NATIONAL TECHNICAL UNIVERSITY OF ATHENS
DEPARTMENT OF NAVAL ARCHITECTURE AND MARINE ENGINEERING

SHIP – DESIGN LABORATORY

Shape-preserving interpolation in $\mathbb{R}^3$

by

P.D. Kaklis and M.I. Karavelas

Athens, February 1996 (Second Revision)

Contents

ABSTRACT **1**

1. Introduction **2**

2. A notion of shape-preserving interpolation in $\mathbb{R}^3$ **4**

3. The family $\mathbb{\Gamma}(\mathrm{K})$ of curvature- and torsion-continuous polynomial splines of non-uniform degree **8**

4. The asymptotic properties of the family $\mathbb{\Gamma}(\mathrm{K})$ for large degrees **10**

5. Shape-preserving interpolation in $\mathbb{R}^3$ with the aid of $\mathbb{\Gamma}(\mathrm{K})$ **20**

5.1. The convexity criterion 21
5.2. The torsion criterion 23
5.3. The coplanarity criterion 23
5.4. The collinearity criterion 30
5.5. The shape-preserving algorithm 33

6. Numerical results **38**

REFERENCES **41**

FIGURES **42**

Shape-preserving interpolation in $\mathbb{R}^3$

by

P.D. Kaklis[(*)] and M.I. Karavelas[(**)]

Abstract

In this paper we develop and test a simple automatic algorithm for constructing curvature- and torsion-continuous interpolants in $\mathbb{R}^3$, which are shape-preserving in a sense that takes into account the convexity, torsion, coplanarity and collinearity information contained in the polygonal line connecting the interpolation points. This algorithm exploits the asymptotic properties of a family of C^2-continuous polynomial splines of non-uniform degree, which tend to the above-mentioned polygonal line, as the segment degrees tend to infinity. The performance of the algorithm is tested for a three-dimensional data set, containing coplanar and collinear groups of points as well.

[(*)] Assistant Professor, address: National Technical University of Athens, Department of Naval Architecture and Marine Engineering, Ship-Design Laboratory, 9 Heroon Polytechneiou Street, GR-157 73 Zografou, Athens, Greece, E-mail address: kaklis@naval.ntua.gr, Tel.: +30-1-7721419, Telefax: +30-1-7721408.

[(**)] Graduate Student, address: SCCM Program, CS Dept., Gates Building 2B, Stanford University, Stanford, CA 94305-9025, U.S.A., E-mail address: menelaos@sccm.stanford.edu .

1. Introduction

Shape preservation for two- and three-dimensional interpolating curves is a subject of major importance for the area of CAGD. But important as this subject is in its three-dimensional version, the research activity has been hitherto restricted to the two-dimensional problem. As a result, the CAD designer can appeal to numerous two-dimensional splines, which use various basis functions, such as polynomials, rationals, exponentials, etc., and possess elegant shape-preserving properties; for a brief literature review see, e.g., Kaklis and Pandelis (1990) for the functional, and Kaklis and Sapidis (1995) for the parametric case.

In contrast to the two-dimensional case, the literature dealing with the problem of constructing interpolating curves in $\mathbb{R}^3$, which preserve the shape of the data to be interpolated, is apparently poor. In this connection, reference should be made to Goodman (1991), which introduces the notion of *inflection count* as the maximum number of inflections that a curve can appear to have when viewed from any direction. A "good" scheme for constructing shape-preserving curves in three dimensions might then be one that produces curves with low inflection count. However, Goodman does not proceed to construct a concrete shape-preserving interpolation scheme, but simply suggests a Hermite scheme involving quadratic splines, which have a lower inflection count than that of cubic splines. Further, Clements (1992) discusses the possibility of extending his rational cubic interpolation scheme in three dimensions, in such a way that the sign of the torsion of the constructed interpolant is preserved in each parameter segment.

The intention of this paper is to present an automatic algorithm for constructing shape-preserving interpolants in $\mathbb{R}^3$, which are curvature- and torsion-continuous. Two major tools will achieve this intention, the one of geometric and the other of analytic nature. The geometrical tool is a notion of shape-preserving interpolation in $\mathbb{R}^3$, resulting from the quantification, with the aid of the "Discrete Geometry" in Sauer (1970), of the shape information, namely the convexity, torsion, coplanarity and collinearity information, contained in the polygonal line connecting the interpolation points (see Definition 2.1). The analytical tool, which provides the working function space, is the natural 3D-extension of the family of polynomial splines of non-uniform degree, introduced in Kaklis and Pandelis (1990) for functional C^2-continuous convexity-preserving interpolation. The basic property of this family is that, as the segment degrees increase, the spline tends to the polygonal interpolant, which contains the very basic shape information of our problem (see Theorem 4.2).

The paper is divided into six sections. In Section 2 we define a concept of shape-preserving interpolation in $\mathbb{R}^3$ (see Def. 2.1), and give its geometrical interpretation. This concept applies to all curvature- and torsion-continuous curves, more accurately all Frénet-frame-continuous curves of order 3, which interpolate a spatial point set with specified parametrization and appropriate boundary conditions. The proposed shape-preserving notion is a local one, ensuring that the distribution of the binormal and the sign of the torsion of a spatial interpolating curve behaves in conformity with the corresponding "discrete" properties of the polygonal interpolant (see Parts *(i)* and *(ii)* of Def. 2.1). In addition, this notion ensures that, if the data contain subsets of coplanar or collinear points, then the interpolant exhibits analogous behaviour in a user-specified closed subinterval of the parametric domain, that corresponds to the coplanar or collinear interpolation points (see Parts *(iii)* and *(iv)* of Def. 2.1).

In Section 3 we introduce the natural 3D-extension $\mathbb{\Gamma}$ of the 2D-family of C^2-continuous polynomial splines of non-uniform degree, introduced in Kaklis and Pandelis (ibid.), and prove that interpolation in $\mathbb{\Gamma}$, under type-I, type-II' or periodic boundary conditions, and specified parametrization, is well-posed. Moreover, we prove that curves in $\mathbb{\Gamma}$ are Frénet-frame-continuous of order 3 (see Theorem 3.1). It is worth pointing out that torsion continuity is due to an intrinsic property of curves in $\mathbb{\Gamma}$, namely their torsion vanishes at the parameter nodes.

In Section 4 we investigate the asymptotic properties of the family $\mathbb{\Gamma}$ as the segment degrees tend to infinity in various manners. Investigation focuses on the asymptotic behaviour of the curvature, torsion and Frénet frame of a curve in $\mathbb{\Gamma}$. More specifically, curvature tends to zero in the interior of a parameter segment, while the nodal curvature tends to infinity, in the case of non-collinear triplets of interpolation points (see Theorem 4.5). As far as torsion is concerned, it tends to zero in the interior of a parameter segment, with the exception of the midpoint, where it tends to infinity for fully 3D data, or remains bounded in the case of coplanar quadruples of interpolation points (see Theorem 4.7). Finally, with respect to the Frénet frame, the unit-tangent vector tends to the corresponding "discrete" unit-tangent vector in the interior of a parameter segment (see Theorem 4.8*(i)*). The binormal of the curve in the first half of a parameter segment tends to the "discrete" binormal corresponding to the left node of the segment, whereas in the second half it tends to the "discrete" binormal corresponding to the right node (see Theorem 4.8*(iii)*). The asymptotic behaviour of the principal normal is then a direct consequence of the asymptotic behaviour of the unit-tangent and the binormal (see Theorem 4.8*(ii)*).

Exploiting the asymptotic results of the preceding section, Section 5 develops an automatic algorithm for constructing spatial shape-preserving interpolants with the aid of $\mathbb{\Gamma}$. These interpolants obey a shape-preserving criterion (see Def. 2.1*), slightly weaker than that introduced in Section 2 (compare Parts *(ii)* and *(iii)* of Defs. 2.1 and 2.1*). After stating Definition 2.1*, the rest of the section is divided into five subsections. The first four are devoted to the derivation of discrete sufficient conditions for each one of the four criteria comprising Definition 2.1*, namely the convexity, torsion, coplanarity and collinearity criterion. Each subsection consists of pairs of lemmata and theorems, with the lemmata providing discrete sufficient conditions and the theorems establishing that these conditions are met for appropriately large segment degrees (see Lemma 5.1 and Theorem 5.2 for the convexity criterion, Lemma 5.3 and Theorem 5.4 for the torsion criterion, Lemma 5.5 and Theorem 5.6 for Part 1 of the coplanarity criterion, Lemma 5.7 and Theorem 5.8 for Part 2 of the same criterion, and, finally, Lemma 5.9 and Theorem 5.10 for the collinearity criterion). The last subsection contains the promised algorithm, which is iterative and whose convergence is established by appealing to the theorems of the previous subsections. Furthermore, this algorithm is computationally simple, since each iteration involves the solution of a linear system with symmetric, positive-definite and tridiagonal or cyclic-tridiagonal matrix.

This work ends with Section 6, in which we present and discuss the performance of the algorithm for a three-dimensional data set containing coplanar and collinear groups of points. The graphical output of the algorithm is collected in Figures 6.1–6.5, where the shape-preserving interpolant provided by the algorithm is compared with the standard C^4-Quintic, interpolating the same data set, with the same boundary conditions and parametrization. These results, along with the numerical experience so far, permit us to assert that, for reasonable parametrizations (e.g., chord-length), the proposed algorithm yields relatively small final segment degrees in the non-collinear regions of the data, which produces visually-pleasing curves. Moreover, it should be noticed that, in general, the standard C^4-Quintic fails to be shape-preserving, which further justifies the need for the algorithm. We end this section by numerically investigating and commenting on the effect of the parametrization on the shape quality of the outcome of the algorithm. The graphical output of this investigation is presented in Figures 6.6 and 6.7.

2. A notion of shape-preserving interpolation in $\mathbb{R}^3$

In this section we introduce the henceforth adopted notion of shape-preserving interpolation in $\mathbb{R}^3$. We start with some preliminary notation. Let $\mathcal{D} = \{\mathbf{I}_m, m = 1(1)N\}$ be a set of points in $\mathbb{R}^3$ with $\mathbf{I}_m \neq \mathbf{I}_{m+1},\ m = 1(1)N-1$, $\mathcal{L}_\mathcal{D}$ be the polygonal line connecting the points of $\mathcal{D}$, $\mathbf{L}_m = \mathbf{I}_{m+1} - \mathbf{I}_m$, $\mathbf{P}_m = \mathbf{L}_{m-1} \times \mathbf{L}_m$ (see Fig. 2.1) and $\Delta_m = |\mathbf{L}_{m-1} \quad \mathbf{L}_m \quad \mathbf{L}_{m+1}| := \det([\mathbf{L}_{m-1} \quad \mathbf{L}_m \quad \mathbf{L}_{m+1}])$.

Appealing to the "Discrete Geometry" in Sauer (1970, Ch. I, § 2.2), the vectors $\mathbf{P}_m$ and the scalars Δ_m can be used to quantify the shape properties of the polygonal line $\mathcal{L}_\mathcal{D}$. More specifically, $\mathbf{P}_m/|\mathbf{P}_m|$ is the so-called discrete binormal at $\mathbf{I}_m$, whereas Δ_m has the same sign as the so-called discrete torsion

$$\overset{\wedge}{\tau}_m = \frac{\Delta_m}{|\mathbf{P}_m||\mathbf{P}_{m+1}|} = \operatorname{sgn}(\Delta_m)\,\frac{\sin(\phi_m)}{|\mathbf{L}_m|}, \quad 0 \le \phi_m < \pi, \tag{2.1}$$

along the segment $\mathbf{I}_m\mathbf{I}_{m+1}$ of $\mathcal{L}_\mathcal{D}$. Here ϕ_m is the bihedral angle of the discrete osculating planes E_m and E_{m+1} at $\mathbf{I}_m$ and $\mathbf{I}_{m+1}$, respectively. In view of the definition of the discrete binormal $\mathbf{P}_m$ at $\mathbf{I}_m$, E_m is defined by the triplet $\mathbf{I}_{m-1}, \mathbf{I}_m$ and $\mathbf{I}_{m+1}$ (see Fig. 2.1).

Let $\mathbf{Q}(u)$, $u \in [u_1, u_N]$, be a sufficiently smooth regular curve interpolating $\mathcal{D}$ with parametrization $\mathcal{U} = \{u_1, u_2, \ldots, u_N : u_1 < u_2 < \ldots < u_N\}$ ($\mathbf{Q}(u_m) = \mathbf{I}_m$), and satisfying appropriate boundary conditions $\mathcal{B}$. The smoothness assumptions imposed on $\mathbf{Q}(u)$ are the following :

(i)

$$\mathbf{Q}(u) \in C^3[u_m, u_{m+1}], \quad m = 1(1)N-1, \tag{2.2}$$

(ii)

$$\mathbf{v}_t(u),\ \mathbf{v}_b(u),\ \kappa(u),\ \tau(u)\ \in C[u_1, u_N], \tag{2.3}$$

where

$$\mathbf{v}_t(u) = \frac{\dot{\mathbf{Q}}(u)}{|\dot{\mathbf{Q}}(u)|}, \quad \dot{\mathbf{Q}}(u) = \frac{d\mathbf{Q}(u)}{du}, \quad |\dot{\mathbf{Q}}(u)| \neq 0, \tag{2.4}$$

is the unit-tangent vector, with $|\cdot|$ denoting the Euclidean norm,

$$\mathbf{v}_b(u) = \frac{\mathbf{w}(u)}{|\mathbf{w}(u)|}, \quad \mathbf{w}(u) = \dot{\mathbf{Q}}(u) \times \ddot{\mathbf{Q}}(u), \quad |\mathbf{w}(u)| \neq 0, \tag{2.5}$$

is the binormal,

$$\kappa(u) = \frac{|\mathbf{w}(u)|}{|\dot{\mathbf{Q}}(u)|^3} \tag{2.6}$$

is the curvature and, finally,

$$\tau(u) = \frac{\det(\mathbb{T}(u))}{|\mathbf{w}(u)|^2}, \quad \mathbb{T}(u) = [\dot{\mathbf{Q}}(u)\ \ddot{\mathbf{Q}}(u)\ \dddot{\mathbf{Q}}(u)]^T, \tag{2.7}$$

is the torsion of the interpolating curve $\mathbf{Q}(u)$. The class of curves characterized by the smoothness assumptions *(i)* and *(ii)* will be hereafter denoted by $\mathcal{F}^3(\mathcal{U})$. Note that the continuity of the vectors $\mathbf{v}_t(u)$, $\mathbf{v}_b(u)$ implies that the principal normal vector

$$\mathbf{v}_n(u) = \mathbf{v}_b(u) \times \mathbf{v}_t(u) \tag{2.8}$$

of $\mathbf{Q}(u)$ will also be continuous in $[u_1, u_N]$. As a consequence the elements of $\mathcal{F}^3(\mathcal{U})$ exhibit Frénet-frame continuity of order 3 (i.e., F^3-continuity or F^3-contact; see Mazure (1994)). Furthermore, it can be shown that the continuity assumption *(ii)* can be replaced by the following matrix equation at the interior parameter nodes :

$$\left[\dot{\mathbf{Q}}(u_m+)\ \ddot{\mathbf{Q}}(u_m+)\ \dddot{\mathbf{Q}}(u_m+)\right] = \left[\dot{\mathbf{Q}}(u_m-)\ \ddot{\mathbf{Q}}(u_m-)\ \dddot{\mathbf{Q}}(u_m-)\right] \cdot \mathbb{C}, \quad m = 2(1)N-1, \qquad (2.9a)$$

where $\mathbb{C}$ is the so-called *connection matrix* defined by

$$\mathbb{C} = \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & \delta \\ 0 & 0 & \alpha^3 \end{bmatrix}, \quad \alpha > 0,\ \beta, \gamma, \delta \in \mathbb{R}. \qquad (2.9b)$$

It is worth noticing that, if $\delta = 3\alpha\beta$, then $\mathbf{Q}(u)$ exhibits contact of order 3 (G^3-continuity or G^3-contact) at the interior parameter nodes, which means that there exists a reparametrization that renders the curve C^3-continuous in $[u_1, u_N]$; see Boehm (1988).

Concerning, now, the boundary conditions $\mathcal{B}$ associated with the interpolation problem, three types of boundary conditions will be employed in this work :

(i) Type-I boundary conditions : $\dot{\mathbf{Q}}_1 = \mathbf{s}_0$, $\dot{\mathbf{Q}}_N = \mathbf{s}_N$, where $\dot{\mathbf{Q}}_m := \dot{\mathbf{Q}}(u_m)$ and $\mathbf{s}_0$, $\mathbf{s}_N$ are given vectors in $\mathbb{R}^3$.

(ii) Type-II' boundary conditions : $\ddot{\mathbf{Q}}_1 = \ddot{\mathbf{Q}}_N = (0,0,0)^T$, which induces zero curvature at the endpoints $\mathbf{I}_1$ and $\mathbf{I}_N$ of the curve.
Finally, in the case of closed data ($\mathbf{I}_1 \equiv \mathbf{I}_N$),

(iii) Periodic boundary conditions : $\dot{\mathbf{Q}}_1 = \dot{\mathbf{Q}}_N$, $\ddot{\mathbf{Q}}_1 = \ddot{\mathbf{Q}}_N$.

In the case of type-I boundary conditions $\mathcal{L}_{\mathcal{D}}$ is extended to $\mathcal{L}_{\mathcal{DB}}$, which connects the points $\mathbf{I}_0, \mathbf{I}_1, \ldots, \mathbf{I}_N, \mathbf{I}_{N+1}$, where $\mathbf{I}_0 = \mathbf{I}_1 - h_0\mathbf{s}_0$, $\mathbf{I}_{N+1} = \mathbf{I}_N + h_N\mathbf{s}_N$, with h_0, h_N being arbitrary positive numbers. $\mathcal{L}_{\mathcal{DB}}$ permits us to include in one polygonal line both internal and boundary shape information. In the case of periodic boundary conditions $\mathcal{L}_{\mathcal{DB}}$ connects the points $\mathbf{I}_0, \mathbf{I}_1, \ldots, \mathbf{I}_N, \mathbf{I}_{N+1}$, where $\mathbf{I}_0 \equiv \mathbf{I}_{N-1}$ and $\mathbf{I}_{N+1} \equiv \mathbf{I}_2$. Finally, $\mathcal{L}_{\mathcal{DB}} \equiv \mathcal{L}_{\mathcal{D}}$ for type-II' boundary conditions.

We are now ready to precisely define the herein adopted notion of shape-preserving interpolation in $\mathbb{R}^3$:

Definition 2.1. A curve $\mathbf{Q}(u) \in \mathcal{F}^3(\mathcal{U})$, which interpolates the data set $\mathcal{D}$ and satisfies boundary conditions $\mathcal{B}$, will be called *shape-preserving* provided that :

(i) *(convexity criterion)* If $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, then

$$\mathbf{w}(u) \cdot \mathbf{P}_n > 0, \quad u \in [u_m, u_{m+1}], \quad n = m, m+1. \qquad (2.10)$$

(ii.1) *(torsion criterion)* If $\Delta_m \neq 0$, then

$$\tau(u)\Delta_m > 0, \quad u \in (u_m, u_{m+1}). \qquad (2.11a)$$

(ii.2) If $\Delta_{m-1}\Delta_m > 0$, then

$$\tau(u_m)\Delta_n > 0, \quad n = m-1, m. \qquad (2.11b)$$

(iii) (coplanarity criterion) If $\Delta_m = 0$ and $|\mathbf{P}_m||\mathbf{P}_{m+1}| \neq 0$, then

$$\frac{|\mathbf{w}(u) \times \mathbf{P}_n|}{|\mathbf{w}(u)||\mathbf{P}_n|} < \varepsilon_1, \quad |\mathbf{w}(u)| \neq 0, \quad u \in \omega_m, \quad n = m, m+1, \tag{2.12}$$

where ε_1 is a user-specified small positive number in $(0, 1]$, and ω_m a user-specified closed interval such that $[u_m, u_{m+1}] \subseteq \omega_m \subset (u_{m-1}, u_{m+2})$.

(iv) (collinearity criterion) If $|\mathbf{P}_m| = 0$ and $\mathbf{L}_{m-1} \cdot \mathbf{L}_m > 0$, then

$$\frac{|\dot{\mathbf{Q}}(u) \times \mathbf{L}_n|}{|\dot{\mathbf{Q}}(u)||\mathbf{L}_n|} < \varepsilon_0, \quad u \in \eta_m, \quad n = m-1, m, \tag{2.13}$$

where ε_0 is a user-specified small positive number in $(0, 1]$, and η_m a user-specified closed subinterval of (u_{m-1}, u_{m+1}) that includes u_m as an interior point.

The range of the index m that occurs in the definition of the various quantities given above, e.g., $\mathbf{L}_m$, $\mathbf{P}_m$, Δ_m, depends on the imposed boundary conditions. More specifically, in the case of type-I and periodic boundary conditions, $0 \leq m \leq N$ for $\mathbf{L}_m$, $1 \leq m \leq N$ for $\mathbf{P}_m$ and $1 \leq m \leq N-1$ for Δ_m, while in the case of type-II' boundary conditions, $1 \leq m \leq N-1$ for $\mathbf{L}_m$, $2 \leq m \leq N-1$ for $\mathbf{P}_m$ and $2 \leq m \leq N-2$ for Δ_m.

We complete this section by giving the geometrical interpretation of the various parts of Definition 2.1. The motivation for introducing Part *(i)* of this definition, also referred to as the convexity criterion, is that, whenever the inner product of two consecutive discrete binormals $\mathbf{P}_m/|\mathbf{P}_m|$, $\mathbf{P}_{m+1}/\,|\mathbf{P}_{m+1}|$ is positive, the inner product of the binormal $\mathbf{v}_b(u)$ in $[u_m, u_{m+1}]$, with each one of $\mathbf{P}_m/|\mathbf{P}_m|$, $\mathbf{P}_{m+1}/|\mathbf{P}_{m+1}|$, should be positive too (see (2.10)). As a result, the spherical image of the binormal $\mathbf{v}_b(u)$ in $[u_m, u_{m+1}]$ is confined by the oblique bihedral angle of the discrete osculating planes E_m and E_{m+1} at $\mathbf{I}_m$ and $\mathbf{I}_{m+1}$, respectively. The intersection of this angle with the plane, defined by $\mathbf{P}_m$ and $\mathbf{P}_{m+1}$, is the conjugate cone of $\mathbf{P}_m/|\mathbf{P}_m|$ and $\mathbf{P}_{m+1}/|\mathbf{P}_{m+1}|$. A further consequence of Part *(i)* of the definition is that, whenever $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, the curve $\mathbf{Q}(u)$ has no inflection points in $[u_m, u_{m+1}]$. From this point of view, Part *(i)* of Definition 2.1 resembles the convexity-preserving criteria employed in the two-dimensional case (see, e.g., Goodman and Unsworth (1988)), which justifies its characterization as a convexity criterion.

Part *(ii.1)* of the definition compels the torsion $\tau(u)$ of $\mathbf{Q}(u)$ in (u_m, u_{m+1}) to have the same sign as the corresponding discrete torsion $\hat{\tau}_m$ (see (2.1) and (2.11a)). This constraint is applied to the nodal torsion $\tau(u_m)$ as well, if the neighbouring discrete torsions $\hat{\tau}_{m-1}$, $\hat{\tau}_m$ share the same sign (see (2.11b)). A consequence of relation (2.11a) and the continuity of the torsion is that, if $\hat{\tau}_{m-1}\hat{\tau}_m < 0$, then $\tau(u_m) = 0$.

In the case of four consecutive coplanar points $\mathbf{I}_{m-1}$, $\mathbf{I}_m$, $\mathbf{I}_{m+1}$ and $\mathbf{I}_{m+2}$, without collinear triplets, Part *(iii)* of the definition enforces the osculating plane of $\mathbf{Q}(u)$ to be adequately close to the plane defined by the four coplanar points (see (2.12)). The degree of closeness is governed by the user-specified constant ε_1 that constrains the absolute value of the sine of the bihedral angle of these two planes, whereas the user-specified interval ω_m determines the scope of this constraint within the parameter segment (u_{m-1}, u_{m+2}).

Finally, in the case of collinear triplets $\mathbf{I}_{m-1}$, $\mathbf{I}_m$, $\mathbf{I}_{m+1}$, with $\mathbf{I}_m$ lying between $\mathbf{I}_{m-1}$ and $\mathbf{I}_{m+1}$ ($\mathbf{L}_{m-1} \cdot \mathbf{L}_m > 0$), Part *(iv)* of the definition enforces the curve to be sufficiently close to the linear interpolant in the user-specified closed interval $\eta_m \subset (u_{m-1}, u_{m+1})$ (see (2.13)). More accurately, the user-specified constant ε_0 bounds the absolute value of the sine of the angle of the tangent vector of $\mathbf{Q}(u)$ and the line defined by $\mathbf{I}_{m-1}, \mathbf{I}_m$ and $\mathbf{I}_{m+1}$. The case $|\mathbf{P}_m| = 0$ with $\mathbf{L}_{m-1} \cdot \mathbf{L}_m < 0$ (i.e., $\mathbf{I}_{m-1}$ or $\mathbf{I}_{m+1}$ lying between the other two points) should not be associated with linear-like behaviour, since such triplets are more likely to imply cusp- or loop-like behaviour.

