_\kappa\operatorname{div}_x\big(\rho\nabla_x\big(h'(\rho) - C_\kappa\triangle_x\rho\big)\big)\nabla_x\rho - C_\kappa\rho\nabla_x\big(h'(\rho) - C_\kappa\triangle_x\rho\big)\triangle_x\rho\Big] \\ &= -\rho\left|\nabla_x\left(h'(\rho) - C_\kappa\triangle_x\rho\right)\right|^2 \le 0\,. \end{aligned} \tag{4.5}$$

The above relations serve as starting point to perform the relative energy computation in the next section.

4.1. **Relative energy estimate.** Next, we devise a relative energy estimate valid between a weak solution of (4.2) and a strong solution of (4.3). This will be used to prove rigorously the diffusive relaxation limit for the case of constant capillarity coefficient $\kappa(\rho) = C_\kappa > 0$ and for $\gamma$–law gases $p(\rho) = k\rho^\gamma$. To this end, following ideas from Section 3 and [21, 22], we rewrite the Cahn-Hilliard equation (4.3) as a correction of the Euler-Korteweg system (4.2),

$$\begin{cases} \bar\rho_t + \dfrac{1}{\varepsilon}\operatorname{div}_x \bar m = 0 \\ \bar m_t + \dfrac{1}{\varepsilon}\operatorname{div}_x \dfrac{\bar m \otimes \bar m}{\bar\rho} = -\dfrac{1}{\varepsilon^2}\bar m - \dfrac{1}{\varepsilon}\bar\rho\nabla_x\big(h'(\bar\rho) - C_\kappa\triangle_x\bar\rho\big) + e(\bar\rho,\bar m) \\ \qquad\qquad\qquad\qquad = -\dfrac{1}{\varepsilon^2}\bar m + \dfrac{1}{\varepsilon}\operatorname{div}_x \bar S + e(\bar\rho,\bar m)\,, \end{cases} \tag{4.6}$$

by introducing

$$\bar m = -\varepsilon\bar\rho\nabla_x\big(h'(\bar\rho) - C_\kappa\triangle_x\bar\rho\big)\,,$$

and setting the error term $e(\bar\rho,\bar m)$ to be given by

$$\begin{aligned} \bar e := e(\bar\rho,\bar m) &= \frac{1}{\varepsilon}\operatorname{div}_x\left(\frac{\bar m\otimes\bar m}{\bar\rho}\right) + \bar m_t \\ &= \varepsilon\operatorname{div}_x\Big(\bar\rho\nabla_x\big(h'(\bar\rho) - C_\kappa\triangle_x\bar\rho\big)\otimes\nabla_x\big(h'(\bar\rho) - C_\kappa\triangle_x\bar\rho\big)\Big) \\ &\quad - \varepsilon\big(\bar\rho\nabla_x\big(h'(\bar\rho) - C_\kappa\triangle_x\bar\rho\big)\big)_t \\ &= O(\varepsilon). \end{aligned} \tag{4.7}$$

Accordingly, (4.5) is rewritten as

$$\begin{aligned} &\partial_t\left(\eta(\bar\rho,\bar m) + \frac{C_\kappa}{2}|\nabla_x\bar\rho|^2\right) + \frac{1}{\varepsilon}\operatorname{div}_x\big(q(\bar\rho,\bar m) - C_\kappa\bar m\triangle_x\bar\rho + C_\kappa\nabla_x\bar\rho\operatorname{div}_x\bar m\big) \\ &= -\frac{1}{\varepsilon^2}\frac{|\bar m|^2}{\bar\rho} + \frac{\bar m}{\bar\rho}\cdot\bar e\,. \end{aligned} \tag{4.8}$$

Following the general recipe (2.12) and (2.16), the relative energy is defined via

$$\eta(\rho,m\,|\,\bar\rho,\bar m) + \frac{C_\kappa}{2}|\nabla_x(\rho-\bar\rho)|^2 = \frac12\rho\left|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right|^2 + h(\rho\,|\,\bar\rho) + \frac{C_\kappa}{2}|\nabla_x(\rho-\bar\rho)|^2\,.$$

Before proving our main result, we recall the notion of weak solutions considered here, either conservative or dissipative [13].

