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Revivals in engineered waveguide arrays

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Abstract

We show that perfect and fractional revivals are possible in finite waveguide arrays with engineered parameters. We consider potential applications of waveguide lattices utilizing operations such as multi-couplers and multi-beam-splitters. Specific examples of finite lattices with four and five elements that allow such operations are presented.

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1. Introduction

Optical couplers are widely used in optical communications for switching applications [1,2]. They can be implemented in different settings such as optical fibers, waveguides, photonic crystals, and photonic crystal fibers [3]. To describe propagation of optical waves in coupled waveguides, coupled-mode theory can be employed [4–6]. The equations obtained using coupled mode theory, describe the evolution of the amplitude of each waveguide.

The simplest system associated with coupling that can be considered is a two port directional coupler in which each waveguide or fiber in isolation is single mode. In such a device a continuous wave excitation will periodically revive along z to the input state (complete revival) independent of the initial waveform. Furthermore, when only a single port or waveguide (say A) is initially excited and at a characteristic propagation length, which is called the coupling length, all the power is coupled to port B. At half the coupling length the power is equally distributed between the two ports and, due to that, a coupler can be used as a beam splitter. These phenomena are associated with the existence of complete and fractional revivals which were originally used to describe Rydberg electron wavepackets excited by short laser pulses [7–10]. Complete revivals are field or intensity restorations of the original excitation, whereas fractional revivals represent coherent superpositions of the original wavefunction. To

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better understand what a fractional revival is, we assume that the intensity of a wave in a system is fully restored after some specific time, i.e., the complete revival time. This system possesses fractional revivals if at some fraction of that time the wave consists of superpositions of the original wave. Each of these superimposed waves is a scaled translation or reflection of the original wave. Thus, the effects of beam switching from port A to port B and beam splitting can be considered as typical cases of fractional revivals.

Going beyond the case of two port couplers to a waveguide configuration with three elements ($N = 3$), complete and fractional revivals are still possible: the power now couples from port A to port C and at half the coupling length the power is equally distributed between these two ports. However, this is true only when port B is not excited at the input. By further increasing the number of waveguides in the array, i.e., in the case of four or more waveguides, the behavior of the system changes drastically. The input pattern now propagates in an aperiodic way and does not revive at a characteristic distance. Thus, the question that naturally arises is: What is the effect that allows such a drastic change in the behavior of a finite waveguide array from the case of 3 waveguides to the case $N = 4$? Furthermore, is it possible to engineer the parameters of the lattice in such a way that beam coupling and splitting (or fractional revivals) are possible for waveguide arrays with $N > 3$. Considering applications, this type of lattice could be used to implement multi-couplers and multi-beam-splitters. In a multi-coupler the input field, originally at port j is directed to port $N - j + 1$ at the output. Similarly, a multi-beam-splitter equally distributes the input power from port j to ports j and $N - j + 1$. Such devices can be used in two-dimensional topologies, such as 2D photonic crystals and in linear finite waveguide arrays where it is desirable for the input signal to cross a specified number of waveguides in a single element without overlap.

In this paper we show that under appropriate conditions perfect and fractional revivals are possible in finite waveguide arrays. This can happen by appropriately engineering the coupling length among successive waveguides. We consider potential applications of fractional revivals in waveguide lattices utilizing operations such as multi-couplers and multi-beam-splitters. Our analysis is based on the classification of the eigenvalues using coupled mode theory. In Sections 2 and 3 of the paper we present the general theory of revivals in waveguide arrays with N elements and in Section 4 we present specific examples for the cases of four and five waveguides (the cases $N = 2,3$ are also presented for comparison). These results can be extended to waveguide lattices with more than five waveguide elements however, due to the complexity of the problem, numerical computations might be necessary.

2. Coupled mode theory

Using coupled mode theory [4–6], the normalized evolution of the amplitude of the optical field of the jth waveguide in a finite array of N waveguides satisfies

$$i\dot{\psi}_j + \alpha_j\psi_j + \kappa_{j-1}\psi_{j-1} + \kappa_j\psi_{j+1} = 0, \tag{1}$$

where $j = 1,\ldots,N$, $\dot{\psi}_j = \mathrm{d}\psi_j/\mathrm{d}z$, z is the propagation distance, κ_j is the coupling coefficient of the field between waveguides j and $j + 1$, and α_j is a local eigenvalue detuning. The boundary conditions are obtained by taking into account that energy can not leak outside the lattice, i.e., $\kappa_0 = \kappa_N = 0$. As it is expected for waveguide and fiber couplers the coupling coefficients are positive, $\kappa_j > 0$. Notice that if some of the values of α_j are negative, then, by applying the trivial transformation $\psi_j = \hat{\psi}_j \exp(-i\lambda_0 z)$ along with $\alpha_j = \hat{\alpha}_j - \lambda_0$, where $\lambda_0 \geqslant -\min(\alpha_j)$, all $\hat{\alpha}_j$ become positive (or zero). Thus, without loss of generality, $\alpha_j \geqslant 0$.

By proper design, it is possible to have lattice coefficients κ_j, α_j that vary with the index j. For example, κ_j and α_j can be tuned by changing only the waveguide geometry (that in turn modifies the effective refractive index and the coupling of the waveguides).

Eq. (1) has a Hamiltonian structure

$$i\dot{\psi}_j + \frac{\partial \mathscr{H}}{\partial \psi_j^*} = 0, \tag{2}$$

where

$$\mathscr{H} = \sum_j \Big\{\alpha_j|\psi_j|^2 + \kappa_{j-1}[\psi_{j-1}\psi_j^* + \psi_{j-1}^*\psi_j] + \kappa_j[\psi_{j+1}\psi_j^* + \psi_{j+1}^*\psi_j]\Big\} \quad (3)$$

and, in addition, the total power is conserved

$$\mathscr{P} = \sum_j |\psi_j|^2. \quad (4)$$

One can rewrite Eq. (1) as

$$\mathrm{i}\dot{\psi}_n + L_n\psi_n = 0, \quad (5)$$

where the operator L_j is defined

$$L_j\psi_j = \alpha_j\psi_j + \kappa_{j-1}\psi_{j-1} + \kappa_j\psi_{j+1}. \quad (6)$$