3. The family $\mathbb{\Gamma}(\mathrm{K})$ of curvature- and torsion-continuous polynomial splines of non-uniform degree

In this section we introduce the natural three-dimensional extension of the two-dimensional family of polynomial splines of non-uniform degree, used in Kaklis and Pandelis (1990) for constructing functional, locally convex, C^2-continuous interpolants. This family will be denoted, henceforth, as $\mathbb{\Gamma}(\mathrm{K})$, $\mathrm{K} = \{k_m,\, m = 1(1)N-1\}$, with k_m being the degree of the polynomial spline in the m-th parameter segment. The parametric representation of an element $\mathbf{Q}(u) = (Q_1(u), Q_2(u), Q_3(u))^T \in \mathbb{\Gamma}(\mathrm{K})$ is given by the following formulae :

$$\mathbf{Q}(u) = \mathbf{l}(u) + h_m^2 \ddot{\mathbf{Q}}_m F_m(1-t) + h_m^2 \ddot{\mathbf{Q}}_{m+1} F_m(t), \quad u \in [u_m, u_{m+1}], \quad 1 \le m \le N-1, \tag{3.1}$$

where

$$\mathbf{l}(u) = (l_1(u), l_2(u), l_3(u))^T = \mathbf{I}_m(1-t) + \mathbf{I}_{m+1}t, \quad t = \frac{u - u_m}{h_m}, \quad h_m = u_{m+1} - u_m, \tag{3.2}$$

and

$$F_m(t) := F(t; k_m) = \frac{t^{k_m} - t}{k_m(k_m - 1)}, \quad t \in [0,1], \quad k_m \ge 4. \tag{3.3}$$

The parameter $t \in [0,1]$ is referred to as the "local" parameter, in contrast to the so-called "global" parameter $u \in [u_1, u_N]$. As can easily be seen from (3.1), $\mathbf{Q}(u)$ interpolates the data set $\mathcal{D}$, i.e.,

$$\mathbf{Q}(u_m) = \mathbf{I}_m, \quad m = 1(1)N. \tag{3.4}$$

Note that, in addition to the conditions of Section 2, we require $\ddot{\mathbf{Q}}(u_m-) = \ddot{\mathbf{Q}}(u_m+) = \ddot{\mathbf{Q}}_m$. Therefore, $\mathbf{Q}(u) \in C^2[u_1, u_N]$, provided that $\dot{\mathbf{Q}}(u_m-) = \dot{\mathbf{Q}}(u_m+)$, $m = 2(1)N-1$. These conditions, along with the chosen boundary conditions $\mathcal{B}$, yield a linear system that defines the second-order nodal derivatives $\ddot{\mathbf{Q}}_m$. Depending on the boundary conditions imposed, these first derivative continuity equations are as follows :

(i) Type-I boundary conditions :

$$d_1 \ddot{\mathbf{Q}}_1 + e_1 \ddot{\mathbf{Q}}_2 = \mathbf{b}_1 = \mathbf{s}_1 - \mathbf{s}_0, \tag{3.5a}$$

$$e_{m-1} \ddot{\mathbf{Q}}_{m-1} + (d_{m-1} + d_m) \ddot{\mathbf{Q}}_m + e_m \ddot{\mathbf{Q}}_{m+1} = \mathbf{b}_m = \mathbf{s}_m - \mathbf{s}_{m-1}, \quad m = 2(1)N-1, \tag{3.5b}$$

$$e_{N-1} \ddot{\mathbf{Q}}_{N-1} + d_{N-1} \ddot{\mathbf{Q}}_N = \mathbf{b}_N = \mathbf{s}_N - \mathbf{s}_{N-1}, \tag{3.5c}$$

where

$$e_m = \frac{h_m}{k_m(k_m-1)}, \quad d_m = \frac{h_m}{k_m}, \quad \mathbf{s}_m = \frac{\mathbf{I}_{m+1} - \mathbf{I}_m}{h_m} = \frac{\mathbf{L}_m}{h_m}, \quad m = 1(1)N-1. \tag{3.5d}$$

(ii) Type-II' boundary conditions :

$$(d_1 + d_2) \ddot{\mathbf{Q}}_2 + e_2 \ddot{\mathbf{Q}}_3 = \mathbf{b}_2, \tag{3.6a}$$

$$e_{m-1} \ddot{\mathbf{Q}}_{m-1} + (d_{m-1} + d_m) \ddot{\mathbf{Q}}_m + e_m \ddot{\mathbf{Q}}_{m+1} = \mathbf{b}_m, \quad m = 3(1)N-2, \tag{3.6b}$$

$$e_{N-2} \ddot{\mathbf{Q}}_{N-2} + (d_{N-2} + d_{N-1}) \ddot{\mathbf{Q}}_{N-1} = \mathbf{b}_{N-1}. \tag{3.6c}$$

(iii) Periodic boundary conditions :

$$(d_1 + d_{N-1})\ddot{\mathbf{Q}}_1 + e_1\ddot{\mathbf{Q}}_2 + e_{N-1}\ddot{\mathbf{Q}}_{N-1} = \mathbf{s}_1 - \mathbf{s}_{N-1}, \tag{3.7a}$$

$$e_{m-1}\ddot{\mathbf{Q}}_{m-1} + (d_{m-1} + d_m)\ddot{\mathbf{Q}}_m + e_m\ddot{\mathbf{Q}}_{m+1} = \mathbf{b}_m, \quad m = 2(1)N-2, \tag{3.7b}$$

$$e_{N-1}\ddot{\mathbf{Q}}_1 + e_{N-2}\ddot{\mathbf{Q}}_{N-2} + (d_{N-2} + d_{N-1})\ddot{\mathbf{Q}}_{N-1} = \mathbf{b}_{N-1}. \tag{3.7c}$$

It is straightforward to prove that the above linear systems are solvable, because their matrices are strongly diagonally dominant. Note that the matrices of systems (3.5) and (3.6) are symmetric and tridiagonal, whereas the matrix of system (3.7) is symmetric and cyclic-tridiagonal.

Since $\mathbf{Q}(u) \in C^2[u_1, u_N]$, its unit tangent $\mathbf{v}_t(u)$, binormal $\mathbf{v}_b(u)$ and curvature $\kappa(u)$ are continuous at every regular point ($|\dot{\mathbf{Q}}(u)| \neq 0$). Regarding now the torsion $\tau(u)$, we first note that $\mathbf{Q}(u)$ is not, in general, three times continuously differentiable with respect to u at the interior nodes u_m of the parametrization $\mathcal{U}$. In view of this fact, we first restrict ourselves to the open interval (u_m, u_{m+1}), where $\mathbf{Q}(u)$ is a polynomial, and thus $\tau(u)$ is continuous, provided that $\kappa(u) \neq 0$. Then, combining (2.7) with (3.1), we get the following expression for the matrix :

$$\mathbb{T}(u)^T = \mathbb{A}_m \cdot \mathbb{B}(t), \quad u \in (u_m, u_{m+1}), \tag{3.8}$$

where

$$\mathbb{A}_m = \begin{bmatrix} \mathbf{s}_m & \ddot{\mathbf{Q}}_m & \ddot{\mathbf{Q}}_{m+1} \end{bmatrix}, \tag{3.9}$$

and

$$\mathbb{B}(t) = \begin{bmatrix} 1 & 0 & 0 \\ -h_m F'_m(1-t) & F''_m(1-t) & -h_m^{-1} F'''_m(1-t) \\ h_m F'_m(t) & F''_m(t) & h_m^{-1} F'''_m(t) \end{bmatrix}, \tag{3.10}$$

the accent denoting differentiation with respect to t. From (3.10) we readily get

$$\det(\mathbb{B}(t)) = (k_m - 2) h_m^{-1} [t(1-t)]^{k_m - 3} \geq 0, \tag{3.11}$$

which implies that $\tau(u)$ retains the sign of $\det(\mathbb{A}_m)$ in (u_m, u_{m+1}) and vanishes at the nodes $u = u_m, u_{m+1}$ $(t = 0, 1)$. As a consequence, the torsion $\tau(u)$ of an interpolant $\mathbf{Q}(u)$ in $\Gamma(\mathrm{K})$ is continuous as well. This fact, in conjunction with the continuity of $\mathbf{v}_t(u)$, $\mathbf{v}_b(t)$ and $\kappa(u)$, implies that interpolants in $\Gamma(\mathrm{K})$ are F^3-continuous (see (2.2) and (2.3)). This result could also be derived by calculating the elements of the connection matrix $\mathbb{C}$ (see (2.9)). Indeed, using (3.1) and taking into account that elements in $\Gamma(\mathrm{K})$ are already C^2-continuous, it can be shown that

$$\mathbb{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{bmatrix}, \qquad \delta = -\frac{k_{m-1} - 2}{h_{m-1}} - \frac{k_m - 2}{h_m}. \tag{3.12}$$

Note that, since $\delta \neq 0 = 3\alpha\beta$, curves in $\Gamma(\mathrm{K})$ are not G^3-continuous.

Collecting the results of the present section, we state

Theorem 3.1. Let $\mathcal{D} = \{\mathbf{I}_m,\ m = 1(1)N\}$ be a given set of points in $\mathbb{R}^3$, $\mathcal{B}$ a set of boundary conditions ($\mathcal{B}$=type-I, type-II' or periodic) and $\mathcal{U} = \{u_1, u_2, \ldots, u_N :\ u_1 < u_2 < \ldots < u_N\}$ a parametrization. The problem of constructing a C^2-continuous spline $\mathbf{Q}(u)$ in $\Gamma(\mathrm{K})$ which interpolates the data set $\mathcal{D}$ with parametrization $\mathcal{U}$ and satisfies the boundary conditions $\mathcal{B}$, results in a well-posed linear system for the second-order nodal derivatives $\ddot{\mathbf{Q}}_m$ of $\mathbf{Q}(u)$ at $u = u_m$ (see system (3.5) for $\mathcal{B}$=type-I, system (3.6) for $\mathcal{B}$=type-II' and system (3.7) for $\mathcal{B}$=periodic). Finally, if $\mathbf{Q}(u)$ is regular and $\kappa(u) \neq 0$ in $[u_1, u_N]$, then $\mathbf{Q}(u)$ is F^3-continuous in $[u_1, u_N]$, with $\tau(u_m) = 0,\ m = 1(1)N$.

4. The asymptotic properties of the family $\mathbb{\Gamma}(\mathrm{K})$ for large degrees

This section is devoted to the investigation of the asymptotic properties of the family $\mathbb{\Gamma}(\mathrm{K})$, as the degrees k_m, $m = 1(1)N-1$, tend to infinity in various manners. Roughly speaking, four degree-increase patterns will occur in the sequel, namely *local* increase, i.e., $k_m \to \infty$ for some fixed m, *left-right* increase, i.e., $k_n \to \infty$, $n = m-1, m$, *semilocal* increase, i.e., $k_n \to \infty$, $n = m-1, m, m+1$, and, finally, *global* increase, i.e., $k_1, k_2, \ldots, k_{N-1} \to \infty$. The ensuing investigation will focus on the asymptotic behaviour of the geometric invariants of a curve $\mathbf{Q}(u) \in \mathbb{\Gamma}(\mathrm{K})$, namely its curvature, torsion and Frénet frame.

We start by stating the following lemma, which will be used intensively within this section :

Lemma 4.1. There exists a positive number M, depending on the data set $\mathcal{D}$, the parametrization $\mathcal{U}$ and the imposed boundary conditions $\mathcal{B}$, but independent of k_m, $1 \le m \le N-1$, such that

$$|R_{im}| \le M, \quad R_{im} = q_m \ddot{Q}_{im}, \quad q_m = d_{m-1} + d_m, \quad d_m = \frac{h_m}{k_m}, \quad 1 \le m \le N-1, \tag{4.1a}$$

where $i = 1, 2, 3$ [(1)] and where $\ddot{Q}_{im}$ (resp. R_{im}) is the i-th component of $\ddot{\mathbf{Q}}_m$ (resp. $\mathbf{R}_m$). Furthermore, $d_0 := d_N := 0$ for type-I and type-II' boundary conditions, and $d_0 := d_{N-1}$, $d_N := d_1$ for periodic boundary conditions. Finally,

$$\lim_{k_{m-1}, k_m \to \infty} \mathbf{R}_m = \mathbf{b}_m. \tag{4.1b}$$

The proof of the above lemma is a direct consequence of Lemmata 3.1 and 4.2 in Kaklis and Pandelis (1990).

Using, now, Lemma 4.1 and the asymptotic estimates :

$$F_m(t) = O(k_m^{-2}), \quad F'_m(t) = O(k_m^{-1}), \qquad t \in [0,1], \tag{4.2a}$$

$$F''_m(t) = o(1), \quad t \in [0,1), \qquad F''_m(t) = O(1), \quad t \in [0,1], \tag{4.2b}$$

we arrive at the following theorem, which is directly analogous to Theorem 3.1 in Kaklis and Pandelis (ibid.).

Theorem 4.2. Let,

$$D_r U_m = \max(|d^r Q_i(u)/du^r - d^r l_i(u)/du^r|, i = 1,2,3), \quad u \in U_m, \quad r = 0,1,2, \tag{4.3}$$

where U_m denotes a closed subinterval of $[u_m, u_{m+1}]$. Furthermore, let $[u_{n-1}, u_n)_c$ be an arbitrary but fixed closed subinterval of $[u_{n-1}, u_n)$ and similarly for $(u_n, u_{n+1})_c$ and $(u_{n+1}, u_{n+2}]_c$. Then :

(i) If the degrees increase locally, i.e., $k_m \to \infty$ for some fixed m [(2)] and $k_n, n \ne m$, remains bounded, then

$$D_0[u_m, u_{m+1}] = O(k_m^{-2}), \quad D_1(u_m, u_{m+1})_c = O(k_m^{-2}), \quad D_2(u_m, u_{m+1})_c = o(1). \tag{4.4}$$

[(1)] The index i will always range over the set $\{1,2,3\}$.

[(2)] The range of index m appearing in the various theorems and lemmata of this section depends on the boundary conditions imposed (see the paragraph that follows Definition 2.1).

(ii) If the degrees increase semi-locally, i.e., $k_{m-1}, k_m, k_{m+1} \to \infty$ for some fixed m and $k_n, n \neq m-1, m, m+1$, remain bounded, then

$$D_0[u_\ell, u_{\ell+1}] = O(k_\ell^{-1}), \quad \ell = m-1, m, m+1, \tag{4.5a}$$

$$D_1[u_{m-1}, u_m)_c = O(k_{m-1}^{-1}), \quad D_1(u_m, u_{m+1})_c = O(k_m^{-1}), \quad D_1(u_{m+1}, u_{m+2}]_c = O(k_{m+1}^{-1}), \tag{4.5b}$$

$$D_2(u_\ell, u_{\ell+1})_c = o(1), \quad \ell = m-1, m, m+1. \tag{4.5c}$$

(iii) If the degrees increase globally, i.e., $k_1, k_2, \ldots, k_{N-1} \to \infty$, then

$$D_0[u_m, u_{m+1}] = O(k_m^{-1}),\ D_1(u_m, u_{m+1})_c = O(k_m^{-1}),\ D_2(u_m, u_{m+1})_c = o(1), \tag{4.6}$$

for $m = 1(1)N-1$.

We shall now proceed to investigate the asymptotic behaviour of the curvature $\kappa(u)$ of an element $\mathbf{Q}(u) \in \mathbb{\Gamma}(\mathrm{K})$. For this purpose we state and prove the following lemma, which will also be useful in studying the asymptotic behaviour of the torsion $\tau(u)$.

Lemma 4.3.

(i) If $k_m, k_{m-1} \to \infty$ with $k_m/k_{m-1} = O(1)$, then

$$\left\{\mathbf{w}(u) q_m (1-t)^{2-k_m}\right\} \to \frac{\mathbf{P}_m}{h_{m-1} h_m}, \quad t \in [0, \frac{1}{2}). \tag{4.7a}$$

(ii) If $k_m, k_{m+1} \to \infty$ with $k_m/k_{m+1} = O(1)$, then

$$\left\{\mathbf{w}(u) q_{m+1} t^{2-k_m}\right\} \to \frac{\mathbf{P}_{m+1}}{h_m h_{m+1}}, \quad t \in (\frac{1}{2}, 1]. \tag{4.7b}$$

(iii) If $k_m, k_n \to \infty$, $n = m-1$ or $m+1$, with $k_m/k_n = O(1)$, then

$$|\mathbf{w}(\bar{u}_m)|(d_m + d_n)(\frac{1}{2})^{2-k_m} = O(1), \tag{4.7c}$$

where $\bar{u}_m = \frac{1}{2}(u_m + u_{m+1})$. Moreover, if $k_{m-1}, k_m, k_{m+1} \to \infty$ with $k_m/k_{m-1} \to c_1 \geq 0$ and $k_m/k_{m+1} \to c_2 \geq 0$, then

$$\left\{\mathbf{w}(\bar{u}_m) q_m (\frac{1}{2})^{2-k_m}\right\} \to \frac{\mathbf{P}_m}{h_{m-1} h_m} + \xi_m \frac{\mathbf{P}_{m+1}}{h_m h_{m+1}}, \tag{4.7d}$$

where $\xi_m = (c_1 h_{m-1} + h_m)/(h_m + c_2 h_{m+1})$.

Proof. Let $t \in [0, \frac{1}{2})$. Observing formulae (2.5) and (2.7) we have :

$$\mathbf{w}(u) = (\mathbb{T}_{31}(t), -\mathbb{T}_{32}(t), \mathbb{T}_{33}(t))^T, \tag{4.8}$$

where $\mathbb{T}_{3i}(t)$ denotes the minor of the element t_{3i} of $\mathbb{T}(t)$. Let us focus on the minor $\mathbb{T}_{33}(t)$. Using formula (3.8) we find :

$$\mathbb{T}_{33}(t) = \sum_{i=1}^{3} \mathbb{A}_{m,3i} \mathbb{B}_{i3}(t), \tag{4.9}$$

with $\mathbb{A}_{m,3i}$, $\mathbb{B}_{i3}(t)$ also denoting minors of the matrices $\mathbb{A}_m$ and $\mathbb{B}(t)$ respectively. Evaluating $\mathbb{B}_{i3}(t)$ in (4.9) with the aid of (3.10), and recalling that $d_m = h_m/k_m$, we get :

$$\begin{aligned}\mathbb{T}_{33}(t)q_m(1-t)^{2-k_m} =& \mathbb{A}_{m,33}\,q_m + \mathbb{A}_{m,32}\,q_m\left(\frac{t}{1-t}\right)^{k_m-2}\\ &-\mathbb{A}_{m,31}\,\frac{q_m d_m}{k_m-1}\left[k_m t^{k_m-2} - \left(\frac{t}{1-t}\right)^{k_m-2} - 1\right],\ t\in[0,\frac{1}{2}].\end{aligned} \tag{4.10}$$

We shall now derive bounds for the quantities $|\mathbb{A}_{m,32}\,q_m|$ and $|\mathbb{A}_{m,31}\,q_m d_m|$. For the first quantity we arrive at the following inequalities :

$$\begin{aligned}|\mathbb{A}_{m,32}\,q_m| &\le \left|s_{1m}R_{2,m+1}\frac{q_m}{q_{m+1}}\right| + \left|s_{2m}R_{1,m+1}\frac{q_m}{q_{m+1}}\right|\\ &\le \left|s_{1m}R_{2,m+1}\frac{q_m}{d_m}\right| + \left|s_{2m}R_{1,m+1}\frac{q_m}{d_m}\right|.\end{aligned} \tag{4.11}$$

Given relation (4.1a) of Lemma 4.1 and since $k_m/k_{m-1} = O(1)$, we conclude that the right-hand side quantity in (4.11) remains bounded. Hence,

$$\mathbb{A}_{m,32}\,q_m = O(1), \quad \text{as} \quad k_{m-1}, k_m \to \infty \quad \text{with} \quad k_m/k_{m-1} = O(1). \tag{4.12}$$

For the second quantity $|\mathbb{A}_{m,31}\,q_m d_m|$ we similarly obtain :

$$|\mathbb{A}_{m,31}\,q_m d_m| \le |R_{1m}||R_{2,m+1}| + |R_{2m}||R_{1,m+1}|, \tag{4.13}$$

which, in view of Lemma 4.1, implies that

$$\mathbb{A}_{m,31}\,q_m d_m = O(1), \quad \text{as} \quad k_{m-1}, k_m \to \infty. \tag{4.14}$$

Regarding now the quantity $\mathbb{A}_{m,33}\,q_m$, the limiting relation (4.1b) provides

$$\lim_{k_{m-1},k_m\to\infty} \mathbb{A}_{m,33}\,q_m = s_{1,m-1}s_{2m} - s_{1m}s_{2,m-1}, \tag{4.15}$$

where s_{im} are defined by (3.5d). In view of (4.12), (4.14) and (4.15), equation (4.10) gives :

$$\left\{\mathbb{T}_{33}(t)q_m(1-t)^{2-k_m}\right\} \to s_{1,m-1}s_{2m} - s_{1m}s_{2,m-1}, \quad t\in[0,\frac{1}{2}), \tag{4.16a}$$

as $k_{m-1}, k_m \to \infty$ with $k_m/k_{m-1} = O(1)$. Working analogously we get :

$$\left\{\mathbb{T}_{32}(t)q_m(1-t)^{2-k_m}\right\} \to s_{1,m-1}s_{3m} - s_{1m}s_{3,m-1}, \quad t\in[0,\frac{1}{2}), \tag{4.16b}$$

and

$$\left\{\mathbb{T}_{31}(t)q_m(1-t)^{2-k_m}\right\} \to s_{2,m-1}s_{3m} - s_{2m}s_{3,m-1}, \quad t\in[0,\frac{1}{2}), \tag{4.16c}$$

as $k_{m-1}, k_m \to \infty$ with $k_m/k_{m-1} = O(1)$. The validity of Part *(i)* of the lemma then follows from (4.8) and (4.16). Part *(ii)* of the lemma can be proved similarly. Regarding now the asymptotic estimate (4.7$c, n = m-1$) of Part *(iii)*, it readily follows by setting $t = \frac{1}{2}$ in (4.10) and taking into account (4.12), (4.14) and (4.15). The asymptotic estimate (4.7c) for $n = m+1$ can be proved similarly. Finally, $(4.7d)$ is a direct consequence of (4.10) for $t = \frac{1}{2}$, (4.14), (4.15) and the following limiting relations :

$$\lim_{k_{m-1},k_m,k_{m+1}\to\infty} \frac{q_m}{q_{m+1}} = \xi_m, \quad k_m/k_{m-1} \to c_1, \quad k_m/k_{m+1} \to c_2, \tag{4.17}$$

and

$$\lim_{k_{m-1},k_m,k_{m+1}\to\infty} \mathbb{A}_{m,32}\, q_m = \xi_m(s_{1m}s_{2,m+1} - s_{1,m+1}s_{2m}). \tag{4.18}$$

■

For $t = 0$ equation (4.10) degenerates to the condition

$$\mathbb{T}_{33}(0)q_m = \mathbb{A}_{m,33}\, q_m + \mathbb{A}_{m,31}\, q_m \frac{d_m}{k_m - 1}. \tag{4.19}$$

Using then (4.14) and (4.15) we get :

$$\mathbb{T}_{33}(0)q_m \to s_{1,m-1}s_{2m} - s_{1m}s_{2,m-1}, \tag{4.20}$$

as $k_{m-1}, k_m \to \infty$, and similarly for $\mathbb{T}_{32}(0)$ and $\mathbb{T}_{31}(0)$. Therefore, the following property is obtained.

Corollary 4.4. If $k_{m-1}, k_m \to \infty$, then

$$q_m \mathbf{w}_m \to \frac{\mathbf{P}_m}{h_{m-1}h_m}, \quad \mathbf{w}_m := \mathbf{w}(u_m). \tag{4.21}$$

It is noticeable that, in contrast to Lemma 4.3, the above corollary is free from any restrictions on the relative rate of growth of k_{m-1} and k_m.

We are now ready to focus on the asymptotic behaviour of the curvature $\kappa(u)$, which is the subject of the next theorem.

Theorem 4.5.

