# Guarding curvilinear art galleries with edge or mobile guards via 2-dominance of triangulation graphs☆

Menelaos I. Karavelas[a,b,*]

[a] *University of Crete, Department of Applied Mathematics, GR-714 09 Heraklion, Greece*
[b] *Foundation for Research and Technology - Hellas, Institute of Applied and Computational Mathematics, P.O. Box 1385, GR-711 10 Heraklion, Greece*

**Abstract**

In this paper we consider the problem of monitoring an art gallery modeled as a polygon, the edges of which are arcs of curves, with edge or mobile guards. Our focus is on piecewise-convex polygons, i.e., polygons that are locally convex, except possibly at the vertices, and their edges are convex arcs.

We transform the problem of monitoring a piecewise-convex polygon to the problem of *2-dominating* a properly defined triangulation graph with edges or diagonals, where 2-dominance requires that every triangle in the triangulation graph has at least two of its vertices in its 2-dominating set. We show that: (1) $\lfloor\frac{n+1}{3}\rfloor$ diagonal guards are always sufficient and sometimes necessary, and (2) $\lfloor\frac{2n+1}{5}\rfloor$ edge guards are always sufficient and sometimes necessary, in order to 2-dominate a triangulation graph. Furthermore, we show how to compute: (1) a diagonal 2-dominating set of size $\lfloor\frac{n+1}{3}\rfloor$ in linear time and space, (2) an edge 2-dominating set of size $\lfloor\frac{2n+1}{5}\rfloor$ in $O(n^2)$ time and $O(n)$ space, and (3) an edge 2-dominating set of size $\lfloor\frac{3n}{7}\rfloor$ in $O(n)$ time and space.

Based on the above-mentioned results, we prove that, for piecewise-convex polygons, we can compute: (1) a mobile guard set of size $\lfloor\frac{n+1}{3}\rfloor$ in $O(n\log n)$ time, (2) an edge guard set of size $\lfloor\frac{2n+1}{5}\rfloor$ in $O(n^2)$ time, and (3) an edge guard set of size $\lfloor\frac{3n}{7}\rfloor$ in $O(n\log n)$ time. All space requirements are linear. Finally, we show that $\lfloor\frac{n}{3}\rfloor$ mobile or $\lceil\frac{n}{3}\rceil$ edge guards are sometimes necessary.

When restricting our attention to monotone piecewise-convex polygons, the bounds mentioned above drop: $\lceil\frac{n+1}{4}\rceil$ edge or mobile guards are always sufficient and sometimes necessary; such an edge or mobile guard set, of size at most $\lceil\frac{n+1}{4}\rceil$, can be computed in $O(n)$ time and space.

*Keywords:* art gallery, curvilinear polygons, triangulation graphs, 2-dominance, edge guards, diagonal guards, mobile guards, piecewise-convex polygons, monotone piecewise-convex polygons

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## 1. Introduction

In recent years Computational Geometry has made a shift towards curvilinear objects. Recent works have addressed both combinatorial properties and algorithmic aspects of such problems, as well as the necessary algebraic techniques required to tackle the predicates used in the algorithms involving these objects. The pertinent literature is quite extensive; the interested reader may consult the recent book edited by Boissonnat and Teillaud [2] for a collection of recent results for various classical Computational Geometry

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*Corresponding author
*Email address:* `mkaravel@tem.uoc.gr` (Menelaos I. Karavelas)
*URL:* `http://www.tem.uoc.gr/~mkaravel/` (Menelaos I. Karavelas)

problems involving curvilinear objects. Despite the apparent shift towards the curvilinear world, and despite the vast range of application areas for art gallery problems, including robotics [3, 4], motion planning [5, 6], computer vision [7–10], graphics [11, 12], CAD/CAM [13, 14] and wireless networks [15], there are very few works dealing with the well-known art gallery and illumination class of problems when the objects involved are curvilinear [16–22].

The original art gallery problem was posted by Klee to Chvátal: given a simple polygon $P$ with $n$ vertices, what is the minimum number of point guards that are required in order to monitor the interior of $P$? Chvátal [23] proved that $\lfloor \frac{n}{3} \rfloor$ vertex guards are always sufficient and sometimes necessary, while Fisk [24], a few years later, gave exactly the same result using a much simpler proof technique based on polygon triangulation and coloring the vertices of the triangulated polygon with three colors. Lee and Lin [25] showed that computing the minimum number of vertex guards for a simple polygon is NP-hard, which is also the case for point guards as shown by Aggarwal [26]. In the context of curvilinear polygons, i.e., polygons the edges of which may be linear segments or arcs of curves, Karavelas, Tóth and Tsigaridas [21] have shown that $\lfloor \frac{2n}{3} \rfloor$ vertex guards are always sufficient and sometimes necessary in order to monitor piecewise-convex polygons (i.e., locally convex polygons, except possibly at the vertices, the edges of which are convex arcs), whereas $\lceil \frac{n}{2} \rceil$ point guards are sometimes necessary. In the same paper it is also shown that $2n - 4$ point guards are always sufficient and sometimes necessary in order to monitor piecewise-concave polygons, i.e., locally concave polygons, except possibly at the vertices, the edges of which are convex arcs. In the special case of monotone piecewise-convex polygons, i.e., polygons for which there exists a line $L$ such that every line $L^{\perp}$ perpendicular to $L$ intersects the polygon at at most two connected components, then $\lfloor \frac{n}{2} \rfloor + 1$ vertex or $\lfloor \frac{n}{2} \rfloor$ point guards are always sufficient and sometimes necessary [20]. Cano-Vila, Longi and Urrutia [22] have also studied the problem of monitoring piecewise-convex polygons with vertex or point guards. More precisely, they have indicated an alternative way for proving the upper bound in [21] for the case of vertex guards, and have improved the upper bound for the case of point guards to $\lfloor \frac{5n}{8} \rfloor$.