Alternatively, a matrix form can be used for Eq. (5)

$$\mathrm{i}\dot{\mathbf{U}} + A\mathbf{U} = 0, \quad (7)$$

where

$$A = \begin{pmatrix} \alpha_1 & \kappa_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \kappa_1 & \alpha_2 & \kappa_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & \kappa_2 & \alpha_3 & \kappa_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \kappa_{N-2} & \alpha_{N-1} & \kappa_{N-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & \kappa_{N-1} & \alpha_N \end{pmatrix}, \quad (8)$$

$\mathbf{U} = (\psi_1,\psi_2,\dots,\psi_N)^{\mathrm{T}}$, and T denotes the transpose. Notice that the matrix (8) is tridiagonal and symmetric ($A^{\mathrm{T}} = A$). We are interested to find the eigenmodes of Eq. (1). To do so, we assume that

$$\psi_j^{(l)} = u_j^{(l)} \exp(\mathrm{i}\lambda_l z) \quad (9)$$

or in vector form

$$\mathbf{U}_l = \mathbf{x}_l \exp(\mathrm{i}\lambda_l z). \quad (10)$$

The resulting eigenvalue problem is

$$A\mathbf{x}_l = \lambda_l\mathbf{x}_l \quad (11)$$

or

$$L_j u_j^{(l)} = \lambda_l u_j^{(l)}. \quad (12)$$

Due to the tridiagonal nature of matrix (8), its characteristic polynomial is obtained by the following recursion formula:

$$F_0 = 1, \quad (13)$$

$$F_1 = \alpha_1 - \lambda, \quad (14)$$

$$F_n = (\alpha_{n-1} - \lambda)F_{n-1} - \kappa_{n-1}^2 F_{n-2}. \quad (15)$$

Eq. (1) has some interesting properties that we are going to use in the following sections. For the coherence presentation of our results, we summarize these properties in Appendix A.

3. Revivals

An input field can always be decomposed into the orthogonal and complete set of eigenvectors $\mathbf{x}_j$ of matrix (8)

$$\mathbf{U}_0 = \mathbf{U}(z=0) = \sum_j c_j\mathbf{x}_j, \quad (16)$$

which represent the array eigenmodes. The weights c_j can be found by projection to the orthogonal basis of eigenvectors

$$c_j = \langle \mathbf{x}_j | \mathbf{U}_0 \rangle, \quad (17)$$

where we denote by

$$\langle \mathbf{x}|\mathbf{y}\rangle = \mathbf{x}^{\dagger}\mathbf{y} = \sum_j (x_j)^* y_j. \quad (18)$$

The evolution of an arbitrary waveguide excitation in the lattice is then given by

$$\mathbf{U}(z) = \sum_j c_j\mathbf{x}_j \exp(\mathrm{i}\lambda_j z). \quad (19)$$

According to the relation between the eigenvalues λ_j, the initial pattern can periodically reappear at multiples of the revival length $z = m\,L_{\mathbf{REV}}$. Except from these perfect revivals, fractional revivals, representing a coherent superposition of the original wavefunction are known to occur in systems with purely quadratic spectrum [7]. As we will show, such revivals are also possible in finite arrays by appropriately choosing the parameters of the lattice. A major difference between the continuous models that support fractional revivals and the waveguide lattices considered here is that, in the latter case, the number of eigenmodes is finite. Also the spectrum of the array that is required for fractional revivals is not parabolic as it is in the continuous case. We will consider both types of revivals: perfect revivals of the initial state and fractional revivals.

3.1. Perfect revivals

For the sake of generality, in this section we do not make any assumptions about the lattice parameters (α_j,κ_j). We distinguish two different cases of perfect revivals, i.e., perfect field revivals where the input field at $z = L_{\mathbf{REV}}$ is identical to the original as well as intensity revivals, where an additional phase difference can accumulate between each revival.

The condition for field revivals of the original pattern is

$$\mathbf{U}(z=0) = \mathbf{U}(L_{\mathrm{F}}),$$

that results to

$$\lambda_j L_{\mathrm{F}} = 2m_j\pi, \quad j = 1,\ldots,N, \tag{20}$$

where L_{F} denotes the field revival length or time. The revival length is then given by the least common multiplier of all the $2\pi/\lambda_j$, i.e.

$$L_{\mathrm{F}} = \mathrm{LCM}\left(\left\{\frac{2\pi}{\lambda_j}\right\}\right). \tag{21}$$

We can now define an effective λ_{lcm} such that

$$L_{\mathrm{F}} = 2\pi/\lambda_{\mathrm{lcm}}. \tag{22}$$

The eigenvalues should then satisfy

$$\lambda_j = m_j\lambda_{\mathrm{lcm}}. \tag{23}$$

Considering now intensity revivals, we notice that the lth element of the field is given by

$$U_l = \sum_j c_j u_l^{(j)} \exp(\mathrm{i}\lambda_j z), \tag{24}$$

from which the corresponding intensity is obtained

$$\begin{aligned} I_l &= U_l^* U_l \\ &= \sum_{j>k} 4c_j c_k u_l^j u_l^k \cos[(\lambda_k - \lambda_j)z] + \sum_j c_j^2 (u_l^j)^2. \end{aligned} \tag{25}$$

One can see that the condition for intensity revival $I(L_{\mathrm{I}}) = I(0)$ (where L_{I} is the intensity revival length) leads to

$$L_{\mathrm{I}}(\lambda_i - \lambda_j) = L_{\mathrm{I}}\Delta\lambda_{i,j} = 2m\pi.$$

The intensity revival length is then given by

$$L_{\mathrm{I}} = \mathrm{LCM}\left(\left\{\frac{2\pi}{\Delta\lambda_{i,j}}\right\}\right). \tag{26}$$