(i) If $k_m \to \infty$, then

$$\kappa(u) \to 0, \quad u \in (u_m, u_{m+1}). \tag{4.22}$$

(ii) Let $|\mathbf{P}_m| \neq 0$. If $k_{m-1}, k_m \to \infty$, then

$$\kappa(u_m) \to \infty. \tag{4.23}$$

Proof. *(i)* Differentiating both sides of formula (3.1) we find

$$\dot{Q}_i(u) - s_{im} = -R_{im}\frac{h_m}{q_m}F'_m(1-t) + R_{i,m+1}\frac{h_m}{q_{m+1}}F'_m(t), \tag{4.24}$$

where R_{im} is given by $(4.1a)$. Thus relation $(4.1a)$ provides the inequality :

$$\left|\dot{Q}_i(u) - s_{im}\right| \leq M\left\{\left|\frac{k_m(1-t)^{k_m-1}-1}{k_m-1}\right| + \left|\frac{k_m t^{k_m-1}-1}{k_m-1}\right|\right\}. \tag{4.25}$$

Since :

$$\lim_{k_m \to \infty} \frac{k_m v^{k_m - 1} - 1}{k_m - 1} = 0, \quad v = t \text{ or } 1 - t, \quad t \in (0,1), \tag{4.26}$$

it follows that

$$\lim_{k_m \to \infty} \dot{\mathbf{Q}}(u) = \mathbf{s}_m, \quad t \in (0,1). \tag{4.27}$$

Analogously we get :

$$\lim_{k_m \to \infty} \ddot{\mathbf{Q}}(u) = (0,0,0)^T, \quad t \in (0,1). \tag{4.28}$$

On the other hand, from (2.6) we have :

$$\kappa(u) \leq \frac{\left|\dot{\mathbf{Q}}(u)\right| \left|\ddot{\mathbf{Q}}(u)\right|}{\left|\dot{\mathbf{Q}}(u)\right|^3} = \frac{\left|\ddot{\mathbf{Q}}(u)\right|}{\left|\dot{\mathbf{Q}}(u)\right|^2}, \tag{4.29}$$

which, in conjunction with (4.27), (4.28) and the fact that $|\mathbf{s}_m| \neq 0$, implies the validity of (4.22).
(ii) Setting $t = 0$ in (4.25) we obtain

$$\left|\dot{Q}_i(u_m) - s_{im}\right| \leq M + \frac{M}{k_m - 1} \leq 2M, \tag{4.30a}$$

which implies

$$\left|\dot{Q}_i(u_m)\right| \leq \max\{|s_{im} - 2M|, |s_{im} + 2M|\}. \tag{4.30b}$$

The above inequality in conjunction with formula (2.6) and Corollary 4.4 establish (4.23), provided that the curve remains regular at $u = u_m$, as the degrees k_{m-1}, k_m tend to infinity. This is proved in the sequel.

Since $|\dot{\mathbf{Q}}_m|$ is uniformly bounded, the limiting relation (4.21) yields :

$$\lim_{k_{m-1}, k_m \to \infty} |\dot{\mathbf{Q}}_m \times \mathbf{R}_m| = \lim_{k_{m-1}, k_m \to \infty} |\dot{\mathbf{Q}}_m \times \mathbf{b}_m| = \frac{|\mathbf{P}_m|}{h_{m-1} h_m} > 0. \tag{4.31}$$

Thus, for sufficiently large degrees k_{m-1}, k_m, there exists a positive constant c such that :

$$|\dot{\mathbf{Q}}_m \times \mathbf{b}_m| \geq c, \tag{4.32a}$$

which provides

$$|\dot{\mathbf{Q}}_m||\mathbf{b}_m| \geq c \quad \Rightarrow \quad |\dot{\mathbf{Q}}_m| \geq \frac{c}{|\mathbf{b}_m|}, \tag{4.32b}$$

i.e., $\mathbf{Q}(u)$ is regular at $u = u_m$ for sufficiently large degrees k_{m-1}, k_m. ∎

Having completed the study of the asymptotic behaviour of the curvature, we now turn to study the asymptotic behaviour of the torsion. We start by proving the following lemma :

Lemma 4.6. If $k_{m-1}, k_m, k_{m+1} \to \infty$, then

$$\det(\mathbb{A}_m) q_m q_{m+1} \to \frac{\Delta_m}{h_{m-1} h_m h_{m+1}}, \tag{4.33}$$

where $\Delta_m = |\mathbf{L}_{m-1} \quad \mathbf{L}_m \quad \mathbf{L}_{m+1}|$ (see §2).

Proof. Using formula (3.9) and appealing to the limiting relation (4.1b), we have :

$$\lim_{k_{m-1},k_m,k_{m+1}\to\infty}\{\det(\mathbb{A}_m)q_m q_{m+1}\} = |\mathbf{s}_m \quad \mathbf{b}_m \quad \mathbf{b}_{m+1}|. \tag{4.34}$$

Substituting the last of (3.5d) into (4.34), some straightforward algebra gives

$$\lim_{k_{m-1},k_m,k_{m+1}\to\infty}\{\det(\mathbb{A}_m)q_m q_{m+1}\} = \frac{|\mathbf{L}_{m-1} \quad \mathbf{L}_m \quad \mathbf{L}_{m+1}|}{h_{m-1}h_m h_{m+1}}, \tag{4.35}$$

which proves the lemma. ■

Now comes the basic result :

Theorem 4.7. *(i)* Let $|\mathbf{P}_m| \neq 0$. If $k_{m-1}, k_m, k_{m+1} \to \infty$ with $k_m/k_{m-1} = O(1)$, then

$$\tau(u) \to 0, \quad t \in (0, \frac{1}{2}). \tag{4.36a}$$

(ii) Let $|\mathbf{P}_{m+1}| \neq 0$. If $k_{m-1}, k_m, k_{m+1} \to \infty$ with $k_m/k_{m+1} = O(1)$, then

$$\tau(u) \to 0, \quad t \in (\frac{1}{2}, 1). \tag{4.36b}$$

(iii) Let $\Delta_m \neq 0$. If $k_{m-1}, k_m, k_{m+1} \to \infty$ with $k_m/k_{m-1} \to c_1 \geq 0$ and $k_m/k_{m+1} \to c_2 \geq 0$, then

$$|\tau(\bar{u}_m)| \to \infty, \quad \bar{u}_m = \frac{1}{2}(u_m + u_{m+1}). \tag{4.36c}$$

(iv) Let $\Delta_m = 0$. If $k_{m-1}, k_m, k_{m+1} \to \infty$ with $k_m/k_{m-1} \to c_1 \geq 0$, $k_m/k_{m+1} \to c_2 \geq 0$ and $|\mathbf{P}_m h_{m-1}^{-1} + \xi_m \mathbf{P}_{m+1} h_{m+1}^{-1}| \neq 0$, $\xi_m = (c_1 h_{m-1} + h_m)/(h_m + c_2 h_{m+1})$, then

$$\tau(\bar{u}_m) = O(1). \tag{4.36d}$$

Proof. *(i)* Using formulae (2.7), (3.8) and the expansion (3.11) of the determinant of $\mathbb{B}(t)$, we have :

$$\tau(u) = \frac{\det(\mathbb{A}_m)h_m^{-1}(k_m-2)[t(1-t)]^{k_m-3}}{|\mathbf{w}(u)|^2}. \tag{4.37}$$

Equivalently, we write :

$$\tau(u)\frac{q_m q_{m+1} h_m [t(1-t)]^{3-k_m}}{(k_m-2)q_m^2(1-t)^{4-2k_m}} = \frac{\det(\mathbb{A}_m)q_m q_{m+1}}{[|\mathbf{w}(u)|q_m(1-t)^{2-k_m}]^2}, \tag{4.38}$$

from which, by virtue of Lemmata 4.3 and 4.6, we arrive at the limiting relation :

$$\lim_{k_{m-1},k_m,k_{m+1}\to\infty} \tau(u)\left[\frac{q_m}{q_{m+1}}\frac{k_m-2}{h_m}t^{-2}\left(\frac{t}{1-t}\right)^{k_m-1}\right]^{-1} = \frac{\Delta_m(h_{m-1}h_m h_{m+1})^{-1}}{|\mathbf{P}_m|^2(h_{m-1}h_m)^{-2}}, \tag{4.39}$$

with $k_m/k_{m-1} = O(1)$. Since,

$$\lim_{k_m\to\infty}\left\{\frac{k_m-2}{h_m}t^{-2}\left(\frac{t}{1-t}\right)^{k_m-1}\right\} = 0, \quad t \in (0, \frac{1}{2}), \tag{4.40}$$

and

$$\frac{q_m}{q_{m+1}} < \frac{h_{m-1}}{h_m}\,\frac{k_m}{k_{m-1}} + 1, \tag{4.41}$$

which is bounded above as $k_{m-1}, k_m \to \infty$ with $k_m/k_{m-1} = O(1)$, we conclude that :

$$\lim_{k_{m-1},k_m,k_{m+1}\to\infty} \tau(u) = 0, \quad t \in (0, \frac{1}{2}), \text{ with } k_m/k_{m-1} = O(1). \tag{4.42}$$

We can similarly prove Part *(ii)* of Theorem 4.7.
(iii) By setting $t = \frac{1}{2}$, formula (4.38) degenerates to :

$$|\tau(\bar{u}_m)|\frac{q_{m+1}h_m(k_m-2)^{-1}}{4q_m} = \frac{|\det(\mathbb{A}_m)|q_m q_{m+1}}{\left[|\mathbf{w}(\bar{u}_m)|q_m(\frac{1}{2})^{2-k_m}\right]^2}. \tag{4.43}$$

Moreover, the limit (4.17) implies

$$\lim_{k_{m-1},k_m,k_{m+1}\to\infty} \left\{\frac{q_m}{q_{m+1}}\,\frac{k_m-2}{h_m}\right\} = \infty. \tag{4.44}$$

Regarding the right-hand side fraction of (4.43), Lemma 4.6 ensures that, as $k_{m-1}, k_m, k_{m+1} \to \infty$, the numerator is bounded below by a positive quantity, for $\Delta_m \neq 0$. Moreover, Lemma 4.3*(iii)* guarantees that, as $k_{m-1}, k_m, k_{m+1} \to \infty$ with $k_m/k_{m-1} \to c_1$ and $k_m/k_{m+1} \to c_2$, the denominator of the same fraction is bounded above. Note that, if $\Delta_m \neq 0$, then the vectors $\mathbf{P}_m$ and $\mathbf{P}_{m+1}$ are linearly independent and, thus, the limit of the denominator is different from zero. Thus, the left-hand side of formula (4.43) is bounded below by a positive quantity, as $k_{m-1}, k_m, k_{m+1} \to \infty$ with $k_m/k_{m-1} \to c_1$ and $k_m/k_{m+1} \to c_2$. By virtue of this result and the limiting relation (4.44) we have :

$$\lim_{k_{m-1},k_m,k_{m+1}\to\infty} |\tau(\bar{u}_m)| = \infty, \tag{4.45}$$

as required.
(iv) Let $\mathbf{f}_m = \mathbf{R}_m - \mathbf{b}_m$. Recalling formula (3.9), we can write :

$$\begin{aligned} \det(\mathbb{A}_m)\; q_m q_{m+1} &= |\mathbf{s}_m \quad \mathbf{b}_m + \mathbf{f}_m \quad \mathbf{b}_{m+1} + \mathbf{f}_{m+1}| \\ &= |\mathbf{s}_m\ \ \mathbf{b}_m\ \ \mathbf{b}_{m+1}| + |\mathbf{s}_m\ \ \mathbf{f}_m\ \ \mathbf{b}_{m+1}| + |\mathbf{s}_m\ \ \mathbf{b}_m\ \ \mathbf{f}_{m+1}| + |\mathbf{s}_m\ \ \mathbf{f}_m\ \ \mathbf{f}_{m+1}|. \end{aligned} \tag{4.46}$$

Since $\Delta_m = 0$, we have $|\mathbf{s}_m\ \ \mathbf{b}_m\ \ \mathbf{b}_{m+1}| = 0$. Thus :

$$\det(\mathbb{A}_m)q_m q_{m+1} = k_m^{-1}|\mathbf{s}_m\ \ k_m\mathbf{f}_m\ \ \mathbf{b}_{m+1}| + k_m^{-1}|\mathbf{s}_m\ \ \mathbf{b}_m\ \ k_m\mathbf{f}_{m+1}| + |\mathbf{s}_m\ \ \mathbf{f}_m\ \ \mathbf{f}_{m+1}|. \tag{4.47}$$

Note that :

$$|\mathbf{f}_m| = |\mathbf{R}_m - \mathbf{b}_m| \leq \sqrt{3}\,M\left(\frac{1}{k_{m-1}-1} + \frac{1}{k_m-1}\right), \tag{4.48}$$

which stems from relation $(4.1a)$ and equations $(3.5b)$ and $(3.5d)$. Combining (4.47), (4.48) and the given limits $k_m/k_{m-1} \to c_1,\ k_m/k_{m+1} \to c_2$ as $k_{m-1}, k_m, k_{m+1} \to \infty$, we conclude :

$$\det(\mathbb{A}_m)q_m q_{m+1} = O(k_m^{-1}). \tag{4.49}$$

Let us now rewrite equation (4.43) in the form :

$$|\tau(\bar{u}_m)| = \frac{4}{h_m}\,\frac{q_m}{q_{m+1}}\,\frac{(k_m-2)|\det(\mathbb{A}_m)q_m q_{m+1}|}{\left[|\mathbf{w}(\bar{u}_m)|q_m(\frac{1}{2})^{2-k_m}\right]^2}. \tag{4.50}$$

Combining the asymptotic estimate (4.49) with the limiting relations (4.17) and (4.7d) (see Part *(iii)* of Lemma 4.3), we easily see that (4.36d) holds true, under the assumption $|\mathbf{P}_m h_{m-1}^{-1} + \xi_m \mathbf{P}_{m+1} h_{m+1}^{-1}| \neq 0$. It is noticeable that this assumption is always satisfied when the planar data $\mathbf{I}_{m-1}, \mathbf{I}_m, \mathbf{I}_{m+1}, \mathbf{I}_{m+2}$ are locally convex ($\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$). ■

Finally, we state Theorem 4.8, which is concerned with the asymptotic properties of the Frénet frame of an element $\mathbf{Q}(u) \in \mathbb{\Gamma}(\mathrm{K})$:

Theorem 4.8. Let $\mathbf{v}_t(u)$, $\mathbf{v}_n(u)$ and $\mathbf{v}_b(u)$ be the unit-tangent vector, the principal normal and the binormal vector of the Frénet frame of an element $\mathbf{Q}(u) \in \mathbb{\Gamma}(\mathrm{K})$. Then :

(i) If $k_m \to \infty$, then

$$\mathbf{v}_t(u) \to \frac{\mathbf{L}_m}{|\mathbf{L}_m|}, \quad t \in (0,1). \tag{4.51a}$$

If $k_{m-1}, k_m \to \infty$ with $k_m/k_{m-1} \to c$ and $|\mathbf{s}_{m-1}h_m + c\,\mathbf{s}_m h_{m-1}| \neq 0$, then

$$\mathbf{v}_t(u_m) \to \frac{\mathbf{s}_{m-1}h_m + c\,\mathbf{s}_m h_{m-1}}{|\mathbf{s}_{m-1}h_m + c\,\mathbf{s}_m h_{m-1}|}. \tag{4.51b}$$

(ii) Let $|\mathbf{P}_m| \neq 0$. If $k_{m-1}, k_m \to \infty$ with $k_m/k_{m-1} = O(1)$, then

$$\mathbf{v}_n(u) \to \frac{\mathbf{P}_m \times \mathbf{L}_m}{|\mathbf{P}_m||\mathbf{L}_m|}, \quad t \in (0, \frac{1}{2}). \tag{4.52a}$$

Let $|\mathbf{P}_{m+1}| \neq 0$. If $k_m, k_{m+1} \to \infty$ with $k_m/k_{m+1} = O(1)$, then

$$\mathbf{v}_n(u) \to \frac{\mathbf{P}_{m+1} \times \mathbf{L}_m}{|\mathbf{P}_{m+1}||\mathbf{L}_m|}, \quad t \in (\frac{1}{2}, 1). \tag{4.52b}$$

If $k_{m-1}, k_m, k_{m+1} \to \infty$ with $k_m/k_{m-1} \to c_1 \geq 0$, $k_m/k_{m+1} \to c_2 \geq 0$ and $|\mathbf{P}_m h_{m-1}^{-1} + \xi_m \mathbf{P}_{m+1} h_{m+1}^{-1}| \neq 0$, $\xi_m = (c_1 h_{m-1} + h_m)/(h_m + c_2 h_{m+1})$, then

$$\mathbf{v}_n(\bar{u}_m) \to \frac{(\mathbf{P}_m \times \mathbf{L}_m)h_{m-1}^{-1} + \xi_m(\mathbf{P}_{m+1} \times \mathbf{L}_m)h_{m+1}^{-1}}{|\mathbf{P}_m h_{m-1}^{-1} + \xi_m \mathbf{P}_{m+1} h_{m+1}^{-1}||\mathbf{L}_m|}. \tag{4.52c}$$

Let $|\mathbf{P}_m| \neq 0$. If $k_{m-1}, k_m \to \infty$ with $k_m/k_{m-1} \to c \geq 0$ and $|\mathbf{s}_{m-1}h_m + c\,\mathbf{s}_m h_{m-1}| \neq 0$, then

$$\mathbf{v}_n(u_m) \to \frac{\mathbf{P}_m \times (\mathbf{s}_{m-1}h_m + c\,\mathbf{s}_m h_{m-1})}{|\mathbf{P}_m||\mathbf{s}_{m-1}h_m + c\,\mathbf{s}_m h_{m-1}|}. \tag{4.52d}$$

(iii) Let $|\mathbf{P}_m| \neq 0$. If $k_{m-1}, k_m \to \infty$ with $k_m/k_{m-1} = O(1)$, then

$$\mathbf{v}_b(u) \to \frac{\mathbf{P}_m}{|\mathbf{P}_m|}, \quad t \in [0, \frac{1}{2}). \tag{4.53a}$$

Let $|\mathbf{P}_{m+1}| \neq 0$. If $k_m, k_{m+1} \to \infty$ with $k_m/k_{m+1} = O(1)$, then

$$\mathbf{v}_b(u) \to \frac{\mathbf{P}_{m+1}}{|\mathbf{P}_{m+1}|}, \quad t \in (\frac{1}{2}, 1]. \tag{4.53b}$$

If $k_{m-1}, k_m, k_{m+1} \to \infty$ with $k_m/k_{m-1} \to c_1 \geq 0$, $k_m/k_{m+1} \to c_2 \geq 0$ and $|\mathbf{P}_m h_{m-1}^{-1} + \xi_m \mathbf{P}_{m+1} h_{m+1}^{-1}| \neq 0$, $\xi_m = (c_1 h_{m-1} + h_m)/(h_m + c_2 h_{m+1})$, then

$$\mathbf{v}_b(\bar{u}_m) \to \frac{\mathbf{P}_m h_{m-1}^{-1} + \xi_m \mathbf{P}_{m+1} h_{m+1}^{-1}}{|\mathbf{P}_m h_{m-1}^{-1} + \xi_m \mathbf{P}_{m+1} h_{m+1}^{-1}|}. \tag{4.53c}$$

Proof. The limits $(4.51a)$, $(4.52a - c)$ and (4.53) can be proved easily, using the defining formulae (2.4), (2.5) and (2.8) of $\mathbf{v}_t(u)$, $\mathbf{v}_b(u)$ and $\mathbf{v}_n(u)$, respectively, in conjunction with Lemma 4.3 and the limiting relation (4.27). Regarding now $(4.51b)$, setting $t = 0$ in formula (4.24) gives :

$$\dot{\mathbf{Q}}(u_m) = \mathbf{s}_m - \mathbf{R}_m \frac{h_m}{k_m q_m} - \mathbf{R}_{m+1} \frac{h_m}{k_m(k_m - 1) q_{m+1}}. \tag{4.54}$$

Using the limit $(4.1b)$ and the fact that $k_m/k_{m-1} \to c$ as $k_{m-1}, k_m \to \infty$, we find

$$\lim_{k_{m-1}, k_m \to \infty} \dot{\mathbf{Q}}(u_m) = \mathbf{s}_m - h_m \mathbf{b}_m \frac{1}{c\, h_{m-1} + h_m} = \frac{h_m \mathbf{s}_{m-1} + c\, h_{m-1} \mathbf{s}_m}{c\, h_{m-1} + h_m}. \tag{4.55}$$

In view of (2.4), (4.55) and the fact that by assumption $|\mathbf{s}_{m-1} h_m + c\, \mathbf{s}_m h_{m-1}| \neq 0$, the limit $(4.51b)$ is true. Note that $|\mathbf{s}_{m-1} h_m + c\, \mathbf{s}_m h_{m-1}| \neq 0$ is always satisfied if $\mathbf{L}_{m-1} \cdot \mathbf{L}_m > 0$. Finally, the limiting relation $(4.52d)$ is a direct consequence of the defining formula (2.8) and the limits $(4.51b)$ and $(4.53a)$. ■

Noting that, if $\Delta_m = 0$ and $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, then $\mathbf{P}_m$ and $\mathbf{P}_{m+1}$ are parallel, Theorem 4.8 provides the following property.

Corollary 4.9. Let $\Delta_m = 0$ and $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$. If $k_{m-1}, k_m, k_{m+1} \to \infty$ with $k_m/k_{m-1} \to c_1 \geq 0$ and $k_m/k_{m+1} \to c_2 \geq 0$, then

$$\mathbf{v}_n(u) \to \frac{\mathbf{P}_m \times \mathbf{L}_m}{|\mathbf{P}_m||\mathbf{L}_m|} = \frac{\mathbf{P}_{m+1} \times \mathbf{L}_m}{|\mathbf{P}_{m+1}||\mathbf{L}_m|}, \quad t \in (0, 1), \tag{4.56}$$

and

$$\mathbf{v}_b(u) \to \frac{\mathbf{P}_m}{|\mathbf{P}_m|} = \frac{\mathbf{P}_{m+1}}{|\mathbf{P}_{m+1}|}, \quad t \in [0, 1]. \tag{4.57}$$

We end this section with some qualitative remarks on the various results obtained above. At first, by virtue of Theorem 4.2, we have that a spline $\mathbf{Q}(u) \in \mathbb{\Gamma}(\mathrm{K})$ tends to the linear interpolant $\mathrm{l}(u)$, as the degrees increase. Moreover, its curvature tends to vanish in (u_m, u_{m+1}) (see Theorem

4.5*(i)*), but it tends to infinity at the node $u = u_m$, provided that the points $\mathbf{I}_{m-1}, \mathbf{I}_m, \mathbf{I}_{m+1}$ are not collinear (see Theorem 4.5*(ii)*).

Regarding now the asymptotic behaviour of the Frénet frame, Theorem 4.8*(iii)* implies that, as the degrees increase, the binormal of $\mathbf{Q}(u)$ tends to the discrete binormal at $\mathbf{I}_m$ for $u \in [u_m, \bar{u}_m)$, $\bar{u}_m = \frac{1}{2}(u_m + u_{m+1})$. In $(\bar{u}_m, u_{m+1}]$, however, the binormal tends to the discrete binormal at $\mathbf{I}_{m+1}$. These results are in agreement with the asymptotic behaviour of the torsion $\tau(u)$ of $\mathbf{Q}(u)$, which tends to zero in $(u_m, \bar{u}_m) \cup (\bar{u}_m, u_{m+1})$ (see Theorem 4.7*(i)*,*(ii)*). However, if $\mathbf{I}_{m-1}, \mathbf{I}_m, \mathbf{I}_{m+1}, \mathbf{I}_{m+2}$ are not coplanar, then the torsion becomes unbounded at the midpoint $\bar{u}_m$ (see Theorem 4.7*(iii)*). On the other hand, if $\mathbf{I}_{m-1}, \mathbf{I}_m, \mathbf{I}_{m+1}, \mathbf{I}_{m+2}$ belong to the same plane, then the torsion remains finite at $u = \bar{u}_m$ (see Theorem 4.7*(iv)*).

5. Shape-preserving interpolation in $\mathbb{R}^3$ with the aid of $\mathbb{\Gamma}(\mathrm{K})$

Based on the results obtained in the preceding sections we shall develop an automatic algorithm for constructing F^3-continuous interpolants in $\mathbb{\Gamma}(\mathrm{K})$, which are shape-preserving in a slightly different sense from that of Definition 2.1. Specifically, the algorithm to be constructed obeys the following shape-preserving criterion.