**Definition 4.1.** *(i) A function $(\rho,m)$ with $\rho\in C([0,\infty);L^1(\mathbb{T}^n))$, $m\in C\big(([0,\infty);\big(L^1(\mathbb{T}^n)\big)^n\big)$, $\rho\ge 0$, is a weak solution of* (4.2) *if $\frac{m\otimes m}{\rho}$, $S\in L^1_{loc}\left(((0,\infty)\times\mathbb{T}^n)\right)^{n\times n}$, and $(\rho,m)$ satisfy*

$$-\iint\Big[\rho\psi_t + m\cdot\nabla_x\psi\Big]dxdt = \int\rho(x,0)\psi(x,0)dx\,, \tag{4.9}$$

$$\begin{aligned} &-\iint\left[m\cdot\varphi_t + \frac{1}{\varepsilon}\left(\frac{m\otimes m}{\rho} : \nabla_x\varphi - S:\nabla_x\varphi\right)\right]dxdt \\ &\qquad = -\frac{1}{\varepsilon^2}\iint m\cdot\varphi dxdt + \int m(x,0)\cdot\varphi(x,0)dx\,, \end{aligned} \tag{4.10}$$

*for all $\psi\in C^1_c\left([0,\infty);C^1_p(\mathbb{T}^n)\right)$ and $\varphi\in C^1_c\left([0,\infty);\left(C^1_p(\mathbb{T}^n)\right)^n\right)$.*

*(ii) If, in addition, $\eta(\rho,m)+\frac12 C_\kappa|\nabla_x\rho|^2\in C([0,\infty);L^1(\mathbb{T}^n))$ and $(\rho,m)$ satisfies*

$$\begin{aligned} -\iint\left(\eta(\rho,m)+\tfrac12 C_\kappa|\nabla_x\rho|^2\right)\dot\theta(t) \le &\int\left(\eta(\rho,m)+\tfrac12 C_\kappa|\nabla_x\rho|^2\right)\Big|_{t=0}\theta(0)dx \\ &-\frac{1}{\varepsilon^2}\iint\frac{|m|^2}{\rho}\theta(\tau)\,dxd\tau \end{aligned} \tag{4.11}$$

*for any non-negative $\theta \in W^{1,\infty}[0,\infty)$ compactly supported on $[0,\infty)$, then $(\rho, m)$ is called a dissipative weak solution.*

*(iii) By contrast, if $\eta(\rho,m) + \frac{1}{2}C_\kappa|\nabla_x\rho|^2 \in C([0,\infty); L^1(\mathbb{T}^n))$ and it satisfies (4.11) as an equality, then $(\rho, m)$ is called a conservative weak solution.*

*(iv) We say that a dissipative (or conservative) weak periodic solution of (4.2) $(\rho, m)$ with $\rho \geq 0$ has finite total mass and energy if*

$$\sup_{t\in(0,T)} \int_{\mathbb{T}^n} \rho dx \leq K_1 < \infty\,,$$

$$\sup_{t\in(0,T)} \int_{\mathbb{T}^n} \left(\eta(\rho,m) + \tfrac{1}{2}C_\kappa|\nabla_x\rho|^2\right) dx \leq K_2 < \infty\,,$$

The issue of existence of solutions for the Euler-Korteweg system (4.2) is a subject of active current study and open in its full generality. We refer to [5] for local well-posedness in Sobolev spaces, and to the recent paper [4] for global well-posedness of strong solutions for small irrotational data in dimensions greater than three. Results on global existence for finite energy solutions are available for the system of quantum hydrodynamics ([2, 3], [12]) due to its special connection to the linear Schroedinger equation via the Madelung transformation. Finally, we refer to [10], [13] for various well or ill posedness results applying to general Euler-Korteweg systems.