We can now define an effective $\Delta\lambda_{\mathrm{lcm}}$ such that

$$L_{\mathrm{I}} = 2\pi/\Delta\lambda_{\mathrm{lcm}}. \tag{27}$$

The eigenvalues should then satisfy

$$\Delta\lambda_{i,j} = m_{i,j}\Delta\lambda_{\mathrm{lcm}}. \tag{28}$$

3.2. Fractional revivals

Considering fractional revivals, we first assume that the local detuning α_j of all the waveguides is zero ($\alpha_i = 0, i = 1,\ldots,N$). In this specific case (see the detailed discussion in Appendix A), the evolution of optical field is given by

$$\mathbf{U}(z) = \sum_{\lambda_j>0} [c_j \mathbf{x}_j \exp(\mathrm{i}\lambda_j z) + \tilde{c}_j \tilde{\mathbf{x}}_j \exp(-\mathrm{i}\lambda_j z)] + c_0 \mathbf{x}_0. \tag{29}$$

Notice that if λ is an eigenvalue then $-\lambda$ is also an eigenvalue of the problem. Our second assumption is that the array is symmetric ($\kappa_j = \kappa_{N-j}$). Since the array is symmetric, all the eigenmodes of this system are either even or odd. The eigenvectors $\mathbf{x}_j$ and $\tilde{\mathbf{x}}_j$ with eigenvalues λ and $-\lambda$, respectively, are related via $\tilde{u}_k^{(j)} = (-1)^k u_k^{(j)}$ (see Appendix A). The last term on the right hand side of Eq. (29) exists only when the dimension of the lattice is odd $N = 2M + 1$ and corresponds to zero eigenvalue. The existence of fractional revivals is related to the existence of pairs of eigenvectors with opposite sign eigenvalues (λ,$-\lambda$). A zero eigenvalue can, in general, break the symmetry of the spectrum that permits fractional revivals. To simplify the results of this section we will assume that the spectrum is such that there is a minimum eigenvalue $\lambda_{\min} = \lambda_j$ and the rest of the eigenvalues are multiples of $\lambda_{\min}$, i.e.

$$\lambda_i = m_i\lambda_{\min}. \tag{30}$$

We will separately consider two different cases: (a) all the eigenvalues are non-zero (even lattice size or $N = 2M$); (b) one zero eigenvalue exists (odd lattice size or $N = 2M + 1$). When $N = 2M$ the input field evolves according to

$$\mathbf{U}(z) = \sum_{j\,\mathrm{even}} [c_j \mathbf{x}_j \exp(\mathrm{i}\lambda_j z) + \tilde{c}_j \tilde{\mathbf{x}}_j \exp(-\mathrm{i}\lambda_j z)]. \quad (31)$$

Fractional revivals require even or odd parity eigenvectors. Thus, using Property 7 in Appendix A, the associated lattice should be symmetric. The summation in Eq. (31) is over the eigenvalues λ_j that correspond to even parity eigenvectors (see Definition 5 in Appendix A). Because the array has an even number of lattice sites, $N = 2M$, $\mathbf{x}$ and $\tilde{\mathbf{x}}$ have opposite parity. Since we assumed that the eigenvectors $\mathbf{x}_j$ are even all $\tilde{\mathbf{x}}_j$ are odd. The complete revival lengths can be found by using the expressions obtained in the previous section $L_F = 2\pi/\lambda_{\mathrm{lcm}}$ and $L_I = 2\pi/\Delta\lambda_{\mathrm{lcm}}$, where in the evaluation of these expressions it should be taken into account that for each λ_j, $-\lambda_j$ is also an eigenvalue. Here, we will consider the two simpler types of fractional revivals, namely, reflection images and mirror states.

A vector U_j and its reflection image V_j, are related by $V_j = U_{N-j+1}$. A waveguide lattice in which the input field U_j can evolve to its mirror image U_{N-j+1} can realize a multi-coupler: the power can switch from waveguide j to waveguide $N - j + 1$.

We express $\mathbf{U}$ as a superposition of the eigenvectors of the problem

$$\mathbf{U} = \sum [c_j \mathbf{x}_j + \tilde{c}_j \tilde{\mathbf{x}}_j]. \quad (32)$$

Then, since $\mathbf{x}$ and $\tilde{\mathbf{x}}$ have opposite parity

$$\mathbf{V} = \sum [c_j \mathbf{x}_j - \tilde{c}_j \tilde{\mathbf{x}}_j] \quad (33)$$

is the reflection image of $\mathbf{U}$. This can be shown, by noting that the parity of

$$\mathbf{U} + \mathbf{V} = 2\sum c_j \mathbf{x}_j \quad (34)$$

is even, whereas

$$\mathbf{U} - \mathbf{V} = 2\sum \tilde{c}_j \tilde{\mathbf{x}}_j \quad (35)$$

is odd. Thus, $U_{N-j+1} + V_{N-j+1} = U_j + V_j$ and $U_{N-j+1} - V_{N-j+1} = -(U_j - V_j)$. Adding these two quantities we get $U_{N-j+1} = V_j$ (and $U_j = V_{N-j+1}$). A generalized condition for reflection images is given by

$$U_j = V_{N-j+1} \exp(\mathrm{i}\phi), \quad (36)$$

where ϕ is an arbitrary phase shift.

We rewrite Eq. (31) in the form

$$\mathbf{U}(z) = \exp(\mathrm{i}\lambda_m z) \sum_{n\,\mathrm{even}} [c_n \mathbf{x}_n \mathrm{e}^{\mathrm{i}(\lambda_n - \lambda_m)z} + \tilde{c}_n \tilde{\mathbf{x}}_j \mathrm{e}^{-\mathrm{i}(\lambda_n + \lambda_m)z}]. \quad (37)$$

From Eq. (37), the conditions for the occurrence of reflections are given by

$$L(\lambda_n - \lambda_m) = 2j_1\pi, \quad L(\lambda_n + \lambda_m) = \pi + 2j_2\pi. \quad (38)$$

Solving these equations, we obtain the following relations between the eigenvalues

$$\lambda_j = (1 + 4m_j)\lambda_{\min}, \quad (39)$$

where $\lambda_{\min}$ is the minimum eigenvalue, and the revival length L_R [such that $I_j(L_R/2) = I_{N-j+1}(L_R/2 + mL_R)$] is

$$L_R = \pi/\lambda_{\min} = L_I. \quad (40)$$

We would like to notice that condition (39) is necessary for the existence of fractional revivals but not sufficient. One should also take into account that all λ_j should correspond to eigenvectors with even (or odd) parity. Arranging the eigenvalues that correspond to even (or odd) eigenvectors as

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_m|, \quad (41)$$

a second relation between the eigenvalues is derived

$$\lambda_j \lambda_{j-1} < 0. \quad (42)$$

Conditions (39), (41) and (42) are now sufficient. As an example, we consider the case $N = 4$. We assume that the eigenvalue with the smallest absolute value is $-\lambda_{\min}$. The second eigenvalue should then be given by $-\lambda_{\min} + 4m\lambda_{\min}$, $m \geqslant 1$.