Definition 2.1*. A curve $\mathbf{Q}(u) \in \mathcal{F}^3(\mathcal{U})$, which interpolates the data set $\mathcal{D}$ and satisfies boundary conditions $\mathcal{B}$, will be called *shape-preserving* provided that :

(i) (convexity criterion) If $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, then

$$\mathbf{w}(u) \cdot \mathbf{P}_n > 0, \quad u \in [u_m, u_{m+1}], \quad n = m, m+1. \tag{5.1}$$

(ii) (torsion criterion) If $\Delta_m \neq 0$, then

$$\tau(u)\Delta_m > 0, \quad u \in (u_m, u_{m+1}). \tag{5.2}$$

(iii.1) (coplanarity criterion) If $\Delta_m = 0$ and $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, then

$$\frac{|\mathbf{w}(u) \times \mathbf{P}_n|}{|\mathbf{w}(u)||\mathbf{P}_n|} < \varepsilon_1, \quad |\mathbf{w}(u)| \neq 0, \quad u \in \omega_m, \quad n = m, m+1, \tag{5.3a}$$

where ε_1 is a user-specified small positive number in $(0,1]$, and ω_m a user-specified closed interval such that $[u_m, u_{m+1}] \subseteq \omega_m \subset (\bar{u}_{m-1}, \bar{u}_{m+1})$, $\bar{u}_n = \frac{1}{2}(u_n + u_{n+1})$, $n = m-1, m+1$.

(iii.2) If $\Delta_m = 0$ and $\mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0$, then

$$\frac{|\mathbf{w}(u) \times \mathbf{P}_n|}{|\mathbf{w}(u)||\mathbf{P}_n|} < \varepsilon_1, \quad |\mathbf{w}(u)| \neq 0, \quad u \in \vartheta_m \cup \vartheta_{m+1}, \quad n = m, m+1, \tag{5.3b}$$

where $\vartheta_m \subset [\hat{u}^\ell_m, \bar{u}_m)$ and $\vartheta_{m+1} \subset (\bar{u}_m, \hat{u}^r_{m+1}]$ are user-specified closed intervals with $\hat{u}^\ell_m$ and $\hat{u}^r_{m+1}$ being user-specified constants such that $\bar{u}_{m-1} < \hat{u}^\ell_m \leq u_m$, and $u_{m+1} \leq \hat{u}^r_{m+1} < \bar{u}_{m+1}$.

(iv) (collinearity criterion) If $|\mathbf{P}_m| = 0$ and $\mathbf{L}_{m-1} \cdot \mathbf{L}_m > 0$, then

$$\frac{|\dot{\mathbf{Q}}(u) \times \mathbf{L}_n|}{|\dot{\mathbf{Q}}(u)||\mathbf{L}_n|} < \varepsilon_0, \quad u \in \eta_m, \quad n = m-1 \text{ and/or } m, \tag{5.4}$$

where ε_0 is a user-specified small positive number in $(0,1]$, and η_m a user-specified closed subinterval of (u_{m-1}, u_{m+1}) that includes u_m as an interior point.

Definition 2.1* lacks Part *(ii.2)* of Definition 2.1, due to an intrinsic property of curves in $\mathbb{\Gamma}(\mathrm{K})$, namely $\tau(u_m) = 0, m = 1(1)N$. As far as Part *(iii.1)* is concerned, it is again the structure of $\mathbb{\Gamma}(\mathrm{K})$ that requires ω_m to lie inside $(\bar{u}_{m-1}, \bar{u}_{m+1})$, which is clearly contained in (u_{m-1}, u_{m+2}), appearing in Definition 2.1. In particular, Lemma 4.3 implies that, as the segment degrees $k_n, n = (m-2)(1)(m+2)$, increase, the limiting osculating planes corresponding to $[u_{m-1}, \bar{u}_{m-1})$ and $(\bar{u}_{m+1}, u_{m+2}]$ are in general different from the limiting osculating plane of $(\bar{u}_{m-1}, \bar{u}_{m+1})$, which is exactly the common plane of the points $\mathbf{I}_{m-1}, \mathbf{I}_m, \mathbf{I}_{m+1}$ and $\mathbf{I}_{m+2}$. This remark applies also to Part *(iii.2)* of Definition 2.1*, with the additional characteristic that the midpoint $\bar{u}_m$ of $[u_m, u_{m+1}]$ has to be excluded from the parameter domain of the coplanarity criterion. Indeed, for $\Delta_m = 0$ with $\mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0$, the distribution of the limiting binormal exhibits a discontinuity at the midpoint $\bar{u}_m$ (see expressions $(4.7a)$ and $(4.7b)$).

We shall now proceed to derive conditions ensuring that an element $\mathbf{Q}(u) \in \mathbb{\Gamma}(\mathrm{K})$ satisfies each one of the five parts of Definition 2.1*. More specifically, for each part we shall provide a lemma and a theorem. The lemma will contain sufficient, and if possible necessary, conditions for satisfying the corresponding part, whereas the theorem will guarantee the validity of these conditions for appropriately large degrees in suitably chosen parameter intervals. These conditions will be discrete and will depend on nodal quantities only, so that they can be tested efficiently in practice.

To improve the readability of the text, the rest of this section is divided into five subsections. The first four address the convexity, torsion, coplanarity and collinearity criteria of Definition 2.1*, while the last one is devoted to the shape-preserving algorithm.

5.1. The convexity criterion

Lemma 5.1. Let $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$ and $\mathbf{g}_m := \ddot{\mathbf{Q}}_{m+1} \times \ddot{\mathbf{Q}}_m$. Furthermore, we introduce the assertions

(i) $\mathbf{w}_m \cdot \mathbf{P}_n > 0$, $\mathbf{w}_{m+1} \cdot \mathbf{P}_n > 0$, $n = m, m+1$.

(ii) $\mathbf{g}_m \cdot \mathbf{P}_n \geq 0$ or

$$|\mathbf{g}_m \cdot \mathbf{P}_n| < h_m^{-1} 2^{k_m - 1}(k_m - 1) \min\{|\mathbf{w}_m \cdot \mathbf{P}_n|, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n|\}, \tag{5.5}$$

for $n = m, m+1$.

Then, *(i)* is necessary, whereas *(i)* along with *(ii)* are sufficient conditions for satisfying the convexity criterion *(i)* of Definition 2.1*.

Proof. *Necessity of (i)* : The necessity comes directly from relation (5.1) for $u = u_m, u_{m+1}$.
Sufficiency of (i) and (ii) : Using formulae (3.1) and (2.5), we get after some straightforward algebra the following expression for $\mathbf{w}(u)$:

$$\mathbf{w}(u) = \mathbf{w}_m F_m''(1-t) + \mathbf{w}_{m+1} F_m''(t) + h_m \mathbf{g}_m \varphi_m(t), \quad u \in [u_m, u_{m+1}], \tag{5.6}$$

where

$$\varphi_m(t) = \frac{F_m''(t)\, F_m''(1-t)}{k_m - 1} = \frac{t^{k_m-2}(1-t)^{k_m-2}}{k_m - 1}, \quad t \in [0,1]. \tag{5.7}$$

Let $\mathbf{g}_m \cdot \mathbf{P}_n \geq 0$. Then, as $F_m''(t)$ and $F_m''(1-t)$ are non-negative for $t \in [0,1]$ and do not vanish simultaneously, we readily conclude from *(i)* that $\mathbf{w}(u) \cdot \mathbf{P}_n > 0$, $u \in [u_m, u_{m+1}]$.

Turning to (5.5), we also deduce from *(i)* and (5.6) that the condition

$$\max_{t\in[0,1]}\{|\mathbf{g}_m \cdot \mathbf{P}_n| h_m \varphi_m(t)\} < \min_{t\in[0,1]}\{|(\mathbf{w}_m \cdot \mathbf{P}_n) F_m''(1-t) + (\mathbf{w}_{m+1} \cdot \mathbf{P}_n) F_m''(t)|\}. \tag{5.8}$$

is sufficient for the validity of $\mathbf{w}(u) \cdot \mathbf{P}_n > 0$ For the left-hand side of the above inequality, we have :

$$\max_{t\in[0,1]}\{|\mathbf{g}_m \cdot \mathbf{P}_n| h_m \varphi_m(t)\} = |\mathbf{g}_m \cdot \mathbf{P}_n| h_m \varphi_m(\frac{1}{2}) = |\mathbf{g}_m \cdot \mathbf{P}_n| h_m \frac{2^{4-2k_m}}{k_m - 1}. \tag{5.9a}$$

Recalling that $\mathbf{w}_m \cdot \mathbf{P}_n$ and $\mathbf{w}_{m+1} \cdot \mathbf{P}_n$ are positive as well as $F''_m(t) \geq 0$, we can write

$$\begin{aligned}
&|(\mathbf{w}_m \cdot \mathbf{P}_n)F''_m(1-t) + (\mathbf{w}_{m+1} \cdot \mathbf{P}_n)F''_m(t)| \\
&\geq \min\{|\mathbf{w}_m \cdot \mathbf{P}_n|, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n|\}(F''_m(1-t) + F''_m(t)) \\
&\geq \min\{|\mathbf{w}_m \cdot \mathbf{P}_n|, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n|\} \min_{t\in[0,1]}\{F''_m(1-t) + F''_m(t)\} \\
&= 2F''_m(\frac{1}{2}) \min\{|\mathbf{w}_m \cdot \mathbf{P}_n|, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n|\} \\
&= 2^{3-k_m} \min\{|\mathbf{w}_m \cdot \mathbf{P}_n|, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n|\}, \qquad t \in [0,1].
\end{aligned} \tag{5.9b}$$

In view of (5.9a) and (5.9b), we readily conclude that a sufficient condition ensuring the validity of (5.8) is :

$$h_m \frac{2^{4-2k_m}}{k_m - 1} |\mathbf{g}_m \cdot \mathbf{P}_n| < 2^{3-k_m} \min\{|\mathbf{w}_m \cdot \mathbf{P}_n|, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n|\}, \tag{5.10}$$

which obviously coincides with (5.5). Since, $n = m, m+1$, the above results ensure that Part *(i)* of Definition 2.1* is fulfilled. ■

The following theorem provides a manner to satisfy inequalities (5.5).

Theorem 5.2. If $\mathbf{P}_m \cdot \mathbf{P}_{m+1} \neq 0$ and k_{m-1}, k_m, k_{m+1} are sufficiently large, then

$$|\mathbf{g}_m \cdot \mathbf{P}_n| < h_m^{-1} 2^{k_m - 1}(k_m - 1) \min\{|\mathbf{w}_m \cdot \mathbf{P}_n|, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n|\}, \ n = m, m+1. \tag{5.11}$$

Proof. For the left-hand side of (5.11), Lemma 4.1 (rel. (4.1a)) gives :

$$q_m q_{m+1} |\mathbf{g}_m \cdot \mathbf{P}_n| \leq 3M^2 |\mathbf{P}_n|. \tag{5.12}$$

For the right-hand side of (5.11), it can easily be seen that :

$$\begin{aligned}
&q_m q_{m+1} \ h_m^{-1} 2^{k_m - 1}(k_m - 1) \min\{|\mathbf{w}_m \cdot \mathbf{P}_n|, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n|\} \\
&\qquad \geq h_m^{-1} 2^{k_m - 1}(k_m - 1) d_m \min\{|\mathbf{w}_m \cdot \mathbf{P}_n| q_m, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n| q_{m+1}\}.
\end{aligned} \tag{5.13}$$

Recalling now Corollary 4.4, we have :

$$\lim_{k_{m-1}, k_m, k_{m+1} \to \infty} \min\{|\mathbf{w}_m \cdot \mathbf{P}_n| q_m, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n| q_{m+1}\} = \min\left\{\frac{|\mathbf{P}_m \cdot \mathbf{P}_n|}{h_{m-1} h_m}, \frac{|\mathbf{P}_{m+1} \cdot \mathbf{P}_n|}{h_m h_{m+1}}\right\}, \tag{5.14}$$

which, in conjunction with the fact that $d_m = h_m / k_m$ and the assumption $\mathbf{P}_m \cdot \mathbf{P}_{m+1} \neq 0$, gives :

$$h_m^{-1} 2^{k_m - 1} d_m (k_m - 1) \min\{|\mathbf{w}_m \cdot \mathbf{P}_n| q_m, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n| q_{m+1}\} \to \infty, \tag{5.15}$$

as $k_{m-1}, k_m, k_{m+1} \to \infty$. In view of (5.12), (5.15) and the positivity of the multiplier $q_m q_{m+1}$, it is inferred that the lemma in question holds true. ■

5.2. The torsion criterion

As far as the torsion criterion *(ii)* of Definition 2.1* is concerned, formulae (2.7) and (3.8) in combination with inequality (3.11) yield

Lemma 5.3. Let $\Delta_m \neq 0$. Then $\mathbf{Q}(u)$ satisfies the torsion criterion *(ii)* of Definition 2.1*, if and only if $\det(\mathbb{A}_m)\Delta_m > 0$.

Using now Lemma 4.6, we readily establish

Theorem 5.4. If $\Delta_m \neq 0$ and k_{m-1}, k_m, k_{m+1} are sufficiently large, then $\det(\mathbb{A}_m)\Delta_m > 0$.

5.3. The coplanarity criterion

Having treated the convexity and torsion criteria of Definition 2.1*, we now turn to the coplanarity criterion of the definition. We shall, at first, deal with the case $\Delta_m = 0$ and $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$ (see Part *(iii.1)* of Def. 2.1*). In this case the following lemma is valid.

Lemma 5.5. Let $\Delta_m = 0$, $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$ and $\omega_m = [\hat{u}^\ell_m, \hat{u}^r_{m+1}]$, $\bar{u}_{m-1} < \hat{u}^\ell_m \leq u_m$, $u_{m+1} \leq \hat{u}^r_{m+1} < \bar{u}_{m+1}$, be the user-specified interval involved in the coplanarity criterion *(iii.1)* of Definition 2.1*. This criterion is satisfied in ω_m if there exist constants α_m, α^ℓ_m and α^r_{m+1} in $(0, \sqrt{2}-1)$ such that the following seven inequalities are true :

$$\mathbf{w}_m \cdot \mathbf{w}_{m+1} > 0, \tag{5.16a}$$

$$|\mathbf{g}_m| \leq \sqrt{2}\,\alpha_m h_m^{-1}(k_m - 1)2^{k_m-2}\min\{|\mathbf{w}_m|, |\mathbf{w}_{m+1}|\}, \tag{5.16b}$$

$$\begin{aligned} \mathsf{cpr}(m,n) := {} & \frac{\sqrt{2}\,\max\{|\mathbf{w}_m \times \mathbf{P}_n|, |\mathbf{w}_{m+1} \times \mathbf{P}_n|\}}{\sqrt{\beta_m}\,\min\{|\mathbf{w}_m|, |\mathbf{w}_{m+1}|\}\,|\mathbf{P}_n|} \\ & + \frac{\sqrt{2}\,(k_m-1)^{-1}|\mathbf{g}_m \times \mathbf{P}_n| h_m}{\sqrt{\beta_m}\,\min\{|\mathbf{w}_m|, |\mathbf{w}_{m+1}|\}\,|\mathbf{P}_n|}\left(\frac{1}{2}\right)^{k_m-1} < \varepsilon_1, \end{aligned} \tag{5.16c}$$

$$n = m, m+1, \qquad \beta_m = 1 - \alpha_m^2 - 2\,\alpha_m,$$

$$\begin{aligned} \mathsf{cplb}(m) := |\mathbf{w}_{m-1}|\left(\frac{1-t^\ell_m}{t^\ell_m}\right)^{k_{m-1}-2} + h_{m-1}|\mathbf{g}_{m-1}|\,\frac{(1-t^\ell_m)^{k_{m-1}-2}}{k_{m-1}-1} \leq \alpha^\ell_m|\mathbf{w}_m|, \\ t^\ell_m := \frac{\hat{u}^\ell_m - u_{m-1}}{h_{m-1}} \in (\frac{1}{2}, 1], \end{aligned} \tag{5.16d}$$

$$\begin{aligned} \mathsf{cplr}(m,n) := {} & \frac{|\mathbf{w}_{m-1} \times \mathbf{P}_n|}{\sqrt{\beta^\ell_m}\,|\mathbf{w}_m||\mathbf{P}_n|}\left(\frac{1-t^\ell_m}{t^\ell_m}\right)^{k_{m-1}-2} + \frac{|\mathbf{w}_m \times \mathbf{P}_n|}{\sqrt{\beta^\ell_m}\,|\mathbf{w}_m||\mathbf{P}_n|} \\ & + \frac{|\mathbf{g}_{m-1} \times \mathbf{P}_n| h_{m-1}(k_{m-1}-1)^{-1}(1-t^\ell_m)^{k_{m-1}-2}}{\sqrt{\beta^\ell_m}\,|\mathbf{w}_m|\,|\mathbf{P}_n|} < \varepsilon_1, \end{aligned} \tag{5.16e}$$

$$n = m, m+1, \qquad \beta^\ell_m = 1 - (\alpha^\ell_m)^2 - 2\,\alpha^\ell_m,$$

$$\mathsf{cprb}(m+1) := |\mathbf{w}_{m+2}| \left(\frac{t^r_{m+1}}{1-t^r_{m+1}}\right)^{k_{m+1}-2} + h_{m+1}|\mathbf{g}_{m+1}| \frac{(t^r_{m+1})^{k_{m+1}-2}}{k_{m+1}-1} \leq \alpha^r_{m+1}|\mathbf{w}_{m+1}|,$$

$$t^r_{m+1} := \frac{\hat{u}^r_{m+1} - u_{m+1}}{h_{m+1}} \in [0, \frac{1}{2}), \qquad \text{and} \tag{5.16f}$$

$$\mathsf{cprr}(m+1,n) := \frac{|\mathbf{w}_{m+1} \times \mathbf{P}_n|}{\sqrt{\beta^r_{m+1}}\,|\mathbf{w}_{m+1}||\mathbf{P}_n|} + \frac{|\mathbf{w}_{m+2} \times \mathbf{P}_n|}{\sqrt{\beta^r_{m+1}}\,|\mathbf{w}_{m+1}||\mathbf{P}_n|} \left(\frac{t^r_{m+1}}{1-t^r_{m+1}}\right)^{k_{m+1}-2}$$
$$+ \frac{|\mathbf{g}_{m+1} \times \mathbf{P}_n| h_{m+1}(k_{m+1}-1)^{-1}(t^r_{m+1})^{k_{m+1}-2}}{\sqrt{\beta^r_{m+1}}\,|\mathbf{w}_{m+1}|\,|\mathbf{P}_n|} < \varepsilon_1, \tag{5.16g}$$

$$n = m, m+1, \qquad \beta^r_{m+1} = 1 - (\alpha^r_{m+1})^2 - 2\,\alpha^r_{m+1}.$$

Proof. The technique of the proof derives lower bounds on the denominator and upper bounds on the numerator of the left-hand side fraction of inequality (5.3a). Furthermore, this technique handles separately the three intervals $[\hat{u}^\ell_m, u_m]$, $[u_m, u_{m+1}]$ and $[u_{m+1}, \hat{u}^r_{m+1}]$, which partition ω_m.

Let us start with $[u_m, u_{m+1}]$. From equation (5.6) we obtain

$$\begin{aligned} |\mathbf{w}(u)|^2 \;=&\; |\mathbf{w}_m F''_m(1-t) + \mathbf{w}_{m+1}F''_m(t)|^2 + |\mathbf{g}_m h_m \varphi_m(t)|^2 \\ &+2\,(\mathbf{w}_m F''_m(1-t) + \mathbf{w}_{m+1}F''_m(t)) \cdot (\mathbf{g}_m h_m \varphi_m(t)) \\ \geq&\; |\mathbf{w}_m F''_m(1-t) + \mathbf{w}_{m+1}F''_m(t)|^2 - |\mathbf{g}_m h_m \varphi_m(t)|^2 \\ &-2\,|\mathbf{w}_m F''_m(1-t) + \mathbf{w}_{m+1}F''_m(t)||\mathbf{g}_m h_m \varphi_m(t)|. \end{aligned} \tag{5.17}$$

Assuming that

$$|\mathbf{g}_m| h_m \varphi_m(t) \leq \alpha_m |\mathbf{w}_m F''_m(1-t) + \mathbf{w}_{m+1}F''_m(t)|, \tag{5.18}$$

(5.17) implies

$$\begin{aligned} |\mathbf{w}(u)|^2 \geq&\; \beta_m \min\{|\mathbf{w}_m|^2, |\mathbf{w}_{m+1}|^2\} \left[(F''_m(1-t))^2 + (F''_m(t))^2\right] + \\ &+2\,\beta_m(\mathbf{w}_m \cdot \mathbf{w}_{m+1}) F''_m(1-t)F''_m(t). \end{aligned} \tag{5.19}$$

Since, by hypothesis, $\mathbf{w}_m \cdot \mathbf{w}_{m+1} > 0$, and $\beta_m > 0$, due to the restriction $0 < \alpha_m < \sqrt{2}-1$, we get the lower bound

$$|\mathbf{w}(u)| \geq \sqrt{\beta_m}\, \min\{|\mathbf{w}_m|, |\mathbf{w}_{m+1}|\} \sqrt{(F''_m(1-t))^2 + (F''_m(t))^2}. \tag{5.20}$$

On the other hand, we deduce from equation (5.6) that the numerator in (5.3a) satisfies

$$|\mathbf{w}(u) \times \mathbf{P}_n| \leq \max\{|\mathbf{w}_m \times \mathbf{P}_n|, |\mathbf{w}_{m+1} \times \mathbf{P}_n|\}\; [F''_m(1-t) + F''_m(t)] + |\mathbf{g}_m \times \mathbf{P}_n| h_m \varphi_m(\frac{1}{2}), \tag{5.21}$$

taking account of the fact that $\varphi_m(\frac{1}{2}) = \max_{t\in[0,1]} \varphi_m(t)$. Using now (5.20) and (5.21), we find that the left-hand side fraction of inequality (5.3a) is bounded by

$$\begin{aligned}\frac{|\mathbf{w}(u)\times\mathbf{P}_n|}{|\mathbf{w}(u)||\mathbf{P}_n|} \le \frac{\max\{|\mathbf{w}_m\times\mathbf{P}_n|, |\mathbf{w}_{m+1}\times\mathbf{P}_n|\}}{\sqrt{\beta_m}\,\min\{|\mathbf{w}_m|, |\mathbf{w}_{m+1}|\}|\mathbf{P}_n|} & \frac{F''_m(1-t)+F''_m(t)}{\sqrt{(F''_m(1-t))^2+(F''_m(t))^2}} \\ +\frac{|\mathbf{g}_m\times\mathbf{P}_n|h_m}{\sqrt{\beta_m}\,\min\{|\mathbf{w}_m|, |\mathbf{w}_{m+1}|\}|\mathbf{P}_n|} & \frac{\varphi_m(\frac{1}{2})}{\sqrt{(F''_m(1-t))^2+(F''_m(t))^2}}.\end{aligned} \tag{5.22}$$

Since

$$\min_{t\in[0,1]}\left\{\sqrt{(F''_m(1-t))^2+(F''_m(t))^2}\right\} = \sqrt{2}\left(\frac{1}{2}\right)^{k_m-2}, \tag{5.23a}$$

$$\frac{F''_m(1-t)+F''_m(t)}{\sqrt{(F''_m(1-t))^2+(F''_m(t))^2}} \le \sqrt{2} \tag{5.23b}$$

and $\varphi_m(\frac{1}{2}) = (k_m-1)^{-1}\,(\frac{1}{2})^{2k_m-4}$, the right-hand side of (5.22) provides

$$\frac{|\mathbf{w}(u)\times\mathbf{P}_n|}{|\mathbf{w}(u)||\mathbf{P}_n|} \le \mathsf{cpr}(m,n). \tag{5.24}$$

In view of (5.24) and (5.16c), we can summarize the hitherto obtained results by stating that, if (5.16a), (5.18) and (5.16c) hold true, then the coplanarity criterion *(iii.1)* is valid in $[u_m, u_{m+1}]$. Next, we derive a discrete sufficient condition for the validity of (5.18). Making use of (5.16a), we obtain

$$\begin{aligned}|\mathbf{w}_m F''_m(1-t) + \mathbf{w}_{m+1}F''_m(t)|^2 &\ge \min\{|\mathbf{w}_m|^2, |\mathbf{w}_{m+1}|^2\}\ \min_{t\in[0,1]}\{[F''_m(1-t)]^2+[F''_m(t)]^2\} \\ &= 2\,\min\{|\mathbf{w}_m|^2, |\mathbf{w}_{m+1}|^2\}\left(\frac{1}{2}\right)^{2k_m-4}.\end{aligned} \tag{5.25}$$

On the other hand,

$$|\mathbf{g}_m|h_m\varphi_m(t) \le |\mathbf{g}_m|h_m\max_{t\in[0,1]}\varphi_m(t) = |\mathbf{g}_m|h_m(k_m-1)^{-1}\left(\frac{1}{2}\right)^{2k_m-4}. \tag{5.26}$$

Thus, a sufficient condition for (5.18) is :

$$|\mathbf{g}_m|h_m(k_m-1)^{-1}\left(\frac{1}{2}\right)^{2k_m-4} \le \sqrt{2}\,\alpha_m\left(\frac{1}{2}\right)^{k_m-2}\min\{|\mathbf{w}_m|, |\mathbf{w}_{m+1}|\}, \tag{5.27}$$

which is the same as inequality (5.16b). Thus, (5.16a), (5.16b) and (5.16c), if true, ensure the validity of the coplanarity criterion *(iii.1)* in $[u_m, u_{m+1}]$.