Soon after the first results on monitoring polygons with vertex or point guards, other types of guarding models where considered. Toussaint introduced in 1981 the notion of *edge guards*. A point $p$ in the interior of the polygon is considered to be monitored if it is visible from at least one point of an edge in the guard set. Edge guards where introduced as a guarding model in which guards where allowed to move along the edges of the polygon. Another variation, dating back to 1983, is due to O'Rourke: guards are allowed to move along any diagonal of the polygon. This type of guards has been called *mobile guards*. Toussaint conjectured that, except for a few polygons, $\lfloor \frac{n}{4} \rfloor$ edge guards are always sufficient. There are only two known counterexamples to this conjecture, with $n = 7, 11$, due to Paige and Shermer (cf. [27]), requiring $\lfloor \frac{n+1}{4} \rfloor$ edge guards. The first step towards Toussaint's conjecture was made by O'Rourke [28, 29] who proved that $\lfloor \frac{n}{4} \rfloor$ mobile guards are always sufficient and occasionally necessary in order to monitor any polygon with $n$ vertices. The technique by O'Rourke amounts to reducing the problem of monitoring a simple polygon to that of dominating a *triangulation graph* of the polygon. A triangulation graph is a maximal outerplanar graph, all internal faces of which are triangles. Dominance in this context means that at least one of the vertices of each triangle in the triangulation graph is an endpoint of a mobile guard. Shermer [27] settled the problem of monitoring triangulation graphs with edge guards by showing that $\lfloor \frac{3n}{10} \rfloor$ edge guards are always sufficient and sometimes necessary, except for $n = 3, 6$ or 13, in which case one extra edge guard may be necessary. When considering orthogonal polygons, i.e., polygons the edges of which are axes-aligned, the afore-mentioned upper and lower bounds drop. Aggarwal [26] showed that $\lfloor \frac{3n+4}{16} \rfloor$ mobile guards are sufficient and sometimes necessary in order to monitor orthogonal polygons with $n$ vertices, a bound that was later on matched for edge guards by Bjorling-Sachs [30]. Finally, Györi, Hoffmann, Kriegel and Shermer [31] showed that when an orthogonal polygon with $n$ vertices contains $h$ holes, $\lfloor \frac{3n+4h+4}{16} \rfloor$ mobile guards are sufficient and sometimes necessary in order to monitor it.

In this paper we consider the problem of monitoring piecewise-convex polygons with edge or mobile guards. In our context an edge guard is an edge of the polygon, whereas a mobile guard is an edge or a diagonal of the polygon (a diagonal is a straight-line segment inside the polygon connecting two polygon vertices). Our proof technique capitalizes on the technique used by O'Rourke to prove tight bounds on the number of mobile guards that are necessary and sufficient for monitoring linear polygons [29]. As we have already mentioned above, O'Rourke's paradigm reduces the geometric guarding problem to a problem of diagonal dominance for the triangulation graph of the linear polygon; the solution for the dominance problem is also a solution for the original geometric mobile guarding problem. In our case, the paradigm involves two steps: firstly the reduction of the geometric problem to an appropriately defined combinatorial problem, and secondly mapping the solution of the combinatorial problem to a solution for the geometric problem. More precisely, in order to monitor piecewise-convex polygons with mobile or edge guards, we first reduce the problem of monitoring our piecewise-convex polygon $P$ to the problem of 2-dominating an appropriately defined triangulation graph. Given a triangulation graph $T_P$ of a polygon $P$, a set of edges/diagonals of $T_P$ is a 2-dominating set of $T_P$ if every triangle in $T_P$ has at least two of its vertices incident to an edge/diagonal in the 2-dominating set. We prove that $\lfloor \frac{n+1}{3} \rfloor$ diagonal guards or $\lfloor \frac{2n+1}{5} \rfloor$ edge guards are always sufficient and sometimes necessary in order to 2-dominate $T_P$. The proofs of sufficiency are inductive on the number of vertices of $P$. In the case of diagonal 2-dominance, our proof yields a linear time and space algorithm.