Another type of fractional revivals consists of two mirror images, i.e., an equal amplitude superposition of the original waveform (32) and its reflection (33). In a waveguide array this corresponds to a multi-beam-splitter. At a specific length the input power from port j is equally divided between ports j and $N - j + 1$. Interestingly, mirror fractional revivals are required for this type of operation.

A mirror image can be obtained from Eq. (31) by assuming that $\lambda_j z = \pi/4$ and so

$$\mathbf{U} = \frac{1}{2}\sum[c_j\mathbf{x}_j + \tilde{c}_j\tilde{\mathbf{x}}_j] + \frac{\mathrm{i}}{2}\sum[c_j\mathbf{x}_j - \tilde{c}_j\tilde{\mathbf{x}}_j]. \quad (43)$$

Using Eq. (37) the relations between the eigenvalues become

$$L(\lambda_n - \lambda_m) = 2j_1\pi, \quad L(\lambda_n + \lambda_m) = \pi/2 + 2j_2\pi, \quad (44)$$

Solving Eq. (44), one can find that the eigenvalues should satisfy

$$\lambda_j = (1 + 8m_j)\lambda_{\min} \quad (45)$$

and the revival length L_M, such that $I(L_M/2) = I(L_M/2 + mL_M)$, is given by

$$L_M = \pi/2\lambda_{\min} = L_I/2. \quad (46)$$

Again Eqs. (41), (42) should be satisfied, to guarantee that λ_j are eigenvalues with even (or odd) parity eigenvectors.

If the dimension of the lattice is odd $N = 2M + 1$ then, in general, fractional revivals are not possible. This is due to the existence of a zero eigenvalue in the lattice spectrum. However, reflection fractional revivals can still occur. In contrast to the even lattice case, here, since $N = 2M + 1$, both $\mathbf{x}$ and $\tilde{\mathbf{x}}$ have the same parity. Thus, we can write the evolution of an initial vector as

$$\mathbf{U} = c_0\mathbf{x}_0 + \sum_{j=1}^{M}\left[c_j\mathbf{x}_j e^{\mathrm{i}\lambda_j z} + \tilde{c}_j\tilde{\mathbf{x}}_j e^{-\mathrm{i}\lambda_j z}\right], \quad (47)$$

where the eigenvalues have the arrangement

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_M. \quad (48)$$

Notice that if the eigenvector with zero eigenvalue is even (odd) then all the eigenvectors with $\lambda = 0, \lambda_2, \lambda_4, \ldots$ are even (odd) and the eigenvectors with $\lambda = \lambda_1, \lambda_3, \ldots$ are odd (even). Following similar arguments as in the even case ($N = 2M$), a reflection can happen if at some distance the waveguide eigenmode becomes

$$\mathbf{U} = c_0\mathbf{x}_0 + \sum_{j=2,4,\ldots}[c_j\mathbf{x}_j + \tilde{c}_j\tilde{\mathbf{x}}_j] - \sum_{j=1,3,\ldots}[c_j\mathbf{x}_j + \tilde{c}_j\tilde{\mathbf{x}}_j]. \quad (49)$$

Eq. (47) will evolve to Eq. (49) if

$$\lambda_j = \lambda_{j-1} + 2m_j\pi + \pi, \quad m \geqslant 0. \quad (50)$$

The fractional revival length in this case is

$$L_R = \pi/\lambda_{\min}. \quad (51)$$

4. Examples

We are now going to consider some specific examples of perfect and fractional revivals in waveguide lattices. We will start from the simplest cases $N = 2$ and $N = 3$ which have already been examined in the literature. We will then proceed to the cases of arrays with 4 and 5 lattice elements. In these latter cases, we will show that by appropriately engineering the lattice parameters fractional revivals are possible.

4.1. Array with two waveguides

This case, consisting of two linearly coupled optical fibers or waveguides, has been studied extensively. The coupled-mode equations describing the amplitude evolution are:

$$\mathrm{i}\dot{\psi}_1 + \kappa\psi_2 = 0, \quad (52)$$

$$\mathrm{i}\dot{\psi}_2 + \alpha\psi_2 + \kappa\psi_1 = 0. \quad (53)$$

These equations are associated with the matrix

$$A = \begin{pmatrix} \kappa & 0 \\ \alpha & \kappa \end{pmatrix} \quad (54)$$

with a characteristic polynomial

$$\lambda^2 - \lambda\alpha - \kappa^2 = 0. \quad (55)$$

The eigenvalues of Eq. (55) are given by

$$\lambda_{1,2} = \lambda_\pm = \frac{\alpha \pm \sqrt{\alpha^2 + 4\kappa^2}}{2}$$

and the corresponding eigenvectors are:

$$u_+ = \begin{pmatrix} 1 \\ \lambda_+/\kappa \end{pmatrix} = \begin{pmatrix} -\lambda_-/\kappa \\ 1 \end{pmatrix},$$
$$u_- = \begin{pmatrix} 1 \\ \lambda_-/\kappa \end{pmatrix} = \begin{pmatrix} -\lambda_+/\kappa \\ 1 \end{pmatrix} \quad (56)$$

and, thus, $\psi_+ = u_+\exp(\mathrm{i}\lambda_+ z)$ and $\psi_- = u_-\exp(\mathrm{i}\lambda_- z)$ are the eigenmodes of the original problem. The evolution of the field along z is given by:

$$\psi_1 = -(\lambda_-/\kappa)a\mathrm{e}^{\mathrm{i}\lambda_+ z} - (\lambda_+/\kappa)b\mathrm{e}^{\mathrm{i}\lambda_- z}, \tag{57}$$

