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Spatial shape-preserving interpolation using ν-splines

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We present a global iterative algorithm for constructing spatial G^2-continuous interpolating ν-splines, which preserve the shape of the polygonal line that interpolates the given points. Furthermore, the algorithm can handle data exhibiting two kinds of degeneracy, namely coplanar quadruples and collinear triplets of points. The convergence of the algorithm stems from the asymptotic properties of the curvature, torsion and Frénet frame of ν-splines for large values of the tension parameters, which are thoroughly investigated and presented. The performance of our approach is tested on two data sets, one of synthetic nature and the other of industrial interest.

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1. Introduction

Shape-preserving interpolation, via functional, as well as parametric splines, can be regarded as a well studied topic for the planar case. Indeed, there exists an abundance of papers, that provide schemes for constructing monotonicity- and/or convexity-preserving planar splines. On the contrary, the literature on shape-preserving interpolation in the three-dimensional space is apparently limited. This may be partially attributed to the fact that, introducing and validating a notion of shape-preserving interpolation in 3D is not as straightforward as it is in 2D. As a consequence, the first relevant papers in the literature have been devoted to introducing and investigating alternative notions of shape-preserving spatial interpolation, such as the *inflection count of a curve* in Goodman [4] and the shape properties of *coils* in Labenski and Piper [9]. In the sequel, the literature was enriched with schemes that enable the automatic construction of spatial shape-preserving interpolants using rational cubic splines (see Goodman & Ong [5], [6]), polynomial splines of non-uniform degree (see Kaklis & Karavelas [8]) and sextic splines (see Asaturyan et al. [1]).

Our present intent is to investigate the ability of ν-splines, introduced by Nielson [10], to construct G^2-continuous spatial interpolants that conform, to the largest possible extent, with the shape-preservation criteria adopted in [8]. This investigation leads eventually to a fully automatic algorithm that is capable to provide, after a finite number of iterations, appropriate values for the ν-parameters that determine shape-preserving G^2-cubic spline interpolants. Given that working with low-degree polynomial splines seems to be a standard choice for the CAD/CAM community, the present work offers the practical advantage of using the most widely used low-degree polynomial spline, namely the cubic spline, for spatial shape-preserving interpolation.

The contents of the paper are structured as follows. In section 2, after introducing some preliminary material, we formulate the *shape-preserving interpolation problem* (see Problem $(\mathcal{P})$) for the class of G^2-continuous splines that, for a fixed parametrization $\mathcal{U}$, interpolate a spatial point set $\mathcal{D}$ and satisfy appropriate (unit tangent vector, zero curvature or periodic) boundary conditions $\mathcal{B}$. Section 3 provides the ν-spline basics, namely the ν-spline representation in which we have introduced the novelty of dimensionless ν-parameters, and the strictly diagonally dominant linear systems we get when solving the G^2 interpolation problem under boundary conditions $\mathcal{B}$ with ν-splines.

Section 4 focuses on investigating the asymptotic behavior of a G^2 interpolatory ν-spline for large (positive) values of the ν-parameters. This investigation leads to a series of results, including the asymptotic behavior of the curve and its tangent vector (see Th. 4.7), as well as its intrinsic characteristics. More specifically, Theorems 4.10 and 4.11 deal with the asymptotic behavior of the curvature, while Theorems 4.12 and 4.13 deal with the asymptotic behavior of the torsion and the Frénet frame of a ν-spline, respectively. These theorems constitute the apparently novel part of §4.

Section 5 consists of two subsections. In §5.1 we derive appropriate increase patterns of the ν-parameters, ensuring the fulfillment of all the criteria of the shape-preservation notion introduced in §2, i.e., the *convexity* criterion (see Lemma 5.4), the *torsion* criterion (see Lemma 5.6), the *coplanarity* criteria (see Lemmata 5.7 and 5.8) and the *collinearity* criterion (see Lemma 5.9). Subsection 5.2 exploits these lemmata for composing an iterative algorithm, dubbed NUSSP (**S**hape **P**reserving **NU** **S**plines), for spatial shape-preserving interpolation via G^2-continuous ν-splines. The question of convergence along with the issues regarding the running time and the storage requirements of NUSSP are also discussed at the end of §5.2.

The paper concludes in §6, where the performance of NUSSP is tested and graphically illustrated for two data sets. The first data comprises of points taken from a helix (see Figs. 1-4), while the second one is industrial data provided by General Motors Design Center (see Figs. 5-8).

2. Problem Formulation

Let $\mathcal{D} = \{\mathbf{I}_m, m = 1(1)N\}$ be a set of points in the three-dimensional affine space $\mathbb{E}^3$, with $\mathbf{I}_m \neq \mathbf{I}_{m+1}$, $m = 1(1)N-1$, and $\mathcal{L}_{\mathcal{D}}$ be the polygonal line connecting the points of $\mathcal{D}$, $\mathbf{L}_m = \mathbf{I}_{m+1} - \mathbf{I}_m$, $\mathbf{P}_m = \mathbf{L}_{m-1} \times \mathbf{L}_m$ and $\Gamma_m = |\mathbf{L}_{m-1}\ \mathbf{L}_m\ \mathbf{L}_{m+1}|$. According to Sauer ([11], Chapter I, §2.2), the vectors $\mathbf{P}_m$ and the quantities Γ_m can be used for establishing a kind of *Discrete Differential Geometry* of the polygonal line $\mathcal{D}$. More specifically, $\mathbf{P}_m/\|\mathbf{P}_m\|$ is the so-called *discrete binormal* at $\mathbf{I}_m$, while Γ_m specifies the sign of the so-called *discrete torsion:*

$$\hat{\tau}_m = \frac{\Gamma_m}{\|\mathbf{P}_m\|\,\|\mathbf{P}_{m+1}\|}, \tag{2.1}$$

along the leg $\mathbf{I}_m\mathbf{I}_{m+1}$ of $\mathcal{D}$.

Furthermore let $\mathbf{Q}(u)$, $u \in [u_1, u_N]$ be a sufficiently smooth curve, interpolating $\mathcal{D}$ with parametrization $\mathcal{U} = \{u_1, u_2, \dots, u_N :\ u_1 < u_2 < \dots < u_N\}$ and satisfying appropriate boundary conditions $\mathcal{B}$. In this work, we shall deal with one of the following three types of boundary conditions:

1. Given unit-tangent vectors: $\dot{\mathbf{Q}}_1/\|\dot{\mathbf{Q}}_1\| = \mathbf{s}_0$, $\dot{\mathbf{Q}}_N/\|\dot{\mathbf{Q}}_N\| = \mathbf{s}_N$, where $\dot{\mathbf{Q}}_m := \dot{\mathbf{Q}}(u_m)$ and $\mathbf{s}_0$, $\mathbf{s}_N$ are given unit vectors in $\mathbb{R}^3$.
2. Zero curvatures: $\kappa(u_1) = \kappa(u_N) = 0$.
3. Periodic conditions for closed data sets ($\mathbf{I}_1 \equiv \mathbf{I}_N$): $\dot{\mathbf{Q}}_1 = \dot{\mathbf{Q}}_N$, $\ddot{\mathbf{Q}}_1 = \ddot{\mathbf{Q}}_N$.

Let also $\kappa(u)$ be the curvature and be $\tau(u)$ the torsion of $\mathbf{Q}(u)$, defined as follows :

$$\kappa(u) = \frac{\|\dot{\mathbf{Q}}(u) \times \ddot{\mathbf{Q}}(u)\|}{\|\dot{\mathbf{Q}}(u)\|^3}, \qquad \tau(u) = \frac{|\dot{\mathbf{Q}}(u)\ \ddot{\mathbf{Q}}(u)\ \dddot{\mathbf{Q}}(u)|}{\|\dot{\mathbf{Q}}(u) \times \ddot{\mathbf{Q}}(u)\|^2}. \tag{2.2}$$

We now proceed to accurately formulate the *shape-preserving interpolation problem,* that we shall attempt to handle in the forthcoming sections with the aid of ν-$splines$:

Problem ($\mathcal{P}$). Find a G^2-continuous curve $\mathbf{Q}(u)$, $u \in [u_1, u_N]$ that interpolates the point set $\mathcal{D}$ with parametrization $\mathcal{U}$, satisfies boundary conditions $\mathcal{B}$ and is *shape preserving* in the sense:

1. (convexity criterion) If $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, then:

$$\mathbf{w}(u) \cdot \mathbf{P}_n > 0, n = m, m+1, \quad \mathbf{w}(u) = \dot{\mathbf{Q}}(u) \times \ddot{\mathbf{Q}}(u), \quad u \in [u_m, u_{m+1}]. \tag{2.3}$$

2. (torsion criterion) If $\Gamma_m \neq 0$, then

$$\tau(u)\,\Gamma_m > 0, \quad u \in [u_m^+, u_{m+1}^-]. \tag{2.4}$$

3. (coplanarity criterion) If $\Gamma_m = 0$ and

   (a) $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, then, for $n = m$ and/or $m+1$:

$$\frac{\|\mathbf{w}(u) \times \mathbf{P}_n\|}{\|\mathbf{w}(u)\|\,\|\mathbf{P}_n\|} < \varepsilon_1, \quad \|\mathbf{w}(u)\| \neq 0, \quad u \in \omega_m, \tag{2.5}$$

   where ε_1 is a user-specified small positive number in $(0,1]$, and ω_m a user-specified closed interval such that $[u_m, u_{m+1}] \subseteq \omega_m \subset (u_{m-1}, u_{m+2})$.

(b) $\mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0$, then, for $n = m$ and/or $m + 1$:

$$\frac{\|\mathbf{w}(u) \times \mathbf{P}_n\|}{\|\mathbf{w}(u)\| \, \|\mathbf{P}_n\|} < \varepsilon_1, \quad \|\mathbf{w}(u)\| \neq 0, \quad u \in \vartheta_m \cup \varphi_m, \tag{2.6}$$

where ε_1 is as above, and $\vartheta_m = [\theta_{m1}, \theta_{m2}]$ and $\varphi_m = [\phi_{m1}, \phi_{m2}]$ are user-specified closed intervals such that $\theta_{m1} \leq u_m$, $u_m < \theta_{m2} < u_m^*$ and $u_m^* < \phi_{m1} < u_{m+1}$, $u_{m+1} \leq \phi_{m2}$, for some user-specified separating point u_m^* in (u_m, u_{m+1}).

4. (collinearity criterion) If $\|\mathbf{P}_m\| = 0$ and $\mathbf{L}_{m-1} \cdot \mathbf{L}_m > 0$, then, for $n = m - 1, m$:

$$\frac{\|\dot{\mathbf{Q}}(u) \times \mathbf{L}_n\|}{\|\dot{\mathbf{Q}}(u)\| \, \|\mathbf{L}_n\|} < \varepsilon_0, \quad u \in \eta_m, \tag{2.7}$$

where ε_0 is a user-specified small positive number in $(0, 1]$, and η_m a user-specified closed subinterval of (u_{m-1}, u_{m+1}) that includes u_m as an interior point.

The notion of shape-preservation, adopted in Problem $(\mathcal{P})$, is essentially the one introduced in Kaklis & Karavelas [8], though slightly modified in order to be applicable to G^2-continuous curves that exhibit, in general, torsion discontinuities at the nodes of the parametrization $\mathcal{U}$; compare the *torsion criterion* in $(\mathcal{P})$ with that in [8].

Let us now provide the motivation underlying the above adopted shape-preservation *criteria*. To start with, the *convexity criterion* requires that the projections of the interpolating curve $\mathbf{Q}(u)$ onto the two planes that are normal on $\mathbf{P}_m$ and $\mathbf{P}_{m+1}$, should be locally convex along $[u_m, u_{m+1}]$. Next, the *torsion criterion* requires that the sign of torsion of $\mathbf{Q}(u)$ onto $[u_m^+, u_{m+1}^-]$ is the same with that of Γ_m, representing the sign of the so-called *discrete torsion* of the polygonal line connecting the points $\mathbf{I}_{m-1}$, $\mathbf{I}_m, \mathbf{I}_{m+1}$ and $\mathbf{I}_{m+2}$. If the previous quadruple is coplanar, then the *coplanarity criterion* suggest that $\mathbf{Q}(u)$ should be as coplanar as the user needs over a closed subinterval ω_m of (u_{m-1}, u_{m+2}), which is also user-specified. Coplanarity is achieved by restricting the deviation between the binormal of the interpolant and either of the parallel *discrete binormals* $\mathbf{P}_m/\|\mathbf{P}_m\|$ and $\mathbf{P}_{m+1}/\|\mathbf{P}_{m+1}\|$ at the vertices $\mathbf{I}_m$ and $\mathbf{I}_{m+1}$, respectively, of the polygonal interpolant. When the quadruple $\mathbf{I}_m$, $i = m - 1, ..., m + 2$, defines a convex polygon *(case a:* $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$*)*, then ω_m contains $[u_m, u_{m+1}]$. When the quadruple of points does not define a convex polygon, ω_m is split in two disjoint intervals, ϑ_m and ϕ_m, in order to account for the fact that the discrete binormals at $\mathbf{I}_m$ and $\mathbf{I}_{m+1}$ now possess opposite directions. Finally, if three consecutive interpolation points, $\mathbf{I}_i$, $i = m - 1, m, m + 1$, are collinear, then the *collinearity criterion* requires that the unit-tangent vector of $\mathbf{Q}(u)$ should be, along a user-specified closed subinterval η_m of (u_{m-1}, u_{m+1}), as parallel as the user needs to the line connecting the collinear triplet.

Of course, the four criteria, introduced in Problem $(\mathcal{P})$, do not constitute the only possible definition of shape preservation. One could impose more stringent shape-preserving criteria. For example, in [6], the *convexity criterion* requires that, when $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, the projections of the curve segment be convex rather than merely locally convex, or exhibit only one inflection point within $[u_m, u_{m+1}]$ if $\mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0$.

In the same work, the *coplanarity criterion* says that $\mathbf{Q}(u)$ should be exactly, not approximately, coplanar with the planar quadruple $\mathbf{I}_m$, $i = m-1, ..., m+2$.