Following [13, Section 3.2], we prove a relative energy estimate between a weak periodic solution of (4.2) and a strong periodic solution of (4.3). We point out that that the main difference between the present case and the analysis in [13, Section 3.2] is due to the presence of the friction term and the error term induced by embedding the Cahn-Hilliard equation within the Euler-Korteweg system, and is manifested in the relative kinetic energy identity analyzed below. It is worth noting that the regularity requirements needed to justify the calculation below and the definition of the stress $S$ in (4.1) are satisfied, since the uniform bound for the energy yields

$$\rho \in C([0,T]; L^\gamma(\mathbb{T}^n)) \text{ and } \nabla_x\rho \in C([0,T]; L^2(\mathbb{T}^n))$$

Note that $\big(\rho F_q\big)(t,\cdot) = \big(\rho\nabla_x\rho\big)(t,\cdot) \in L^1(\mathbb{T}^n)$ since $\rho(t,\cdot) \in L^2(\mathbb{T}^n)$, and the latter property comes from $\rho(t,\cdot) \in L^1(\mathbb{T}^n) \cap L^\gamma(\mathbb{T}^n)$ and, if $\gamma < 2$, the $L^2$ integrability of $\nabla_x\rho$ and the Gagliardo–Niremberg–Sobolev inequality. For further details, see [13] and in particular condition **(A)** of page 22 and Remark 3.3.

**Theorem 4.2.** *Let $(\rho, m)$ be a dissipative or conservative weak solution of (4.2) with finite total energy according to Definition 4.1, and let $\bar\rho$ be a smooth solution of (4.3). Then*

$$\begin{aligned}
&\int_{\mathbb{T}^n} \left(\eta(\rho,m|\,\bar\rho,\bar m) + \tfrac{1}{2}C_\kappa|\nabla_x(\rho-\bar\rho)|^2\right) dx\Big|_t \\
&\leq \int_{\mathbb{T}^n} \left(\eta(\rho,m|\,\bar\rho,\bar m) + \tfrac{1}{2}C_\kappa|\nabla_x(\rho-\bar\rho)|^2\right) dx\Big|_0 \\
&- \frac{1}{\varepsilon^2}\iint_{[0,t]\times\mathbb{T}^n} \rho\left|\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right|^2 dxd\tau - \iint_{[0,t]\times\mathbb{T}^n} e(\bar\rho,\bar m)\cdot\frac{\rho}{\bar\rho}\left(\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right) dxd\tau \\
&- \frac{1}{\varepsilon}\iint_{[0,t]\times\mathbb{T}^n} \mathrm{div}_x\left(\frac{\bar m}{\bar\rho}\right)\left[p(\rho\,|\bar\rho) + \tfrac{1}{2}C_\kappa|\nabla_x(\rho-\bar\rho)|^2\right] dxd\tau \\
&- \frac{C_\kappa}{\varepsilon}\iint_{[0,t]\times\mathbb{T}^n} \Bigg[\nabla_x\left(\frac{\bar m}{\bar\rho}\right) : \nabla_x(\rho-\bar\rho)\otimes\nabla_x(\rho-\bar\rho) \\
&+\nabla_x\,\mathrm{div}_x\left(\frac{\bar m}{\bar\rho}\right)\cdot(\rho-\bar\rho)\nabla_x(\rho-\bar\rho)\Bigg] dxd\tau \\
&- \frac{1}{\varepsilon}\iint_{[0,t]\times\mathbb{T}^n} \rho\nabla_x\left(\frac{\bar m}{\bar\rho}\right) : \left(\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right)\otimes\left(\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right) dxd\tau\,.
\end{aligned} \tag{4.12}$$

*Proof.* Let $(\rho, m)$ a weak dissipative (or conservative) solution of (4.2), let $(\bar\rho, \bar m)$ be a strong solution or (4.6), and as already noted embed the Cahn-Hilliard equation into the Euler-Korteweg system with an error term (4.6). In (4.11) we consider $\theta(\tau)$ given in (3.22) and let $\mu \downarrow 0$ to conclude

$$\int_{\mathbb{T}^n} \Big(\eta(\rho, m) + \tfrac{1}{2} C_\kappa |\nabla_x \rho|^2\Big) dx \Big|_{\tau=0}^{t} \le -\frac{1}{\varepsilon^2} \iint_{[0,t]\times\mathbb{T}^n} \frac{|m|^2}{\rho}\, dx d\tau \tag{4.13}$$

Moreover, time integration of (4.8) gives

$$\begin{aligned} &\int_{\mathbb{T}^n} \Big(\eta(\bar\rho, \bar m) + \tfrac{1}{2} C_\kappa \, |\nabla_x \bar\rho|^2\Big) dx \Big|_{\tau=0}^{t} \\ &= -\frac{1}{\varepsilon^2} \iint_{[0,t]\times\mathbb{T}^n} \frac{|\bar m|^2}{\bar\rho} dx d\tau + \iint_{[0,t]\times\mathbb{T}^n} \frac{\bar m}{\bar\rho} \cdot \bar e dx d\tau \,. \end{aligned} \tag{4.14}$$