In the case of edge 2-dominance, the inductive step incorporates edge contraction operations, thus yielding an $O(n^2)$ time and $O(n)$ space algorithm, where $n$ is the number of vertices of $P$. A linear time and space algorithm can be attained by slightly relaxing the size of the edge 2-dominating set. More precisely, we show inductively that we can 2-dominate $T_P$ with $\lfloor \frac{3n}{7} \rfloor$ edges; the proof is similar, though more complicated, to the proof presented for the case of diagonal 2-dominance. As in the diagonal 2-dominance case, it does not make use of edge contraction operations, thus permitting us to transform it to a linear time and space algorithm. As a final note, the proof of sufficiency for the diagonal 2-dominance problem is not the simplest possible; in Section A of the Appendix we present a much simpler alternate proof. The drawback of this alternate proof is that it makes use of edge contractions, rendering it unsuitable as the basis for a time-efficient algorithm; we present it, however, for the sake of completeness.

Focusing back to the geometric guarding problem, the triangulation graph $T_P$ of the piecewise-convex polygon $P$ is a constrained triangulation graph: based on the geometry of $P$, we require that certain diagonals of $T_P$ are present; the remaining non-triangular subpolygons of $T_P$ may be triangulated arbitrarily. For the edge guarding problem, we prove that any edge 2-dominating set computed for $T_P$ is also an edge guard set for $P$. Unlike edge guards, a diagonal 2-dominating set computed for $T_P$ is mapped to a set of mobile guards of $P$, since the 2-dominating set for $T_P$ may contain diagonals of $T_P$ that are not embeddable as straight-line diagonals of $P$. Using our results on 2-dominance of triangulation graphs, we then prove that: (1) we can compute a mobile guard set for $P$ of size at most $\lfloor \frac{n+1}{3} \rfloor$ in $O(n \log n)$ time and $O(n)$ space, (2) we can compute an edge guard set for $P$ of size at most $\lfloor \frac{2n+1}{5} \rfloor$ in $O(n^2)$ time and $O(n)$ space, and (3) we can compute an edge guard set for $P$ of size at most $\lfloor \frac{3n}{7} \rfloor$ in $O(n \log n)$ time and $O(n)$ space. Finally, we show that $\lfloor \frac{n}{3} \rfloor$ mobile or $\lceil \frac{n}{3} \rceil$ edge guards are sometimes necessary in order to monitor a piecewise-convex polygon $P$.

In the special case of monotone piecewise-convex polygons, i.e., piecewise-convex polygons with the property that there exists a line $L$ such that any line perpendicular to $L$ intersects the piecewise-convex polygon at at most two connected components, the upper and lower bounds on the number of edge/mobile guards presented above can be further improved. We show that $\lceil \frac{n+1}{4} \rceil$ edge or mobile guards are always sufficient and sometimes necessary, while an edge or mobile guard set of that size can be computed in linear time and space. The same results also hold for monotone locally convex polygons. Tables 1 and 2 summarize the known results relevant to the problems considered in this paper, as well as our results.

The rest of the paper is structured as follows. In Section 2 we prove our matching upper and lower bounds on the number of diagonals required in order to 2-dominate a triangulation graph and show how such a 2-dominating set can be computed in linear time and space. The next section, Section 3 deals with the problem of 2-dominance of triangulation graphs with edge guards. We first prove our matching upper and lower bounds on the number of edges required in order to 2-dominate a triangulation graph. We then prove our relaxed bound and show how the proof is transformed into a linear time and space algorithm. In Section 4 we show how to construct the triangulation graph $T_P$ of a piecewise-convex polygon $P$. We describe how a diagonal 2-dominating set of $T_P$ is mapped to a mobile guard set for $P$. We also show that an edge 2-dominating set for $T_P$ is also an edge guard set for $P$. Algorithmic considerations are also discussed. We end this section by providing lower bound constructions for both guarding problems. The special case of monotone piecewise-convex polygons is treated in Section 5. Finally in Section 6 we conclude