3. ν-spline Basics

Introduced by Nielson [10], a G^2-cubic ν-spline can be written in parametric form as follows:

$$\mathbf{Q}(u) = \mathbf{I}_m H_0^3(t) + \mathbf{q}_m \Delta_m H_1^3(t) + \mathbf{q}_{m+1} \Delta_m H_2^3(t) + \mathbf{I}_{m+1} H_3^3(t), \tag{3.1}$$

$u \in [u_m, u_{m+1}]$, where $t = \frac{u-u_m}{\Delta_m}$ with $\Delta_m = u_{m+1} - u_m > 0$ is the so-called local parameter $(t \in [0,1])$, $H_i^3(t), i = 0(1)3$, are the standard cubic Hermite polynomials, and $\mathbf{q}_m$ are the tangent vectors of $\mathbf{Q}(u)$ at the interpolation points $\mathbf{I}_m$, $m = 1, ..., N$. If these vectors are specified, then the interpolant is, in general, C^1-continuous. To achieve G^2-continuity, the following conditions should be fulfilled at the interior nodes:

$$\ddot{\mathbf{Q}}(u_m^+) - \ddot{\mathbf{Q}}(u_m^-) = \tilde{\nu}_m \, \mathbf{q}_m, \quad \tilde{\nu}_m = \frac{\nu_m}{\sqrt{\Delta_{m-1}\Delta_m}}, \quad m = 2(1)N-1, \tag{3.2}$$

for some constants ν_m, the so-called ν-parameters. In the above conditions, these parameters are normalized by the quantity $\sqrt{\Delta_{m-1}\Delta_m}$, so that they become dimensionless. This differs from the approach used in the literature, where ν's represent inverses of lengths; see [10] and Farin [3]. The normalizing quantity has been chosen so that $\tilde{\nu}$'s are invariant with respect to the ordering of the points in $\mathcal{D}$. Furthermore, if $\mathcal{U}$ is the chord-length parametrization, then the $\tilde{\nu}$'s are invariant with respect to uniform scaling.

Using the above relation and incorporating the boundary conditions $\mathcal{B}$, we arrive at a linear system for the quantities $\mathbf{q}_m, m = 1(1)N$, which is of the form:

$$C_m \mathbf{q}_{m-1} + A_m \mathbf{q}_m + D_m \mathbf{q}_{m+1} = \mathbf{B}_m, \quad m = 1(1)S, \tag{3.3}$$

where $S = N$ for the unit-tangent vector and zero-curvature boundary conditions, and $S = N-1$, $\mathbf{q}_0 := \mathbf{q}_S$ and $\mathbf{q}_{S+1} := \mathbf{q}_1$ for the periodic boundary conditions. In particular, for $m = 2(1)S-1$, we have:

$$C_m = \Delta_m, \tag{3.4}$$

$$A_m = 2\Delta_{m-1} + 2\Delta_m + \frac{1}{2}\sqrt{\Delta_{m-1}\Delta_m}\, \nu_m, \tag{3.5}$$

$$D_m = \Delta_{m-1}, \tag{3.6}$$

$$\mathbf{B}_m = 3\left(\frac{\Delta_m}{\Delta_{m-1}}\mathbf{L}_{m-1} + \frac{\Delta_{m-1}}{\Delta_m}\mathbf{L}_m\right). \tag{3.7}$$

The remaining coefficients and right-hand side vectors depend on the boundary conditions. More accurately:

1. Unit-tangent vector boundary conditions: $C_n = D_n = 0$, $A_n = 1 + \nu_n$, $n = 1, N$, and

$$\mathbf{B}_1 = \mathbf{s}_0, \qquad \mathbf{B}_N = \mathbf{s}_N. \tag{3.8}$$

2. Zero-curvature boundary conditions: In this case we require the following conditions that yield zero curvature at the end points:

$$\ddot{\mathbf{Q}}(u_1^+) = \frac{\nu_1}{\Delta_1}\,\mathbf{q}_1, \tag{3.9a}$$

$$\ddot{\mathbf{Q}}(u_N^-) = \frac{\nu_N}{\Delta_{N-1}}\mathbf{q}_N. \tag{3.9b}$$

As a result, we get that: $C_1 = D_N = 0$, $C_N = D_1 = 1$, $A_n = 2 + \frac{1}{2}\nu_n$, $n = 1, N$ and

$$\mathbf{B}_1 = \frac{3}{\Delta_1}\,\mathbf{L}_1, \qquad \mathbf{B}_{N-1} = \frac{3}{\Delta_{N-1}}\,\mathbf{L}_{N-1}. \tag{3.10}$$

3. Periodic boundary conditions: $C_1 = \Delta_1$, $D_1 = \Delta_{N-1}$, $A_1 = 2\Delta_1 + 2\Delta_{N-1} + \frac{1}{2}\sqrt{\Delta_1\Delta_{N-1}}\nu_1$ and

$$\mathbf{B}_1 = 3\left(\frac{\Delta_{N-1}}{\Delta_1}\,\mathbf{L}_1 + \frac{\Delta_1}{\Delta_{N-1}}\,\mathbf{L}_{N-1}\right). \tag{3.11}$$

It can be easily seen that all the above linear systems are symmetric, under a diagonal scaling, and strictly diagonally dominant for $\nu_m \geq 0, m = 1(1)N$, thus uniquely solvable.

4. Asymptotic Properties of ν-splines

In this section we shall investigate the asymptotic behavior of ν-splines $\mathbf{Q}(u)$ for large (positive) ν-parameters. In particular, we will focus on deriving the asymptotic behavior of $\mathbf{Q}(u)$ and its tangent vector $\dot{\mathbf{Q}}(u)$, the co-directional to its binormal vector $\mathbf{w}(u) = \dot{\mathbf{Q}}(u) \times \ddot{\mathbf{Q}}(u)$, its curvature, $\kappa(u)$, and torsion, $\tau(u)$, distributions and, finally, its Frénet frame $\mathbf{v}_t(u)$, $\mathbf{v}_b(u)$ and $\mathbf{v}_n(u)$, where $\mathbf{v}_{t/b/n}(u)$ denotes the unit-tangent / binormal / principal normal of the curve, respectively.

Depending on the type of the imposed boundary conditions, we shall make use of the following conventions regarding the boundary values of the vectors $\mathbf{P}_m$ and the quantities $\Delta_m = u_{m+1} - u_m$ and ν_m.

1. Given unit-tangent vectors: $\Delta_0 := \Delta_1$, $\Delta_N := \Delta_{N-1}$, $\mathbf{P}_1 = \mathbf{s}_0 \times \mathbf{L}_1$, $\mathbf{P}_N = \mathbf{L}_{N-1} \times \mathbf{s}_N$ and $\nu_0 = \nu_{N+1} = 0$.

2. Zero curvatures: $\Delta_0 := \Delta_1$, $\Delta_N := \Delta_{N-1}$, $\mathbf{P}_1 = \mathbf{P}_N = (0,0,0)^T$ and $\nu_0 = \nu_{N+1} = 0$.

3. Periodic conditions for closed data sets: $\Delta_0 := \Delta_{N-1}$, $\Delta_N := \Delta_1$, $\mathbf{P}_1 = \mathbf{P}_N = \mathbf{L}_{N-1} \times \mathbf{L}_1$ and $\nu_0 := \nu_{N-1}, \nu_{N+1} := \nu_1$.

Since we are interested in the asymptotic behavior of our curve, we shall repeatedly need and use the Θ- and $\mathcal{O}$-notation. The definitions that follow are those in Cormen, Leiserson and Rivest [2].

Definition (Θ-notation) For a given function $g(n)$, we denote by $\Theta(g(n))$ the set of functions

$$\Theta(g(n)) = \{f(n) : \text{ there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\}$$

Definition ($\mathcal{O}$-notation) For a given function $g(n)$, we denote by $\mathcal{O}(g(n))$ the set of functions

$$\mathcal{O}(g(n)) = \{f(n) : \text{ there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\}$$

In the sequel we shall abuse the notation a little and instead of writing, e.g., $g(n) \in \mathcal{O}(f(n))$, we shall write $g(n) = \mathcal{O}(f(n))$.

4.1. Preparatory Results

We shall now derive three lemmas and three corollaries, that will be used in the subsequent subsection for obtaining the main results of this section.

Lemma 4.1. There exists a positive number M, depending on the set $\mathcal{D}$, the parameterization $\mathcal{U}$ and the boundary conditions $\mathcal{B}$ such that

$$\|\mathbf{q}_m\| \le M, \quad m = 1, ..., S. \tag{4.1}$$

Proof. The linear system (3.3) can be recast as:

$$\frac{C_m}{A_m} q_{i,m-1} + q_{i,m} + \frac{D_m}{A_m} q_{i,m+1} = \frac{B_{i,m}}{A_m}, \quad i = 1, 2, 3, \tag{4.2}$$

or, in matrix form:

$$(\mathbb{I} + \mathbb{A})\,\tilde{\mathbf{q}}_i = \hat{\mathbf{B}}_i, \quad i = 1, 2, 3, \tag{4.3}$$

where $\tilde{\mathbf{q}}_i = (q_{i,1} \ \dots \ q_{i,N})^T$, $\hat{\mathbf{B}}_i = (\frac{B_{i,1}}{A_1} \ \frac{B_{i,2}}{A_2} \ \dots \ \frac{B_{i,S-1}}{A_{S-1}} \ \frac{B_{i,S}}{A_S})^T$ and $\mathbb{A}$ is the matrix of the linear system (4.2) with its diagonal elements set to zero. Grounded on the defining formulae of A_m, C_m, D_m (see equs. (3.4)-(3.6) and items 1-3 in §3), one can write that, for all boundary conditions $\mathcal{B}$:

$$\left|\frac{C_m}{A_m}\right| + \left|\frac{D_m}{A_m}\right| \le \frac{1}{2}, \quad m = 1(1)S, \tag{4.4}$$

which gives: $\|\mathbb{A}\|_\infty \le \frac{1}{2}$. Combining this result with (4.3), we have:

$$\|\tilde{\mathbf{q}}_i\|_\infty = \|(\mathbb{I} + \mathbb{A})^{-1}\hat{\mathbf{B}}_i\|_\infty \le \frac{\|\hat{\mathbf{B}}_i\|_\infty}{1 - \|\mathbb{A}\|_\infty} \le \frac{2\|\tilde{\mathbf{B}}_i\|_\infty}{\min_{1 \le m \le S}\{A_m\}}, \tag{4.5}$$

where $\tilde{\mathbf{B}} = (B_{i,1} \ \dots \ B_{i,S})^T$. But

$$A_m \geq \min\{2\Delta_m, 1\} \geq \min_{1 \leq m \leq N-1}\{2\Delta_m, 1\}, \quad \forall m. \tag{4.6}$$

Thus,

$$\|\tilde{\mathbf{q}}_i\|_\infty \leq \frac{2\,\|\tilde{\mathbf{B}}_i\|_\infty}{\min_{1 \leq m \leq N-1}\{2\Delta_m, 1\}}. \tag{4.7}$$

Setting

$$M = \frac{2\sqrt{\|\tilde{\mathbf{B}}_1\|_\infty^2 + \|\tilde{\mathbf{B}}_2\|_\infty^2 + \|\tilde{\mathbf{B}}_3\|_\infty^2}}{\min_{1 \leq m \leq N-1}\{2\Delta_m, 1\}}, \tag{4.8}$$

(4.7) leads readily to the desired result. □

Using Lemma 4.1 we can now prove a well-known property of ν-splines:

Lemma 4.2.

$$\lim_{\nu_m \to \infty} \mathbf{q}_m = (0, 0, 0)^T \tag{4.9}$$

Proof. At first, we rewrite equations (3.3) as:

$$\mathbf{q}_m = A_m^{-1}(\mathbf{B}_m - C_m \mathbf{q}_{m-1} - D_m \mathbf{q}_{m+1}). \tag{4.10}$$

Secondly, appealing to formulae (3.4) and (3.6) as well as items 1-3 in §3, we easily get the following upper bound:

$$C_m, D_m \leq \max_{1 \leq i \leq N-1}\{\Delta_i\} + 1. \tag{4.11}$$

Then, combining (4.10) with bounds (4.1) and (4.11) and taking into account that $A_m = \Theta(\nu_m)$, the validity of the Lemma follows readily. □

Applying the above lemma for $\mathbf{q}_{m-1}$ and $\mathbf{q}_{m+1}$ and using the bounds (4.11), equality (4.10) yields the following important corollary:

Corollary 4.3.

$$\lim_{\nu_{m-1}, \nu_{m+1} \to \infty} A_m \mathbf{q}_m = \mathbf{B}_m. \tag{4.10}$$

Corollary 4.3 gives rise, in its turn, to another pair of corollaries:

Corollary 4.4.

$$\lim_{\nu_{m-1}, \nu_{m+1} \to \infty} A_m(\mathbf{q}_m \times \mathbf{L}_m) = 3\,\frac{\Delta_m}{\Delta_{m-1}}\,\mathbf{P}_m \tag{4.11}$$

$$\lim_{\nu_{m-1},\nu_{m+1}\to\infty} A_m(\mathbf{L}_{m-1}\times\mathbf{q}_m) = 3\,\frac{\Delta_{m-1}}{\Delta_m}\,\mathbf{P}_m. \tag{4.12}$$

Corollary 4.5.

$$\lim_{\nu_s\to\infty} A_n(\mathbf{q}_m\times\mathbf{q}_{m+1}) = (0,0,0)^T, \quad n = m, m+1, \tag{4.13}$$

where $s = m$, if $n = m+1$ and $s = m+1$ if $n = m$.

Proof. By virtue of Lemma 4.1, inequalities (4.11) and equations (3.3), we get the estimate $A_n\|\mathbf{q}_n\| = \mathcal{O}(1)$, $n = m, m+1$, which yields:

$$A_m A_{m+1}\|\mathbf{q}_m\times\mathbf{q}_{m+1}\| \le A_m A_{m+1}\|\mathbf{q}_m\|\|\mathbf{q}_{m+1}\| = \mathcal{O}(1). \tag{4.14}$$

The validity of the Corollary then follows from (4.14) and the fact that $A_n = \Theta(\nu_n)$, $n = m, m+1$. □

Lemma 4.6. If $\nu_i\to\infty$, $i = m-1(1)m+2$, then:

$$A_m A_{m+1}\left|\dot{\mathbf{Q}}(u)\ \ddot{\mathbf{Q}}(u)\ \dddot{\mathbf{Q}}(u)\right| \to \frac{108\,\Gamma_m}{\Delta_{m-1}\Delta_m^2\Delta_{m+1}},\ u\in[u_m^+, u_{m+1}^-]. \tag{4.15}$$

Proof. Using (3.1) we arrive, after some straightforward algebra, at the formula:

$$\left|\dot{\mathbf{Q}}(u)\ \ddot{\mathbf{Q}}(u)\ \dddot{\mathbf{Q}}(u)\right| = 12\Delta_m^{-4}|\mathbf{q}_m\ \mathbf{L}_m\ \mathbf{q}_{m+1}|. \tag{4.16}$$

On using (4.16) in conjunction with Corollary 4.3, we are lead to:

$$A_m A_{m+1}|\dot{\mathbf{Q}}(u)\ \ddot{\mathbf{Q}}(u)\ \dddot{\mathbf{Q}}(u)| \to 12\Delta_m^{-4}|\mathbf{B}_m\ \mathbf{L}_m\ \mathbf{B}_{m+1}|, \tag{4.17}$$

as $\nu_{m-1},\nu_m,\nu_{m+1},\nu_{m+2}\to\infty$, which, in view of the fact that:

$$|\mathbf{B}_m\ \mathbf{L}_m\ \mathbf{B}_{m+1}| = \frac{9\,\Delta_m^2}{\Delta_{m-1}\Delta_{m+1}}\,\Gamma_m, \tag{4.18}$$

gives the wanted result. □

4.2. Main Results

We are now fully equipped for materializing the aims of this section. To begin with, the ensuing theorem focuses on the asymptotic behavior of a ν-spline and its tangent vector in the parametric interval $[u_m, u_{m+1}]$.