$$\psi_2 = a\mathrm{e}^{\mathrm{i}\lambda_+ z} + b\mathrm{e}^{\mathrm{i}\lambda_- z}, \tag{58}$$

where the parameters a, b are determined from the initial excitation. The intensities of Eqs. (57) and (58) are given by:

$$|\psi_1|^2 = (1/\kappa^2)(\lambda_-^2 a^2 + \lambda_+^2 b^2) + (2\lambda_+\lambda_- ab/\kappa^2)\cos((\lambda_+ - \lambda_-)z), \tag{59}$$

$$|\psi_2|^2 = (a^2 + b^2) + 2ab\cos((\lambda_+ - \lambda_-)z). \tag{60}$$

In this example, independent from the choice of the parameters, perfect intensity revivals can happen. However, fractional revivals will occur only if the detuning α is zero. In this latter case the eigenvalues will satisfy the condition $\lambda_- = -\lambda_+$. The lack of fractional revivals when the detuning is non-zero can be explained by the fact that when $\alpha \neq 0$ the lattice is not symmetric. In the rest of the examples, we will consider only the zero detuning case $\alpha_j = 0$.

4.2. Array with three waveguides

In a lattice consisting of three waveguides the equations read:

$$\mathrm{i}\dot{\psi}_1 + \kappa_1\psi_2 = 0, \tag{61}$$

$$\mathrm{i}\dot{\psi}_2 + \kappa_1\psi_1 + \kappa_2\psi_3 = 0, \tag{62}$$

$$\mathrm{i}\dot{\psi}_3 + \kappa_2\psi_2 = 0 \tag{63}$$

and the corresponding array is given by

$$A = \begin{pmatrix} 0 & \kappa_1 & 0 \\ \kappa_1 & 0 & \kappa_2 \\ 0 & \kappa_2 & 0 \end{pmatrix}. \tag{64}$$

The characteristic polynomial of Eq. (64) is

$$\lambda^3 - (\kappa_1^2 + \kappa_2^2)\lambda = 0$$

with eigenvalues

$$\lambda_0 = 0, \quad \lambda_{1,2} = \lambda_\pm = \pm\sqrt{\kappa_1^2 + \kappa_2^2}.$$

Since $\lambda_- = -\lambda_+$ this system will always possess perfect revivals as well as fractional reflections. However, the existence of a zero eigenvalue does not permit any higher order fractional revivals to occur.

4.3. Array with four waveguides

The evolution of an array of four waveguides with coupling coefficients that vary along the lattice is described by:

$$\mathrm{i}\dot{\psi}_1 + \kappa_1\psi_2 = 0, \tag{65}$$

$$\mathrm{i}\dot{\psi}_2 + \kappa_1\psi_1 + \kappa_2\psi_3 = 0, \tag{66}$$

$$\mathrm{i}\dot{\psi}_3 + \kappa_2\psi_2 + \kappa_3\psi_4 = 0, \tag{67}$$

$$\mathrm{i}\dot{\psi}_4 + \kappa_3\psi_3 = 0. \tag{68}$$

The associated matrix is

$$A = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_1 & 0 & \kappa_2 & 0 \\ 0 & \kappa_2 & 0 & \kappa_3 \\ 0 & 0 & \kappa_3 & 0 \end{pmatrix}. \tag{69}$$

Matrix (69) possesses a characteristic polynomial

$$\lambda^4 - (\kappa_1^2 + \kappa_2^2 + \kappa_3^2)\lambda^2 + \kappa_1^2\kappa_3^2 = 0$$

with eigenvalues

$$\lambda_{\{1,2\},\pm} = \pm(1/2)\sqrt{\pm 2\sqrt{F_2^2 - 4G_{22}} + 2F_2},$$

where $F_2 = \kappa_1^2 + \kappa_2^2 + \kappa_3^2$ and $G_{2,2} = \kappa_1^2\kappa_3^2$. We arrange the eigenvalues as $\lambda_2 > \lambda_1 > \lambda_{1,-}$ $(=-\lambda_1) > \lambda_{2,-}$ $(=-\lambda_2)$. When $\lambda_2 = m\lambda_1$ perfect and fractional revival can occur. After some algebraic calculations this condition results

$$-(1 + m^2)^2\kappa_1^2\kappa_3^2 + (\kappa_1^2 + \kappa_2^2 + \kappa_3^2)^2 m^2 = 0. \tag{70}$$

Perfect revivals can happen for any non-zero m that is consistent with Eq. (70). However, for fractional revivals additional constraints should be applied to the allowed values of m. Using Eqs. (39) and (42) the required values of m for reflections (permitting switching between waveguides j and $N - j + 1$) are provided

$$m = -1 + 4n, \quad n = 1, 2, \ldots \tag{71}$$

or $m = 3, 7, 11, \ldots$ On the other hand mirror images (that allow beam splitting between waveguide j and $N - j + 1$) can happen if Eqs. (42) and (45) are satisfied, i.e.

$$m = -1 + 8n, \quad n = 1, 2, \ldots \tag{72}$$

or $m = 7,15,23,\ldots$ Notice that if Eq. (72) is satisfied, Eq. (71) is also satisfied. Thus, when mirror revivals happen, reflection revivals will also happen, whereas, the opposite is not always true. In addition a symmetric lattice is required for fractional revivals to occur, i.e., $\kappa_1 = \kappa_3 = \kappa$.

4.3.1. $\kappa_1 = \kappa_3 = \kappa$

In this case, the lattice is symmetric and thus both perfect and fractional revivals are possible. Substituting the condition $\kappa_1 = \kappa_3 = \kappa$ into Eq. (70) we find

$$\kappa_2^2 = \frac{1}{2m^2}\left[-4\kappa^2 m^2 \pm \sqrt{(2\kappa m)^4 + (2m(1-m^2)\kappa^2)^2}\right].$$

In Figs. 1–3 we can see some typical examples of revivals for this example. In Fig. 1 the ratio of the two eigenvalues is $m = 2$ and therefore only perfect revivals are allowed. In Fig. 2, since $m = 3$, reflection fractional revivals occur but mirror revivals are not possible. Finally, in Fig. 3 both reflection and mirror fractional revivals happen since $m = 7$. In this latter case, an initial intensity pattern, say, (1,0,0,0) will revive to its original form at $z = L_{\mathrm{I}}$. In addition, when $z = L_{\mathrm{I}}/2$ the input power switches from waveguide 1 to waveguide 4, i.e., (0,0,0,1). Finally, when $z = L_{\mathrm{I}}/4, 3L_{\mathrm{I}}/4$, the intensity is equally divided between waveguides 1 and 4, i.e., (1/4,0,0,1/4).