<table>
<thead>
<tr><th>Polygon type</th><th>Guard type</th><th>Upper bound</th><th>Lower bound</th><th>Reference</th></tr>
</thead>
<tbody>
<tr><td rowspan="3">linear</td><td>vertex/point</td><td colspan="2">$\lfloor \frac{n}{3} \rfloor$</td><td>[23, 24]</td></tr>
<tr><td>edge</td><td>$\lfloor \frac{3n}{10} \rfloor^{\dagger}$</td><td>$\lfloor \frac{n}{4} \rfloor$</td><td>[27],[28]</td></tr>
<tr><td>mobile</td><td colspan="2">$\lfloor \frac{n}{4} \rfloor$</td><td>[28]</td></tr>
<tr><td rowspan="2">orthogonal</td><td>mobile</td><td colspan="2">$\lfloor \frac{3n+4}{16} \rfloor$</td><td>[26]</td></tr>
<tr><td>edge</td><td colspan="2">$\lfloor \frac{3n+4}{16} \rfloor$</td><td>[30]</td></tr>
<tr><td>orthogonal with $h$ holes</td><td>mobile</td><td colspan="2">$\lfloor \frac{3n+4h+4}{16} \rfloor$</td><td>[31]</td></tr>
<tr><td rowspan="2">piecewise-convex</td><td>vertex</td><td colspan="2">$\lfloor \frac{2n}{3} \rfloor$</td><td>[21]</td></tr>
<tr><td>point</td><td>$\lfloor \frac{5n}{8} \rfloor$</td><td>$\lceil \frac{n}{2} \rceil$</td><td>[22],[21]</td></tr>
<tr><td rowspan="2">monotone piecewise-convex</td><td>vertex</td><td colspan="2">$\lfloor \frac{n}{2} \rfloor + 1$</td><td rowspan="2">[20]</td></tr>
<tr><td>point</td><td colspan="2">$\lfloor \frac{n}{2} \rfloor$</td></tr>
<tr><td>piecewise-concave</td><td>point</td><td colspan="2">$2n - 4$</td><td>[21]</td></tr>
<tr><td rowspan="2">piecewise-convex</td><td>edge</td><td>$\lfloor \frac{2n+1}{5} \rfloor^{\ddagger}$</td><td>$\lceil \frac{n}{3} \rceil$</td><td rowspan="4">this paper</td></tr>
<tr><td>mobile</td><td>$\lfloor \frac{n+1}{3} \rfloor$</td><td>$\lfloor \frac{n}{3} \rfloor$</td></tr>
<tr><td>monotone piecewise-convex</td><td>edge/mobile</td><td colspan="2">$\lceil \frac{n+1}{4} \rceil$</td></tr>
<tr><td>monotone locally convex</td><td>edge/mobile</td><td colspan="2">$\lceil \frac{n+1}{4} \rceil$</td></tr>
</tbody>
</table>

Table 1: Upper and lower bounds for the number of guards required to monitor a polygon with $n$ vertices. We focus on types of polygons and types of guards that are relevant to this paper. The upper part of the table contains previous results, whereas the lower part contains the results in this paper.

<table>
<thead>
<tr><th>Dominance type</th><th>Guard type</th><th>Upper & lower bound</th><th>Reference</th></tr>
</thead>
<tbody>
<tr><td rowspan="2">dominance</td><td>diagonal</td><td>$\lfloor \frac{n}{4} \rfloor$</td><td>[28]</td></tr>
<tr><td>edge</td><td>$\lfloor \frac{3n}{10} \rfloor^{\dagger}$</td><td>[27]</td></tr>
<tr><td rowspan="2">2-dominance</td><td>diagonal</td><td>$\lfloor \frac{n+1}{3} \rfloor$</td><td rowspan="2">this paper</td></tr>
<tr><td>edge</td><td>$\lfloor \frac{2n+1}{5} \rfloor^{\ddagger}$</td></tr>
</tbody>
</table>

Table 2: Upper and lower bounds for the number of guards required to dominate or 2-dominate the triangulation graph of a polygon with $n$ vertices. The upper part of the table refers to previously known results, whereas the lower part to the results presented in this paper.

with a discussion of our results and open problems.

$^{\dagger}$Except for $n = 3, 6$ or 13, where an extra guard may be required.
$^{\ddagger}$Except for $n = 4$, where an additional guard is required.

## 2. 2-dominance of triangulation graphs: diagonal guards

A *triangulation graph* $T$ is a maximal outerplanar graph, i.e., a Hamiltonian planar graph with $n$ vertices and $2n-3$ edges, all internal faces of which are triangles (see Fig. 1(top left)). The unique Hamiltonian cycle in $T$ is the cycle that bounds the outer face. The edges that do not belong to the Hamiltonian cycle are called *diagonals*, whereas the term *edge* is used to refer to the edges of the Hamiltonian cycle. Given an $n$-vertex linear polygon $P$, i.e., a polygon the edges of which are line segments, its triangulation graph, denoted by $T_P$, is the planar graph we get when the polygon has been triangulated.

A *dominating set* $D$ of a triangulation graph $T$ is a set of vertices, edges or diagonals of $T$ such that at least one of the vertices of each triangle in $T$ belongs to $D$ (see Fig. 1(top right)[1]). An *edge (resp., diagonal) dominating set* of $T$ is a dominating set of $T$ consisting of only edges (resp., edges or diagonals) of $T$. A *2-dominating set* $D$ of $T$ is a dominating set of $T$ that has the property that every triangle in $T$ has at least two of its vertices in $D$. In a similar manner, an *edge (resp., diagonal) 2-dominating set* of $T$ is a 2-dominating set of $T$ consisting only of edges (resp., edges or diagonals) of $T$ (see Fig. 1(bottom row)).

[1]Unless otherwise stated, in all figures, edges/diagonals in a dominating/guard set are shown as thick solid/dashed lines, while vertices in a dominating/guard set are transparent.