Theorem 4.7. For $u \in [u_m, u_{m+1}]$,

$$\lim_{\nu_m,\nu_{m+1}\to\infty} \mathbf{Q}(u) = \mathbf{I}_m(1-t)^2(1+2t) + \mathbf{I}_{m+1}t^2(3-2t), \tag{4.19}$$

$$\lim_{\nu_m,\nu_{m+1}\to\infty} \dot{\mathbf{Q}}(u) = 6\Delta_m^{-1}\mathbf{L}_m t(1-t). \tag{4.20}$$

Proof. Appealing to (3.1) and applying Lemma 4.2 for $\mathbf{q}_m$ and $\mathbf{q}_{m+1}$, we readily get the limiting relation (4.19). Next, direct differentiation of (3.1) yields:

$$\dot{\mathbf{Q}}(u) = 6\Delta_m^{-1}\mathbf{L}_m t(1-t) + \mathbf{q}_m(1-t)(1-3t) + \mathbf{q}_{m+1}t(3t-2), \tag{4.21}$$

which, in conjunction with Lemma 4.2 again, gives (4.20). Obviously, for $u = u_m, u_{m+1}$, (4.20) degenerates to Lemma 4.2. □

The next two results summarize the asymptotic behavior of the vector $\mathbf{w}(u) = \dot{\mathbf{Q}}(u)\times\ddot{\mathbf{Q}}(u)$, that shares the same direction with the binormal $\mathbf{v}_b(u)$ of the ν-spline $\mathbf{Q}(u)$.

Theorem 4.8. If $\nu_i\to\infty$, $i = m-1(1)m=2$, with $\nu_m/\nu_{m+1} \to c \geq 0$, then:

$$A_n\,\mathbf{w}(u) \to \frac{18}{\Delta_m}\left[\frac{d_{nm}}{\Delta_{m-1}}\,\mathbf{P}_m\,(1-t)^2 + \frac{d_{n,m+1}}{\Delta_{m+1}}\,\mathbf{P}_{m+1}\,t^2\right], \quad u \in [u_m, u_{m+1}], \tag{4.22}$$

for $n = m, m+1$ and $d_{nm} = \lim_{\nu_n,\nu_m\to\infty} \frac{A_n}{A_m}$.

Proof. Using the defining formula (3.1) for $\mathbf{Q}(u)$, we arrive after some straightforward algebra at the expression:

$$\begin{aligned} A_n\,\mathbf{w}(u) = {} & 6\Delta_m^{-2}\frac{A_n}{A_m}(A_m\mathbf{q}_m \times \mathbf{L}_m)(1-t)^2 \\ & + 6\Delta_m^{-2}\frac{A_n}{A_{m+1}}(\mathbf{L}_m \times A_{m+1}\mathbf{q}_{m+1})t^2 \\ & - 2\Delta_m^{-1}A_n(\mathbf{q}_m \times \mathbf{q}_{m+1})(1-3t+3t^2). \end{aligned} \tag{4.23}$$

Appealing to Corollaries 4.4 and 4.5 and exploiting the hypothesis that $\nu_m/\nu_{m+1} \to c$, the validity of the Theorem follows readily from (4.23). □

A much simpler result can be derived for the nodal quantity $A_m\,\mathbf{w}_m$, $\mathbf{w}_m := \mathbf{w}(u_m)$. In particular, we have:

Corollary 4.9.

$$\lim_{\nu_{m-1},\nu_{m+1}\to\infty} \{A_m\,\mathbf{w}_m\} = \frac{18}{\Delta_{m-1}\Delta_m}\,\mathbf{P}_m. \tag{4.24}$$

Proof. For $n = m$ and $t = 0$, (4.23) degenerates to

$$A_m \,\mathbf{w}_m = 6\Delta_m^{-2} A_m(\mathbf{q}_m \times \mathbf{L}_m) - 2\,\Delta_m^{-1} A_m(\mathbf{q}_m \times \mathbf{q}_{m+1}), \tag{4.25}$$

which, in view of Corollaries 4.4 and 4.5 gives the wanted result. □

In fine, we turn to derive asymptotic results for the actual geometric quantities of a ν-spline, such as the curvature, the torsion and the Frénet frame. For the behavior of the curvature of a ν-spline in the open interval (u_m, u_{m+1}) we have:

Theorem 4.10. If $\nu_i \to \infty$, $i = m-1(1)m-2$, with $\nu_m/\nu_{m+1} \to c \geq 0$, then:

$$\kappa(u)A_n \to \frac{\Delta_m^2}{12\|\mathbf{L}_m\|^3} \left\| \frac{d_{nm}}{\Delta_{m-1}} \frac{\mathbf{P}_m}{(1-t)t^3} + \frac{d_{n,m+1}}{\Delta_{m+1}} \frac{\mathbf{P}_{m+1}}{(1-t)^3 t} \right\|, \; u \in (u_m, u_{m+1}), \tag{4.26}$$

for $n = m, m+1$ and $d_{nm} = \lim_{\nu_n, \nu_m \to \infty} \frac{A_n}{A_m}$.

Proof. Combining Theorems 4.8 and 4.7 we arrive at the following relation:

$$\kappa(u)A_n = \frac{A_n\,\|\mathbf{w}(u)\|}{\|\dot{\mathbf{Q}}(u)\|^3} \to \frac{\frac{18}{\Delta_m} \left\| \frac{d_{nm}}{\Delta_{m-1}} \mathbf{P}_m (1-t)^2 + \frac{d_{n,m+1}}{\Delta_{m+1}} \mathbf{P}_{m+1} t^2 \right\|}{\left[6\Delta_m^{-1} \|\mathbf{L}_m\| t(1-t) \right]^3}, \tag{4.27}$$

the right-hand side of which is equal to that of relation (4.26). □

The corresponding result for the nodal curvature, which is a direct consequence of Corollaries 4.3 and 4.9, has as follows.

Theorem 4.11.

$$\lim_{\nu_{m-1}, \nu_{m+1} \to \infty} \kappa(u_m) A_m^{-2} = \frac{18\|\mathbf{P}_m\|}{\Delta_{m-1}\Delta_m \|\mathbf{B}_m\|^3}. \tag{4.28}$$

We now proceed to state and prove the corresponding asymptotic result for the torsion:

Theorem 4.12. If $\nu_i \to \infty$, $i = m-1(1)m+2$, with $\nu_m/\nu_{m+1} \to c \geq 0$ and $\|\Delta_{m-1}^{-3/2} \mathbf{P}_m (1-t)^2 + c\Delta_{m+1}^{-3/2} \mathbf{P}_{m+1} t^2\| \neq 0$, then for $u \in [u_m^+, u_{m+1}^-]$:

$$\tau(u) \to \frac{c\Gamma_m}{3\Delta_{m-1}^{3/2}\Delta_{m+1}^{3/2}} \|\Delta_{m-1}^{-3/2} \mathbf{P}_m (1-t)^2 + c\Delta_{m+1}^{-3/2} \mathbf{P}_{m+1} t^2\|^{-2}. \tag{4.29}$$

Proof. Using Theorem 4.8, Lemma 4.6 and the fact that:

$$\lim_{\substack{\nu_m, \nu_{m+1} \to \infty \\ \nu_m/\nu_{m+1} \to c}} \frac{A_m}{A_{m+1}} = c\sqrt{\frac{\Delta_{m-1}}{\Delta_{m+1}}}, \tag{4.30}$$

we easily get that the torsion $\tau(u)$ of a ν-spline can be expressed as:

$$\tau(u) = \frac{A_m}{A_{m+1}} \frac{A_m A_{m+1} \left|\dot{\mathbf{Q}}(u)\ \ddot{\mathbf{Q}}(u)\ \dddot{\mathbf{Q}}(u)\right|}{\left(A_m \left\|\dot{\mathbf{Q}}(u) \times \ddot{\mathbf{Q}}(u)\right\|\right)^2}. \tag{4.31}$$

It is then easily shown that, under the hypotheses of the Theorem, the right-hand side of the second equality above, tends to the quantity:

$$c\sqrt{\frac{\Delta_{m-1}}{\Delta_{m+1}}} \frac{108\,\Gamma_m}{\Delta_{m-1}\Delta_m^2\Delta_{m+1}} \left\| \frac{18\,\mathbf{P}_m(1-t)^2}{\Delta_{m-1}\Delta_m} + c\sqrt{\frac{\Delta_{m-1}}{\Delta_{m+1}}} \frac{18\,\mathbf{P}_{m+1}t^2}{\Delta_m\Delta_{m+1}} \right\|^{-2}, \tag{4.32}$$

which proves the wanted result. □

The last theorem of the section is concerned with the asymptotic behavior of the Frénet frame of $\mathbf{Q}(u)$:

Theorem 4.13. 1. Unit-tangent vector:

$$\lim_{\nu_m,\nu_{m+1}\to\infty} \mathbf{v}_t(u) = \frac{\mathbf{L}_m}{\|\mathbf{L}_m\|},\ u\in(u_m, u_{m+1}), \tag{4.33}$$

Moreover, if $\|\mathbf{B}_m\| \neq 0$, then:

$$\lim_{\nu_{m-1},\nu_{m+1}\to\infty} \mathbf{v}_t(u_m) = \frac{\mathbf{B}_m}{\|\mathbf{B}_m\|} \tag{4.34}$$

2. Binormal:

$$\mathbf{v}_b(u) \to \frac{\Delta_{m-1}^{-3/2}\mathbf{P}_m(1-t)^2 + c\Delta_{m+1}^{-3/2}\mathbf{P}_{m+1}t^2}{\|\Delta_{m-1}^{-3/2}\mathbf{P}_m(1-t)^2 + c\Delta_{m+1}^{-3/2}\mathbf{P}_{m+1}t^2\|},\ u \in [u_m, u_{m+1}], \tag{4.35}$$

as $\nu_i \to \infty$, $i = m-1(1)m+2$, with $\nu_m/\nu_{m+1} \to c \geq 0$ and $\|\Delta_{m-1}^{-3/2}\mathbf{P}_m\ (1-t)^2 + c\Delta_{m+1}^{-3/2}\mathbf{P}_{m+1}t^2\| \neq 0$. (4.35) retains its validity for $u = u_m$ ($u = u_{m+1}$) as well, provided that $\nu_{m-1}, \nu_{m+1}\to\infty$ ($\nu_m, \nu_{m+2}\to\infty$) only.

3. Principal normal:

$$\mathbf{v}_n(u)\to \frac{(\Delta_{m-1}^{-3/2}\mathbf{P}_m(1-t)^2 + c\Delta_{m+1}^{-3/2}\mathbf{P}_{m+1}t^2) \times \mathbf{L}_m}{\|\Delta_{m-1}^{-3/2}\mathbf{P}_m(1-t)^2 + c\Delta_{m+1}^{-3/2}\mathbf{P}_{m+1}t^2\|\,\|\mathbf{L}_m\|}, u \in (u_m, u_{m+1}), \tag{4.36}$$

as $\nu_i \to \infty$, $i = m-1(1)m+2$, with $\nu_m/\nu_{m+1} \to c \geq 0$ and $\|\Delta_{m-1}^{-3/2}\mathbf{P}_m\ (1-t)^2 + c\Delta_{m+1}^{-3/2}\mathbf{P}_{m+1}t^2\| \neq 0$. Also, if $\|\mathbf{B}_m\| \neq 0$, then:

$$\lim_{\nu_{m-1},\nu_{m+1}\to\infty} \mathbf{v}_n(u_m) = \frac{\mathbf{P}_m \times \mathbf{B}_m}{\|\mathbf{P}_m\|\,\|\mathbf{B}_m\|}. \tag{4.37}$$

Proof.

1. Limiting relation (4.33) is a direct consequence of Theorem 4.7, whereas limiting relation (4.34) stems from Corollary 4.3.

2. Relation (4.35) is a result of Theorem 4.8 and limiting relation (4.30); relation (4.37) can be derived from Corollary 4.9.

3. The results for the principal normal $\mathbf{v}_n(u)$ stem readily from those of the tangent vector $\mathbf{v}_t(u)$ and the binormal $\mathbf{v}_b(u)$, since $\mathbf{v}_n(u) = \mathbf{v}_b(u) \times \mathbf{v}_t(u)$.

□

We shall conclude this subsection by describing in more qualitative terms the asymptotic results obtained above. To start with, Theorem 4.7 implies that, as ν-parameters increase, a ν-spline tends to a C^1 cubic parametrization of the polygonal line that connects the vertices of the point-set $\mathcal{D}$. From this point of view, ν-splines resemble to *tension splines,* the major difference being that the *tension parameters* in ν-splines are not associated with parametric intervals, as it is the case for the *exponential splines in tension*, but with parametric nodes. As a result, convergence to the polygonal line over the whole interval $[u_m, u_{m+1}]$ requires that both ν_m and ν_{m+1} tend to infinity. Furthermore, as one should expect, the limit parametrization of the polygonal line ceases to be regular at the nodes of $\mathcal{U}$ (correspondingly: vertices of $\mathcal{D}$), where the tangent vector vanishes; see eq. (4.20) for $t = 0, 1$.

Theorem 4.8 and Corollary 4.9 summarize the asymptotic behavior of the vector $\mathbf{w}(u) = \dot{\mathbf{Q}}(u)\times\ddot{\mathbf{Q}}(u)$ that, as already stated at the beginning of this section, is co-directional to the binormal of $\mathbf{Q}(u)$. Taking into account that the quantities A_m are positive (see eq. (3.5)), Corollary 4.9 guarantees that, for sufficiently large values of the ν-parameters ν_{m-1} and ν_{m+1}, the vector $\mathbf{w}(u)$ at the node $u = u_m$ (equivalently, at the vertex $\mathbf{I}_m$) will tend to be parallel to the so-called *discrete binormal* $\mathbf{P}_m/\|\mathbf{P}_m\|$ at $\mathbf{I}_m$. To obtain a similar result for the whole parametric interval $[u_m, u_{m+1}]$, Theorem 4.8 requires that ν_m and ν_{m+2} should increase as well and that the growth rates of ν_m and ν_{m+1} should be asymptotically equivalent $(\nu_m/\nu_{m+1} \to c \geq 0)$. Under these hypotheses, (4.22) implies that $\mathbf{w}(u)$, $u\in[u_m, u_{m+1}]$, will remain in the acute angle of the discrete binormals at $\mathbf{P}_m$ and $\mathbf{P}_{m+1}$.