Let us now turn our attention to the neighbouring interval $[\hat{u}^\ell_m, u_m]$. In this case we introduce the hypothesis

$$|\mathbf{r}^*_{m-1}(u)| \le \alpha^\ell_m|\mathbf{w}_m F''_{m-1}(t)|, \quad u\in[\hat{u}^\ell_m, u_m], \tag{5.28}$$

where the replacement of m by $m-1$ in (5.6) provides

$$\mathbf{r}^*_{m-1}(u) := \mathbf{w}(u) - \mathbf{w}_m F''_{m-1}(t) = \mathbf{w}_{m-1}F''_{m-1}(1-t) + \mathbf{g}_{m-1}h_{m-1}\varphi_{m-1}(t). \tag{5.29}$$

Based on this hypothesis we find

$$
\begin{aligned}
|\mathbf{w}(u)|^2 &= |\mathbf{w}_m F''_{m-1}(t)|^2 + |\mathbf{r}^*_{m-1}(u)|^2 + 2\,(\mathbf{w}_m \cdot \mathbf{r}^*_{m-1}(u)) F''_{m-1}(t) \\
&\geq |\mathbf{w}_m|^2 \left(F''_{m-1}(t)\right)^2 - |\mathbf{r}^*_{m-1}(u)|^2 - 2\,|\mathbf{w}_m||\mathbf{r}^*_{m-1}(u)| F''_{m-1}(t) \\
&\geq (1 - (\alpha^\ell_m)^2 - 2\,\alpha^\ell_m)\,|\mathbf{w}_m|^2\,\left(F''_{m-1}(t)\right)^2 = \beta^\ell_m\,|\mathbf{w}_m|^2\,\left(F''_{m-1}(t)\right)^2 .
\end{aligned}
\tag{5.30}
$$

On the other hand, recalling (5.6) and (5.7), we get :

$$
\begin{aligned}
|\mathbf{w}(u) \times \mathbf{P}_n| &\leq |\mathbf{w}_{m-1} \times \mathbf{P}_n| F''_{m-1}(1-t) + |\mathbf{w}_m \times \mathbf{P}_n| F''_{m-1}(t) + |\mathbf{g}_{m-1} \times \mathbf{P}_n| h_{m-1} \varphi_{m-1}(t) \\
&\leq F''_{m-1}(t) \left\{ |\mathbf{w}_{m-1} \times \mathbf{P}_n| \left(\frac{1-t}{t} \right)^{k_{m-1}-2} + |\mathbf{w}_m \times \mathbf{P}_n| \right\} \\
&\quad + |\mathbf{g}_{m-1} \times \mathbf{P}_n| h_{m-1} (k_{m-1} - 1)^{-1} F''_{m-1}(t) \max_{t \in [t^\ell_m, 1]} F''_{m-1}(1-t) \\
&\leq F''_{m-1}(t) \left\{ |\mathbf{w}_{m-1} \times \mathbf{P}_n| \left(\frac{1-t^\ell_m}{t^\ell_m} \right)^{k_{m-1}-2} + |\mathbf{w}_m \times \mathbf{P}_n| \right. \\
&\quad \left. + |\mathbf{g}_{m-1} \times \mathbf{P}_n| h_{m-1} \frac{(1-t^\ell_m)^{k_{m-1}-2}}{k_{m-1}-1} \right\}, \qquad u \in [\hat{u}^\ell_m, u_m],
\end{aligned}
\tag{5.31}
$$

where t^ℓ_m is defined in (5.16d). Combining (5.30) and (5.31), it is readily seen that, under hypothesis (5.28), (5.16e) is a sufficient condition for the validity of the coplanarity criterion *(iii.1)* in $[\hat{u}^\ell_m, u_m]$. It remains to establish (5.28). To do so, we deduce the bound

$$
\begin{aligned}
|\mathbf{r}^*_{m-1}(u)| &\leq |\mathbf{w}_{m-1}| F''_{m-1}(1-t) + |\mathbf{g}_{m-1}| h_{m-1} \varphi_{m-1}(t) \\
&= F''_{m-1}(t) \left\{ |\mathbf{w}_{m-1}| \left(\frac{1-t}{t} \right)^{k_{m-1}-2} + |\mathbf{g}_{m-1}| h_{m-1} \frac{F''_{m-1}(1-t)}{k_{m-1}-1} \right\} \\
&\leq F''_{m-1}(t) \left\{ |\mathbf{w}_{m-1}| \left(\frac{1-t^\ell_m}{t^\ell_m} \right)^{k_{m-1}-2} + |\mathbf{g}_{m-1}| h_{m-1} \frac{(1-t^\ell_m)^{k_{m-1}-2}}{k_{m-1}-1} \right\}.
\end{aligned}
\tag{5.32}
$$

Therefore (5.28) is satisfied if (5.16d) holds true. Summarizing, (5.16d) and (5.16e) imply the validity of the coplanarity criterion *(iii.1)* in $[\hat{u}^\ell_m, u_m]$.

In an analogous manner it can be proved that inequalities (5.16f) and (5.16g) suffice to guarantee that Part *(iii.1)* of the coplanarity criterion holds true in $[u_{m+1}, \hat{u}^r_{m+1}]$, as well. ■

The following theorem confirms that, for sufficiently large degrees k_{m-1}, k_m, k_{m+1}, inequalities (5.16) of Lemma 5.5 become true.

Theorem 5.6. Let $\Delta_m = 0$, $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, ω_m be a user-specified interval as in Lemma 5.5 and $\alpha^\ell_m, \alpha_m, \alpha^r_{m+1} \in (0, \sqrt{2} - 1)$. Then :

(i) If k_{m-1}, k_m, k_{m+1} are sufficiently large, then inequalities (5.16a) and (5.16b) hold true.

(ii) If k_{m-1}, k_m, k_{m+1} are sufficiently large with $k_{m-1} = k_m = k_{m+1}$, then inequality (5.16c) holds true.

(iii) If k_{m-1}, k_m are sufficiently large with $k_{m-1} = k_m$, then inequalities (5.16d) and (5.16e) hold true, and

(iv) If k_m, k_{m+1} are sufficiently large with $k_m = k_{m+1}$, then inequalities (5.16f) and (5.16g) hold true.

Proof. *(i)* Since $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$ and $q_n > 0$, $n = m, m+1$, Corollary 4.4 readily implies that inequality (5.16a) will be satisfied for sufficiently large k_{m-1}, k_m, k_{m+1}. Regarding, now, the left-hand side of inequality (5.16b), Lemma 4.1 and $\mathbf{g}_m := \ddot{\mathbf{Q}}_{m+1} \times \ddot{\mathbf{Q}}_m$ give

$$q_m q_{m+1} |\mathbf{g}_m| \le |\mathbf{R}_m||\mathbf{R}_{m+1}| \le 3M^2. \tag{5.33}$$

Moreover, Corollary 4.4 and $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$ provide

$$\begin{aligned} &\sqrt{2}\,\alpha_m\ h_m^{-1}(k_m-1)2^{k_m-2} q_m q_{m+1}\ \min\{|\mathbf{w}_m|, |\mathbf{w}_{m+1}|\} \\ &\qquad \ge \sqrt{2}\,\alpha_m h_m^{-1}(k_m-1)2^{k_m-2} d_m\ \min\{|q_m\mathbf{w}_m|, |q_{m+1}\mathbf{w}_{m+1}|\} \to \infty, \end{aligned} \tag{5.34}$$

as $k_{m-1}, k_m, k_{m+1} \to \infty$. It follows from (5.33) and (5.34) that inequality (5.16b) holds true for large enough k_{m-1}, k_m, k_{m+1}.
(ii) Since $k_{m-1} = k_m = k_{m+1}$, the common denominator of the two fractions on the left-hand side of (5.16c), multiplied by q_m, has the value

$$q_m \sqrt{\beta_m}\ \min\{|\mathbf{w}_m|, |\mathbf{w}_{m+1}|\}|\mathbf{P}_n| = \sqrt{\beta_m}\ \min\{|q_m\mathbf{w}_m|, |\zeta_m q_{m+1}\mathbf{w}_{m+1}|\}|\mathbf{P}_n|, \quad \zeta_m = \frac{h_{m-1}+h_m}{h_m+h_{m+1}}, \tag{5.35}$$

which, in view of Corollary 4.4, tends to

$$\sqrt{\beta_m}\ \min\left\{\frac{|\mathbf{P}_m|}{h_{m-1}h_m}, \zeta_m \frac{|\mathbf{P}_{m+1}|}{h_m h_{m+1}}\right\} |\mathbf{P}_n| > 0, \tag{5.36}$$

as $k_{m-1}, k_m, k_{m+1} \to \infty$. We now examine the asymptotic behaviour of the numerators of the fractions in (5.16c), also multiplied by q_m. For the first numerator we have :

$$\sqrt{2}\, q_m \max\{|\mathbf{w}_m \times \mathbf{P}_n|, |\mathbf{w}_{m+1} \times \mathbf{P}_n|\} = \sqrt{2}\ \max\{|q_m\mathbf{w}_m \times \mathbf{P}_n|, |\zeta_m q_{m+1}\mathbf{w}_{m+1} \times \mathbf{P}_n|\}, \tag{5.37}$$

which, by virtue of Corollary 4.4 once again, tends to

$$\sqrt{2}\ \max\left\{\frac{|\mathbf{P}_m \times \mathbf{P}_n|}{h_{m-1}h_m}, \zeta_m \frac{|\mathbf{P}_{m+1} \times \mathbf{P}_n|}{h_m h_{m+1}}\right\} = 0, \tag{5.38}$$

as $k_{m-1} = k_m = k_{m+1} \to \infty$, because $\Delta_m = 0$ implies $|\mathbf{P}_m \times \mathbf{P}_n| = |\mathbf{P}_{m+1} \times \mathbf{P}_n| = 0$, $n = m, m+1$. For the second numerator, using Lemma 4.1 and the definition of $\mathbf{g}_m$, we find

$$\begin{aligned} \frac{\sqrt{2}\, q_m}{k_m-1}|\mathbf{g}_m \times \mathbf{P}_n| h_m \left(\frac{1}{2}\right)^{k_m-1} &\le \frac{\sqrt{2}}{d_m(k_m-1)}|\mathbf{R}_m||\mathbf{R}_{m+1}||\mathbf{P}_n| h_m \left(\frac{1}{2}\right)^{k_m-1} \\ &\le \sqrt{2}\,\frac{k_m}{k_m-1}\, 3M^2 |\mathbf{P}_n| \left(\frac{1}{2}\right)^{k_m-1}, \end{aligned} \tag{5.39}$$

which tends to zero as $k_m \to \infty$. Thus the numerators of the fractions in (5.16c), multiplied by q_m, tend to zero as $k_{m-1} = k_m = k_{m+1} \to \infty$, whereas the common denominator, multiplied by the same quantity, tends to a positive constant (see (5.36)). This completes the proof of Part *(ii)* of this theorem.

(iii) We now turn to inequalities (5.16d) and (5.16e) for sufficiently large k_{m-1}, k_m, with $k_{m-1} = k_m$. As far as the first term on the left-hand side of (5.16d) is concerned, the definition (2.5) of $\mathbf{w}(u)$ and Lemma 4.1 give the bound

$$\begin{aligned} q_{m-1}|\mathbf{w}_{m-1}| \left(\frac{1-t_m^\ell}{t_m^\ell}\right)^{k_{m-1}-2} &\leq |\dot{\mathbf{Q}}_{m-1}||\mathbf{R}_{m-1}| \left(\frac{1-t_m^\ell}{t_m^\ell}\right)^{k_{m-1}-2} \\ &\leq |\dot{\mathbf{Q}}_{m-1}|\sqrt{3}M \left(\frac{1-t_m^\ell}{t_m^\ell}\right)^{k_{m-1}-2}, \end{aligned} \tag{5.40}$$

which tends to zero as $k_{m-1} \to \infty$, since $|\dot{\mathbf{Q}}_{m-1}|$ is bounded (see (4.30b)) and $t_m^\ell \in (\frac{1}{2}, 1]$. Regarding the second term on the left-hand side of (5.16d), some recent arguments provide

$$\begin{aligned} q_{m-1}|\mathbf{g}_{m-1}|h_{m-1} \frac{(1-t_m^\ell)^{k_{m-1}-2}}{k_{m-1}-1} &\leq \frac{h_{m-1}}{d_{m-1}} |\mathbf{R}_{m-1}||\mathbf{R}_m| \frac{(1-t_m^\ell)^{k_{m-1}-2}}{k_{m-1}-1} \\ &\leq 3M^2 \frac{k_{m-1}}{k_{m-1}-1} (1-t_m^\ell)^{k_{m-1}-2}, \end{aligned} \tag{5.41}$$

which also tends to zero as $k_{m-1} \to \infty$. Thus, the left-hand side of inequality (5.16d), multiplied by q_{m-1}, tends to zero as $k_{m-1} \to \infty$. On the other hand, its right-hand side, also multiplied by q_{m-1}, has the lower bound

$$q_{m-1}\alpha_m^\ell|\mathbf{w}_m| \geq \frac{d_{m-1}}{q_m} \alpha_m^\ell|q_m\mathbf{w}_m| = \frac{h_{m-1}}{h_{m-1}+h_m} \alpha_m^\ell|q_m\mathbf{w}_m|, \tag{5.42}$$

where we have used $k_{m-1} = k_m$. Recalling now Corollary 4.4, we obtain

$$\frac{h_{m-1}}{h_{m-1}+h_m} \alpha_m^\ell|q_m\mathbf{w}_m| \to \frac{\alpha_m^\ell h_{m-1}}{h_{m-1}+h_m} \frac{|\mathbf{P}_m|}{h_{m-1}h_m} > 0, \tag{5.43}$$

as $k_{m-1}, k_m \to \infty$. Based on the above results, the validity of inequality (5.16d) for sufficiently large k_{m-1}, k_m with $k_{m-1} = k_m$, is secured.

Let us now consider inequality (5.16e), and recast its left-hand side in the form

$$\begin{aligned} &\frac{q_m q_{m-1}^{-1}|q_{m-1}\mathbf{w}_{m-1} \times \mathbf{P}_n|}{\sqrt{\beta_m^\ell}\,|q_m\mathbf{w}_m||\mathbf{P}_n|} \left(\frac{1-t_m^\ell}{t_m^\ell}\right)^{k_{m-1}-2} + \frac{|q_m\mathbf{w}_m \times \mathbf{P}_n|}{\sqrt{\beta_m^\ell}\,|q_m\mathbf{w}_m||\mathbf{P}_n|} \\ &+\frac{q_{m-1}^{-1}|(\mathbf{R}_m \times \mathbf{R}_{m-1}) \times \mathbf{P}_n|h_{m-1}(k_{m-1}-1)^{-1}(1-t_m^\ell)^{k_{m-1}-2}}{\sqrt{\beta_m^\ell}\,|q_m\mathbf{w}_m||\mathbf{P}_n|}. \end{aligned} \tag{5.44}$$

Treating the numerators of the first and the third fractions of (5.44) as in (5.40) and (5.41), respectively, and noting that $q_{m-1}^{-1} < d_{m-1}^{-1}$ and $q_m d_{m-1}^{-1} = 1 + h_m/h_{m-1}$, it becomes clear that both numerators tend to zero, as $k_{m-1}, k_m \to \infty$ with $k_{m-1} = k_m$. The same conclusion can be

drawn for the numerator of the second fraction of (5.44), by simply applying Corollary 4.4 and recalling that $|\mathbf{P}_m \times \mathbf{P}_n| = 0$, $n = m, m+1$. Finally, appealing once again to Corollary 4.4, we can establish that, for sufficiently large degrees k_{m-1}, k_m, the common denominator in (5.44) is bounded below by a positive constant. Therefore Part *(iii)* of this theorem holds true.
(iv) This part can be proved in a directly analogous manner to that employed for proving Part *(iii)*. ∎

Lemma 5.5 and Theorem 5.6 treat in detail the case of coplanar data ($\Delta_m = 0$) with $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$. In the case $\mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0$, the analogous properties are as follows.

Lemma 5.7. Let $\Delta_m = 0$, $\mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0$ and $\vartheta_m = [\hat{u}^\ell_m, \hat{u}^r_m]$, $\vartheta_{m+1} = [\hat{u}^\ell_{m+1}, \hat{u}^r_{m+1}]$ be the user-specified intervals in the coplanarity criterion *(iii.2)* of Definition 2.1*, where $\bar{u}_{m-1} < \hat{u}^\ell_m \le u_m < \hat{u}^r_m < \bar{u}_m < \hat{u}^\ell_{m+1} < u_{m+1} \le \hat{u}^r_{m+1} < \bar{u}_{m+1}$. This criterion is satisfied, if inequalities (5.16d), (5.16e), (5.16f), (5.16g) of Lemma 5.5 and the following four inequalities hold true :

$$\mathsf{cprb}(m) := |\mathbf{w}_{m+1}| \left(\frac{t^r_m}{1-t^r_m}\right)^{k_m-2} + |\mathbf{g}_m| h_m (k_m-1)^{-1} (t^r_m)^{k_m-2} \le \alpha^r_m |\mathbf{w}_m|, \qquad t^r_m := \frac{\hat{u}^r_m - u_m}{h_m} \in (0, \frac{1}{2}), \tag{5.45a}$$

$$\begin{aligned}\mathsf{cprr}(m,n) := \; & \frac{|\mathbf{w}_m \times \mathbf{P}_n|}{\sqrt{\beta^r_m}\,|\mathbf{w}_m||\mathbf{P}_n|} + \frac{|\mathbf{w}_{m+1} \times \mathbf{P}_n|}{\sqrt{\beta^r_m}\,|\mathbf{w}_m||\mathbf{P}_n|} \left(\frac{t^r_m}{1-t^r_m}\right)^{k_m-2} \\ & + \frac{|\mathbf{g}_m \times \mathbf{P}_n| h_m (k_m-1)^{-1} (t^r_m)^{k_m-2}}{\sqrt{\beta^r_m}\,|\mathbf{w}_m||\mathbf{P}_n|} < \varepsilon_1, \quad n = m, m+1,\end{aligned} \tag{5.45b}$$

$$\mathsf{cplb}(m+1) := |\mathbf{w}_m| \left(\frac{1-t^\ell_{m+1}}{t^\ell_{m+1}}\right)^{k_m-2} + |\mathbf{g}_m| h_m \frac{(1-t^\ell_{m+1})^{k_m-2}}{k_m-1} \le \alpha^\ell_{m+1} |\mathbf{w}_{m+1}|, \qquad t^\ell_{m+1} := \frac{\hat{u}^\ell_{m+1} - u_m}{h_m} \in (\frac{1}{2}, 1), \tag{5.45c}$$

$$\begin{aligned}\mathsf{cplr}(m+1,n) := \; & \frac{|\mathbf{w}_m \times \mathbf{P}_n|}{\sqrt{\beta^\ell_{m+1}}\,|\mathbf{w}_{m+1}||\mathbf{P}_n|} \left(\frac{1-t^\ell_{m+1}}{t^\ell_{m+1}}\right)^{k_m-2} + \frac{|\mathbf{w}_{m+1} \times \mathbf{P}_n|}{\sqrt{\beta^\ell_{m+1}}\,|\mathbf{w}_{m+1}||\mathbf{P}_n|} \\ & + \frac{|\mathbf{g}_m \times \mathbf{P}_n| h_m (k_m-1)^{-1} (1-t^\ell_{m+1})^{k_m-2}}{\sqrt{\beta^\ell_{m+1}}\,|\mathbf{w}_{m+1}||\mathbf{P}_n|} < \varepsilon_1, \quad n = m, m+1,\end{aligned} \tag{5.45d}$$

where α^r_m and α^ℓ_{m+1} are constants from $(0, \sqrt{2}-1)$ and where β^r_m and β^ℓ_{m+1} take the values $1-(\alpha^r_m)^2 - 2\,\alpha^r_m$ and $1-(\alpha^\ell_{m+1})^2 - 2\,\alpha^\ell_{m+1}$, respectively.

Theorem 5.8. Let $\Delta_m = 0$, $\mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0$, ϑ_m and ϑ_{m+1} be user-specified intervals as in Lemma 5.7 and $\alpha^\ell_n, \alpha^r_n \in (0, \sqrt{2}-1)$, $n = m, m+1$.

(i) If k_{m-1}, k_m are sufficiently large with $k_{m-1} = k_m$, then inequalities (5.16d), (5.16e), (5.45a) and (5.45b) hold true, and

(ii) If k_m, k_{m+1} are sufficiently large with $k_m = k_{m+1}$, then inequalities (5.45c), (5.45d), (5.16f) and (5.16g) hold true.

The truth of Lemma 5.7 and Theorem 5.8 can be obtained by employing the techniques used in the proofs of Lemma 5.5 and Theorem 5.6. More specifically, one can treat conditions (5.45a) and (5.45b) as (5.16f) and (5.16g), respectively. The same stands for the pairs (5.45c), (5.45d) and (5.16d), (5.16e). Note that conditions (5.45a), (5.45b) and (5.45c), (5.45d) ensure the validity of the coplanarity criterion in the "interior" intervals $[u_m, \hat{u}_m^r]$ and $[\hat{u}_{m+1}^\ell, u_{m+1}]$, whereas (5.16$d$), (5.16$e$) and (5.16$f$), (5.16$g$) ensure the validity of the coplanarity criterion in the "exterior" intervals $[\hat{u}_m^\ell, u_m]$ and $[u_{m+1}, \hat{u}_{m+1}^r]$.

5.4. The collinearity criterion

We complete the investigation of Definition 2.1*, by deriving sufficient conditions for the collinearity criterion.

Lemma 5.9. Let $|\mathbf{P}_m| = 0$, $\mathbf{L}_{m-1} \cdot \mathbf{L}_m > 0$ and $\eta_m = [\tilde{u}_m^\ell, \tilde{u}_m^r]$, $u_{m-1} < \tilde{u}_m^\ell < u_m < \tilde{u}_m^r < u_{m+1}$, be the user-specified interval involved in the collinearity criterion *(iv)* of Definition 2.1*. This criterion is satisfied, if the four inequalities

$$\mathsf{cllb}(m) := \left[|\mathbf{R}_{m-1}|(1-t_m^\ell)^{k_{m-1}-1} + |\mathbf{R}_m| \frac{d_{m-1}}{q_m}(1-(t_m^\ell)^{k_{m-1}-1}) \right] \frac{k_{m-1}}{k_{m-1}-1} \le \gamma_m^\ell |\dot{\mathbf{Q}}_m|,$$
$$t_m^\ell := \frac{\tilde{u}_m^\ell - u_{m-1}}{h_{m-1}} \in (0,1), \tag{5.46a}$$

$$\mathsf{cllr}(m,n) := \frac{|\dot{\mathbf{Q}}_m \times \mathbf{L}_n|}{\sqrt{\delta_m^\ell}\,|\dot{\mathbf{Q}}_m||\mathbf{L}_n|} + \left[\frac{|\mathbf{R}_{m-1} \times \mathbf{L}_n|(1-t_m^\ell)^{k_{m-1}-1}}{\sqrt{\delta_m^\ell}\,|\dot{\mathbf{Q}}_m||\mathbf{L}_n|} + \frac{q_m^{-1} d_{m-1}|\mathbf{R}_m \times \mathbf{L}_n|(1-(t_m^\ell)^{k_{m-1}-1})}{\sqrt{\delta_m^\ell}\,|\dot{\mathbf{Q}}_m||\mathbf{L}_n|} \right] \frac{k_{m-1}}{k_{m-1}-1} < \varepsilon_0, \tag{5.46b}$$

$$n = m-1, m, \quad \delta_m^\ell = 1 - (\gamma_m^\ell)^2 - 2\,\gamma_m^\ell,$$

$$\mathsf{clrb}(m) := \left[|\mathbf{R}_m| \frac{d_m}{q_m}(1-(1-t_m^r)^{k_m-1}) + |\mathbf{R}_{m+1}|(t_m^r)^{k_m-1} \right] \frac{k_m}{k_m-1} \le \gamma_m^r |\dot{\mathbf{Q}}_m|, \tag{5.46c}$$
$$t_m^r := \frac{\tilde{u}_m^r - u_m}{h_m} \in (0,1), \quad \text{and}$$

$$\mathsf{clrr}(m,n) := \frac{|\dot{\mathbf{Q}}_m \times \mathbf{L}_n|}{\sqrt{\delta^r_m}\,|\dot{\mathbf{Q}}_m||\mathbf{L}_n|} + \Bigg[\frac{q_m^{-1} d_m |\mathbf{R}_m \times \mathbf{L}_n|(1-(1-t^r_m)^{k_m-1})}{\sqrt{\delta^r_m}\,|\dot{\mathbf{Q}}_m||\mathbf{L}_n|}$$

$$+\frac{|\mathbf{R}_{m+1} \times \mathbf{L}_n|(t^r_m)^{k_m-1}}{\sqrt{\delta^r_m}\,|\dot{\mathbf{Q}}_m||\mathbf{L}_n|}\Bigg] \frac{k_m}{k_m-1} < \varepsilon_0, \tag{5.46d}$$

$$n = m-1, m, \quad \delta^r_m = 1 - (\gamma^r_m)^2 - 2\,\gamma^r_m,$$

hold true, where γ^ℓ_m and γ^r_m are constants from $(0, \sqrt{2}-1)$.