We evaluate now the dynamics of the linear part of the relative energy, starting from the weak formulation for the equations satisfied by the differences $(\rho - \bar\rho, m - \bar m)$:

$$-\iint_{[0,+\infty)\times\mathbb{T}^n} \Big(\psi_t(\rho - \bar\rho) + \frac{1}{\varepsilon}\psi_{x_i}(m_i - \bar m_i)\Big) dx dt - \int_{\mathbb{T}^n} \psi(\rho - \bar\rho)\Big|_{t=0} dx = 0 \,, \tag{4.15}$$

$$\begin{aligned} &-\iint_{[0,+\infty)\times\mathbb{T}^n} \left(\varphi_t \cdot (m - \bar m) + \frac{1}{\varepsilon}\partial_{x_i}\varphi_j \left(\frac{m_i m_j}{\rho} - \frac{\bar m_i \bar m_j}{\bar\rho}\right) - \frac{1}{\varepsilon}\partial_{x_i}\varphi_j \Big(S_{ij} - \bar S_{ij}\Big)\right) dx dt \\ &-\int_{\mathbb{T}^n} \varphi \cdot (m - \bar m)\Big|_{t=0} dx = -\frac{1}{\varepsilon^2} \iint_{[0,+\infty)\times\mathbb{T}^n} (m - \bar m) \cdot \varphi dx dt - \iint_{[0,+\infty)\times\mathbb{T}^n} \bar e \cdot \varphi dx dt \,, \end{aligned} \tag{4.16}$$

where, as before, $\varphi$, $\psi$ are Lipschitz test functions compactly supported in $[0, +\infty)$ in time and periodic in space.

In the above relations, we select the test functions:

$$\psi = \theta(\tau) \left(h'(\bar\rho) - C_\kappa \triangle_x \bar\rho - \frac{1}{2}\frac{|\bar m|^2}{\bar\rho^2}\right), \quad \varphi = \theta(\tau) \frac{\bar m}{\bar\rho} \,,$$

with $\theta(\tau)$ as in (3.22). Then (4.15) with $\mu \downarrow 0$ gives the linear part of the potential energy:

$$\begin{aligned} &\int_{\mathbb{T}^n} \left(h'(\bar\rho)(\rho - \bar\rho) + C_\kappa \nabla_x \bar\rho \cdot \nabla_x(\rho - \bar\rho) - \frac{1}{2}\frac{|\bar m|^2}{\bar\rho^2}(\rho - \bar\rho)\right)\Bigg|_{\tau=0}^{t} dx \\ &- \iint_{[0,t]\times\mathbb{T}^n} \left[\partial_\tau \left(h'(\bar\rho) - \frac{1}{2}\frac{|\bar m|^2}{\bar\rho^2}\right)(\rho - \bar\rho) + C_\kappa \partial_\tau\big(\nabla_x \bar\rho\big) \cdot \nabla_x(\rho - \bar\rho)\right] dx d\tau \\ &- \frac{1}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} \nabla_x \left(h'(\bar\rho) - C_\kappa \triangle_x \bar\rho - \frac{1}{2}\frac{|\bar m|^2}{\bar\rho^2}\right) \cdot (m - \bar m) dx d\tau = 0 \,. \end{aligned}$$

Moreover, from (4.16) one has

$$\begin{aligned} &\int_{\mathbb{T}^n} \frac{\bar m}{\bar\rho} \cdot (m - \bar m)\Big|_{\tau=0}^{t} dx - \iint_{[0,t]\times\mathbb{T}^n} \partial_\tau \left(\frac{\bar m}{\bar\rho}\right) \cdot (m - \bar m) dx d\tau \\ &- \frac{1}{\varepsilon} \iint_{[0,t]\times\mathbb{T}^n} \partial_{x_i}\left(\frac{\bar m_j}{\bar\rho}\right) \left(\Big(\frac{m_i m_j}{\rho} - \frac{\bar m_i \bar m_j}{\bar\rho}\Big) - (S_{ij} - \bar S_{ij})\right) dx d\tau \\ &= -\frac{1}{\varepsilon^2} \iint_{[0,t]\times\mathbb{T}^n} \frac{\bar m}{\bar\rho} \cdot (m - \bar m) dx d\tau - \iint_{[0,t]\times\mathbb{T}^n} \frac{\bar m}{\bar\rho} \cdot \bar e dx d\tau \,. \end{aligned}$$