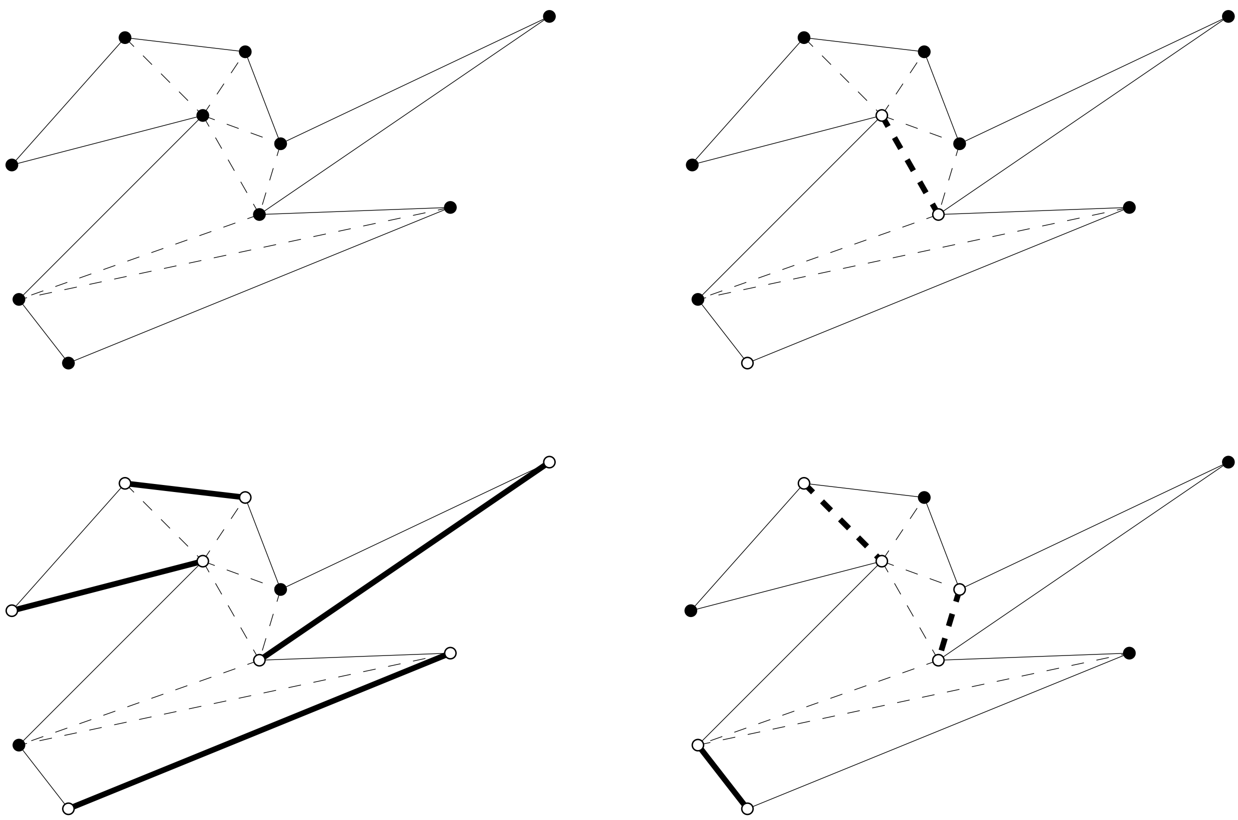

Figure 1: A triangulation graph $T$ with $n = 10$ vertices and various dominating sets. The diagonals of $T$ are shown with dashed lines, whereas the edges of the Hamiltonian cycle in $T$ are shown with solid lines. Vertices in a dominating set are transparent, whereas edges (resp., diagonals) in a dominating set are shown with thick solid (resp., dashed) lines. Top left: the triangulation graph $T$. Top right: a dominating set of $T$ consisting of a vertex and a diagonal. Bottom left: an edge 2-dominating set of $T$. Bottom right: a diagonal 2-dominating set of $T$.

In the rest of the paper we shall only refer to triangulation graphs of polygons. Let us, initially, state the following lemma, which is a direct generalization of Lemmas 1.1 and 3.6 in [29].

**Lemma 1.** *Consider an integer* $\lambda \geq 2$*. Let* $P$ *be a polygon of* $n \geq 2\lambda$ *vertices, and* $T_P$ *a triangulation graph of* $P$*. There exists a diagonal* $d$ *in* $T_P$ *that partitions* $T_P$ *into two pieces, one of which contains* $k$ *arcs corresponding to edges of* $P$*, where* $\lambda \leq k \leq 2(\lambda - 1)$*.*

*Proof.* Choose $d$ to be a diagonal of $T_P$ that separates off a *minimum* number of polygon edges that is at least $\lambda$. Let $k \geq \lambda$ be this minimum number, and label the vertices of $P$ with the labels $0, 1, \ldots, n-1$, such that $d$ is $(0, k)$. The diagonal $d$ supports a triangle whose apex is at vertex $t$, $0 \leq t \leq k$. Since $k$ is minimal $t \leq \lambda - 1$ and $k - t \leq \lambda - 1$. Thus, $\lambda \leq k \leq 2(\lambda - 1)$. □

Before proceeding with the first main result of this section, we state an intermediate lemma dealing with the diagonal 2-dominance problem for small values of $n$.