Proof. We seek upper bounds on the numerator and lower bounds on the denominator of the left-hand side fraction of inequality (5.4). Furthermore, the proof handles separately the intervals $[\tilde{u}^\ell_m, u_m]$ and $[u_m, \tilde{u}^r_m]$, which partition η_m, though qualitatively it is the same for both of them.

In the first case $u \in [\tilde{u}^\ell_m, u_m]$, we introduce the hypothesis

$$|\mathbf{r}_{m-1}(u)| \le \gamma^\ell_m |\dot{\mathbf{Q}}_m|, \tag{5.47}$$

where

$$\mathbf{r}_{m-1}(u) := \dot{\mathbf{Q}}(u) - \dot{\mathbf{Q}}_m = -h_{m-1}\ddot{\mathbf{Q}}_{m-1}\psi_1(t) + h_{m-1}\ddot{\mathbf{Q}}_m\psi_2(t), \tag{5.48a}$$

$$\psi_1(t) = \frac{(1-t)^{k_{m-1}-1}}{k_{m-1}-1}, \quad \psi_2(t) = \frac{t^{k_{m-1}-1}-1}{k_{m-1}-1}. \tag{5.48b}$$

It follows from (5.47) and the first part of (5.48a) that $|\dot{\mathbf{Q}}(u)|^2$ is bounded below by

$$|\dot{\mathbf{Q}}(u)|^2 \ge |\dot{\mathbf{Q}}_m|^2(1-(\gamma^\ell_m)^2 - 2\,\gamma^\ell_m) = \delta^\ell_m |\dot{\mathbf{Q}}_m|^2. \tag{5.49}$$

We now come to the numerator on the left-hand side of (5.4). Using the definition (5.48a) of the residual $\mathbf{r}_{m-1}(u)$, we deduce the inequalities :

$$\begin{aligned}
|\dot{\mathbf{Q}}(u) \times \mathbf{L}_n| &\le |\dot{\mathbf{Q}}_m \times \mathbf{L}_n| + |\mathbf{r}_{m-1}(u) \times \mathbf{L}_n| \\
&\le |\dot{\mathbf{Q}}_m \times \mathbf{L}_n| + \frac{h_{m-1}}{q_{m-1}}|\mathbf{R}_{m-1} \times \mathbf{L}_n||\psi_1(t)| + \frac{h_{m-1}}{q_m}|\mathbf{R}_m \times \mathbf{L}_n||\psi_2(t)| \\
&= |\dot{\mathbf{Q}}_m \times \mathbf{L}_n| + \Bigg[\frac{d_{m-1}}{q_{m-1}}|\mathbf{R}_{m-1} \times \mathbf{L}_n|(1-t)^{k_{m-1}-1} \\
&\quad + \frac{d_{m-1}}{q_m}|\mathbf{R}_m \times \mathbf{L}_n|(1-t^{k_{m-1}-1})\Bigg]\frac{k_{m-1}}{k_{m-1}-1} \\
&\le |\dot{\mathbf{Q}}_m \times \mathbf{L}_n| + \Bigg[|\mathbf{R}_{m-1} \times \mathbf{L}_n|(1-t^\ell_m)^{k_{m-1}-1} \\
&\quad + \frac{d_{m-1}}{q_m}|\mathbf{R}_m \times \mathbf{L}_n|(1-(t^\ell_m)^{k_{m-1}-1})\Bigg]\frac{k_{m-1}}{k_{m-1}-1}, \qquad \tilde{u}^\ell_m \le u \le u_m.
\end{aligned} \tag{5.50}$$

In view of (5.49) and (5.50), we conclude that (5.47) and (5.46b) ensure the validity of the collinearity criterion in $[\tilde{u}^\ell_m, u_m]$. We only have to establish (5.47), which can be achieved as in (5.50). This leads to the bound

$$|\mathbf{r}_{m-1}(u)| \le \Bigg[|\mathbf{R}_{m-1}|(1-t^\ell_m)^{k_{m-1}-1} + \frac{d_{m-1}}{q_m}|\mathbf{R}_m|(1-(t^\ell_m)^{k_{m-1}-1})\Bigg]\frac{k_{m-1}}{k_{m-1}-1}. \tag{5.51}$$

It is now obvious that $(5.46a)$ and $(5.46b)$ are sufficient conditions for securing the validity of the collinearity criterion in $[\tilde{u}_m^\ell, u_m]$.

It can be proved similarly that $(5.46c)$ and $(5.46d)$ guarantee that the collinearity criterion holds true in $[u_m, \tilde{u}_m^r]$. ■

The question that now arises is whether inequalities (5.46) of Lemma 5.9 can be satisfied for sufficiently large degrees. The following theorem provides an answer to this question.

Theorem 5.10. Let $|\mathbf{P}_m| = 0$, $\mathbf{L}_{m-1} \cdot \mathbf{L}_m > 0$ and η_m be a user-specified interval as in Lemma 5.9. If

$$\lambda_m^\ell = \frac{h_{m-1}|\mathbf{s}_m - \mathbf{s}_{m-1}|}{|h_m\mathbf{s}_{m-1} + h_{m-1}\mathbf{s}_m|} < \sqrt{2} - 1, \quad \lambda_m^r = \frac{h_m}{h_{m-1}}\lambda_m^\ell < \sqrt{2} - 1 \tag{5.52}$$

and $\gamma_m^\ell \in (\lambda_m^\ell, \sqrt{2}-1)$, $\gamma_m^r \in (\lambda_m^r, \sqrt{2}-1)$, then inequalities (5.46) are satisfied for sufficiently large degrees k_{m-1}, k_m with $k_{m-1} = k_m$. In the case of type-I boundary conditions, the above result is also true for $m = 1$ and $m = N-1$ if

$$\lambda_1^r = \frac{|\mathbf{s}_1 - \mathbf{s}_0|}{|\mathbf{s}_0|} < \sqrt{2} - 1, \quad \gamma_1^r \in (\lambda_1^r, \sqrt{2}-1), \quad \text{and} \tag{5.53a}$$

$$\lambda_N^\ell = \frac{|\mathbf{s}_N - \mathbf{s}_{N-1}|}{|\mathbf{s}_N|} < \sqrt{2} - 1, \quad \gamma_N^\ell \in (\lambda_N^\ell, \sqrt{2}-1), \tag{5.53b}$$

respectively.

Proof. First we consider condition $(5.46a)$ as $k_{m-1} = k_m \to \infty$. Lemma 4.1 and $t_m^\ell \in (0,1)$ imply that the left-hand side of $(5.46a)$ tends to

$$\frac{h_{m-1}|\mathbf{b}_m|}{h_{m-1} + h_m} = \frac{h_{m-1}|\mathbf{s}_m - \mathbf{s}_{m-1}|}{h_{m-1} + h_m}. \tag{5.54}$$

On the other hand, appealing to the limiting relation (4.55) with $c = k_m/k_{m-1} = 1$, we find that

$$\gamma_m^\ell \, \frac{|h_m\mathbf{s}_{m-1} + h_{m-1}\mathbf{s}_m|}{h_{m-1} + h_m} \tag{5.55}$$

is the limit of the right-hand side of $(5.46a)$. Thus, a sufficient condition for $(5.46a)$ to be true asymptotically is that (5.54) be strictly less than (5.55), which is true since $\gamma_m^\ell \in (\lambda_m^\ell, \sqrt{2}-1)$ by assumption.

We now turn to $(5.46b)$. Setting $c = 1$ in (4.55) again and using Lemma 4.1, we deduce

$$|\dot{\mathbf{Q}}_m \times \mathbf{L}_n| \to \frac{|h_m(\mathbf{s}_{m-1} \times \mathbf{L}_n) + h_{m-1}(\mathbf{s}_m \times \mathbf{L}_n)|}{h_{m-1} + h_m} = 0, \tag{5.56a}$$

$$|\mathbf{R}_{m-1} \times \mathbf{L}_n|(1 - t_m^\ell)^{k_{m-1}-1} \leq \sqrt{3}M|\mathbf{L}_n|(1 - t_m^\ell)^{k_{m-1}-1} \to 0, \tag{5.56b}$$

$$q_m^{-1} d_{m-1}|\mathbf{R}_m \times \mathbf{L}_n|(1 - (t_m^\ell)^{k_{m-1}-1}) \to \frac{h_{m-1}}{h_{m-1} + h_m}\,|\mathbf{b}_m \times \mathbf{L}_n| = 0, \tag{5.56c}$$

which imply that the numerators of the left-hand side fractions of $(5.46b)$ tend to zero, as $k_{m-1} = k_m \to \infty$. This result and the fact that the common denominator $\sqrt{\delta_m^\ell}\,|\dot{\mathbf{Q}}_m||\mathbf{L}_n|$ in $(5.46b)$ tends to

$$\sqrt{\delta_m^\ell}\,\frac{|h_m\mathbf{s}_{m-1} + h_{m-1}\mathbf{s}_m|}{h_{m-1} + h_m}\,|\mathbf{L}_n| > 0, \tag{5.57}$$

as $k_{m-1} = k_m \to \infty$, leads to the conclusion that condition (5.46b) is fulfilled for $k_{m-1} = k_m$ large enough. The remaining conditions (5.46c) and (5.46d) can be handled similarly.

In the case of type-I boundary conditions with, e.g., $|\mathbf{P}_N| = 0$, $\mathbf{L}_{N-1} \cdot \mathbf{L}_N > 0$, the proof above has only to be modified at the following points. Instead of expression (5.54), the limit of the left-hand side of (5.46a) is

$$|\mathbf{b}_N| = |\mathbf{s}_N - \mathbf{s}_{N-1}|, \tag{5.58}$$

whereas, the right-hand side of (5.46a) is equal to $\gamma_N^\ell |\mathbf{s}_N|$. Secondly, with regard to the left-hand side of (5.46b), we have

$$|\dot{\mathbf{Q}}_N \times \mathbf{L}_n| = |\mathbf{s}_N \times \mathbf{L}_n| = 0, \quad \text{and} \tag{5.59a}$$

$$q_N^{-1} d_{N-1} |\mathbf{R}_N \times \mathbf{L}_n| (1 - (t_N^\ell)^{k_{N-1}-1}) = |\mathbf{R}_N \times \mathbf{L}_n| (1 - (t_N^\ell)^{k_{N-1}-1}) \to |\mathbf{b}_N \times \mathbf{L}_N| = 0, \tag{5.59b}$$

while (5.56b) remains valid. Finally, the common denominator in (5.46b) is equal to $\sqrt{\delta_N^\ell}|\mathbf{s}_N||\mathbf{L}_n|$, which is a positive constant. ■

Remark: Obviously, there exist data sets for which the assumptions (5.52) are not satisfied. Whenever this happens, one can modify the parametrization locally, so that $h_n = |\mathbf{L}_n|$, $n = m-1, m$ (chord-length parametrization), which implies $\lambda_m^\ell = \lambda_m^r = 0 < \sqrt{2} - 1$. Moreover, for type-$I$ boundary conditions the nullification of λ_1^r and λ_N^ℓ can be achieved by setting $h_1 = |\mathbf{L}_1|/|\mathbf{s}_0|$ and $h_{N-1} = |\mathbf{L}_{N-1}|/|\mathbf{s}_N|$, respectively.

5.5. The shape-preserving algorithm

Collecting the conditions in Lemmata 5.1, 5.3, 5.5, 5.7 and 5.9, we propose the following automatic algorithm for constructing Frénet-frame-continuous interpolants in $\mathbb{\Gamma}(\mathrm{K})$, which are shape-preserving in the sense of Definition 2.1*. In order to lighten its complexity, we shall restrict ourselves to data which contain only disjoint coplanar quadruples and/or collinear triplets of interpolation points. Furthermore, the algorithm below is valid for type-I boundary conditions. In the case of type-II' and periodic boundary conditions, the range of the index m has to be modified slightly.

The algorithm SPIN3D[(3)]

STEP 0.0:

(i) Calculate the vectors $\mathbf{L}_m$, $m = 1(1)N-1$, $\mathbf{L}_0 = \mathbf{s}_0$, $\mathbf{L}_N = \mathbf{s}_N$, $\mathbf{P}_m$, $m = 1(1)N$, and the scalars Δ_m, $m = 1(1)N-1$.

(ii) Determine the sets :

$$\mathcal{I}_1 = \{m \in \mathcal{T}_{N-1} : \Delta_m \neq 0\}, \quad \mathcal{T}_{N-1} = \{1, 2, \ldots, N-1\}, \tag{5.60}$$

$$\mathcal{J}_0 = \{m \in \mathcal{T}_N : |\mathbf{P}_m| = 0 \ \wedge \ \mathbf{L}_{m-1} \cdot \mathbf{L}_m > 0\}, \tag{5.61}$$

$$\mathcal{K}_0 = \{m \in \mathcal{T}_{N-1} : \mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0\}, \tag{5.62}$$

In the case of planar sets, set $\mathcal{M}_0 = \mathcal{M}_1 = \emptyset$ and **GOTO STEP 0.2**. Otherwise,

$$\mathcal{M}_0 = \{m \in \mathcal{T}_{N-1} : \Delta_m = 0 \ \wedge \ \mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0\}, \tag{5.63}$$

$$\mathcal{M}_1 = \{m \in \mathcal{T}_{N-1} : \Delta_m = 0 \ \wedge \ \mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0\}. \tag{5.64}$$

[(3)]The acronym **SPIN3D** stands for **S**hape-**P**reserving **IN**terpolation in **3** **D**imensions.

STEP 0.1: If $\mathcal{M}_0 \cup \mathcal{M}_1 = \emptyset$ **GOTO STEP 0.2**. Otherwise :

(i) Specify the constant $\varepsilon_1 \in (0, 1]$.

(ii) For $m \in (\mathcal{M}_0 \cup \mathcal{M}_1) \setminus \{1\}$, specify the parameter value $\hat{u}^\ell_m \in (\bar{u}_{m-1}, u_m]$, $\bar{u}_{m-1} = \frac{1}{2}(u_{m-2} + u_{m-1})$ and set $\alpha^\ell_m = (\sqrt{2}-1)/2$. For $m \in \mathcal{M}_0$, set $\alpha_m = (\sqrt{2}-1)/2$. For $m \in (\mathcal{M}_0 \cup \mathcal{M}_1) \setminus \{N-1\}$, specify the parameter value $\hat{u}^r_{m+1} \in [u_{m+1}, \bar{u}_{m+1})$ and set $\alpha^r_{m+1} = (\sqrt{2}-1)/2$. For $m \in \mathcal{M}_1$, specify the parameter values $\hat{u}^r_m \in (u_m, \bar{u}_m)$, $\hat{u}^\ell_{m+1} \in (\bar{u}_m, u_{m+1})$ and set $\alpha^r_m = \alpha^\ell_{m+1} = (\sqrt{2}-1)/2$.

STEP 0.2: If $\mathcal{J}_0 = \emptyset$ **GOTO STEP 0.3**. Otherwise :

(i) Specify the constant $\varepsilon_0 \in (0, 1]$.

(ii) Calculate the ratios $\lambda^\ell_m, \lambda^r_m, m \in \mathcal{J}_0$, by formula (5.52), where $\lambda^\ell_1 := \lambda^r_1$ if $1 \in \mathcal{J}_0$ and $\lambda^r_N := \lambda^\ell_N$ if $N \in \mathcal{J}_0$. Furthermore, let

$$\rho_m = \begin{cases} |\mathbf{s}_0|^{-1}, & m = 1, \\ |\mathbf{s}_N|^{-1}, & m = N, \\ 1, & \text{otherwise.} \end{cases} \tag{5.65}$$

For $m \in \mathcal{J}_0$, if

$$\max\{\lambda^\ell_m, \lambda^r_m\} < \sqrt{2} - 1, \tag{5.66}$$

then set $\gamma^\ell_m = (\lambda^\ell_m + \sqrt{2} - 1)/2$, $\gamma^r_m = (\lambda^r_m + \sqrt{2} - 1)/2$. Otherwise, set $h_n = \rho_n |\mathbf{L}_n|$, $n = m-1, m$, and $\gamma^\ell_m = \gamma^r_m = (\sqrt{2}-1)/2$.

(iii) For $m \in \mathcal{J}_0 \setminus \{1\}$, specify the parameter values $\tilde{u}^\ell_m \in (u_{m-1}, u_m)$. For $m \in \mathcal{J}_0 \setminus \{N\}$, specify the parameter values $\tilde{u}^r_m \in (u_m, u_{m+1})$.

STEP 0.3: Set $j = 0$ and initialize the degrees $k^{(0)}_m = 4$, $m = 1(1)N-1$ or $k^{(0)}_m = 3$, $m = 1(1)N-1$ in the case of planar sets.

STEP 1: Let $\mathbf{Q}(\mathrm{K}^{(j)})$ be the curve that has the parameter values $k_m = k^{(j)}_m$, $m = 1(1)N-1$, where $\mathrm{K}^{(j)}$ denotes the set $\{k^{(j)}_m : m = 1(1)N-1\}$. Calculate $\ddot{\mathbf{Q}}_m(\mathrm{K}^{(j)})$, $\dot{\mathbf{Q}}_m(\mathrm{K}^{(j)})$, $m = 1(1)N$, $\mathbf{g}_m(\mathrm{K}^{(j)})$, $m = 1(1)N-1$, $\mathbf{w}_m(\mathrm{K}^{(j)}) = \dot{\mathbf{Q}}_m \times \ddot{\mathbf{Q}}_m$, $m = 1(1)N$ and $\det(\mathbb{A}_m) = -\mathbf{s}_m \cdot \mathbf{g}_m$, $m = 1(1)N-1$.

STEP 2: Determine the "failure" sets :

$$\mathcal{I}_{11} = \{m \in \mathcal{I}_1 : \det(\mathbb{A}_m)\Delta_m \le 0\}, \tag{5.67}$$

$$\mathcal{K}_{01} = \{m \in \mathcal{K}_0 : \mathbf{w}_m \cdot \mathbf{P}_m \le 0 \ \ \vee \ \ \mathbf{w}_m \cdot \mathbf{P}_{m+1} \le 0\}, \tag{5.68}$$

$$\mathcal{K}_{11} = \{m-1 \in \mathcal{K}_0 : \mathbf{w}_m \cdot \mathbf{P}_m \le 0 \ \ \vee \ \ \mathbf{w}_m \cdot \mathbf{P}_{m-1} \le 0\}, \tag{5.69}$$

$$\begin{aligned} \mathcal{K}_{02} = \{m \in \mathcal{K}_0 : \mathbf{g}_m \cdot \mathbf{P}_n < 0 \ \wedge \\ |\mathbf{g}_m \cdot \mathbf{P}_n| \ge h_m^{-1} 2^{k_m - 1}(k_m - 1) \min\{|\mathbf{w}_m \cdot \mathbf{P}_n|, |\mathbf{w}_{m+1} \cdot \mathbf{P}_n|\}, \ \ n = m \text{ or } m+1\}, \end{aligned} \tag{5.70}$$

$$\mathcal{M}_{01} = \{m \in \mathcal{M}_0 : \mathbf{w}_m \cdot \mathbf{w}_{m+1} \leq 0 \ \vee \tag{5.71}$$

$$|\mathbf{g}_m| > \sqrt{2}\,\alpha_m h_m^{-1}(k_m - 1)2^{k_m-2}\min\{|\mathbf{w}_m|, |\mathbf{w}_{m+1}|\}\},$$

$$\mathcal{M}_{02} = \{m \in \mathcal{M}_0 : \mathsf{cpr}(m,m) \geq \varepsilon_1\}^{(4)}, \tag{5.72}$$

$$\mathcal{M}_{11} = \{m \in \mathcal{M}_1 : \mathsf{cprb}(m) > \alpha_m^r|\mathbf{w}_m| \ \ \vee \ \ \mathsf{cprr}(m,m) \geq \varepsilon_1\}, \tag{5.73}$$

$$\mathcal{M}_{12} = \Big\{m \in \mathcal{M}_1 : \mathsf{cplb}(m+1) > \alpha_{m+1}^\ell|\mathbf{w}_{m+1}| \ \ \vee \ \ \mathsf{cplr}(m+1,m) \geq \varepsilon_1\Big\}, \tag{5.74}$$

$$\mathcal{M}_\ell = \Big\{m \in \mathcal{M}_0 \cup \mathcal{M}_1 : u_m^\ell < u_m \ \wedge \tag{5.75}$$

$$\Big(\mathsf{cplb}(m) > \alpha_m^\ell|\mathbf{w}_m| \ \ \vee \ \ \mathsf{cplr}(m,m) \geq \varepsilon_1\Big)\Big\},$$

$$\mathcal{M}_r = \Big\{m \in \mathcal{M}_0 \cup \mathcal{M}_1 : u_{m+1}^r > u_{m+1} \ \wedge \tag{5.76}$$

$$(\mathsf{cprb}(m+1) > \alpha_{m+1}^r|\mathbf{w}_{m+1}| \ \ \vee \ \ \mathsf{cprr}(m+1,m) \geq \varepsilon_1)\Big\},$$

$$\mathcal{J}_{01} = \Big\{m \in \mathcal{J}_0 \setminus \{1\} : \mathsf{cllb}(m) > \gamma_m^\ell|\dot{\mathbf{Q}}_m| \ \ \vee \ \ \mathsf{cllr}(m,m) \geq \varepsilon_0\Big\}, \tag{5.77}$$

$$\mathcal{J}_{02} = \Big\{m \in \mathcal{J}_0 \setminus \{N\} : \mathsf{clrb}(m) > \gamma_m^r|\dot{\mathbf{Q}}_m| \ \ \vee \ \ \mathsf{clrr}(m,m) \geq \varepsilon_0\Big\}. \tag{5.78}$$

STEP 3: If $\mathcal{I}_{failure} = \mathcal{I}_{11} \cup \mathcal{K}_{01} \cup \mathcal{K}_{11} \cup \mathcal{K}_{02} \cup \mathcal{M}_{01} \cup \mathcal{M}_{02} \cup \mathcal{M}_{11} \cup \mathcal{M}_{12} \cup \mathcal{M}_\ell \cup \mathcal{M}_r \cup \mathcal{J}_{01} \cup \mathcal{J}_{02} = \emptyset$, **STOP**. Otherwise, determine the sets

$$\mathcal{N}_1 = \mathcal{I}_{11} \cup \mathcal{K}_{02} \cup \mathcal{M}_{01}, \tag{5.79a}$$

$$\mathcal{N}_2 = \mathcal{K}_{01} \cup \mathcal{K}_{11}, \tag{5.79b}$$

$$\mathcal{N}_3 = \mathcal{M}_{02} \cup (\mathcal{M}_\ell \cap \mathcal{M}_r) \cup (\mathcal{M}_r \cap \mathcal{M}_{11}) \cup (\mathcal{M}_\ell \cap \mathcal{M}_{12}) \cup (\mathcal{M}_{11} \cap \mathcal{M}_{12}), \tag{5.79c}$$

$$\mathcal{N}_4 = \mathcal{M}_\ell \cup \mathcal{M}_{11} \cup \mathcal{J}_{01} \cup \mathcal{J}_{02}, \tag{5.79d}$$

$$\mathcal{N}_5 = \mathcal{M}_r \cup \mathcal{M}_{12} \tag{5.79e}$$

(note that $I_{failure} = \bigcup_{p=1}^5 \mathcal{N}_p$). For each $\mathcal{N}_p$, $p = 1(1)5$, determine the sets $\{\ell_{p,n}^m,\ n \in \mathcal{T}_{N-1}\}_{m \in \mathcal{N}_p}$ such that :

$$\ell_{1,n}^m = k_n^{(j)} + 1, \ \ n = m-1, m, m+1, \ \ m \in \mathcal{N}_1, \tag{5.80a}$$

$$\ell_{2,n}^m = k_n^{(j)} + 1, \ \ n = m-1, m, \ \ m \in \mathcal{N}_2, \tag{5.80b}$$

$$\ell_{3,n}^m = \max\{k_{m-1}^{(j)} + 1, k_m^{(j)} + 1, k_{m+1}^{(j)} + 1\}, \ \ n = m-1, m, m+1, \ \ m \in \mathcal{N}_3, \tag{5.80c}$$

$$\ell_{4,n}^m = \max\{k_{m-1}^{(j)} + 1, k_m^{(j)} + 1\}, \ \ n = m-1, m, \ \ m \in \mathcal{N}_4, \tag{5.80d}$$

$$\ell_{5,n}^m = \max\{k_m^{(j)} + 1, k_{m+1}^{(j)} + 1\}, \ \ n = m, m+1, \ \ m \in \mathcal{N}_5, \tag{5.80e}$$

---

(4) Since $\mathbf{P}_m/|\mathbf{P}_m| = \mathbf{P}_{m+1}/|\mathbf{P}_{m+1}|$ for $\Delta_m = 0$ with $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, $\mathsf{cpr}(m,m) = \mathsf{cpr}(m,m+1)$ (see rel. (5.16c)). As a consequence, the set $\mathcal{M}_{02}$ would not have been altered if, instead of $\mathsf{cpr}(m,m)$, we had used $\mathsf{cpr}(m,m+1)$. Analogous comments can be said for $\mathsf{cprr}(m,m)$ in $\mathcal{M}_{11}$, $\mathsf{cplr}(m+1,m)$ in $\mathcal{M}_{12}$, $\mathsf{cplr}(m,m)$ in $\mathcal{M}_\ell$, $\mathsf{cprr}(m+1,m)$ in $\mathcal{M}_r$, $\mathsf{cllr}(m,m)$ in $\mathcal{J}_{01}$ and $\mathsf{clrr}(m,m)$ in $\mathcal{J}_{02}$.

$$\ell_{p,n}^m = k_n^{(j)} \text{ for all other } n, \quad p = 1(1)5. \tag{5.80f}$$

Then

$$k_n^{(j+1)} = \max_{\substack{m\in\mathcal{N}_p \\ 1\le p\le 5}} \{\ell_{p,n}^m\}, \quad n = 1(1)N-1. \tag{5.80g}$$

Finally, increase the index j by one and **GOTO STEP 1**.