4.3.2. $\kappa_1 = \kappa_2 = \kappa$

Due to the asymmetric nature of the couplings only perfect revivals are possible. From Eq. (70) the relation between the couplings and m is

$$\kappa_3^2 = \frac{\kappa^2}{2m^2}\left[m^4 - 2m^2 + 1 \pm \sqrt{(1 + m^4 - 2m^2)^2 - 16m^4}\right]. \tag{73}$$

4.3.3. $\kappa_1 = \kappa - x,\ \kappa_2 = \kappa,\ \kappa_3 = \kappa + x$

In this case the relation between the couplings and m is

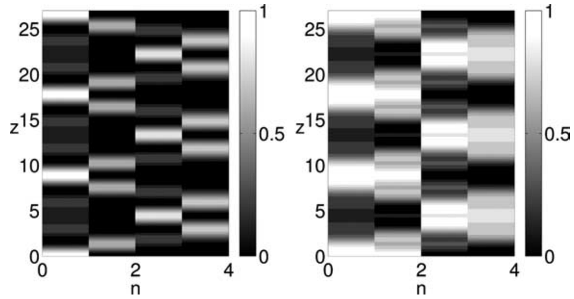

Fig. 1. Intensity evolution in a four element waveguide lattice with $\kappa_1 = \kappa_3 = 1$ and $m = 2$ (corresponding to $\kappa_2 \simeq 0.707$). In the left (right) panel the initial pattern is (1 0 0 0) [(1 1 0 0)]. For this value of m only perfect revivals are possible.

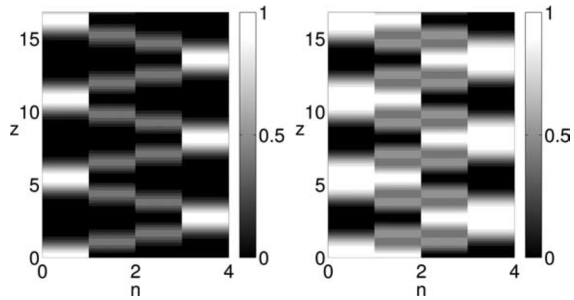

Fig. 2. Intensity evolution in a four element waveguide lattice with $\kappa_1 = \kappa_3 = 1$ and $m = 3$ (corresponding to $\kappa_2 \simeq 1.15$). In the left (right) panel the initial pattern is (1 0 0 0) [(1 1 0 0)]. For this value of m perfect revivals and reflection (but not mirror) fractional revivals are possible.

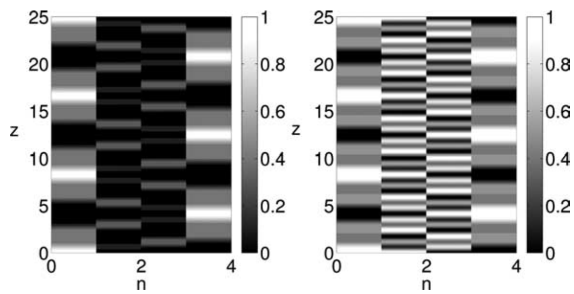

Fig. 3. Intensity evolution in a four element waveguide lattice with $\kappa_1 = \kappa_3 = 1$ and $m = 7$ (corresponding to $\kappa_2 \simeq 2.27$). In the left (right) panel the initial pattern is (1 0 0 0) [(1 1 0 0)]. For this value of m perfect revivals as well as fractional revivals (reflections and mirror images) are possible.

$$x^2 = \frac{(1 \pm 3m + m^2)\kappa^2}{(m \mp 1)^2}. \quad (74)$$

4.4. Array with five waveguides

Since N is odd, only perfect and mirror fractional revivals are allowed by the lattice. The evolution for the five equations is described by:

$$i\dot{\psi}_1 + \kappa_1\psi_2 = 0, \quad (75)$$

$$i\dot{\psi}_2 + \kappa_1\psi_1 + \kappa_2\psi_3 = 0, \quad (76)$$

$$i\dot{\psi}_3 + \kappa_2\psi_2 + \kappa_3\psi_4 = 0, \quad (77)$$

$$i\dot{\psi}_4 + \kappa_3\psi_3 + \kappa_4\psi_5 = 0, \quad (78)$$

$$i\dot{\psi}_5 + \kappa_4\psi_4 = 0. \quad (79)$$

The characteristic polynomial of the associated matrix is

$$\lambda^5 - (\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2)\lambda^3 + (\kappa_1^2\kappa_3^2 + \kappa_1^2\kappa_4^2 + \kappa_2^2\kappa_4^2)\lambda = 0$$

with eigenvalues

$$\lambda = 0, \quad \lambda_{\{1,2\},\pm} = \pm(1/2)\sqrt{\pm 2\sqrt{F_4 + 2F_{22}} + 2F_2},$$

where $F_4 = \kappa_1^4 + \kappa_2^4 + \kappa_3^4 + \kappa_4^4$, $F_2 = \kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2$, $F_{22} = \kappa_1^2\kappa_2^2 + \kappa_2^2\kappa_3^2 + \kappa_3^2\kappa_4^2 - \kappa_1^2\kappa_3^2 - \kappa_1^2\kappa_4^2 - \kappa_2^2\kappa_4^2$. The revival condition $\lambda_2 = m\lambda_1$ then results to

$$(1 + m^2)^2(\kappa_1^2\kappa_3^2 + \kappa_1^2\kappa 4^2 + \kappa_2^2\kappa_4^2) = m^2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2)^2.$$

5. Conclusions

In this paper, we showed that perfect and fractional revivals are possible in finite waveguide lattices. Such revivals are experimentally realizable in different settings, such as, waveguide arrays, multi-fiber couplers, and photonic crystals. Potential applications of fractional revivals utilizing operations such as multi-couplers and multi-beam-splitters have been considered.