**Lemma 2.** *Every triangulation graph* $T_P$ *with* $3 \leq n \leq 7$ *vertices, corresponding to a polygon* $P$*, can be 2-dominated by* $\lfloor \frac{n+1}{3} \rfloor$ *diagonal guards.*

*Proof.* Let $v_i$, $1 \leq i \leq n$ be the vertices of $T_P$, and let $e_i$ be the edge $v_i v_{i+1}$[2]. For each of the five values for $n$ we are going to define a diagonal 2-dominating set $D$ of size $\lfloor \frac{n+1}{3} \rfloor$.

$\boldsymbol{n = 3}$. Trivial: let $D$ consist of any of the three edges of $T_P$.

$\boldsymbol{n = 4}$. Again trivial: let $D$ consist of the unique diagonal $d$ of $T_P$.

$\boldsymbol{n = 5}$. Let $D$ consist of the two diagonals of the pentagon. $D$ is a 2-dominating set for $T_P$, since the two ears have two of their vertices in $D$, whereas the third triangle in $T_P$ has all three vertices in $D$.

$\boldsymbol{n = 6}$. Let $t$ be an ear of $T_P$, and let $e'$ and $e''$ be the edges of $P$ incident to $t$ that do not belong to $t$ (see Fig. 2(left)). Set $D = \{e', e''\}$; $D$ is a diagonal 2-dominating set for $T_P$, since the triangulation graph $T_P \setminus \{t\}$ has all but one of its vertices in $D$, whereas $t$ has two of its vertices in $D$.

$\boldsymbol{n = 7}$. Let $t_1$ and $t_2$ be two ears of $T_P$, and let $d_1$ and $d_2$ be the diagonals of $T_P$ supporting these ears. The two possible relative positions of $t_1$ and $t_2$ are shown in Fig. 2: either $d_1$ and $d_2$ share a vertex, or $d_1$ and $d_2$ do not share any vertices of $P$. In the former case, let $e$ be the edge of $P$ incident to $d_1$ that is not an edge of $t_1$ or $t_2$. Set $D = \{e, d_2\}$; $D$ is a diagonal 2-dominating set for $T_P$, since $t_1$ is 2-dominated by vertices of $e$ and $d_2$, $t_2$ is 2-dominated by the two vertices of $d_2$, whereas the triangulation graph $T_P \setminus \{t_1, t_2\}$ has four of its five vertices in $D$. In the latter case, set $D = \{d_1, d_2\}$; $D$ is a diagonal 2-dominating set for $T_P$, since $t_1$ is 2-dominated by the two vertices of $d_1$, $t_2$ is 2-dominated by the two vertices of $d_2$, whereas the triangulation graph $T_P \setminus \{t_1, t_2\}$ has four of its five vertices in $D$. □

[2]Indices are considered to be evaluated modulo $n$.

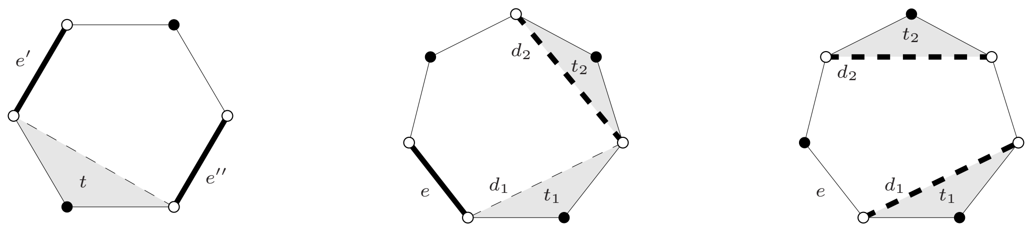


Figure 2: Proof of Lemma 2 for $n = 6, 7$. Left: the case $n = 6$. Middle: the case $n = 7$ and $d_1$, $d_2$ share a vertex. Right: the case $n = 7$ and $d_1$, $d_2$ do not share a vertex.

Using Lemma 1 for $\lambda = 4$, yields the following theorem concerning the worst-case number of diagonals that are sufficient and necessary in order to 2-dominate a triangulation graph. The inductive proof that follows is not the simplest possible. The interested reader may find a much simpler alternative proof in Section A of the Appendix. The proof in Section A, however, makes use of edge contractions (to be discussed in detail in Section 3), which make it unsuitable as a basis for a linear time and space algorithm. On the other hand, the proof presented below can be implemented in linear time and space, as will be discussed in Section 2.1. The proof below is a detailed, rather technical, case-by-case analysis; we present it, however, uncondensed, so as to illustrate the details that pertain to our linear time and space algorithm.

**Theorem 3.** *Every triangulation graph $T_P$ of a polygon $P$ with $n \geq 3$ vertices can be 2-dominated by $\lfloor \frac{n+1}{3} \rfloor$ diagonal guards. This bound is tight in the worst-case.*

*Proof.* In Lemma 2, we have shown the result for $3 \leq n \leq 7$. Let us now assume that $n \geq 8$ and that the theorem holds for all $n'$ such that $3 \leq n' < n$. By means of Lemma 1 with $\lambda = 4$, there exists a diagonal $d$ that partitions $T_P$ into two triangulation graphs $T_1$ and $T_2$, where $T_1$ contains $k$ boundary edges of $T_P$ with $4 \leq k \leq 6$. Let $v_i$, $0 \leq i \leq k$, be the $k+1$ vertices of $T_1$, as we encounter them while traversing $P$ counterclockwise, and let $v_0 v_k$ be the common edge of $T_1$ and $T_2$. For each value of $k$ we are going to define a diagonal 2-dominating set $D$ for $T_P$ of size $\lfloor \frac{n+1}{3} \rfloor$. In what follows $d_{ij}$ denotes the diagonal $v_i v_j$, whereas $e_i$ denotes the edge $v_i v_{i+1}$. Consider each value of $k$ separately.