The convergence of the algorithm SPIN3D is achieved provided that, after a finite number of iterations, $\mathcal{I}_{failure} = \emptyset$. This is the case with the degree increase pattern (5.80) of STEP 3 of the above algorithm. This pattern ensures that the segment degrees will increase by at least one and in conformity with the required patterns in Theorems 5.2, 5.4, 5.6, 5.8 and 5.10, whenever the corresponding failure sets are non-empty.

More accurately, (5.80a) in conjunction with Theorem 5.4 ensures that the torsion failure set $\mathcal{I}_{11}$ eventually becomes the empty set. The same can be said for the convexity failure sets $\mathcal{K}_{01}$ and $\mathcal{K}_{11}$, as a consequence of (5.80b) and Corollary 4.4. As far as $\mathcal{K}_{02}$ is concerned, (5.80a) and Theorem 5.2 imply that this failure set will also become the empty set.

We now move to the failure index sets $\mathcal{M}_{ij}, i = 0,1, j = 1,2$, and $\mathcal{M}_\ell, \mathcal{M}_r$ related to the coplanarity criterion. Parts *(i)* and *(ii)* of Theorem 5.6, in conjunction with (5.80a) and (5.80c), guarantee that the index sets $\mathcal{M}_{01}$ and $\mathcal{M}_{02}$ will become empty. Moreover, Parts *(iii)* and *(iv)* of Theorem 5.6, along with Theorem 5.8 and (5.80c), (5.80d) and (5.80e), ensure that $\mathcal{M}_\ell$ and $\mathcal{M}_r$ will become empty, as well. The same can be said for the sets $\mathcal{M}_{11}$ and $\mathcal{M}_{12}$ on the basis of (5.80c), (5.80d), (5.80e) and Theorem 5.8 alone.

Finally, with regard to the collinearity failure sets $\mathcal{J}_{01}$ and $\mathcal{J}_{02}$, we deduce from (5.80d) and Theorem 5.10 that $\mathcal{J}_{01}$ and $\mathcal{J}_{02}$ will eventually become empty. Note that STEP 0.2*(ii)* of SPIN3D ensures the validity of the assumptions (5.52) and (5.53) (see the remark just after the proof of Theorem 5.10).

We conclude this section by commenting on various aspects of the algorithm. To begin with, the parametrization $\mathcal{U}$ plays a crucial role on the shape quality of the outcome of SPIN3D. Our numerical experience suggests that a good choice, also recommended by the literature on polynomial spline interpolation in general, is the chord-length parametrization, which is the natural parametrization of the polygonal interpolant (see also §6).

Regarding now the values of the parameters $\alpha_m^\ell, \alpha_m^r, \alpha_m, \alpha_{m+1}^\ell, \alpha_{m+1}^r$, the algorithm sets them equal to the midpoint of the admissible interval $(0, \sqrt{2}-1)$ (see STEP 0.1*(ii)* of SPIN3D). This choice avoids values of these parameters that are near the boundaries of $(0, \sqrt{2}-1)$. To clarify the need for this strategy, let us consider the parameter α_m which appears in inequalities (5.16b) and (5.16c). It is seen that, as α_m tends to zero, the right-hand side of (5.16b) tends to zero, whereas the left-hand side $\mathsf{cpr}(m,n)$ of (5.16c) decreases. As a consequence, inequality (5.16b) becomes more difficult to be satisfied, which ultimately leads to larger final segment degrees. On the other hand, when α_m tends to $\sqrt{2}-1$ the degree of difficulty between (5.16b) and (5.16c) is interchanged, which is due to the fact that, in this case, $\mathsf{cpr}(m,n)$ tends to infinity, while the right-hand side of (5.16b) increases. Analogous comments can be made for the chosen values of the parameters $\gamma_m^\ell \in (\lambda_m^\ell, \sqrt{2}-1)$ and $\gamma_m^r \in (\lambda_m^r, \sqrt{2}-1)$ (see STEP 0.2*(ii)* of SPIN3D). However, the user is free to modify interactively the values of these internal parameters in order to achieve lower segment degrees.

The rate of convergence of the algorithm is also affected by the choice of the boundary points of the user-specified intervals $\omega_m, \vartheta_m, \vartheta_{m+1}$ and η_m appearing in the coplanarity and collinearity criteria of Definition 2.1*, as well as the associated Theorems 5.6, 5.8 and 5.10. For example, if the left boundary point $\tilde{u}_m^\ell$ of the interval $\eta_m = [\tilde{u}_m^\ell, \tilde{u}_m^r]$ tends to u_{m-1}, then the left-hand side of inequality $(5.46a)$ increases, thus making the fulfilment of this inequality more difficult. This in its turn leads to larger final segment degrees.

Finally, we note that, since planar data give planar interpolants in $\mathbb{I}\Gamma(\mathrm{K})$, the coplanarity failure sets $\mathcal{M}_{ij}, i = 0, 1, j = 1, 2, \mathcal{M}_\ell$ and $\mathcal{M}_r$ should be empty, which is achieved by simply setting $\mathcal{M}_0 = \mathcal{M}_1 = \emptyset$ in STEP 0.0*(ii)* of SPIN3D.

6. Numerical results

The algorithm SPIN3D described in the previous section has been implemented in the Mathematica Programming Language (see Wolfram (1991)). In this final section we present the results of this implementation in the case of a single three-dimensional data set that contains coplanar as well as collinear groups of points. As a result, all steps of SPIN3D are activated.

The data in question consists of 13 points, the co-ordinates of which are given in the following table :

m	x_m	y_m	z_m
1	0.00	0.00	6.00
2	1.20	0.00	0.00
3	2.50	0.50	0.00
4	3.75	2.50	0.00
5	3.50	6.00	0.00
6	2.50	8.00	-3.00
7	0.00	8.00	-3.00
8	-2.50	8.00	-3.00
9	-3.50	6.00	0.00
10	-3.75	2.50	0.00
11	-2.50	0.50	0.00
12	-1.20	0.00	0.00
13	0.00	0.00	6.00

Table 6.1: The x, y, z co-ordinates of the points of the data set.

This yz-symmetric data contains two distinct quadruples of coplanar points ($\mathcal{M}_0 = \{3, 10\}$) and one triplet of collinear points ($\mathcal{J}_0 = \{7\}$). In this example we impose periodic boundary conditions and employ chord-length parametrization.

The chosen values of the input parameters in STEP 0.1 of SPIN3D, which is related to the coplanarity part of the algorithm, are collected in Table 6.2.

Parameter	Value
ε_1	0.2
$\hat{u}^{\ell}_{m}$	$\frac{1}{4}\, u_{m-1} + \frac{3}{4}\, u_m,\ m = 3, 10$
$\hat{u}^{r}_{m+1}$	$\frac{3}{4}\, u_{m+1} + \frac{1}{4}\, u_{m+2},\ m = 3, 10$

Table 6.2: Values of the parameters related to the coplanarity part of the algorithm.

The choice $\varepsilon_1 = 0.2$ ensures that the angle of the binormal of the curve and the discrete binormal will be less than 11.5370°. Furthermore, the interval $[\frac{1}{4}\, u_{m-1} + \frac{3}{4}\, u_m, \frac{3}{4}\, u_{m+1} + \frac{1}{4}\, u_{m+2}]$ defines the subinterval of $(\bar{u}_{m-1}, \bar{u}_{m+1})$, where the coplanarity criterion *(iii.1)* of Definition 2.1* is to be satisfied by the algorithm.

The chosen values of the input parameters in STEP 0.2 of SPIN3D, linked with the collinearity part of the algorithm, are given in Table 6.3.

PARAMETER	VALUE
ε_0	0.1
$\tilde{u}_7^\ell$	$\frac{3}{4}\, u_6 + \frac{1}{4}\, u_7$
$\tilde{u}_7^r$	$\frac{1}{4}\, u_7 + \frac{3}{4}\, u_8$

Table 6.3: Values of the parameters related to the collinearity part of the algorithm.

Note that these choices ensure that, in the parameter interval $[\tilde{u}_7^\ell, \tilde{u}_7^r] = [\frac{3}{4}\, u_6 + \frac{1}{4}\, u_7, \frac{1}{4}\, u_7 + \frac{3}{4}\, u_8]$, the angle between the unit tangent of the curve and the line connecting the corresponding collinear triplet will be less than $5.73917°$.

The algorithm converges after nine iterations, during which all criteria of Definition 2.1* are activated. Indeed, the convexity criterion is triggered in the parametric intervals $[u_2, u_3]$, $[u_4, u_5]$, $[u_5, u_6]$, $[u_8, u_9]$, $[u_9, u_{10}]$ and $[u_{11}, u_{12}]$, and the torsion one in $[u_1, u_2]$ and $[u_{12}, u_{13}]$. Part 1 of the coplanarity criterion is triggered in $[\frac{1}{4}\, u_{m-1} + \frac{3}{4}\, u_m, \frac{3}{4}\, u_{m+1} + \frac{1}{4}\, u_{m+2}]$, $m = 3, 10$ (see also Table 6.2), and the collinearity one in $[\frac{3}{4}\, u_6 + \frac{1}{4}\, u_7, \frac{1}{4}\, u_7 + \frac{3}{4}\, u_8]$ (see also Table 6.3). Note that Part 2 of the coplanarity criterion is not activated, for the data set does not contain coplanar points ($\Delta_m = 0$) with $\mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0$. The final segment degrees are $\mathrm{K}_{ch}^{(9)} = \{6, 9, 9, 10, 10, 13, 13, 10, 10, 9, 9, 6\}$. In general, the degrees corresponding to the coplanar and collinear subsets of the data can be further reduced by finely tuning the parameters $\alpha_m^\ell, \alpha_m^r, \alpha_m, \alpha_{m+1}^\ell, \alpha_{m+1}^r$ and $\gamma_m^\ell, \gamma_m^r$ of Steps 0.1 and 0.2 of the algorithm. In our case, setting $\alpha_m^\ell = \alpha_m = \alpha_{m+1}^r = 0.16$ (default value= 0.207107), $m = 3, 10$, and $\gamma_7^\ell = \gamma_7^r = 0.17$ (default value= 0.207107) results in $\mathrm{K}_{ch}^{(8)} = \{6, 8, 8, 10, 10, 12, 12, 10, 10, 8, 8, 6\}$.

The graphical output of the algorithm for $\mathrm{K}_{ch}^{(9)} = \{6, 9, 9, 10, 10, 13, 13, 10, 10, 9, 9, 6\}$ is shown in Figures 6.1–6.5. Figure 6.1 depicts the first iteration (thin solid line), the last iteration (thick solid line) and the standard C^4-Quintic spline interpolating the given data set with the same boundary conditions and parametrization as those adopted by the algorithm (dotted line). All three curves retain, as expected, the symmetry of the interpolated data, while the most striking failure of both the first iteration and the C^4-Quintic, from the shape-preservation point of view, is that they are self-intersecting.

Figure 6.2 contains the torsion plots of the curves in Figure 6.1. The left half of this figure depicts the corresponding torsion plot in smaller τ-scale in order to better illustrate the failure of the first iteration and the standard C^4-Quintic, with regard to the torsion criterion. It can be seen that the torsion of both these curves fails to have the correct sign in $[u_1, u_2]$ and $[u_{12}, u_{13}]$.

Figure 6.3 shows the plots of the ratios $(\mathbf{w}(u) \cdot \mathbf{P}_2)/(|\mathbf{w}(u)||\mathbf{P}_2|)$, $(\mathbf{w}(u) \cdot \mathbf{P}_5)/(|\mathbf{w}(u)||\mathbf{P}_5|)$ and $(\mathbf{w}(u) \cdot \mathbf{P}_6)/(|\mathbf{w}(u)||\mathbf{P}_6|)$ in the parameter intervals $[u_2, u_3]$, $[u_4, u_5]$ and $[u_5, u_6]$, respectively. Negative values indicate failure with respect to the convexity criterion of Definition 2.1*, which happens on the first iteration (thin solid line) and for the C^4-Quintic (dotted line). It is worth noticing that this failure appears in the inner product of the binormal of the curve at a parameter node, e.g., $\mathbf{w}(u_3)/|\mathbf{w}(u_3)|$, with the discrete binormal at the other node of the corresponding parameter interval, i.e., $\mathbf{P}_2/|\mathbf{P}_2|$. Analogous results hold true for the parameter intervals $[u_8, u_9]$, $[u_9, u_{10}]$ and $[u_{11}, u_{12}]$.

Figure 6.4 depicts the plot of the coplanarity ratio $|\mathbf{w}(u) \times \mathbf{P}_3|/(|\mathbf{w}(u)||\mathbf{P}_3|)$ in $[\frac{1}{4}\, u_2 + \frac{3}{4}\, u_3, \frac{3}{4}\, u_4 +$

$\frac{1}{4}\,u_5]$. Clearly, the first-iteration and the C^4-Quintic ratios fail to be less than the user-specified level $\varepsilon_1 = 0.2$ (dashed line).

Finally, Figure 6.5 contains the collinearity ratio $|\dot{\mathbf{Q}}(u) \times \mathbf{L}_7|/(|\dot{\mathbf{Q}}(u)||\mathbf{L}_7|)$ in the parameter interval $[\frac{3}{4}\,u_6 + \frac{1}{4}\,u_7, \frac{1}{4}\,u_7 + \frac{3}{4}\,u_8]$. It is again clear that the first-iteration and C^4-Quintic ratios do not satisfy the collinearity constraint, which requires the ratios to be less than $\varepsilon_0 = 0.1$ (dashed line).

We conclude this section by investigating the effect of the parametrization on the shape quality of the outcome of SPIN3D, when applied to the data of Table 6.1 with the same boundary conditions and different parametrizations. For equidistant parametrization the algorithm yielded, after seventeen iterations, the following final segment degrees: $\mathrm{K}_{eq}^{(17)} = \{4, 21, 21, 20, 6, 15, 15, 6, 20, 21, 21, 4\}$. For the so-called centripetal parametrization ($u_{m+1} - u_m = h_m = \sqrt{|\mathbf{L}_m|}$, $m = 1(1)N-1$, $u_1 = 0$), the algorithm gave $\mathrm{K}_{cp}^{(10)} = \{4, 11, 11, 10, 8, 14, 14, 8, 10, 11, 11, 4\}$. Comparing $\mathrm{K}_{eq}^{(17)}$ and $\mathrm{K}_{cp}^{(10)}$ with the final degree distribution $\mathrm{K}_{ch}^{(9)}$, corresponding to chord-length parametrization, it can be seen that the degrees in $\mathrm{K}_{eq}^{(17)}$ and $\mathrm{K}_{cp}^{(10)}$ are, in general, larger than those in $\mathrm{K}_{ch}^{(9)}$. In the parameter intervals $[u_1, u_2]$ and $[u_{12}, u_{13}]$, however, the segment degrees k_1 and k_{12} have not been altered during the execution of the algorithm with equidistant and centripetal parametrization. This is in contrast to the chord-length case, where the torsion criterion is triggered, resulting in $k_{1,ch}^{(9)} = k_{12,ch}^{(9)} = 6$. Noting that $h_{1,eq} = h_{12,eq} = 1.0$ and $h_{1,cp} = h_{12,cp} = 2.47363$ are smaller than $h_{1,ch} = h_{12,ch} = 6.11882$, we have experimented by decreasing $h_{1,ch}$ and $h_{12,ch}$ and retaining unchanged the remaining parameter distances $h_{m,ch}$, $m = 2(1)11$. This experimentation resulted in the following hybrid parametrization : $h_{1,hy} = h_{12,hy} = 5.0$, $h_{m,hy} = h_{m,ch}$, $m = 2(1)11$, which in its turn yielded the final degree distribution: $\mathrm{K}_{hy}^{(9)} = \{4, 9, 9, 10, 10, 13, 13, 10, 10, 9, 9, 4\}$.

Figure 6.6 depicts the shape-preserving interpolant provided by SPIN3D for (a) equidistant, (b) centripetal, (c) chord-length and (d) the above-mentioned hybrid parametrization. Comparing Fig. 6.6(c) with Figs. 6.6(a) and 6.6(b), it is easily seen that chord-length parametrization yields a more visually-pleasing curve in the area of coplanar data, where equidistant and centripetal parametrizations exhibit a rather linear-like behaviour. Moreover, the transition from the coplanar data to the off-plane point is smoother in the case of chord-length parametrization. This is also reflected in the torsion plots 6.7(a), 6.7(b) and 6.7(c), where the maximum of the torsion in the transition intervals $[u_2, u_3]$ and $[u_{11}, u_{12}]$, for equidistant and centripetal parametrization, is considerably larger than that for chord-length parametrization. The remaining local extrema of the torsion plots in question are, in general, of the same order. Regarding now the chord-length and the hybrid parametrization, Figs. 6.6(c), 6.6(d), as well as the corresponding torsion plots 6.7(c) and 6.7(d), imply that the resulting curves are more or less of the same shape quality, even in the parameter intervals $[u_1, u_2]$ and $[u_{12}, u_{13}]$, where $k_{1,hy} = k_{12,hy} = 4 < k_{1,ch} = k_{12,ch} = 6$.

References

[1] Boehm, W. (1988) Letter to the Editor: On the definition of geometric continuity, *CAD* **20**, 370–372.

[2] Clements, J.C. (1992) A convexity-preserving C^2 parametric rational cubic interpolation, *Numer. Math.* **63**, 165–171.

[3] Goodman, T.N.T. (1991) Inflections on curves in two and three dimensions, *CAGD* **8**, 37–50.

[4] Goodman, T.N.T., and Unsworth, K. (1988) Shape-preserving interpolation by parametrically defined curves, *SIAM J. Numer. Anal.* **25**, 1453–1465.

[5] Kaklis, P.D., and Pandelis, D.G. (1990) Convexity-Preserving Polynomial Splines of Non-uniform Degree, *IMA J. Numer. Anal.* **10**, 223–234.

[6] Kaklis, P.D., and Sapidis, N.S. (1995) Convexity-preserving interpolatory parametric splines of non-uniform polynomial degree, *CAGD* **12**, 1–26.

[7] Mazure, M.-L. (1994) Geometric contact for curves and surfaces, *CAGD* **11**, 177–195.

[8] Sauer, R. (1970) Differenzengeometrie, Berlin: *Springer Verlag.*

[9] Wolfram, S. (1991) Mathematica, A System for Doing Mathematics by Computer, New York: *Addison-Wesley Publishing Co.*, Second Edition.

FIGURES

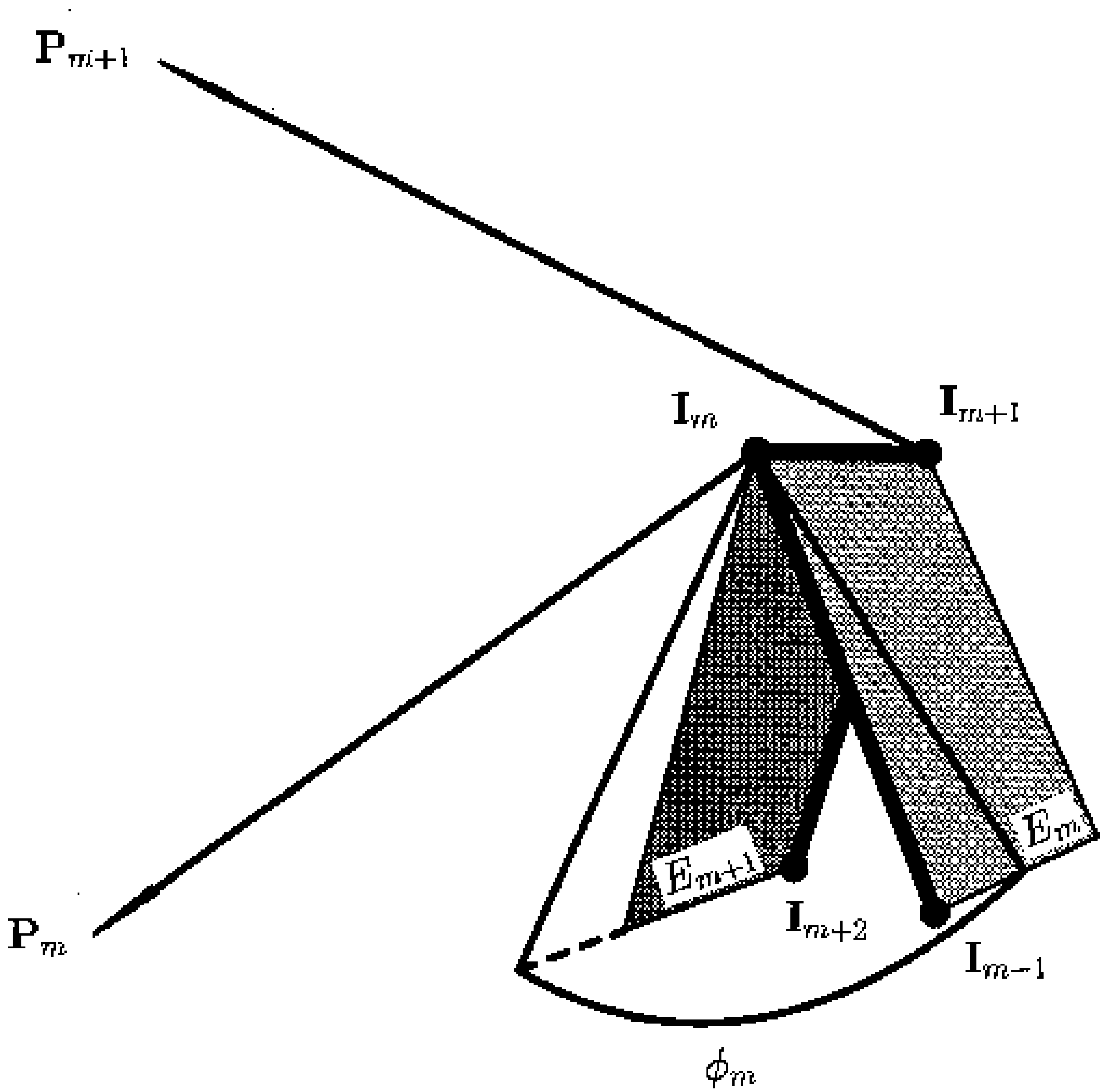

Figure 2.1: "Discrete Geometry" of the polygonal line (thick line) connecting the interpolation points $\mathbf{I}_{m-1}, \mathbf{I}_m, \mathbf{I}_{m+1}$ and $\mathbf{I}_{m+2}$.