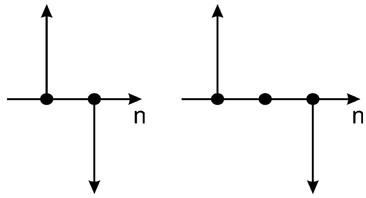

Fig. 4. Schematic illustration of the two possible node types. On the left panel $u_j u_{j+1} < 0$ whereas on the right panel $u_{j-1}u_{j+1} < 0$ along with $u_j = 0$.

Appendix A. Properties of the lattice

For completeness, we present here some of the basic properties of linear waveguide lattices. The properties of the eigenvector and eigenvalues of Eq. (11) are associated with the properties of matrix A. Direct consequences of the symmetric nature of A are:

Property 1. *The eigenvectors and eigenvalues of A are real.*

Property 2. *A has N linearly independent and orthogonal eigenvectors.*

Property 3. *The recursion formula (13)–(15) is a Sturm sequence when $\kappa_j \neq 0$ $\forall j$.*

Property 4. *As a result of Property 3, the eigenvalues of A are distinct when all $\kappa_j \neq 0$.*

The number of nodes of an eigenfunction in continuous eigenvalue problems is directly related to the order of the corresponding eigenvalue [11]. This property can be extended in the discrete domain [12].

Definition 1. An eigenvector u_i has a node on j when (a) $u_j u_{j+1} < 0$ or (b) $u_j = 0$ along with $u_{j+1}u_{j-1} < 0$ (see Fig. 4).

Definition 2. We define $N(u) = l$ as the total number of nodes of the eigenvector u and $N(u, j_1 \leqslant j \leqslant j_2)$ as the number of nodes of u that lie in the regime $j_1 \leqslant j \leqslant j_2$.

Definition 3. If $j_1, j_2, \ldots$ are the nodes of the eigenvector u then $\mathrm{IDX}(u) = \{j_1, j_2, \ldots\}$.

Property 5. *If the nontrivial eigenvector u_i has a zero on j then it has a node on j.*

Proof. Assume that $u_j = 0$. If $u_{j-1} = 0$ then from Eq. (5) $u_{j+1} = 0$. Applying Eq. (5) to adjacent lattice elements results to $u_i = 0$, $i = 1,\ldots,N$. Thus, without loss of generality, we can assume that $u_{j-1} > 0$. From Eq. (5) and keeping in mind that $\kappa_l > 0$, $l = 1,\ldots,N-1$, it is $u_{j+1} = -\kappa_{j-1}u_{j-1}/\kappa_j < 0$. □

Property 6. (*discrete node theorem* [12]). *Assuming that the eigenvalues of A have the order $\lambda_1 > \lambda_2 > \cdots > \lambda_{N-1} > \lambda_N$, and $\kappa_j > 0$, $j = 1,\ldots,N$, then the eigenvector $u^{(j)}$ corresponding to λ_j has exactly $j-1$ nodes.*

Proof. We consider two pairs of eigenvectors, eigenvalues of A, say $(\lambda_1,u^{(1)})$ and $(\lambda_2,u^{(2)})$, with different eigenvalues $\lambda_1 \neq \lambda_2$. Without loss of generality we can assume that $\lambda_1 > \lambda_2$. By contraction, one can show that

$$\kappa_j W\left(u_j^{(1)}, u_j^{(2)}\right) - \kappa_{j-1} W\left(u_{j-1}^{(1)}, u_{j-1}^{(2)}\right) = (\lambda_1 - \lambda_2) u_j^{(1)} u_j^{(2)}, \tag{A.1}$$

where the discrete Wronskian W is defined as

$$W(u_j, v_j) = u_{j+1}v_j - u_j v_{j+1}. \tag{A.2}$$

Summation over Eq. (A.1) from j_1 to j_2 results to

$$\kappa_{j_2} W\left(u_{j_2}^{(1)}, u_{j_2}^{(2)}\right) - \kappa_{j_1-1} W\left(u_{j_1-1}^{(1)}, u_{j_1-1}^{(2)}\right) = (\lambda_1 - \lambda_2) \sum_{j=j_1}^{j_2} u_j^{(1)} u_j^{(2)}. \tag{A.3}$$

We apply Eq. (A.3) into the regime $j_1 \leqslant j \leqslant j_2$. We assume that $\{\mathrm{IDX}(u^{(1)}): j_1 \leqslant j \leqslant j_2\} = \{j_1,j_2\}$, i.e., j_1 and j_2 are two consecutive nodes of $u^{(1)}$ ($u_{j_1}^{(1)} \leqslant 0, u_j^{(1)} > 0$ for $j = j_1,\ldots,j_2-1$ and $u_{j_2}^{(1)} \geqslant 0$). If $u^{(2)}$ does not have a node in the regime $j_1 \leqslant j \leqslant j_2$ then the left hand side of Eq. (A.3) becomes negative whereas the right hand side is positive. This is a contradiction and thus $u^{(2)}$ should have at least one node in the regime $j_1 \leqslant j \leqslant j_2$. The same procedure can be applied in the domain boundaries, where, for example, for the left boundary it can be shown that if the first node of $u^{(1)}$ is on j_1 then $N(u^{(2)},1 \leqslant j \leqslant j_1) \geqslant 1$.

We continue by considering the case $N(u^{(1)},j_1 \leqslant j \leqslant j_3) = 3$ and $\{\mathrm{IDX}(u^{(1)}): j_1 \leqslant j \leqslant j_3\} = \{j_1,j_2,j_3\}$. In this case we will show that $u^{(2)}$ has at least two nodes in the regime $j_1 \leqslant j \leqslant j_3$. If $u^{(2)}$ has zero nodes then by restricting the domain to $j_1 \leqslant j \leqslant j_2$ we come to contradiction. Now assuming that $N(u^{(2)},j_1 \leqslant j \leqslant j_2) = 1$ and $\{\mathrm{IDX}(u^{(2)}): j_1 \leqslant j \leqslant j_3\} \neq j_2$, we can show that $u^{(2)}$ has at least one node in each of the regions $j_1 \leqslant j < j_2$ and $j_2 < j \leqslant j_3$. If $N(u^{(2)},j_1 \leqslant j \leqslant j_2) = 1$ and $\{\mathrm{IDX}(u^{(2)}): j_1 \leqslant j \leqslant j_3\} = j_2$, we can show by contradiction [applying Eq. (A.3)] that $u^{(2)}$ has at least a second node.