Theorems 4.10 and 4.11 exhibit the asymptotic behavior of the curvature $\kappa(u)$ of a ν-spline for large values of the ν-parameters. Since $\mathbf{w}(u)$ appears in the numerator of the rational expression for $\kappa(u)$, Theorems 4.10 and 4.11 inherit the ν-parameter increase patterns adopted by Theorem 4.8 and Corollary 4.9. Thus, given that $A_n = \Theta(\nu_n)$, the limit relation (4.26) implies that $\kappa(u)$ will tend to zero within each open interval (u_m, u_{m+1}). On the contrary, the nodal curvature $\kappa(u_m)$ will tend to infinity at a rate of $\Theta(\nu_m^2)$; see eq. (4.28). Now, if $\mathcal{D}$ is two-dimensional, then one can speak of signed curvature and the limiting relations (4.26) and (4.28) degenerate to the ones below :

$$\kappa(u)A_n \to \frac{\Delta_m^2}{12\|\mathbf{L}_m\|^3}\Big(\frac{d_{nm}}{\Delta_{m-1}}\,\frac{P_m}{(1-t)t^3} + \frac{d_{n,m+1}}{\Delta_{m+1}}\,\frac{P_{m+1}}{(1-t)^3 t}\Big),\ u\in(u_m, u_{m+1}), \tag{4.38}$$

$$\lim_{\nu_{m-1},\nu_{m+1}\to\infty} \kappa(u_m)A_m^{-2} = \frac{18P_m}{\Delta_{m-1}\Delta_m\|\mathbf{B}_m\|^3}. \tag{4.39}$$

with P_m denoting the non-zero component of $\mathbf{P}_m$, that is vertical to the plane of $\mathcal{D}$. The above limiting relations display the convexity-preservation properties of ν-splines in the plane for, if the vectors $\mathbf{P}_m$ and $\mathbf{P}_{m+1}$ share the same direction, it is ensured that, for sufficiently large values of ν_i, $i = m-1(1)m-2$, the curvature will retain constant sign over $[u_m, u_{m+1}]$.

One may legitimately assert that the contents of Theorems 4.10 and 4.11 can be, more or less, anticipated on the basis of the remark that, as the ν-parameters increase, a ν-interpolant tends to the polygonal line with vertices at $\mathcal{D}$. This remark, however, is useless for anticipating the asymptotic behavior of the torsion of a ν-spline, for the torsion of a straight line is not well defined. The sought for behavior is derived in Theorem 4.12, stating that, if ν's increase as in Theorems 4.8 and 4.10, then the restriction of torsion $\tau(u)$ on $[u_m^+, u_{m+1}^-]$ will tend to a well-defined limit, that shares the same sign with the so-called *discrete torsion* Γ_m of the polygonal line that connects the vertices $\mathbf{I}_n$, $n = m-1(1)m+2$.

Finally, we are coming to Theorem 4.13, which collects the asymptotic behavior of the Frénet frame of a ν-spline for large values of the ν-parameters. Thus, if ν_m and ν_{m+1} tend to infinity, then the unit-tangent vector $\mathbf{v}_t(u)$ in the open interval $(u_m, u_{m+1})\ni u$ will tend to the unit-tangent vector of the linear segment, that connects the interpolation points $\mathbf{I}_m$ and $\mathbf{I}_{m+1}$. At the nodes, $u = u_m$, $\mathbf{v}_t(u_m)$ will tend to a positive combination of the unit-tangent vectors of the two neighboring linear segments. As for the binormal $\mathbf{v}_b(u)$ in $[u_m, u_{m+1}]$, if ν's increase as in Theorem 4.8, it will tend to a positive combination of the neighboring discrete binormals, namely $\mathbf{P}_n/\|\mathbf{P}_n\|$, $n = m, m+1$. The limiting relations for the principal normal are a direct consequence of the relations between the vectors of the Frénet frame.

5. A Discrete Algorithm for Constructing Shape-Preserving ν-splines

In this section we shall provide sufficient and possibly necessary conditions for ensuring that a G^2-continuous ν-spline will be shape-preserving in the sense specified in Problem $(\mathcal{P})$ of §2. In addition, we shall present patterns for increasing the ν-parameters so that the corresponding sufficient conditions will eventually be satisfied. This material is collected within the first subsection; the second subsection is devoted to presenting a discrete and iterative algorithm, that is able to construct interpolatory shape-preserving ν-splines after a finite number of iterations.

5.1. Sufficient Conditions and Increase Patterns

We start by stating two lemmata, which will be used in the sequel. We only prove the second lemma; the first one is concerned with the well known necessary and sufficient conditions for a quadratic Bernstein polynomial to be positive in $[0,1]$.

Lemma 5.1. Let $f(t) = \alpha\,(1-t)^2 + \beta\,t^2 + \gamma\,t(1-t)$. Then $f(t) > 0$ for $t \in [0,1]$, if and only if $\alpha, \beta > 0$ and $2\sqrt{\alpha\beta} + \gamma > 0$.

Lemma 5.2. Let $f(t;k) = \alpha\,(1-t)^k + \beta\,t^k$, $t \in [0,1]$, $k \geq 2$, $\alpha, \beta \geq 0$, $|\alpha| + |\beta| \neq 0$. Then:

$$\min_{t\in[0,1]} f(t;k) = \frac{\alpha\beta}{(\alpha^{\frac{1}{k-1}} + \beta^{\frac{1}{k-1}})^{k-1}}. \tag{5.1}$$

Proof. The Lemma can be readily obtained by noting that $f'(t;k) = 0$ admits of a unique solution at:

$$t_0 = \frac{\alpha^{\frac{1}{k-1}}}{\alpha^{\frac{1}{k-1}} + \beta^{\frac{1}{k-1}}}, \tag{5.2}$$

and that $f''(t;k) \geq 0$. □

Convexity Criterion (see item 1 in Problem $(\mathcal{P})$ of §2)

Expanding the defining expression for $\mathbf{w}(u) = \dot{\mathbf{Q}}(u) \times \ddot{\mathbf{Q}}(u)$ we arrive at:

$$\mathbf{w}(u) = \mathbf{w}_m(1-t)^2 + \mathbf{w}_{m+1}t^2 + \mathbf{g}_m t(1-t), \tag{5.3}$$

where

$$\mathbf{w}_m = 6\Delta_m^{-2}(\mathbf{q}_m \times \mathbf{L}_m) - 2\Delta_m^{-1}(\mathbf{q}_m \times \mathbf{q}_{m+1}), \tag{5.4a}$$

$$\mathbf{w}_{m+1} = 6\Delta_m^{-2}(\mathbf{L}_m \times \mathbf{q}_{m+1}) - 2\Delta_m^{-1}(\mathbf{q}_m \times \mathbf{q}_{m+1}), \tag{5.4b}$$

$$\mathbf{g}_m = 2\Delta_m^{-1}(\mathbf{q}_m \times \mathbf{q}_{m+1}). \tag{5.4c}$$

Using now Lemma 5.1 with $\alpha = \mathbf{w}_m \cdot \mathbf{P}_n$, $\beta = \mathbf{w}_{m+1} \cdot \mathbf{P}_n$ and $\gamma = \mathbf{g}_m \cdot \mathbf{P}_n$, expansion (5.3) readily leads to the following necessary and sufficient conditions for the convexity part of the shape-preservation criterion:

Theorem 5.3. If $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, then the convexity criterion is satisfied in $[u_m, u_{m+1}]$ if and only if the ensuing conditions (1) and (2) hold true.

1. $\mathbf{w}_s \cdot \mathbf{P}_n > 0, \quad s = m, m+1, \quad n = m, m+1,$
2. $2\sqrt{(\mathbf{w}_m \cdot \mathbf{P}_n)\,(\mathbf{w}_{m+1} \cdot \mathbf{P}_n)} + \mathbf{g}_m \cdot \mathbf{P}_n > 0, \quad n = m, m+1.$

The ensuing result provides an increase pattern of ν's that permits the fulfillment of the conditions in Theorem 5.3.

Lemma 5.4. (i) Conditions (1) in Theorem 5.3 will hold for sufficiently large values of ν_k, where $k = m-1, m+1$ for $s = m$ and $k = m, m+2$ for $s = m+1$.

(ii) Conditions (2) in Theorem 5.3 will hold for sufficiently large values of ν_k, $k = m-1, m, m+1, m+2$.

Proof.

(i) It is a direct consequence of Corollary 4.9, the positivity of A_s and the fact that, under the hypotheses of Theorem 5.3, $\mathbf{P}_s \cdot \mathbf{P}_n > 0$.

(ii) Using again Corollary 4.9 we have that, as $\nu_k \to \infty$, $k = m-1, m, m+1, m+2$,

$$\sqrt{A_m A_{m+1}}\sqrt{(\mathbf{w}_m \cdot \mathbf{P}_n)(\mathbf{w}_{m+1} \cdot \mathbf{P}_n)} \to \frac{18\sqrt{(\mathbf{P}_m \cdot \mathbf{P}_n)(\mathbf{P}_{m+1} \cdot \mathbf{P}_n)}}{\Delta_m \sqrt{\Delta_{m-1}\,\Delta_{m+1}}}. \quad (5.5)$$

The right-hand side in the limiting relation (5.5) is strictly positive for $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$. On the other hand, we can write:

$$\sqrt{A_m A_{m+1}}(\mathbf{g}_m \cdot \mathbf{P}_n) = \frac{A_m A_{m+1}(\mathbf{g}_m \cdot \mathbf{P}_n)}{\sqrt{A_m A_{m+1}}}, \quad (5.6)$$

which, in view of the estimate (4.14) and the fact that $A_n = \Theta(\nu_n)$, $n = m, m+1$, implies that:

$$\sqrt{A_m A_{m+1}}(\mathbf{g}_m \cdot \mathbf{P}_n) \to 0, \quad (5.7)$$

as $\nu_k \to \infty$, $k = m, m+1$. Combining (5.5) with (5.7), we conclude that the left-hand side quantity in Condition (2) of Theorem 5.3, multiplied by $\sqrt{A_m A_{m+1}}$, tends to a strictly positive quantity, which yields the sought-for result for $\sqrt{A_m A_{m+1}}$ is positive.

□

Torsion Criterion (see item 2 in Problem ($\mathcal{P}$) of §2)

Theorem 5.5. The torsion criterion is satisfied in $[u_m^+, u_{m+1}^-]$, if and only if:

$$E_m \Gamma_m > 0, \quad (5.8)$$

where $E_m = |\mathbf{q}_m\ \mathbf{L}_m\ \mathbf{q}_{m+1}|$.

Proof. In view of (4.31), the sign of the torsion in $[u_m^+, u_{m+1}^-]$ is determined by the sign of the quantity $|\dot{\mathbf{Q}}(u)\ \ddot{\mathbf{Q}}(u)\ \dddot{\mathbf{Q}}(u)|$, which, in view of (4.16), coincides with that of E_m. □

The corresponding pattern for increasing the ν-parameters is suggested by Lemma 4.6, yielding the following

Lemma 5.6. Condition (5.8) will be satisfied for sufficiently large values of the parameters ν_k, $k = m-1, m, m+1, m+2$.

Coplanarity Criterion (see items 3a and 3b in Problem ($\mathcal{P}$) of §2)

The various user-defined quantities, encountered in this criterion, are chosen as follows:

- If $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, then $\omega_m = [u_m, u_{m+1}]$

- If $\mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0$, then the separating point u_m^* is set equal to the unique root in $[u_m, u_{m+1}]$ of the numerator (and also denominator) of the limit of $\mathbf{v}_b(u)$ as $\nu_i \Rightarrow \infty$, $i = m-1(1)m+2$, with $\nu_m = \nu_{m+1}$ (see Th. 4.13, rel. (4.35)) :

$$u_m^* = u_m + \Delta_m \frac{\Delta_{m+1}^{3/4}\sqrt{\|\mathbf{P}_m\|}}{\Delta_{m+1}^{3/4}\sqrt{\|\mathbf{P}_m\|} + \Delta_{m-1}^{3/4}\sqrt{\|\mathbf{P}_{m+1}\|}}, \tag{5.9}$$

while $\vartheta_m = [u_m, u_m^\ell]$ and $\varphi_m = [u_m^r, u_{m+1}]$, with u_m^ℓ and u_m^r being any parameter value in (u_m, u_m^*) and (u_m^*, u_{m+1}), respectively.

The following lemma ensures that part (a) of the coplanarity criterion, dealing with the case when the planar 4-gon $\mathbf{I}_{m-1}, \mathbf{I}_m, \mathbf{I}_{m+1}, \mathbf{I}_{m+2}$ is convex, will be fulfilled for appropriately large values of the ν-parameters.

Lemma 5.7. Let

$$\psi(u) = \|\mathbf{w}(u)\|^2 \|\mathbf{P}_m\|^2 \varepsilon_1^2 - \|\mathbf{w}(u) \times \mathbf{P}_m\|^2 \tag{5.10}$$

and $\psi_n = \psi(u_n)$, $n = m, m+1$. If $\Gamma_m = 0$ and $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$, then:

(i) $\psi_n > 0$ for large enough values of the parameters $\nu_{n-1}, \nu_{n+1}, n = m, m+1$,

(ii) $\psi(u) > 0$, $u \in [u_m, u_{m+1}]$ for large values of the parameters ν_{m-1}, ν_m, ν_{m+1}, ν_{m+2}.

Proof. Part (i) of the Lemma stems directly from Corollary 4.9, the assumption $\mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0$ and the positivity of A_m and A_{m+1}.