$\boldsymbol{k = 4}$**.** In this case $T_2$ contains $n-3$ vertices. By our induction hypothesis we can 2-dominate $T_2$ with $f(n-3) = \lfloor \frac{n+1}{3} \rfloor - 1$ diagonal guards. Let $D_2$ be the diagonal 2-dominating set for $T_2$. At least one of $v_0$ and $v_4$ is in $D_2$. The cases are symmetric, so we can assume without loss of generality that $v_0 \in D_2$. Consider the following cases (see Fig. 3):

$\boldsymbol{d_{13} \in T_1}$**.** Set $D = D_2 \cup \{d_{13}\}$.

$\boldsymbol{d_{24} \in T_1}$**.** Set $D = D_2 \cup \{d_{24}\}$.

$\boldsymbol{d_{02}, d_{03} \in T_1}$**.** Set $D = D_2 \cup \{e_2\}$.

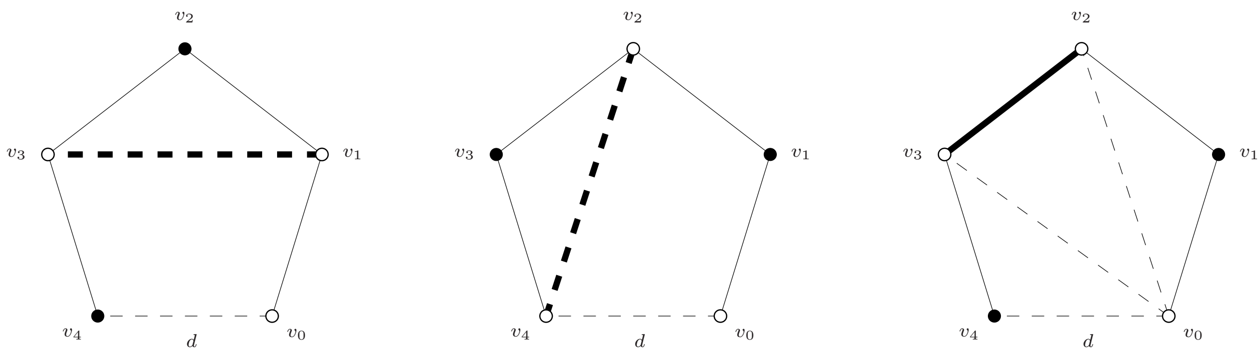


Figure 3: Proof of Theorem 3: the case $k = 4$. Left: $d_{13} \in T_1$. Middle: $d_{24} \in T_1$. Right: $d_{02}, d_{03} \in T_1$.

$\boldsymbol{k = 5}$**.** The presence of diagonals $d_{04}$ and $d_{15}$ would violate the minimality of $k$. Let $t$ be the triangle supported by $d$ in $T_1$. The apex $v$ of this triangle can either be $v_2$ or $v_3$. The two cases are symmetric, so we assume, without loss of generality that the apex of $t$ is $v_2$. Consider the triangulation graph $T' = T_2 \cup \{t\}$. It has $n-3$ vertices, hence, by our induction hypothesis, it can be 2-dominated with $f(n-3) = \lfloor \frac{n+1}{3} \rfloor - 1$ diagonal guards. Let $D'$ be the 2-dominating set for $T'$. Consider the following cases (see Fig. 4):

$\boldsymbol{d_{02} \in D_2}$**.** Set $D = D' \cup \{e_3\}$.

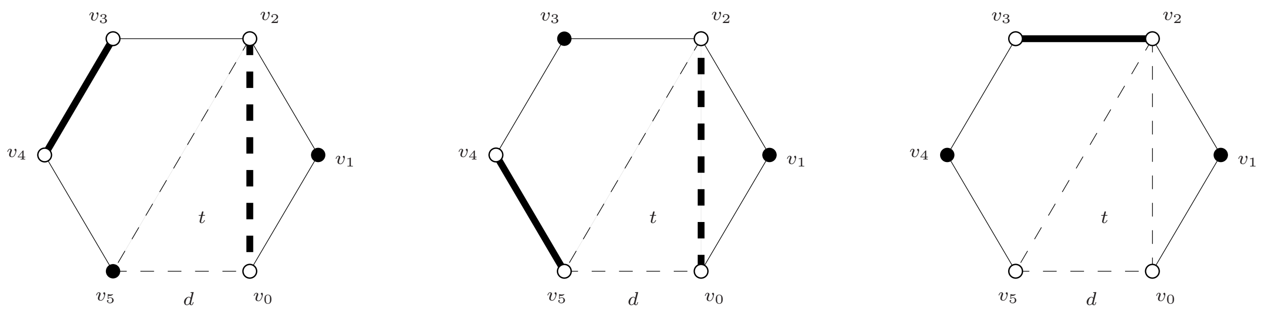


Figure 4: Proof of Theorem 3: the case $k = 5$. Left: $d_{02} \in D'$. Middle: $d_{02} \notin D'$ and $d_{25} \in D'$. Right: $d_{02}, d_{25} \notin D'$.