We can extend this procedure to the general case where $u^{(1)}$ has m nodes in the region $j_1 \leqslant j \leqslant j_m$. By dividing this region into subregions that do not have common nodes in the boundaries it is easy to show that $N(u^{(2)},j_1 \leqslant j \leqslant j_m) \geqslant N(u^{(1)},j_1 \leqslant j \leqslant j_m)-1$.

Taking into account that the eigenvectors $u^{(j)}$ do not have nodes on the boundaries we conclude that

$$N(u^{(2)}) \geqslant N(u^{(1)}) + 1. \tag{A.4}$$

Thus if $\lambda_1 > \lambda_2$, the number of nodes of $u^{(2)}$ are more than the number of nodes of $u^{(1)}$, i.e., $N(u^{(1)}) < N(u^{(2)})$. Considering that the lattice consists of N elements, the maximum number of nodes that an eigenvector can have is $N-1$. Taking this into account, the only possible choice for the number of nodes of the eigenvector $u^{(l)}$ is $N(u^{(l)}) = l-1$. □

Definition 4. A lattice is symmetric if its elements are symmetric with respect to the center of the lattice, i.e., $\kappa_j = \kappa_{N-j}$ along with $\alpha_j = \alpha_{N-j+1}$.

Definition 5. An eigenvector is symmetric or has even parity if $u_j = u_{N-j+1}$ and is antisymmetric or has odd parity if $u_j = -u_{N-j+1}$.

Property 7. *The eigenvectors of a symmetric lattice are either symmetric or antisymmetric.*

Proof. For Eq. (6) it is

$$L_n u_n = \kappa_{n-1} + \kappa_n u_n + \alpha_n u_n = \lambda u_n. \tag{A.5}$$

Since the operator is symmetric

$$L_n u_n = L_{N-n+1} u_n, \tag{A.6}$$

by applying the transformation $n \to N - n + 1$ we obtain

$$L_n u_{N-n+1} = \lambda u_{N-n+1}. \tag{A.7}$$

By inspection of Eqs. (A.5) and (A.7) one can find that $u_{N-n+1} = \pm u_n$. Thus, if $u_{N-n+1} = u_n$ the eigenvalue is symmetric, whereas, if $u_{N-n+1} = -u_n$ the eigenvalue is antisymmetric. □

Property 8. *If the eigenvalues of a symmetric lattice are arranged as $\lambda_1 > \lambda_2 > \cdots > \lambda_N$ then the eigenvector $u^{(l)}$ will be symmetric if $l = 2m + 1$ and antisymmetric if $l = 2m$.*

Proof. The lth eigenvector has $l - 1$ nodes (Property 6). Also, it is either symmetric or antisymmetric (Property 7). Notice that if the number of nodes $l - 1$ is even (odd) the corresponding eigenvector will also be even (odd). As a result, the eigenvector $u^{(l)}$ will be even (odd) if $l = 2m + 1$ ($l = 2m$). □

We will now discuss the properties of matrix A when $\alpha_j = 0$ (zero local detuning). We define matrix B as $B = \{A: \alpha_j = 0, j = 1\ldots,N\}$ or

$$B = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \kappa_1 & 0 & \kappa_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \kappa_2 & 0 & \kappa_3 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \kappa_{N-1} & 0 & \kappa_N \\ 0 & 0 & 0 & 0 & \ldots & 0 & \kappa_N & 0 \end{pmatrix}. \tag{A.8}$$

The characteristic polynomial of (A.8) is

$$\lambda^N - G_{2,1}\lambda^{N-2} + G_{2,2}\lambda^{N-4} - G_{2,3}\lambda^{N-6} + \cdots = 0, \tag{A.9}$$

where its coefficients are explicitly given by:

$$G_2, 1 = \sum_j \kappa_j^2,$$

$$G_{2,2} = \sum_{i<j-1} \kappa_i^2 \kappa_j^2,$$

$$G_{2,3} = \sum_{i<j-1<k-1} \kappa_i^2 \kappa_j^2 \kappa_k^2$$

and so on. Matrix (A.8) has some interesting properties that are absent from the original matrix (8).

Property 9. *We define matrix T as*

$$T \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \end{pmatrix} \tag{A.10}$$

Then if v is an eigenvector of B with eigenvalue λ, $\tilde{v} = Tv$ will also be an eigenvector with eigenvalue $-\lambda$.

Proof. We multiply the original eigenvalue problem

$$Bx = \lambda x \tag{A.11}$$

with T

$$TBx = \lambda Tx. \tag{A.12}$$

However, since $\{B,T\} = 0$, where $\{A,B\}$ is the anticommutator

$$\{A, B\} = AB + BA, \tag{A.13}$$

it is

$$B\tilde{v} = -\lambda\tilde{v} \tag{A.14}$$

□.

Property 10. *The sum of the eigenvalues of B is zero*

Proof. This becomes obvious by noticing that $\mathrm{Tr}(B) = 0$. □

Property 11. *When the dimension of matrix B is odd, $N = 2M + 1$, the matrix will have $2M$ non-zero and one zero eigenvalues. If the dimension of the matrix is even, $N = 2M$, then all the eigenvalues will be non-zero.*

Proof. If matrix B is even, $N = 2M$, one can prove that

$$\prod_j \lambda_j = \kappa_1^2 \kappa_3^2 \ldots \kappa_{2M-1}^2.$$

Since the coupling coefficients are non-zero, the product of all the eigenvalues is positive. As a result all eigenvalues are non-zero $\lambda_j \neq 0$. On the other hand, if the dimension of B is odd, $N = 2M + 1$, since

$$\prod_j \lambda_j = 0, \tag{A.15}$$

at least one eigenvalue will be zero. However, the determinants of all the $2M \times 2M$ submatrices are non-zero. Thus, the remaining $2M$ eigenvalues are non-zero. □

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