Let us now concentrate on proving Part (ii) of the Lemma. Using (5.3) we easily arive at the formula:

$$\begin{aligned} \|\mathbf{w}(u)\|^2 = {} & \|\mathbf{w}_m\|^2(1-t)^4 + \|\mathbf{w}_{m+1}\|^2 t^4 + \|\mathbf{g}_m\|^2 t^2 (1-t)^2 \\ & + 2(\mathbf{w}_m \cdot \mathbf{w}_{m+1}) t^2 (1-t)^2 + 2(\mathbf{w}_m \cdot \mathbf{g}_m)(1-t)^3 t \\ & + 2(\mathbf{w}_{m+1} \cdot \mathbf{g}_m)(1-t) t^3. \end{aligned} \tag{5.11}$$

Moreover, we have :

$$\begin{aligned} \|\mathbf{w}(u) \times \mathbf{P}_m\|^2 \le {} & 2\,\|\mathbf{w}_m \times \mathbf{P}_m\|^2 (1-t)^4 + 2\,\|\mathbf{w}_{m+1} \times \mathbf{P}_m\|^2 t^4 \\ & + 2\,\|\mathbf{g}_m \times \mathbf{P}_m\|^2 t^2 (1-t)^2. \end{aligned} \tag{5.12}$$

Combining (5.10) with (5.11) and (5.12), we deduce the following lower bound for $\psi(u)$:

$$\begin{aligned}\psi(u) \geq\ & [\|\mathbf{w}_m\|^2\|\mathbf{P}_m\|^2\varepsilon_1^2 - 2\,\|\mathbf{w}_m \times \mathbf{P}_m\|^2]\,(1-t)^4 \\ & +[\|\mathbf{w}_{m+1}\|^2\|\mathbf{P}_m\|^2\varepsilon_1^2 - 2\,\|\mathbf{w}_{m+1} \times \mathbf{P}_m\|^2]\,t^4 \\ & +[\|\mathbf{g}_m\|^2\|\mathbf{P}_m\|^2\varepsilon_1^2 - 2\,\|\mathbf{g}_m \times \mathbf{P}_m\|^2]\,t^2(1-t)^2 \\ & +\{2\,(\mathbf{w}_m\cdot\mathbf{w}_{m+1})\,t^2(1-t)^2 + 2\,(\mathbf{w}_m\cdot\mathbf{g}_m)\,(1-t)^3 t \\ & +2\,(\mathbf{w}_{m+1}\cdot\mathbf{g}_m)\,(1-t)t^3\}\,\varepsilon_1^2\,\|\mathbf{P}_m\|^2.\end{aligned} \tag{5.13}$$

Appealing, once again, to Corollary 4.9 and the positiveness of A_m and A_{m+1}, and using the fact that, under the assumptions of this lemma, we have $\mathbf{P}_m = c\mathbf{P}_{m+1}$, where $c = \|\mathbf{P}_m\|/\|\mathbf{P}_{m+1}\| > 0$, we can infer that there exists a positive constant N_1 such that, for every $\nu_n > N_1$, $n = m-1(1)m+2$, the following inequalities hold true:

$$\|\mathbf{w}_n\|^2\|\mathbf{P}_m\|^2\varepsilon_1^2 - 2\,\|\mathbf{w}_n \times \mathbf{P}_m\|^2 > 0, \quad n = m, m+1, \tag{5.14}$$

$$\mathbf{w}_m\cdot\mathbf{w}_{m+1} > 0. \tag{5.15}$$

Next, let t_1 be any number in $(0,1)$. Then for every $t \in [0, t_1]$ and for every $\nu_n > N_1$, $n = m-1(1)m+2$, we can strengthen (5.13) as below:

$$\begin{aligned}\psi(u) \geq\ & \frac{1}{2}\,[\|\mathbf{w}_m\|^2\|\mathbf{P}_m\|^2\varepsilon_1^2 - 2\,\|\mathbf{w}_m \times \mathbf{P}_m\|^2]\,(1-t_1)^4 \\ & +[\|\mathbf{w}_{m+1}\|^2\|\mathbf{P}_m\|^2\varepsilon_1^2 - 2\,\|\mathbf{w}_{m+1} \times \mathbf{P}_m\|^2]\,t^4 \\ & +(1-t)^2\left\{\frac{1}{2}\,[\|\mathbf{w}_m\|^2\|\mathbf{P}_m\|^2\varepsilon_1^2 - 2\,\|\mathbf{w}_m \times \mathbf{P}_m\|^2]\,(1-t_1)^2\right. \\ & \qquad -\big|\|\mathbf{g}_m\|^2\|\mathbf{P}_m\|^2\varepsilon_1^2 - 2\,\|\mathbf{g}_m \times \mathbf{P}_m\|^2\big|\,\varepsilon_1^2\,\|\mathbf{P}_m\|^2 \\ & \qquad \left. -2\,\|\mathbf{w}_m\|\,\|\mathbf{g}_m\|\,\varepsilon_1^2\,\|\mathbf{P}_m\|^2\right\} \\ & +2\,\varepsilon_1^2\,\|\mathbf{P}_m\|^2 t^2(1-t)\,\left[(\mathbf{w}_m\cdot\mathbf{w}_{m+1})\,(1-t_1) - \|\mathbf{w}_{m+1}\|\,\|\mathbf{g}_m\|\right].\end{aligned} \tag{5.16}$$

By virtue of (5.14) we can say that the first term in the right-hand side of (5.16) is positive, whereas the second is non-negative for $\nu_n > N_1$, $n = m-1(1)m+2$. Next, recalling the defining relation (5.4*c*) of the quantity $\mathbf{g}_m$, Corollary 4.5 and, once again, 4.9 and the positivity of A_m, we conclude that there exists a positive number N_2 such that, for every $\nu_n > N_2$, $n = m-1(1)m+2$, the multiplier in the curly brackets of the third term in the right-hand side of (5.16) will be positive. Employing the same argumentation, (5.15) implies the existence of a positive number N_3 such that, for every $\nu_n > N_3$, $n = m-1(1)m+2$, the multiplier in the square brackets of the fourth term in right-hand side of (5.16) will be positive too. Collecting the above results, we can say that the function $\psi(u)$ will be positive on $[0, t_1]$ for $\nu_n > N_\ell = \max\{N_1, N_2, N_3\}$, $n = m-1(1)m+2$. In a completely analogous manner we can prove that there exists a positive number N_r such that if $\nu_n > N_r$, $n = m-1(1)m+2$, we have $\psi(u) > 0$ for every $t \in [t_1, 1]$. Let $N_{max} = \max\{N_\ell, N_r\}$. Then for $\nu_n > N_{max}$, $n = m-1(1)m+2$ the positivity of $\psi(u)$ is ensured in the entire interval $[u_m, u_{m+1}]$. □

The case when the 4-gon $\mathbf{I}_{m-1}, \mathbf{I}_m, \mathbf{I}_{m+1}, \mathbf{I}_{m+2}$ is planar but non-convex is handled by the ensuing lemma, establishing that part (b) of the coplanarity criterion will be fulfilled for appropriately large ν's:

Lemma 5.8. If $\Gamma_m = 0$ and $\mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0$, then:

(i) $\psi_n > 0$, for large enough values of the parameters ν_{n-1}, ν_{n+1}, $n = m, m+1$,

(ii) $\psi(u) > 0$, $u \in [u_m, u_m^\ell] \cup [u_m^r, u_{m+1}]$, for large enough values of the parameters $\nu_{m-1}, \nu_m, \nu_{m+1}, \nu_{m+2}$, with $\nu_m = \nu_{m+1}$.

Proof. The proof of Part (i) of the Lemma coincides with that of Part 1 of Lemma 5.7. Using the expansion of $\mathbf{w}(u)$ in (5.11), as well as the fact that $t \in [0,1]$, we easily get the following lower bound:

$$\begin{aligned}\|\mathbf{w}(u)\|^2 \geq\ & \left[\|\mathbf{w}_m\|\,(1-t)^2 - \|\mathbf{w}_{m+1}\|\,t^2\right]^2 - \|\mathbf{g}_m\|^2 \\ & -2\,\|\mathbf{w}_m\|\,\|\mathbf{g}_m\| - 2\,\|\mathbf{w}_{m+1}\|\,\|\mathbf{g}_m\|.\end{aligned} \tag{5.17}$$

Now, let

$$\Lambda_m = \frac{\sqrt{\|\mathbf{w}_m\|}}{\sqrt{\|\mathbf{w}_m\|} + \sqrt{\|\mathbf{w}_{m+1}\|}}, \tag{5.18}$$

which, if ν_m and ν_{m+1} increase under the constraint: $\nu_m = \nu_{m+1}$, will tend to the quantity $(u_m^* - u_m)/\Delta_m$, as it is readily implied from Corollary 4.9 and the defining formula (5.9) of u_m^*. It is then legal to say that there exists a positive number N_1 such that, for every $\nu_n > N_1$, $n = m-1(1)m+2$, with $\nu_m = \nu_{m+1}$, the quantity Λ_m will lie strictly in between t_m^ℓ and t_m^r, i.e.,

$$t_m^\ell < \Lambda_m < t_m^r. \tag{5.19}$$

Furthermore, one can easily verify that $\|\mathbf{w}_m\|\,(1-t)^2 - \|\mathbf{w}_{m+1}\|\,t^2$ is a non-negative decreasing function of t in $[0, \Lambda_m]$. Combining this with (5.19), we conclude that under the same assumptions for which (5.19) holds, $\|\mathbf{w}(u)\|$ can be bounded from below in $[0, t_m^\ell]$ as follows :

$$\begin{aligned}\|\mathbf{w}(u)\|^2 \geq\ & \left[\|\mathbf{w}_m\|\,(1-t_m^\ell)^2 - \|\mathbf{w}_{m+1}\|\,(t_m^\ell)^2\right]^2 - \|\mathbf{g}_m\|^2 \\ & -2\,\|\mathbf{w}_m\|\,\|\mathbf{g}_m\| - 2\,\|\mathbf{w}_{m+1}\|\,\|\mathbf{g}_m\|.\end{aligned} \tag{5.20}$$

The above inequality, in conjunction with Corollaries 4.5 and 4.9 as well as the assumption: $u_m^\ell < u_m^*$, implies the existence of a positive number $N_2 > N_1$ such that, for every $\nu_n > N_2$, $n = m-1(1)m+2$ and $\nu_m = \nu_{m+1}$, the quantity $A_m^2\,\|\mathbf{w}(u)\|^2$, is uniformly bounded from below in $u \in [u_m, u_m^\ell]$ by a ν-independent strictly positive number. On the other hand, inequality (5.12) along with the fact that $t \in [0,1]$, gives :

$$\|\mathbf{w}(u) \times \mathbf{P}_m\|^2 \leq 2 \left(\|\mathbf{w}_m \times \mathbf{P}_m\|^2 + \|\mathbf{w}_{m+1} \times \mathbf{P}_m\|^2 + \|\mathbf{g}_m \times \mathbf{P}_m\|^2\right), \quad (5.21)$$

which, in view of the afore-mentioned corollaries and assumptions, suggests that $A_m^2\|\mathbf{w}(u) \times \mathbf{P}_m\|^2$ can become arbitrarily small if ν_n, $n = m-1(1)m+2$ are adequately large. Hence, there exists a positive number $N_\ell > N_2$ such that, for every $\nu_n > N_\ell$, $n = m-1(1)m+2$ and $\nu_m = \nu_{m+1}$, we have $\psi(u) > 0$, $u \in [u_m, u_m^\ell]$. In a directly analogous manner, one can secure the positivity of $\psi(u)$ on $u \in [u_m^r, u_{m+1}]$ for $\nu_n > N_r$, $n = m-1(1)m+2$, with N_r being a sufficiently large positive number and $\nu_m = \nu_{m+1}$. Then, the validity of Part (ii) of the Lemma follows for $\nu_n > \max\{N_\ell, N_r\}$, $n = m-1(1)m+2$ and $\nu_m = \nu_{m+1}$. □

Collinearity Criterion (see item 4 in Problem ($\mathcal{P}$) of §2)

The user-specified interval η_m, encountered in this criterion, is chosen as follows: $\eta_m = [u_m^\ell, u_m^r]$, with u_m^ℓ and u_m^r being any parameter value in (u_{m-1}, u_m) and (u_m, u_{m+1}), respectively. Then, the following lemma ensures that the collinearity criterion of Problem ($\mathcal{P}$) will hold in $[u_m^\ell, u_m^r]$ for appropriately large ν's.

Lemma 5.9. Let

$$\phi(u) = \|\dot{\mathbf{Q}}(u)\|^2 \, \|\mathbf{L}_m\|^2 \, \varepsilon_0^2 - \|\dot{\mathbf{Q}}(u) \times \mathbf{L}_m\|^2 \quad (5.22)$$

and $\phi_m = \phi(u_m)$. If $\mathbf{P}_m = 0$ and $\mathbf{L}_{m-1} \cdot \mathbf{L}_m > 0$, then

(i) $\phi_m > 0$ for large enough values of the parameters ν_{m-1}, ν_{m+1},

(ii) $\phi(u_m^\ell) > 0$ for large enough values of the parameters ν_{m-1}, ν_m,

(iii) $\phi(u_m^r) > 0$ for large enough values of the parameters ν_m, ν_{m+1},

(iv) $\phi(u) > 0$, $u \in [u_m^\ell, u_m^r]$, for large enough values of the parameters ν_{m-1}, ν_m, ν_{m+1}.

Proof. Part (i) of the Lemma is obtained by applying Corollary 4.3 and noticing that A_m is positive as well as, under the given assumptions, $\mathbf{B}_m \neq 0$ and $\mathbf{B}_m \times \mathbf{P}_m = 0$. Parts (ii) and (iii) of the Lemma are direct consequences of limiting relation (4.20) of Theorem 4.7 and the fact that, if $\mathbf{P}_m = 0$, then $\mathbf{L}_{m-1} \times \mathbf{L}_m = 0$.

We shall now concentrate on Part (iv) of the lemma. Let:

$$\Phi_1(\mathbf{x}, \mathbf{y}) = \varepsilon_0^2 \, \|\mathbf{x}\|^2 \, \|\mathbf{y}\|^2 - \|\mathbf{x} \times \mathbf{y}\|^2 \quad (5.23)$$

and

$$\Phi_2(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \varepsilon_0^2 \, (\mathbf{x} \cdot \mathbf{y}) \, \|\mathbf{z}\|^2 - (\mathbf{x} \times \mathbf{z}) \cdot (\mathbf{y} \times \mathbf{z}). \quad (5.24)$$

Using the defining expression (4.21) of $\dot{\mathbf{Q}}(u)$, $\phi(u)$ can be written as follows :

$$\begin{aligned}\phi(u) = \ & \phi_m\,(1-t)^2(1-3t)^2 \\ & +t^2\Big\{\,36\,\varepsilon_0^2\,\|\mathbf{L}_m\|^2\,\|\mathbf{s}_m\|^2\,(1-t)^2 + \Phi_1(\mathbf{q}_{m+1},\mathbf{L}_m)\,(3t-2)^2 \\ & \qquad +12\,\varepsilon_0^2\,\|\mathbf{L}_m\|^2\,(\mathbf{s}_m\cdot\mathbf{q}_{m+1})\,(3t-2)(1-t)\Big\} \\ & +t(1-t)(1-3t)\,\Big\{\,12\,\varepsilon_0^2\,\|\mathbf{L}_m\|^2\,(\mathbf{s}_m\cdot\mathbf{q}_m)\,(1-t) \\ & \qquad\qquad +2\,\Phi_2(\mathbf{q}_m,\mathbf{q}_{m+1},\mathbf{L}_m)\,(3t-2)\Big\},\end{aligned} \tag{5.25}$$

where $\mathbf{s}_m = \mathbf{L}_m\Delta_m^{-1}$. Let $t_1 \in (0, \min\{\frac{1}{3}, t_m^r\})$. Then, for $u \in [u_m, u_m + t_1\Delta_m]$, the above expression for $\phi(u)$ can be bounded from below as follows :

$$\begin{aligned}\phi(u) \geq \ & \phi_m\,(1-t)^2(1-3t)^2 \\ & +t^2\Big\{\,36\,\varepsilon_0^2\,\|\mathbf{L}_m\|^2\,\|\mathbf{s}_m\|^2\,(1-t_1)^2 - 4\,|\Phi_1(\mathbf{q}_{m+1},\mathbf{L}_m)| \\ & \qquad -24\,\varepsilon_0^2\,\|\mathbf{L}_m\|^2\,\|\mathbf{s}_m\|\,\|\mathbf{q}_{m+1}\|\Big\} \\ & +4\,t(1-t)(1-3t)\,\Big\{\,3\,\varepsilon_0^2\,\|\mathbf{L}_m\|^2\,(\mathbf{s}_m\cdot\mathbf{q}_m)\,(1-t) \\ & \qquad\qquad -|\Phi_2(\mathbf{q}_m,\mathbf{q}_{m+1},\mathbf{L}_m)|\Big\}\end{aligned} \tag{5.26}$$

By virtue of Part (i) of the Lemma, there exists a positive number N_1 such that, for every $\nu_{m-1}, \nu_{m+1} > N_1$, we have $\phi_m > 0$. Next, Lemma 4.2 implies that there exists a positive number N_2 such that, for every $\nu_{m+1} > N_2$, the multiplier of t^2 in the right-hand side of (5.26) is positive. Finally, Corollary 4.3 implies the existence of a positive number N_3 such that, for every $\nu_{m-1}, \nu_{m+1} > N_3$, we have $\mathbf{s}_m\cdot\mathbf{q}_m > 0$. Hence, for $\nu_{m-1}, \nu_{m+1} > N_3$, the multiplier of $t(1-t)(1-3t)$ in (5.26) is bounded from below by the quantity:

$$3\,\varepsilon_0^2\,\|\mathbf{L}_m\|^2\,(\mathbf{s}_m\cdot\mathbf{q}_m)\,(1-t_1) - \,|\Phi_2(\mathbf{q}_m,\mathbf{q}_{m+1},\mathbf{L}_m)|. \tag{5.27}$$

Using once again Lemma 4.2 and Corollary 4.3, the above quantity becomes positive for ν_{m-1}, ν_{m+1} greater than some positive number N_4. The positivity of $\phi(u)$ in $[u_m, u_m + t_1\Delta_m]$ is then guaranteed by choosing $\nu_{m-1}, \nu_{m+1} > N_\ell$, where $N_\ell = \max_{1\leq i\leq 4} N_i$.