Combining the above equations we conclude

$$\begin{aligned} &\int_{\mathbb{T}^n} \Big[\eta(\rho, m\,|\bar\rho, \bar m\,) + \tfrac{1}{2} C_\kappa |\nabla_x(\rho - \bar\rho)|^2\Big]\Bigg|_{\tau=0}^{t} dx \\ &\le -\frac{1}{\varepsilon^2} \iint_{[0,t]\times\mathbb{T}^n} \Big[\rho|u|^2 - \bar\rho|\bar u|^2 - \bar u \cdot (\rho u - \bar\rho \bar u)\Big] dx d\tau \\ &- \iint_{[0,t]\times\mathbb{T}^n} \Big[\partial_\tau\big(h'(\bar\rho) - \tfrac{1}{2}|\bar u|^2\big)(\rho - \bar\rho) + C_\kappa \partial_\tau\big(\nabla_x \bar\rho\big) \cdot \nabla_x(\rho - \bar\rho) \end{aligned}$$

$$
\begin{aligned}
&\quad + \partial_\tau(\bar u)\cdot(\rho u-\bar\rho\bar u)\Big]dxd\tau \\
&- \frac{1}{\varepsilon}\iint_{[0,t]\times\mathbb{T}^n} \nabla_x\big(h'(\bar\rho)-C_\kappa\triangle_x\bar\rho-\tfrac12|\bar u|^2\big)\cdot(\rho u-\bar\rho\bar u)dxd\tau \\
&- \frac{1}{\varepsilon}\iint_{[0,t]\times\mathbb{T}^n} \partial_{x_i}(\bar u_j)\Big((\rho u_i u_j-\bar\rho\bar u_i\bar u_j)-(S_{ij}-\bar S_{ij})\Big)dxd\tau\,,
\end{aligned}
\tag{4.17}
$$

where $m=\rho u$ and $\bar m=-\varepsilon\bar\rho\nabla_x\big(h'(\bar\rho)-C_\kappa\triangle_x\bar\rho\big)=\bar\rho\bar u$. For this model, $\bar u$ verifies

$$
\partial_t\bar u+\frac{1}{\varepsilon}(\bar u\cdot\nabla_x)\bar u=-\frac{1}{\varepsilon^2}\bar u-\frac{1}{\varepsilon}\nabla_x\big(h'(\bar\rho)-C_\kappa\triangle_x\bar\rho\big)+\frac{\bar e}{\bar\rho}\,,
$$

so that

$$
\begin{aligned}
&\partial_\tau\big(-\tfrac12|\bar u|^2\big)(\rho-\bar\rho)+\partial_\tau(\bar u)\cdot(\rho u-\bar\rho\bar u)+\frac{1}{\varepsilon}\nabla_x\big(-\tfrac12|\bar u|^2\big)\cdot(\rho u-\bar\rho\bar u) \\
&+\frac{1}{\varepsilon}\partial_{x_i}(\bar u_j)(\rho u_iu_j-\bar\rho\bar u_i\bar u_j)=-\frac{1}{\varepsilon^2}\rho\bar u\cdot(u-\bar u)-\frac{1}{\varepsilon}\rho\nabla_x\big(h'(\bar\rho)-C_\kappa\triangle_x\bar\rho\big)\cdot(u-\bar u) \\
&+\bar e\frac{\rho}{\bar\rho}\cdot(u-\bar u)-\frac{1}{\varepsilon}\rho\nabla_x(\bar u):(u-\bar u)\otimes(u-\bar u)\,.
\end{aligned}
$$