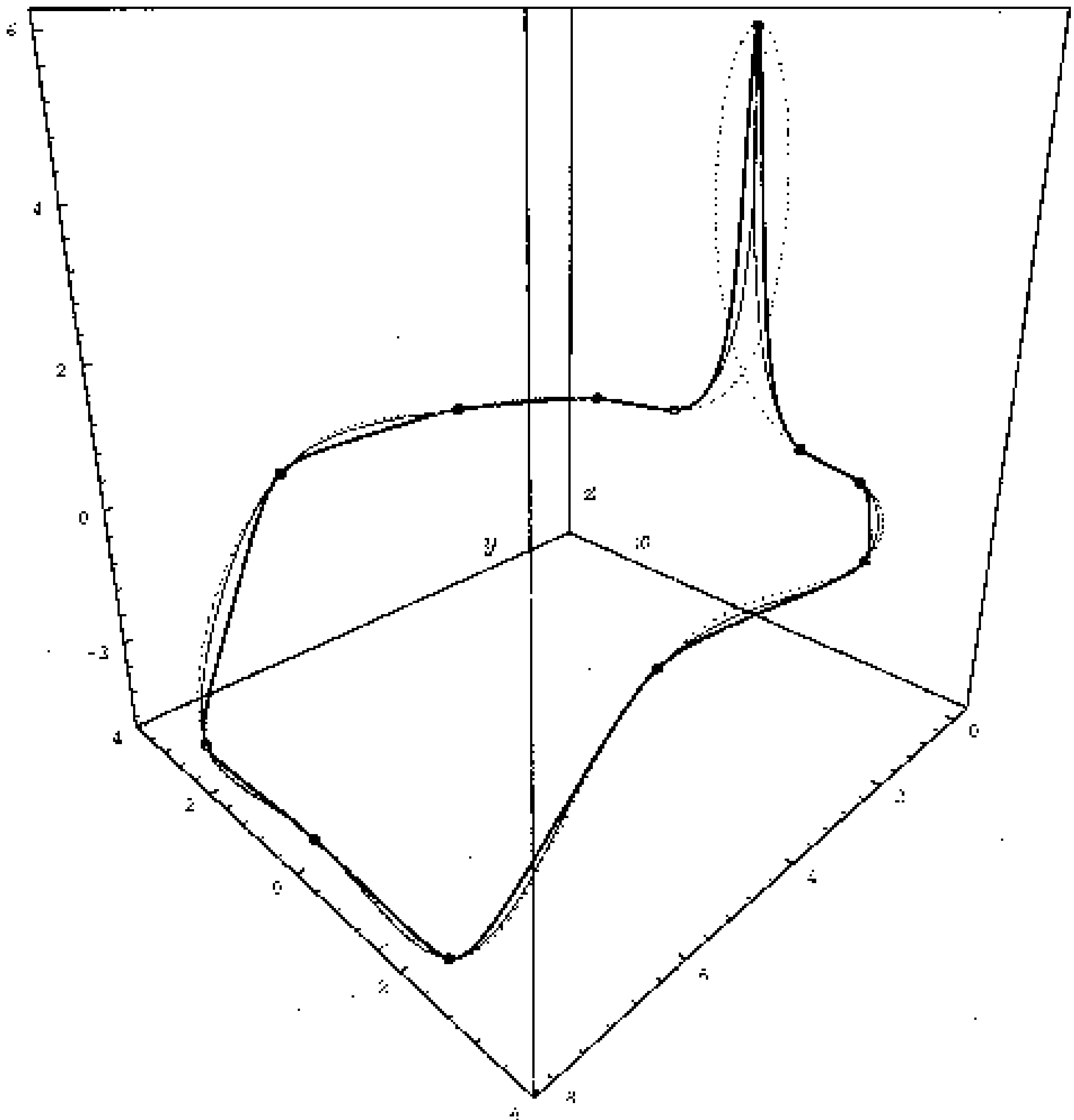

Figure 6.1: The first iteration (thin solid line), the last iteration (thick solid line) and the standard C^4-Quintic (dotted line).

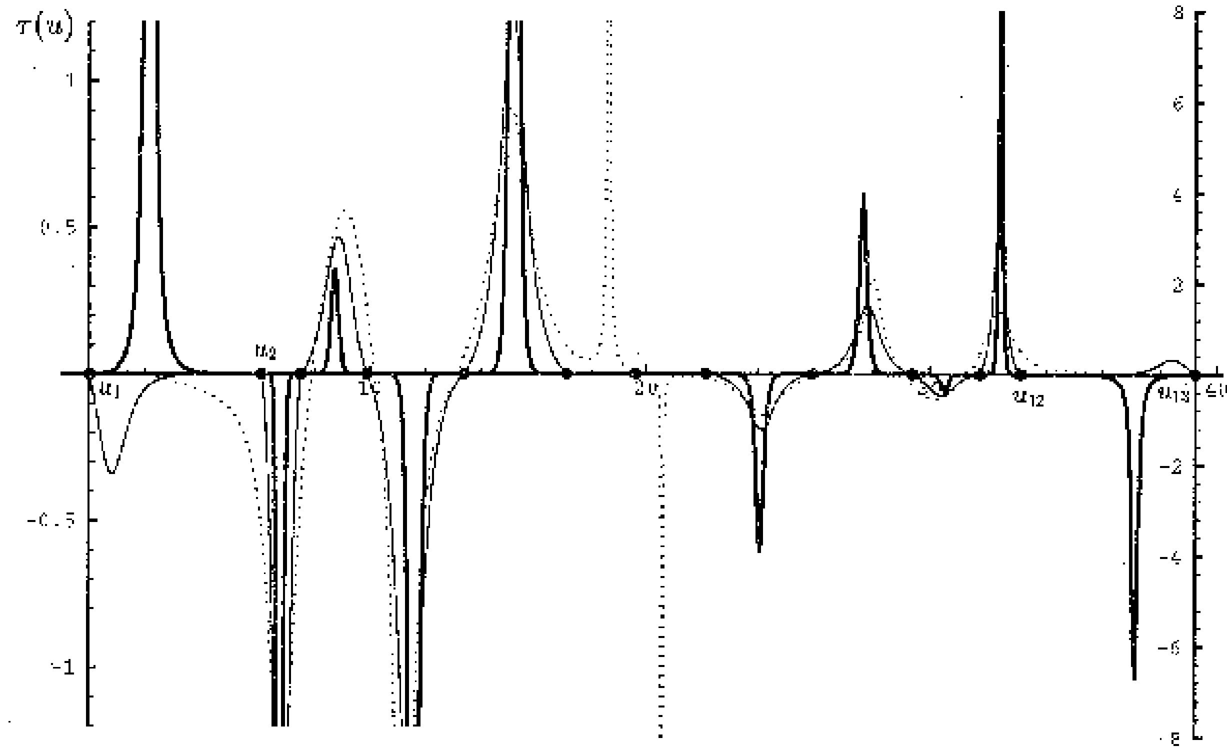

Figure 6.2: The torsion plot $\tau(u)$ of the curves in Fig. 6.1. Left-half scale: $[-1.2, 1.2]$. Right-half scale: $[-8, 8]$.

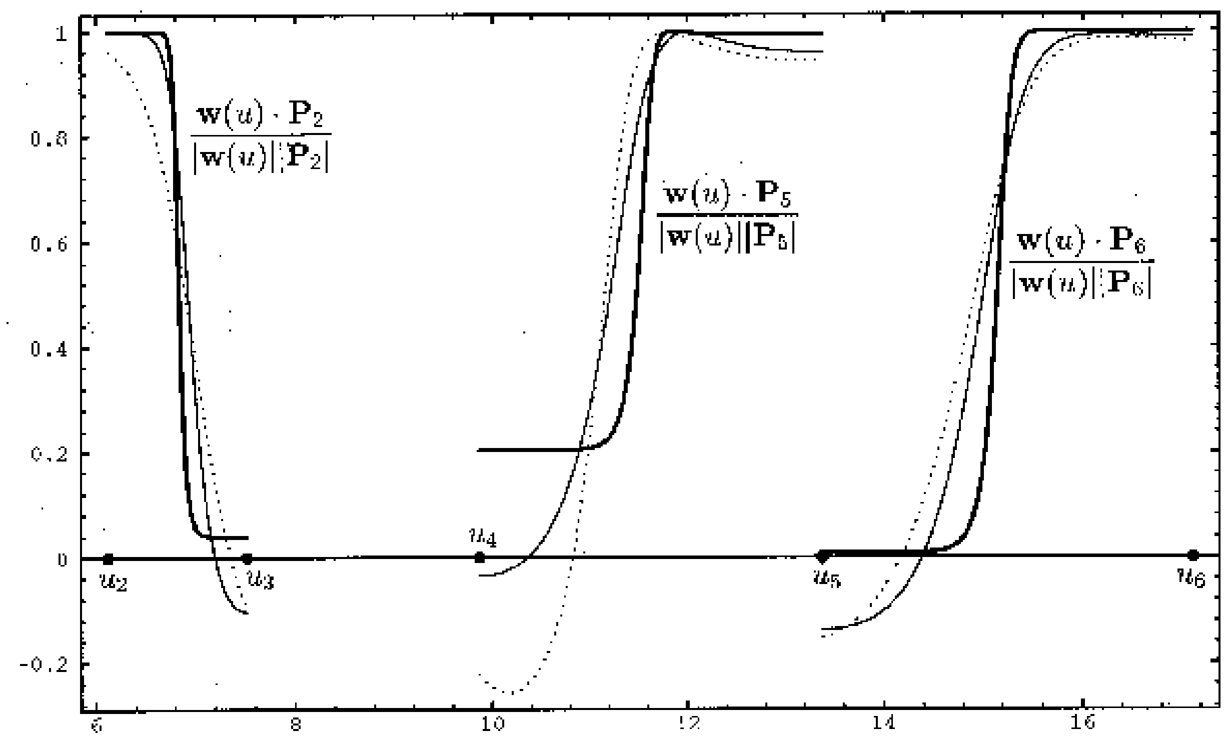

Figure 6.3: Plots of the following convexity ratios of the curves in Fig. 6.1: $(\mathbf{w}(u)\cdot\mathbf{P}_2)/(|\mathbf{w}(u)|\,|\mathbf{P}_2|)$, $u \in [u_2, u_3]$, $(\mathbf{w}(u)\cdot\mathbf{P}_5)/(|\mathbf{w}(u)||\mathbf{P}_5|)$, $u \in [u_4, u_5]$ and $(\mathbf{w}(u)\cdot\mathbf{P}_6)/(|\mathbf{w}(u)||\mathbf{P}_6|)$, $u \in [u_5, u_6]$.

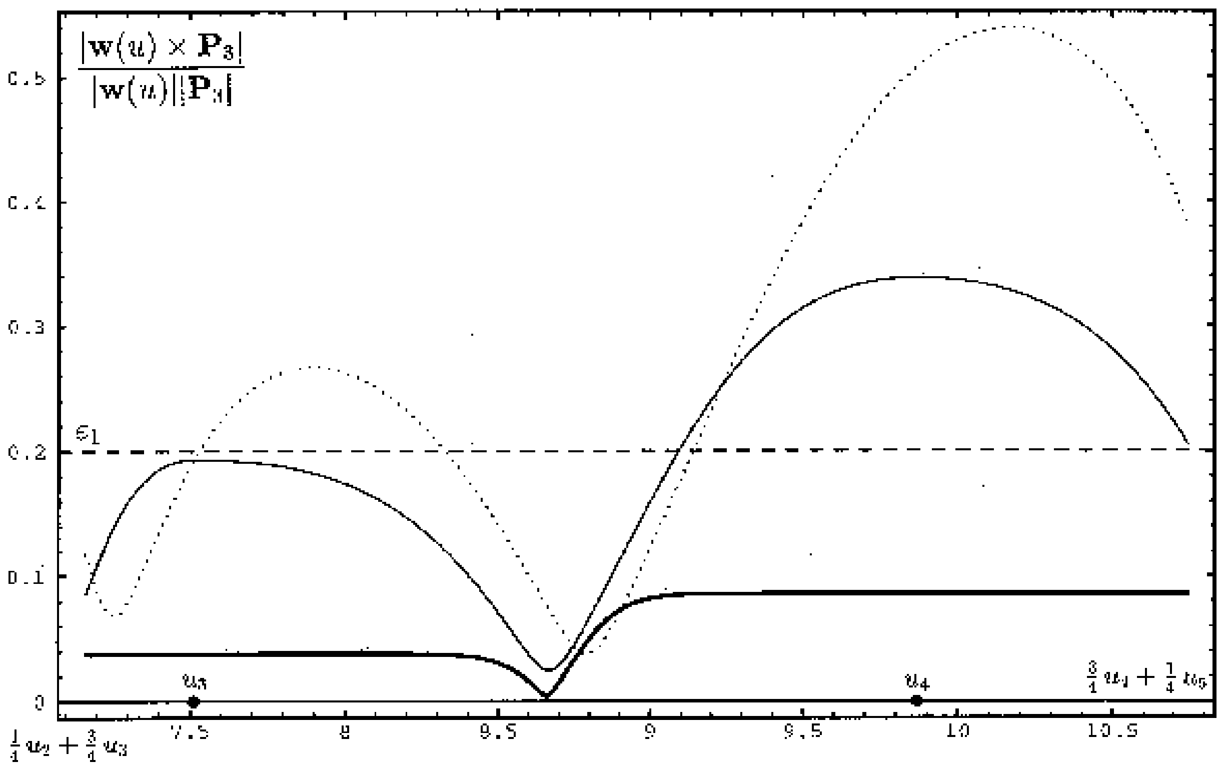

Figure 6.4: Plot of the coplanarity ratio $|\mathbf{w}(u) \times \mathbf{P}_3|/(|\mathbf{w}(u)||\mathbf{P}_3|)$, $u \in [\frac{1}{4}\,u_2 + \frac{3}{4}\,u_3, \frac{3}{4}\,u_4 + \frac{1}{4}\,u_5]$, of the curves in Fig. 6.1.

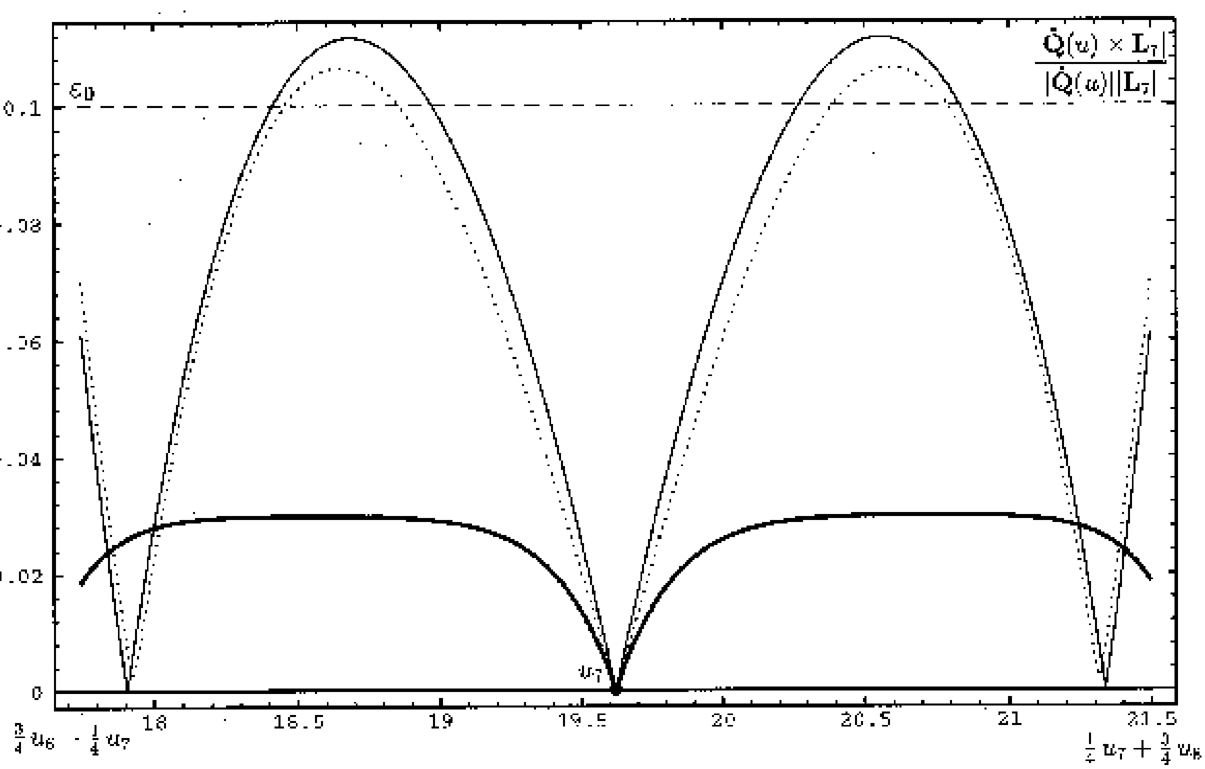

Figure 6.5: Plot of the collinearity ratio $|\dot{\mathbf{Q}}(u) \times \mathbf{L}_7|/(|\dot{\mathbf{Q}}(u)||\mathbf{L}_7|)$, $u \in [\frac{3}{4}\,u_6 + \frac{1}{4}\,u_7, \frac{1}{4}\,u_7 + \frac{3}{4}\,u_8]$, of the curves in Fig. 6.1.

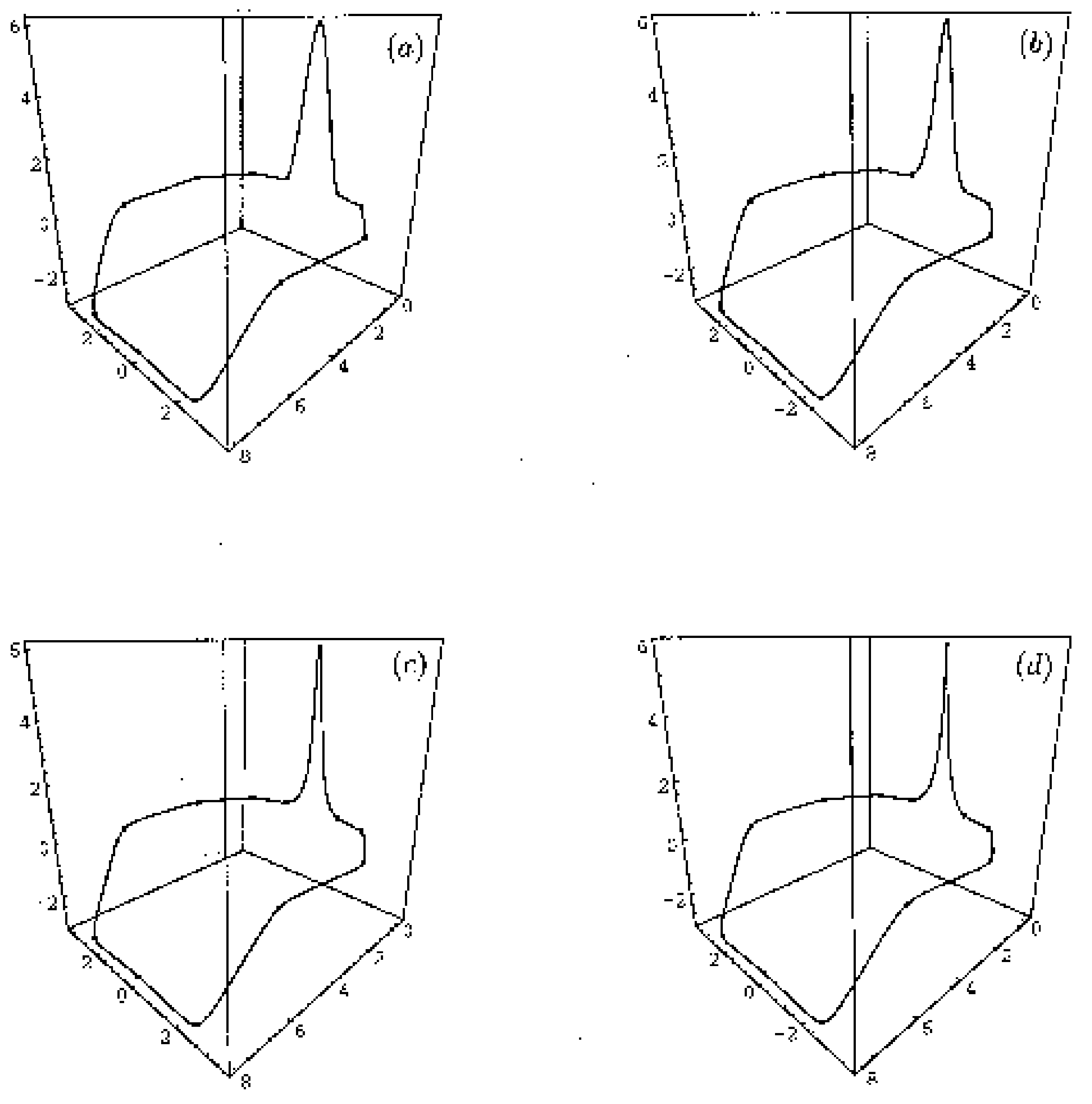

Figure 6.6: The last iteration using : (a) equidistant, (b) centripetal, (c) chord-length and (d) a hybrid parametrization.

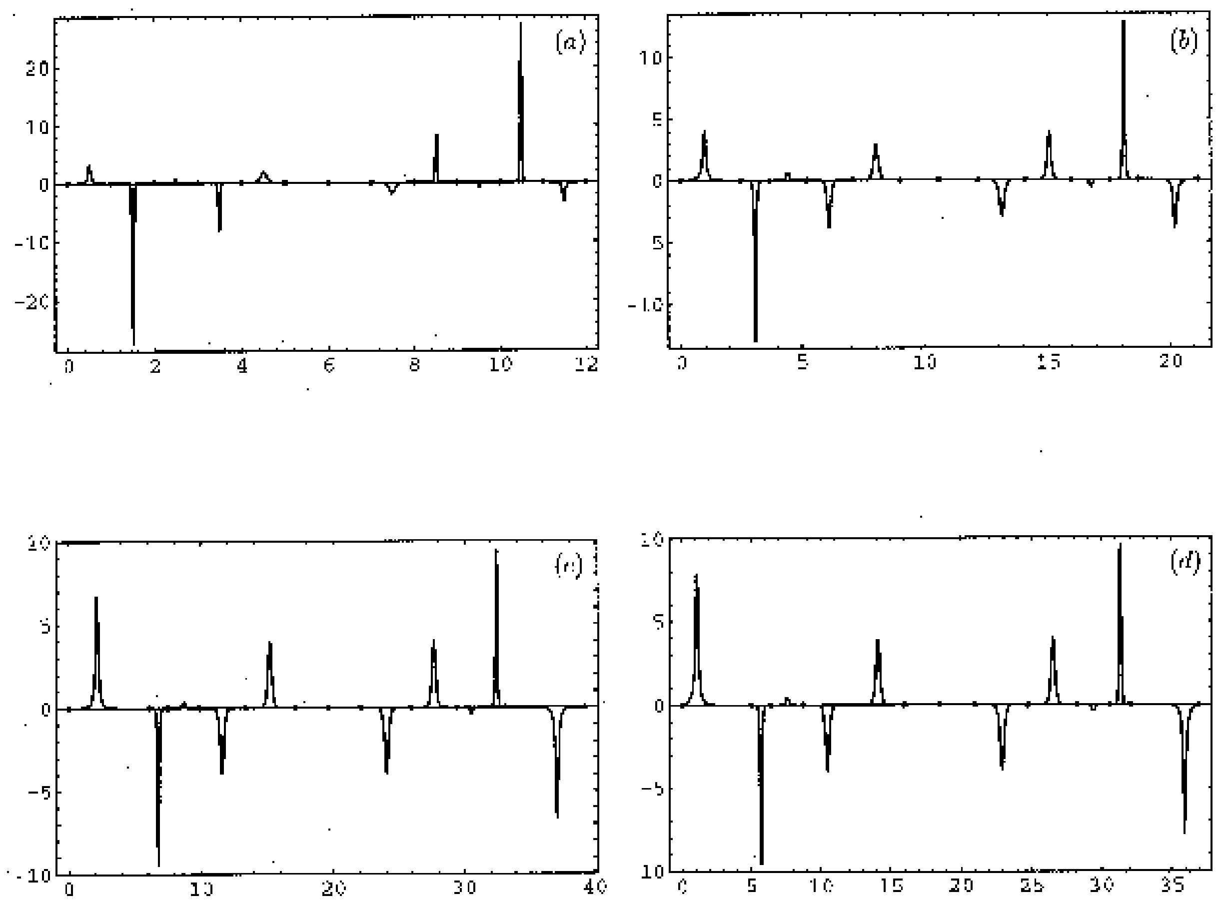

Figure 6.7: The torsion plots of the curves in Fig. 6.6.