For $u \in [u_m + t_1\Delta_m, u_m^r]$, $\phi(u)$ can be bounded from below as:

$$\begin{aligned}\phi(u) \geq \ & 36\,\varepsilon_0^2\,\|\mathbf{L}_m\|^2\,\|\mathbf{s}_m\|^2\,t_1^2(1-t_m^r)^2 \\ & -24\,\varepsilon_0^2\,\|\mathbf{L}_m\|^2\,\|\mathbf{s}_m\|\,(\|\mathbf{q}_m\| + \|\mathbf{q}_{m+1}\|) - 4\,|\Phi_1(\mathbf{q}_m,\mathbf{L}_m)| \\ & -4\,|\Phi_1(\mathbf{q}_{m+1},\mathbf{L}_m)| - 8\,|\Phi_2(\mathbf{q}_m,\mathbf{q}_{m+1},\mathbf{L}_m)|.\end{aligned} \tag{5.28}$$

Grounded once more on Lemma 4.2, we conclude that all but the first of the terms in the right-hand side of the above inequality tend to zero as ν_m, ν_{m+1} tend to infinity. Thus, there exists a positive number N_r such that, for every $\nu_m, \nu_{m+1} > N_r$, we have $\phi(u) > 0$ for $u \in [u_m + t_1\Delta_m, u_m^r]$. This result can be deduced for the entire interval $[u_m, u_m^r]$ by simply choosing $\nu_{m-1}, \nu_m, \nu_{m+1} > \max\{N_\ell, N_r\}$.

The proof that $\phi(u) > 0$ for $u \in [u_m^\ell, u_m]$ is directly analogous to the one presented above for $u \in [u_m, u_m^r]$. □

5.2. The algorithm NUSSP

Based on the theorems and lemmata, derived in the previous subsection, we propose the ensuing algorithm NUSSP for constructing interpolatory G^2-continuous shape-preserving ν-splines. This algorithm is global in the sense that, within each iteration, the ν-spline interpolation problem has to be solved for a different set of ν-parameters. Furthermore, NUSSP makes extensive use of a predicate, called $R(f, [a, b])$, that returns the number of real roots of a polynomial $f(t)$ in $[a, b]$, provided that $f(a)f(b)\neq 0$. This is done with the aid of the following theorem, that exploits the properties of the so-called *Sturm sequence* of a polynomial; see, e.g., [7, §5.2].

Theorem 5.10. Let $f(x)$ be a polynomial with real coefficients and $\{f_0(x), f_1(x), \ldots, f_s(x)\}$ be the standard sequence for $f(x)$, defined as:

$$f_0(x) = f(x), \tag{5.29}$$
$$f_1(x) = f'(x), \tag{5.30}$$
$$f_0(x) = q_1(x)f_1(x) - f_2(x), \tag{5.31}$$
$$\vdots$$
$$f_{i-1}(x) = q_i(x)f_i(x) - f_{i+1}(x), \tag{5.32}$$
$$\vdots$$
$$f_{s-1}(x) = q_s(x)f_s(x). \tag{5.33}$$

Then the number of distint real roots of $f(x)$ in (a, b) is $V_a - V_b$, where V_c denotes the number of variations of the subsequence of $\{f_0(c), f_1(c), \ldots, f_s(c)\}$ obtained after dropping its zeros.

The afore-mentioned predicate is used for calculating the number of roots, over specific parametric intervals, of the quartic polynomials $\psi(u)$ and $\phi(u)$, involved in Lemmata 5.7, 5.8 and 5.9, respectively.

NUSSP always converges after a finite number of iterations. The convergence issue, along with the questions regarding the running time and the storage requirements of NUSSP, are discussed at the end of this subsection.

STEP 0.0.

(i) Evaluate the scalars $\Delta_m = u_{m+1} - u_m$, $m = 1(1)N-1$, and set:
- $\Delta_0 = \Delta_1$ and $\Delta_N = \Delta_{N-1}$ for unit-tangent vector and zero-curvature boundary conditions,
- $\Delta_0 = \Delta_{N-1}$ and $\Delta_N = \Delta_1$ for periodic boundary conditions.
Evaluate the vectors $\mathbf{L}_m$, $m = 1(1)N-1$, and set:

- $\mathbf{L}_0 = \mathbf{s}_0$ and $\mathbf{L}_N = \mathbf{s}_N$ for unit-tangent-vector boundary conditions,
- $\mathbf{L}_0 = \mathbf{L}_1$ and $\mathbf{L}_N = \mathbf{L}_{N-1}$ for zero-curvature boundary conditions, and
- $\mathbf{L}_0 = \mathbf{L}_{N-1}$, $\mathbf{L}_N = \mathbf{L}_1$ for periodic boundary conditions.

Evaluate the vectors $\mathbf{B}_m$, using relations (3.7), (3.8), (3.10) and (3.11).
Evaluate the scalars $\Gamma_m = |\mathbf{L}_{m-1}\ \mathbf{L}_m\ \mathbf{L}_{m+1}|$, $m = 1(1)N-1$.

(ii) Set $\mathcal{T}_N = \{1, 2, \dots, N\}$ and determine the following index sets:
- The *three-dimensionality* index set:

$$\mathcal{I}_1 = \{m \in \mathcal{T}_{N-1} : \Gamma_m \neq 0\}, \tag{5.34}$$

- The *convexity* index set:

$$\mathcal{K}_0 = \{m \in \mathcal{T}_{N-1} : \mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0\}, \tag{5.35}$$

- The *coplanarity* index sets:

If $\mathcal{I}_1 = \emptyset$, set $\mathcal{M}_0 = \mathcal{M}_1 = \emptyset$ and **GOTO STEP 0.2**. Otherwise :

$$\mathcal{M}_0 = \{m \in \mathcal{T}_{N-1} : \Gamma_m = 0 \ \ \wedge \ \ \mathbf{P}_m \cdot \mathbf{P}_{m+1} > 0\}, \tag{5.36}$$

$$\mathcal{M}_1 = \{m \in \mathcal{T}_{N-1} : \Gamma_m = 0 \ \ \wedge \ \ \mathbf{P}_m \cdot \mathbf{P}_{m+1} < 0\}. \tag{5.37}$$

- The *collinearity* index set:

$$\mathcal{J}_0 = \{m \in \mathcal{T}_N : \|\mathbf{P}_m\| = 0 \ \ \wedge \ \ \mathbf{L}_{m-1} \cdot \mathbf{L}_m > 0\} \tag{5.38}$$

for unit-tangent vector and periodic boundary conditions, while

$$\mathcal{J}_0 = \{m \in \mathcal{T}_{N-1} \setminus \{1\} : \|\mathbf{P}_m\| = 0 \ \ \wedge \ \ \mathbf{L}_{m-1} \cdot \mathbf{L}_m > 0\} \tag{5.39}$$

for zero-curvature boundary conditions.

STEP 0.1. If $\mathcal{M}_0 \cup \mathcal{M}_1 = \emptyset$ **GOTO STEP 0.2**. Otherwise :

(i) Specify the *coplanarity tolerance* $\varepsilon_1 \in (0, 1]$.

(ii) For $m \in \mathcal{M}_1$ specify the parameter values $u_m^\ell \in (u_m, u_m^*)$ and $u_m^r \in (u_m^*, u_{m+1})$, where u_m^* is given by equ. (5.9).

STEP 0.2. If $\mathcal{J}_0 = \emptyset$ **GOTO STEP 0.3**. Otherwise :

(i) Specify the *collinearity tolerance* $\varepsilon_0 \in (0, 1]$.

(ii) For $m \in \mathcal{J}_0 \setminus \{1\}$ specify the parameter values $u_m^\ell \in (u_{m-1}, u_m)$. For $m \in \mathcal{J}_0 \setminus \{N\}$ specify the parameter values $u_m^r \in (u_m, u_{m+1})$.

STEP 0.3. Set $j = 0$, initialize the parameters $\nu_m^{(0)} = 0$, $m = 1(1)N$, and specify the constant $\Delta\nu \in (0, \infty)$.

STEP 1. Evaluate $\mathbf{q}_m$, $m = 1(1)N$, by solving the linear systems (3.3).

STEP 2. Determine the “failure” sets :

$$\mathcal{K}_{01} = \{m \in \mathcal{K}_0 : \mathbf{w}_m \cdot \mathbf{P}_m \leq 0 \quad \vee \quad \mathbf{w}_m \cdot \mathbf{P}_{m+1} \leq 0\}, \tag{5.40}$$

$$\mathcal{K}_{02} = \{m \in \mathcal{K}_0 : \mathbf{w}_{m+1} \cdot \mathbf{P}_m \leq 0 \quad \vee \quad \mathbf{w}_{m+1} \cdot \mathbf{P}_{m+1} \leq 0\}, \tag{5.41}$$

$$\begin{aligned}\mathcal{K}_{03} = \{m \in \mathcal{K}_0 \setminus (\mathcal{K}_{01} \cup \mathcal{K}_{02}) :\\ 2\sqrt{(\mathbf{w}_m \cdot \mathbf{P}_n)\,(\mathbf{w}_{m+1} \cdot \mathbf{P}_n)} + \mathbf{g}_m \cdot \mathbf{P}_n \leq 0,\, n = m \,\text{or}\, m+1\},\end{aligned} \tag{5.42}$$

$$\mathcal{I}_{11} = \{m \in \mathcal{I}_1 : E_m \Gamma_m \leq 0\}, \tag{5.43}$$

$$\mathcal{M}_{01} = \{m \in \mathcal{M}_0 : \psi_m \leq 0\}, \tag{5.44}$$

$$\mathcal{M}_{02} = \{m \in \mathcal{M}_0 : \psi_{m+1} \leq 0\}, \tag{5.45}$$

$$\mathcal{M}_{03} = \{m \in \mathcal{M}_0 \setminus (\mathcal{M}_{01} \cup \mathcal{M}_{02}) : R(\psi, [u_m, u_{m+1}]) > 0\}, \tag{5.46}$$

$$\mathcal{M}_{11} = \{m \in \mathcal{M}_1 : \psi_m \leq 0\}, \tag{5.47}$$

$$\mathcal{M}_{12} = \{m \in \mathcal{M}_1 : \psi(u_m^\ell) \leq 0\}, \tag{5.48}$$

$$\mathcal{M}_{13} = \{m \in \mathcal{M}_1 \setminus (\mathcal{M}_{11} \cup \mathcal{M}_{12}) : R(\psi, [u_m, u_m^\ell]) > 0\}, \tag{5.49}$$

$$\mathcal{M}_{21} = \{m \in \mathcal{M}_1 : \psi(u_m^r) \leq 0\}, \tag{5.50}$$

$$\mathcal{M}_{22} = \{m \in \mathcal{M}_1 : \psi_{m+1} \leq 0\}, \tag{5.51}$$

$$\mathcal{M}_{23} = \{m \in \mathcal{M}_1 \setminus (\mathcal{M}_{21} \cup \mathcal{M}_{22}) : R(\psi, [u_m^r, u_{m+1}]) > 0\}, \tag{5.52}$$

$$\mathcal{J}_{01} = \{m \in \mathcal{J}_0 : \phi_m \leq 0\}, \tag{5.53}$$

$$\mathcal{J}_{11} = \{m \in \mathcal{J}_0 \setminus \{1\} : \phi(u_m^\ell) \leq 0\}, \tag{5.54}$$

$$\mathcal{J}_{12} = \{m \in \mathcal{J}_0 \setminus (\mathcal{J}_{11} \cup \mathcal{J}_{01} \cup \{1\}) : R(\phi, [u_m^\ell, u_m]) > 0\}, \tag{5.55}$$

$$\mathcal{J}_{21} = \{m \in \mathcal{J}_0 \setminus \{N\} : \phi(u_m^r) \leq 0\}, \tag{5.56}$$

$$\mathcal{J}_{22} = \{m \in \mathcal{J}_0 \setminus (\mathcal{J}_{21} \cup \mathcal{J}_{01} \cup \{N\}) : R(\phi, [u_m, u_m^r]) > 0\}. \tag{5.57}$$

STEP 3. Set $\nu_m^{(j+1)} = \nu_m^{(j)}, \forall m$. Let :

$$\mathcal{S}_0 = \mathcal{K}_{03} \cup \mathcal{I}_{11} \cup \mathcal{M}_{03}, \tag{5.58a}$$
$$\mathcal{S}_1 = \mathcal{K}_{01} \cup \mathcal{M}_{01} \cup \mathcal{M}_{11} \cup \mathcal{J}_{01}, \tag{5.58b}$$
$$\mathcal{S}_2 = \mathcal{K}_{02} \cup \mathcal{M}_{02} \cup \mathcal{M}_{22}, \tag{5.58c}$$
$$\mathcal{S}_3 = \mathcal{J}_{22} \cup \mathcal{J}_{12}, \tag{5.58d}$$
$$\mathcal{S}_4 = \mathcal{J}_{11}, \tag{5.58e}$$
$$\mathcal{S}_5 = \mathcal{J}_{21}, \tag{5.58f}$$
$$\mathcal{S}_6 = \mathcal{M}_{12} \cup \mathcal{M}_{13} \cup \mathcal{M}_{21} \cup \mathcal{M}_{23}. \tag{5.58g}$$