Then, as in the previous section, using also the reformulation of the stress given by (4.1),

$$
\operatorname{div}_x S=-\rho\nabla_x\big(h'(\rho)-C_\kappa\triangle_x\rho\big)\,,
$$

we rewrite (4.17) as follows:

$$
\begin{aligned}
&\int_{\mathbb{T}^n}\Big[\eta(\rho,m\,|\bar\rho,\bar m)+\tfrac12C_\kappa|\nabla_x(\rho-\bar\rho)|^2\Big]\Big|_{\tau=0}^{t}dx \\
&\le-\frac{1}{\varepsilon^2}\iint_{[0,t]\times\mathbb{T}^n}\rho|u-\bar u|^2dxd\tau-\iint_{[0,t]\times\mathbb{T}^n}\bar e\frac{\rho}{\bar\rho}\cdot(u-\bar u)dxd\tau \\
&-\frac{1}{\varepsilon}\iint_{[0,t]\times\mathbb{T}^n}\rho\nabla_x(\bar u):(u-\bar u)\otimes(u-\bar u)dxd\tau-\frac{1}{\varepsilon}\iint_{[0,t]\times\mathbb{T}^n}\operatorname{div}_x(\bar u)p(\rho\,|\bar\rho)dxd\tau \\
&-\frac{1}{\varepsilon}C_\kappa\iint_{[0,t]\times\mathbb{T}^n}\Big[\tfrac12\operatorname{div}_x(\bar u)\Big(|\nabla_x\rho|^2-|\nabla_x\bar\rho|^2\Big)+\nabla_x\operatorname{div}_x(\bar u)\cdot\Big(\rho\nabla_x\rho-\bar\rho\nabla_x\bar\rho\Big)\Big]dxd\tau \\
&-\frac{1}{\varepsilon}C_\kappa\iint_{[0,t]\times\mathbb{T}^n}\nabla_x(\bar u):\Big[\nabla_x\rho\otimes\nabla_x\rho-\nabla_x\bar\rho\otimes\nabla_x\bar\rho\Big]dxd\tau \\
&+\frac{1}{\varepsilon}C_\kappa\iint_{[0,t]\times\mathbb{T}^n}\Big[\nabla_x\operatorname{div}_x(\bar\rho\bar u)\cdot\nabla_x(\rho-\bar\rho)+\bar u\cdot(\rho-\bar\rho)\nabla_x\triangle_x\bar\rho\Big]dxd\tau\,.
\end{aligned}
\tag{4.18}
$$

Then we compute

$$
\begin{aligned}
\iint_{[0,t]\times\mathbb{T}^n}\nabla_x\operatorname{div}_x(\bar\rho\bar u)\cdot\nabla_x(\rho-\bar\rho)dxd\tau&=\iint_{[0,t]\times\mathbb{T}^n}\Big[\nabla_x\operatorname{div}_x(\bar u)\cdot\bar\rho\nabla_x(\rho-\bar\rho) \\
&\quad+\operatorname{div}_x(\bar u)\nabla_x\bar\rho\cdot\nabla_x(\rho-\bar\rho)+\nabla_x\big(\bar u\cdot\nabla_x\bar\rho\big)\cdot\nabla_x(\rho-\bar\rho)\Big]dxd\tau
\end{aligned}
$$

and integration by parts gives

$$
\begin{aligned}
\iint_{[0,t]\times\mathbb{T}^n}\bar u\cdot(\rho-\bar\rho)\nabla_x\triangle_x\bar\rho dxd\tau&=\iint_{[0,t]\times\mathbb{T}^n}\nabla_x\operatorname{div}_x\big((\rho-\bar\rho)\bar u\big)\cdot\nabla_x\bar\rho dxd\tau \\
&=\iint_{[0,t]\times\mathbb{T}^n}\Big[\nabla_x\operatorname{div}_x(\bar u)\cdot(\rho-\bar\rho)\nabla_x\bar\rho+\operatorname{div}_x(\bar u)\nabla_x\bar\rho\cdot\nabla_x(\rho-\bar\rho) \\
&\quad+\nabla_x\big(\bar u\cdot\nabla_x(\rho-\bar\rho)\big)\cdot\nabla_x\bar\rho\Big]dxd\tau\,.
\end{aligned}
$$