If

$$\bigcup_{i=0}^{6} \mathcal{S}_i = \emptyset, \tag{5.59}$$

then **STOP**. Else set :

$$\mathcal{Q}_{-1} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_6, \tag{5.60a}$$
$$\mathcal{Q}_0 = \mathcal{S}_0 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5 \cup \mathcal{S}_6, \tag{5.60b}$$
$$\mathcal{Q}_1 = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_3 \cup \mathcal{S}_5 \cup \mathcal{S}_6, \tag{5.60c}$$
$$\mathcal{Q}_2 = \mathcal{S}_0 \cup \mathcal{S}_2 \cup \mathcal{S}_6, \tag{5.60d}$$
$$\mathcal{R} = \mathcal{S}_6. \tag{5.60e}$$

Let $\{\mathcal{R}_\mu\}_{\mu=1}^{M}$ be the partition of $\mathcal{R}$ with the properties: (1) if $m, m+1 \in \mathcal{R}$, then $m, m+1 \in \mathcal{R}_\mu$ for some $\mu \in \mathcal{T}_M$ and (2) if $m_1, m_2 \in \mathcal{R}$, with $m_1 < m_2$, then $m_1, m_2 \in \mathcal{R}_\mu$ for some $\mu \in \mathcal{T}_M$ if and only if $m_1 + i \in \mathcal{R}_\mu, \forall i \in \{0, \ldots, m_2 - m_1\}$. Without loss of generality we can assume that if $\mu_1 < \mu_2$, then $m_1 < m_2, \forall m_1 \in \mathcal{R}_{\mu_1}, m_2 \in \mathcal{R}_{\mu_2}$. Then, for unit-tangent and zero-curvature boundary conditions, set :

$$\nu_{m-k}^{(j+1)} = \nu_{m-k}^{(j)} + \Delta\nu, \quad m \in \mathcal{Q}_k, \quad k \in \{-1, 0, 1, 2\}, \tag{5.61}$$

provided that $m - k \in \mathcal{T}_N$, whereas for periodic boundary conditions, set :

$$\nu_{i(m,k)}^{(j+1)} = \nu_{i(m,k)}^{(j)} + \Delta\nu, \quad m \in \mathcal{Q}_k, \quad k \in \{-1, 0, 1, 2\}, \tag{5.62}$$

where $i(m,k) = (m - k + N - 1) \pmod{N} + 1$; also, if $M > 1$ and (at least) one of the following three conditions holds

1. $1 \in \mathcal{R}_1$ and $N \in \mathcal{R}_M$
2. $1 \in \mathcal{R}_1$ and $N - 1 \in \mathcal{R}_M$
3. $2 \in \mathcal{R}_1$ and $N \in \mathcal{R}_M$

set

$$\mathcal{R}_1 = \mathcal{R}_1 \cup \mathcal{R}_M, \tag{5.63}$$
$$M = M - 1. \tag{5.64}$$

Finally for all boundary conditions, set :

$$\nu_m^{(j+1)} = \nu_{m+1}^{(j+1)} = \max_{n\in\mathcal{R}_\mu}\{\nu_n^{(j)}, \nu_{n+1}^{(j)}\} + \Delta\nu, \quad m \in \mathcal{R}_\mu, \quad \mu \in \mathcal{T}_M, \tag{5.65}$$

increase j by one and **GOTO STEP 1**. ♠

Step 0 of NUSSP is the preprocessing step of the algorithm, devoted to computing quantities related to the parametrization and the geometry of the polygonal line that connects the interpolation points. Within this step we furthermore determine the index sets and specify the tolerances associated to the various criteria of the shape-preservation notion introduced in Problem $(\mathcal{P})$ of Section 2 and, finally, initialize the ν-parameters.

In Step 1 we solve the linear system for the current ν-parameter distribution, while Step 2 is responsible for the computation of the failure index sets, i.e., the sets that determine the parameter location where the sufficient, and possibly necessary, conditions of §5 fail to be true. As already mentioned at the beginning of the current subsection, Step 2 relies on the predicate $R(f, [a, b])$, that returns the roots of the polynomial $f(t)$ in $[a, b]$, provided that $f(a)f(b) \neq 0$. However, in order to avoid redundant computations, the algorithm exploits Bolzano's Theorem and appeals to $R(f, [a, b])$ only if $f(a)f(b) > 0$.

The algorithm concludes with Step 3, in which we check if the union of the failure sets is the empty set. If this is the case NUSSP stops; otherwise, the values of the ν-parameters are increased appropriately and we iterate by going back to Step 1.

Steps 0, 1 and 2 are fairly straightforward. On the contrary, Step 3 is rather complicated and full of technical details. Nevertheless, we shall not get into the details; instead, we shall provide the general underlying idea and skip the boundary condition dependent details. The sets $\{\mathcal{S}_i\}_{i=0}^6$ contain the indices for which the corresponding or some neighboring ν-value has to be increased. At this point we do not care about ν-values having to be equal to each other. Which ν-values have to be increased is determined by the lemmata of §5 related to the failure sets comprising each of the $\mathcal{S}_i$'s. Then, depending on which $\mathcal{S}_i$ contains m, we might have to increase $\nu_{m-1}, \nu_m, \nu_{m+1}$ or ν_{m+2}. This is exactly what the sets $\{\mathcal{Q}_k\}_{k=-1}^2$ correspond to: given $m \in \mathcal{S}_i$, $\mathcal{Q}_k$ contains m if ν_{m-k} has to be increased. There are situations, however, related exclusively to the coplanarity criterion, where some ν-values have to be equal. This is the purpose of introducing the set $\mathcal{R}$; it contains all those m's for which ν_m and ν_{m+1} have to be increased under the constraint: $\nu_m = \nu_{m+1}$. The partition $\{\mathcal{R}_\mu\}_{\mu=1}^M$ of $\mathcal{R}$ is done in order to guarantee that if $m, m+1$ are in $\mathcal{R}$, then $\nu_m = \nu_{m+1} = \nu_{m+2}$ after we increase the ν-values. Step 3 ends by increasing the ν's appropriately. Note that a specific $\nu_m^{(j+1)}$ may be assigned a value several times during the execution of Step 3, but at the end of the step it will have the correct value.

The algorithm NUSSP stops only when the set $\cup_{i=1}^6 \mathcal{S}_i = \emptyset$. Therefore, in order to prove convergence for NUSSP, we only need to prove that each one of the failure sets, constituting the $\mathcal{S}_i$'s, becomes the empty set after a finite number of iterations. This can be done by appealing to the various lemmata of §5.1. In particular, the convexity failure sets $\{\mathcal{K}_{0i}\}_{i=1}^2$ and $\mathcal{K}_{03}$ will become empty in view of Part (i) and Part (ii) of Lemma

5.4, respectively. Regarding the torsion failure set $\mathcal{I}_{11}$, it will become empty due to Lemma 5.6. Next, in the case of convex coplanar data, Part (i) of Lemma 5.7 ensures that the failure sets $\mathcal{M}_{01}$ and $\mathcal{M}_{02}$ will become empty, whereas Part (ii) of the same lemma implies that the failure set $\mathcal{M}_{03}$ will become empty too. Now, for non-convex coplanar data, it is Lemma 5.8, Part (i), that secures that the failure sets $\mathcal{M}_{11}$ and $\mathcal{M}_{22}$ will become empty, while the same conclusion for the sets $\mathcal{M}_{12}$, $\mathcal{M}_{21}$, $\mathcal{M}_{13}$ and $\mathcal{M}_{23}$, is inferred from Part (ii) of the same lemma. Finally, as for the collinearity failure sets $\mathcal{J}_{01}$, $\mathcal{J}_{11}$, $\mathcal{J}_{21}$ and $\{\mathcal{J}_{i2}\}_{i=1}^{2}$, they will eventually become empty as Parts (i), (ii), (iii) and (iv) of Lemma 5.9 guarantee, respectively.

In order to compute the running time and space requirements of our algorithm, the following facts should be taken into account:

1. A tridiagonal system of S equations can be solved in $\Theta(S)$ time using $\Theta(S)$ space.
2. The cost of computing the quotient and remainder of the division of two polynomials p and q of degrees m and $n < m$, respectively, is $\mathcal{O}(n(n-m))$. This implies that the cost to compute the standard sequence of a polynomial f of degree d is $\mathcal{O}(d^2)$. Moreover, the cost to compute the value $f(x)$ of f at x is $\mathcal{O}(d)$. Combining the above bounds with the fact that the standard sequence of f contains $\mathcal{O}(d)$ polynomials, of monotonically decreasing degree, we conclude that the total cost to compute the predicate $R(f,[a,b])$ is $\mathcal{O}(d^2)$: $\mathcal{O}(d^2)$ operations to compute the standard sequence of f, $\mathcal{O}(d^2)$ operations to compute the values of the polynomials in the standard sequence at a and b, and $\mathcal{O}(d)$ operations to determine the number of variations in sign of the finite sequences of these values.
3. Let $\mathcal{A}$ be a set and $T(\cdot)$ a boolean function operating on the elements of $\mathcal{A}$. Let also $\mathcal{A}_t = \{\alpha \in \mathcal{A} : T(\alpha) = t\}$, $t \in \{0,1\}$. If $T(\cdot)$ can be computed in $\mathcal{O}(1)$ time and space, then the set $\mathcal{A}_t$ can be computed in $\mathcal{O}(|\mathcal{A}|)$ time and space, where $|\mathcal{A}|$ is the cardinality of $\mathcal{A}$. Moreover, if $\mathcal{A}$ is sorted then $\mathcal{A}_t$ can be computed to be sorted in the above time and space bounds.
4. If $\mathcal{A}, \mathcal{B}$ are two sorted sets, then the time needed to compute $\mathcal{A} \setminus \mathcal{B}$ and $\mathcal{A} \cup \mathcal{B}$ is $\mathcal{O}(|\mathcal{A}| + |\mathcal{B}|)$. The space requirement is $\mathcal{O}(|\mathcal{A}| + |\mathcal{B}|)$ as well, and $\mathcal{A} \setminus \mathcal{B}$, $\mathcal{A} \cup \mathcal{B}$ can be computed to be sorted in the above time and space bounds.
5. Let $\mathcal{A}$ be a sorted integer set. Let $\{\mathcal{A}_\lambda\}_{\lambda=1}^{\Lambda}$ be the (unique) partition of $\mathcal{A}$ with the following properties: (1) if $m, m+1 \in \mathcal{A}$, then $m, m+1 \in \mathcal{A}_\lambda$ for some $\lambda \in \mathcal{T}_\Lambda$ and (2) if $m_1, m_2 \in \mathcal{A}$, then $m_1, m_2 \in \mathcal{A}_\lambda$, for some $\lambda \in \mathcal{T}_\Lambda$, if and only if $m_1 + i \in \mathcal{A}_\lambda$, $\forall i \in \{0, \ldots, m_2 - m_1\}$. This partition can be computed in $\mathcal{O}(|\mathcal{A}|)$ time and space, and the sets $\mathcal{A}_\lambda$ can be computed to be sorted.

In the sequel we shall assume that all sets that are results of the above mentioned set operations are sorted.

As far as algorithm NUSSP is concerned we note the following:

1. The computation of each one of the nodal geometric quantities needs $\mathcal{O}(1)$ space and time.

2. The degree of the polynomials $\phi(u)$ and $\psi(u)$ is four. Hence, the cost to compute the predicates $R(\phi, [a,b])$ and $R(\psi, [c,d])$ for any real numbers a, b, c, d is $\mathcal{O}(1)$.

3. All the sets involved are sorted integer sets and of cardinality $\mathcal{O}(N)$.

4. The assignment (5.65) takes $\mathcal{O}(N)$ total time: we need $|\mathcal{R}_\mu|$ time to compute the $\max_{n\in\mathcal{R}_\mu}\{\nu_n^{(j)}, \nu_{n+1}^{(j)}\}$, and $\sum_{\mu=1}^{M} |\mathcal{R}_\mu| = \mathcal{O}(N)$ time to perform the assignments.

Based on the above, we readily conclude that the space requirements for our algorithm is $\Theta(N)$, whereas the running time is $\Theta(N)$ per iteration or $\Theta(NK_{\Delta\nu})$ in total, where $K_{\Delta\nu}$ is the number of iterations that the algorithm performs.

6. Numerical Examples

In this section we will present the performance of the algorithm NUSSP on two data sets. Each data set comprises a family of data points along with information related to the adopted parametrization and the boundary conditions to be imposed. The first data set is of synthetic nature, obtained from [6], while the second one is an industrial data set, kindly provided by General Motors Design Center (Warren, Michigan).

The NUSSP outcome is documented with the final set of ν-parameters and a set of figures; see Figs. 1-4 for Example 1 and Figs. 5-8 for Example 2. The dashed line in these figures depicts the outcome of NUSSP for zero ν_i's, i.e., the standard C^2-continuous cubic spline interpolant. On the other hand, solid line corresponds to the shape-preserving G^2-continuous ν-spline interpolant, that possesses the final ν-parameter distribution provided by NUSSP. In both examples, the boundary conditions are fixed unit-tangent vectors or fixed tangent vectors for zero ν_i's, while the step $\Delta\nu$ of increasing the ν-parameters, after each iteration, is set equal to 0.5. We shall discuss the choice of $\Delta\nu$ at the end of this section.

Example 1

The data consists of twenty five points, sampled from the helix $(\cos\theta, \sin\theta, \theta)$ with θ-step equal to $\pi/3$. The imposed boundary tangent vectors are inherited from those of the two circular arcs, interpolating the first and last three interpolation points. The chordal parametrization was used, while the coplanarity tolerance ε_1 was set equal to 0.2.

The C^2-cubic spline (see dashed line in Fig. 1) failed to meet the torsion, convexity and coplanarity criteria. The algorithm ended after eight iterations, yielding a shape-preserving ν-spline (see solid line in Fig. 1) that possesses the following ν-parameter distribution: $\nu_i = 3.5$, $i = 1, 3, 23, 25$, $\nu_i = 2$, $i = 2, 4, 22, 24$, and $\nu_i = 0$, for the remaining i's. In particular, the C^2-cubic spline had the wrong torsion sign in $[u_2, u_3]$ and $[u_{23}, u_{24}]$ (see Fig. 2). As far as the convexity criterion is concerned the binormals $\mathbf{w}_2/\|\mathbf{w}_2\|$ and $\mathbf{w}_{24}/\|\mathbf{w}_{24}\|$ of the C^2-cubic failed to comply with it. Finally, the C^2-cubic failed to be shape-preserving with respect to the coplanarity criterion for the chosen tolerance in the first and last parametric intervals; see Fig. 4 for the coplanarity ratio in $[u_1, u_2]$.

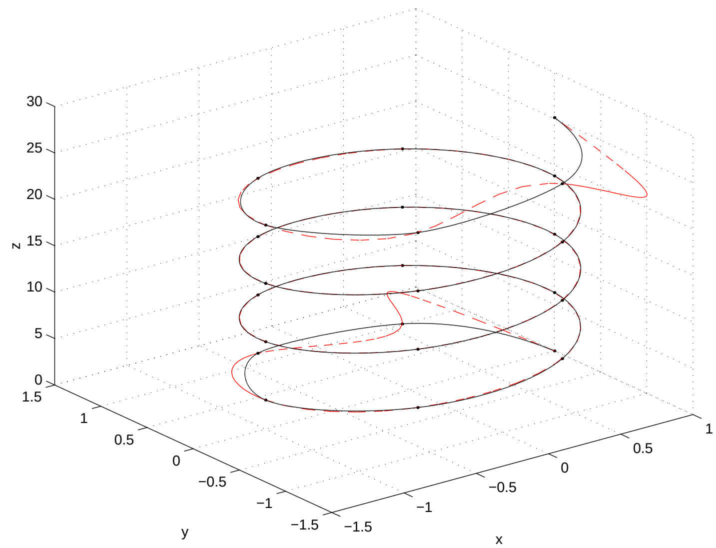

Figure 1. The C^2-cubic (dashed line) and G^2-cubic shape-preserving spline (solid line) interpolating the helical data with the unit-tangent vector boundary conditions described in the text and chordal parametrization.

Example 2

The data are given in Tables 6.1 and 6.2. The C^2-cubic spline interpolant (see dashed line in Fig. 5) failed to satisfy the torsion and convexity criteria. The algorithm ended after five iterations, yielding a shape-preserving ν-spline (see solid line in Fig. 5) that possesses the following ν-parameter distribution: $\nu_i = 0.5$, $i = 1(1)7$, $\nu_i = 2$, $i = 15(1)18$, and $\nu_i = 0$ for the remaining i's. In particular, the C^2-cubic spline exhibited wrong sign for its torsion in $[u_2, u_3]$, $[u_5, u_6]$ and $[u_{16}, u_{17}]$ (see Fig. 6), whereas its binormal $\mathbf{w}_2/\|\mathbf{w}_2\|$ failed to be in conformity with the convexity criterion in $[u_1, u_2]$, since $\mathbf{w}_2 \cdot \mathbf{P}_1 < 0$; see Fig. 8 for the corresponding convexity ratio.

As far as the step $\Delta\nu$ is concerned, we can offer, for the present version of the algorithm NUSSP, but some general guidance based on qualitative remarks and our, so far obtained, numerical experience with NUSSP. Choosing too big a step $\Delta\nu$ results quickly in a shape-preserving curve that is, however, too close to the corresponding linear interpolant and, thus, not visually pleasing (fair). On the other hand, too small a step $\Delta\nu$ increases the runninng time as a result of permitting too many unnecessary iterations. Our numerical experience suggests that values in between 0.1 and 1 seem to be a good

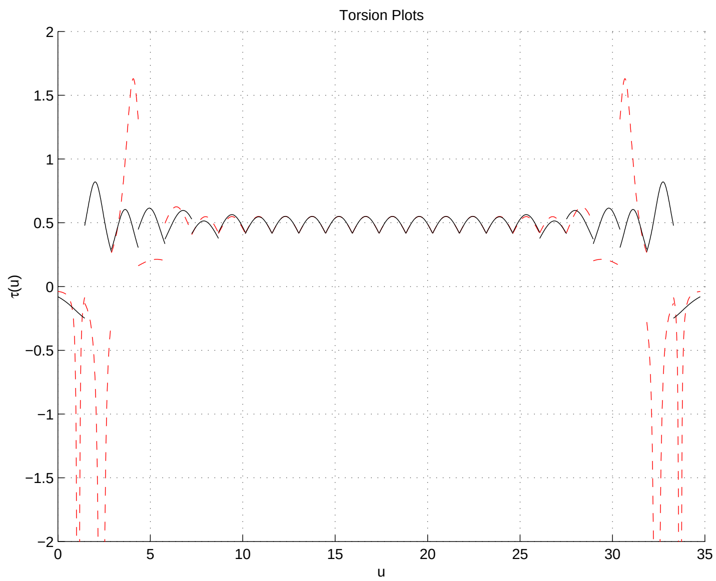

Figure 2. The torsion distributions of the C^2-cubic (dashed line) and G^2-cubic shape-preserving spline (solid line) of Example 1.

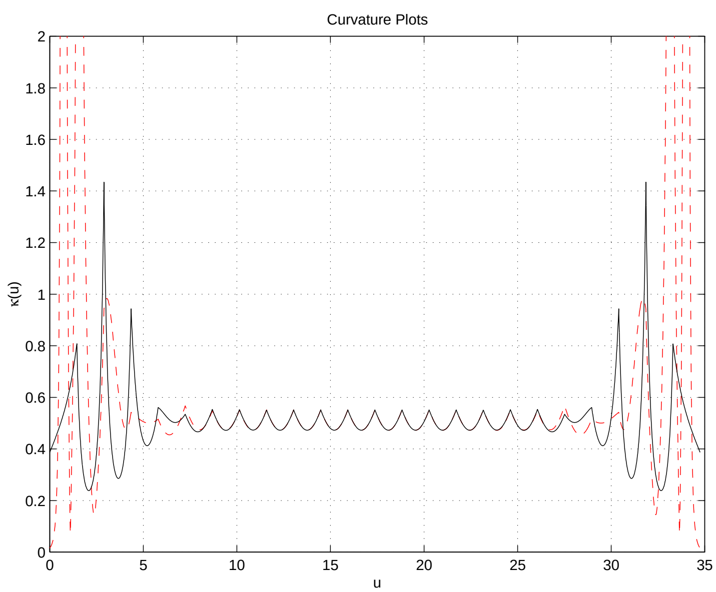

Figure 3. The curvature distributions of the C^2-cubic (dashed line) and G^2-cubic shape-preserving spline (solid line) of Example 1.

compromise between the two conflicting requirements stated previously, namely *improving fairness* and *decreasing running time* – please recall that in our formulation ν's are

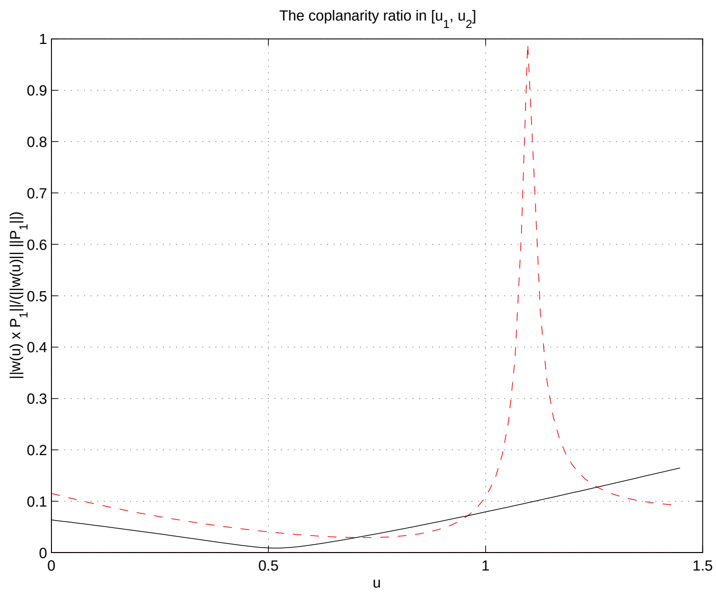

Figure 4. The coplanarity ratio of the C^2-cubic (dashed line) and G^2-cubic shape-preserving spline (solid line) in $[u_1, u_2]$ of Example 1; the coplanarity tolerance ε_1 is equal to 0.2.

non-dimensional (see Section 3). This is the reasoning behind the choice $\Delta\nu = 0.5$, made in the examples presented above.

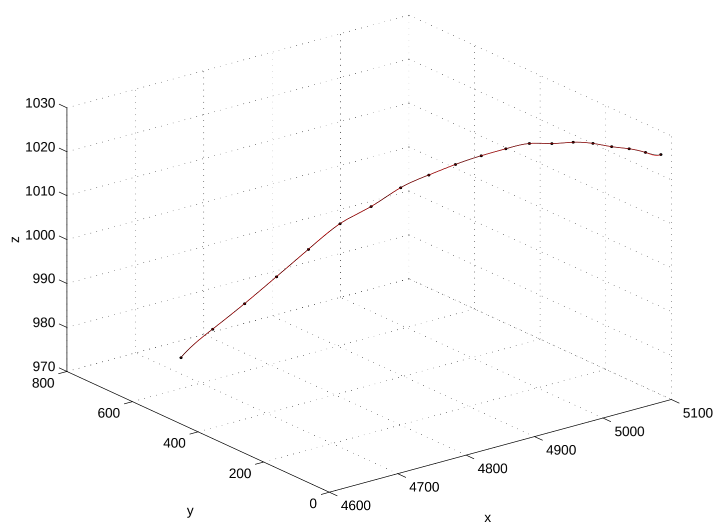

Figure 5. The C^2-cubic (dashed line) and G^2-cubic shape-preserving spline (solid line) interpolating the data of the second example; the two curves are essentially the same.

m	u_m	x_m	y_m	z_m
1	0.0	5088.062500000	7.906158924	1025.970947266
2	0.0526315789473684	5085.933593750	50.088546753	1025.114746094
3	0.10526315789473701	5082.252441406	92.210144043	1024.650390625
4	0.15789473684210501	5076.693847656	134.211181641	1023.926940918
5	0.21052631578947401	5069.430175781	175.916946411	1023.537658691
6	0.26315789473684198	5060.208496094	217.127090454	1022.765319824
7	0.31578947368421101	5048.564941406	257.767242432	1021.563720703
8	0.36842105263157898	5034.916992188	297.822265625	1020.773681641
9	0.42105263157894701	5019.403320313	337.164062500	1018.875549316
10	0.47368421052631598	5001.769531250	375.606842041	1016.703918457
11	0.52631578947368395	4982.209960938	413.023437500	1014.270690918
12	0.57894736842105298	4960.532226563	449.337707520	1011.526733398
13	0.63157894736842101	4936.054199219	483.775268555	1008.486572266
14	0.68421052631578904	4907.513671875	514.686279297	1004.340209961
15	0.73684210526315796	4874.898925781	541.243225098	1000.899230957
16	0.78947368421052599	4839.287109375	563.890319824	995.753234863
17	0.84210526315789502	4802.104003906	583.373291016	990.441406250
18	0.89473684210526305	4764.125488281	601.033081055	985.327575684
19	0.94736842105263197	4725.323730469	617.624389648	980.590759277
20	1.0	4685.979980469	632.172729492	975.273193359

Table 6.1
The x, y, z co-ordinates of the points of Example 2 and the corresponding parameter values u.

	x	y	z
$\mathbf{s}_0$	-11.038033	801.759583	-57.116488
$\mathbf{s}_{20}$	-758.641754	249.218852	-132.963858

Table 6.2
The x, y, z co-ordinates of the initial tangent vectors at the endpoints for Example 2.

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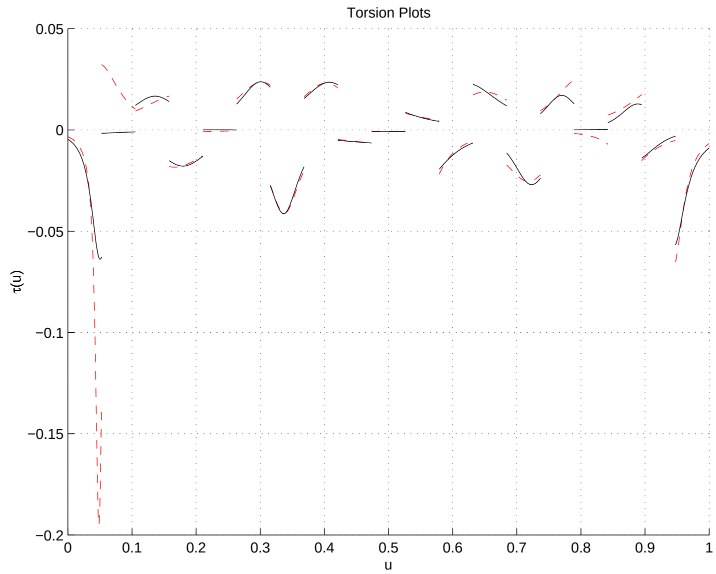

Figure 6. The torsion distributions of the C^2-cubic (dashed line) and G^2-cubic shape-preserving spline (solid line) of Example 2.

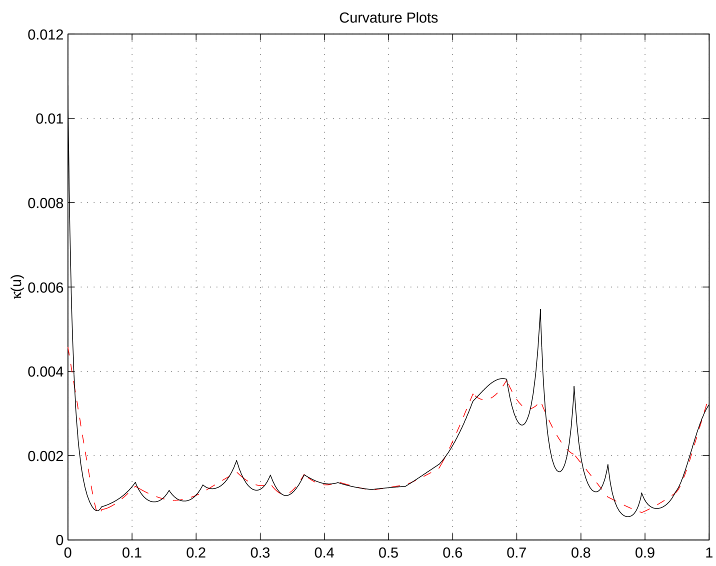

Figure 7. The curvature distributions of the C^2-cubic (dashed line) and G^2-cubic shape-preserving spline (solid line) of Example 2.

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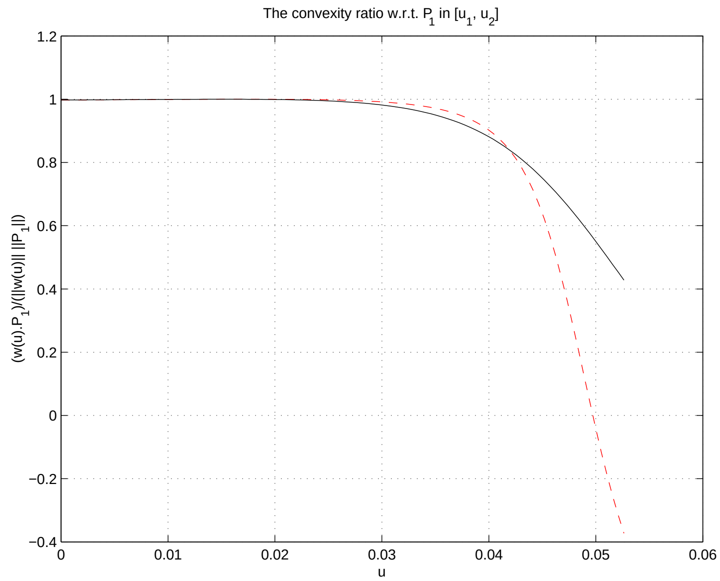

Figure 8. The convexity ratio of the C^2-cubic (dashed line) and G^2-cubic shape-preserving spline (solid line) in $[u_1, u_2]$; the convexity ratio of the C^2-cubic fails to be positive in near u_2.

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