Moreover,

$$
\iint_{[0,t]\times\mathbb{T}^n}\operatorname{div}_x(\bar u)\nabla_x\bar\rho\cdot\nabla_x(\rho-\bar\rho)+\nabla_x\big(\bar u\cdot\nabla_x\bar\rho\big)\cdot\nabla_x(\rho-\bar\rho)+\nabla_x\big(\bar u\cdot\nabla_x(\rho-\bar\rho)\big)\cdot\nabla_x\bar\rho\Big]dxd\tau
$$

$$= \iint_{[0,t]\times\mathbb{T}^n} \partial_{x_i}\bar{u}_j \Big[\partial_{x_j}\bar{\rho}\partial_{x_i}(\rho-\bar{\rho}) + \partial_{x_j}(\rho-\bar{\rho})\partial_{x_i}\bar{\rho}\Big] dxd\tau$$
$$+ \iint_{[0,t]\times\mathbb{T}^n} \bar{u}_j \Big[\partial_{x_i}\partial_{x_j}\bar{\rho}\partial_{x_i}(\rho-\bar{\rho}) + \partial_{x_i}\partial_{x_j}(\rho-\bar{\rho})\partial_{x_i}\bar{\rho}\Big] dxd\tau + \iint_{[0,t]\times\mathbb{T}^n} \partial_{x_j}\bar{u}_j \Big[\partial_{x_i}\bar{\rho}\partial_{x_i}(\rho-\bar{\rho})\Big] dxd\tau$$
$$= \iint_{[0,t]\times\mathbb{T}^n} \partial_{x_i}\bar{u}_j \Big[\partial_{x_j}\bar{\rho}\partial_{x_i}\rho + \partial_{x_j}\rho\partial_{x_i}\bar{\rho} - 2\partial_{x_j}\bar{\rho}\partial_{x_i}\bar{\rho}\Big] dxd\tau\,.$$

Finally, using the above relations in (4.18) we end up with (4.12) and the proof is complete. □

4.2. **Convergence to Cahn-Hilliard in the large friction limit.** Having established (4.12), we proceed as in the previous section to obtain a stability estimate and complete the convergence in the diffusive relaxation limit. Let us introduce the following quantity, based on the relative energy,

$$\Psi(t) = \int_{\mathbb{T}^n} \Big[\eta(\rho, m\,|\bar{\rho},\bar{m}) + \tfrac{1}{2}C_\kappa|\nabla_x(\rho-\bar{\rho})|^2\Big] dx \tag{4.19}$$

to control the distance between the solutions of (4.2) and (4.6). As it is manifest from Lemma 3.4 and Remark 3.5, if $\gamma \geq 2$ the relative potential energy, and thus $\Psi$, in particular controls the $L^2$ norm of the distance $\rho - \bar{\rho}$; see (3.33). Hence, all the terms on the right and side of (4.12) are estimated in terms of $\Psi(t)$ and we can follow the lines of the proof of Theorem 3.9 to prove:

**Theorem 4.3.** *Let $T > 0$ be fixed, let $p(\rho)$ satisfy* (H), (A$_2$) *and* (A$_1$) *with $\gamma \geq 2$, let $(\rho, m)$ be a dissipative (or conservative) weak solution of* (4.2) *with finite total energy according to Definition 4.1, and let $\bar{\rho}$ be a smooth solution of* (4.3) *such that $\bar{\rho} \geq \delta > 0$. Then the stability estimate*

$$\Psi(t) \leq C\big(\Psi(0) + \varepsilon^4\big), \quad t \in [0, T]\,,$$

*holds true, with $C$ a positive constant depending only on $T$, $K_1$, $\bar{\rho}$ and its derivatives. Moreover, if $\Psi(0) \to 0$ as $\varepsilon \downarrow 0$, then*

$$\sup_{t\in[0,T]} \Psi(t) \to 0, \ \ as\ \varepsilon \downarrow 0\,.$$

*Remark* 4.4. The convergence in the general case $\gamma > 1$ can also be established, provided a uniform $L^\infty$ bound for both $\rho$ and $\bar{\rho}$ is available. In the particular case $\gamma = 2$, we have $h(\rho\,|\bar{\rho}) = |\rho - \bar{\rho}|^2$ and Lemma 3.4 is not needed anymore; in that case we may treat the presence of vacuum for both $\rho$ and $\bar{\rho}$. However, the regularity assumptions for the latter are still necessary, and this might be inconsistent with the presence of vacuum for $\bar{\rho}$; see also [13].

**Acknowledgements.** We thank the anonymous referee for suggesting to consider the effects of confinement potentials, what led to Section 2.3. AET was supported by funding from King Abdullah University of Science and Technology (KAUST).

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