$\boldsymbol{d_{02} \notin D_2}$**.** If $d_{25} \in D'$, set $D = (D' \setminus \{d_{25}\}) \cup \{d_{02}, e_4\}$. Otherwise, $v_2$ cannot belong to $D'$ (both edges of $T'$ incident to $v_2$ do not belong to $D'$). However, the triangle $t$ is 2-dominated in $T'$, which implies that both $v_0$ and $v_5$ belong to $D'$. Hence, set $D = D' \cup \{e_2\}$.

$\boldsymbol{k = 6}$**.** The presence of diagonals $d_{04}$, $d_{05}$, $d_{16}$ and $d_{26}$ would violate the minimality of $k$. Let $t$ be the triangle supported by $d$ in $T_1$. The apex $v$ of this triangle must be $v_3$. Let $t'$ be the second triangle in $T_1$ beyond $t$ supported by the diagonal $d_{03}$, and let $v'$ be its vertex opposite to $d_{03}$. Symmetrically, let $t''$ be the second triangle in $T_1$ beyond $t$ supported by the diagonal $d_{36}$, and let $v''$ be its vertex opposite to $d_{36}$. Consider the triangulation graphs $T' = T_2 \cup \{t, t'\}$ and $T'' = T_2 \cup \{t, t''\}$. $T'$ and $T''$ have $n-3$ vertices, hence, by our induction hypothesis, they can be 2-dominated with $f(n-3) = \lfloor \frac{n+1}{3} \rfloor - 1$ diagonal guards. Let $D'$ (resp., $D''$) be the 2-dominating set for $T'$ (resp., $T''$).

Let us first consider the case $v' \equiv v_2$. Let $d''$ be the unique diagonal of the quadrilateral $v_3v_4v_5v_6$. Consider the following cases (see Fig. 5):

$\boldsymbol{d_{02} \in D'}$**.** Set $D = D' \cup \{d''\}$.

$\boldsymbol{d_{02} \notin D'}$**.** We further distinguish between the following two cases:

$\boldsymbol{d_{36} \in D'}$**.** If $v_0 \in D'$, simply set $D = (D' \setminus \{d_{36}\}) \cup \{e_2, e_5\}$. If $v_0 \notin D'$, the diagonal $d_{03}$ cannot belong to $D'$. Therefore, in order for the triangle $t'$ to be 2-dominated by $D'$, we must have that $e_2$ in $D'$. Thus, set $D = (D' \setminus \{d_{36}\}) \cup \{e_0, e_5\}$.

$\boldsymbol{d_{36} \notin D'}$**.** In order for $t'$ to be 2-dominated by $D'$ we must have that either $d_{03} \in D'$ or $e_2 \in D'$. If $d_{03} \in D'$, set $D = (D' \setminus \{d_{03}\}) \cup \{d_{02}, d''\}$; otherwise, set $D = (D' \setminus \{e_2\}) \cup \{d_{02}, d''\}$.

Let us now consider the case $v' \equiv v_1$. We first consider the situation $v'' \equiv v_4$. Consider the following cases (see Fig. 6):

$\boldsymbol{d_{46} \in D''}$**.** Set $D = D'' \cup \{d_{13}\}$.

$\boldsymbol{d_{46} \notin D''}$**.** We further distinguish between the following two cases:

$\boldsymbol{d_{03} \in D''}$**.** If $v_6 \in D''$, simply set $D = (D'' \setminus \{d_{03}\}) \cup \{e_0, e_3\}$. If $v_6 \notin D''$, the diagonal $d_{36}$ cannot belong to $D''$. Therefore, in order for the triangle $t''$ to be 2-dominated by $D''$, we must have that $e_3$ in $D''$. Thus, set $D = (D'' \setminus \{d_{03}\}) \cup \{e_0, e_5\}$.

$\boldsymbol{d_{03} \notin D''}$**.** In order for $t''$ to be 2-dominated by $D''$ we must have that either $d_{36} \in D''$ or $e_3 \in D''$. If $d_{36} \in D''$, set $D = (D'' \setminus \{d_{36}\}) \cup \{d_{13}, d_{46}\}$; otherwise, set $D = (D'' \setminus \{e_3\}) \cup \{d_{13}, d_{46}\}$.

The only remaining case is the case where $v' \equiv v_1$ and $v'' \equiv v_5$. Consider the following cases (see Fig. 7):

$\boldsymbol{d_{13} \in D'}$**.** Set $D = D' \cup \{e_5\}$.

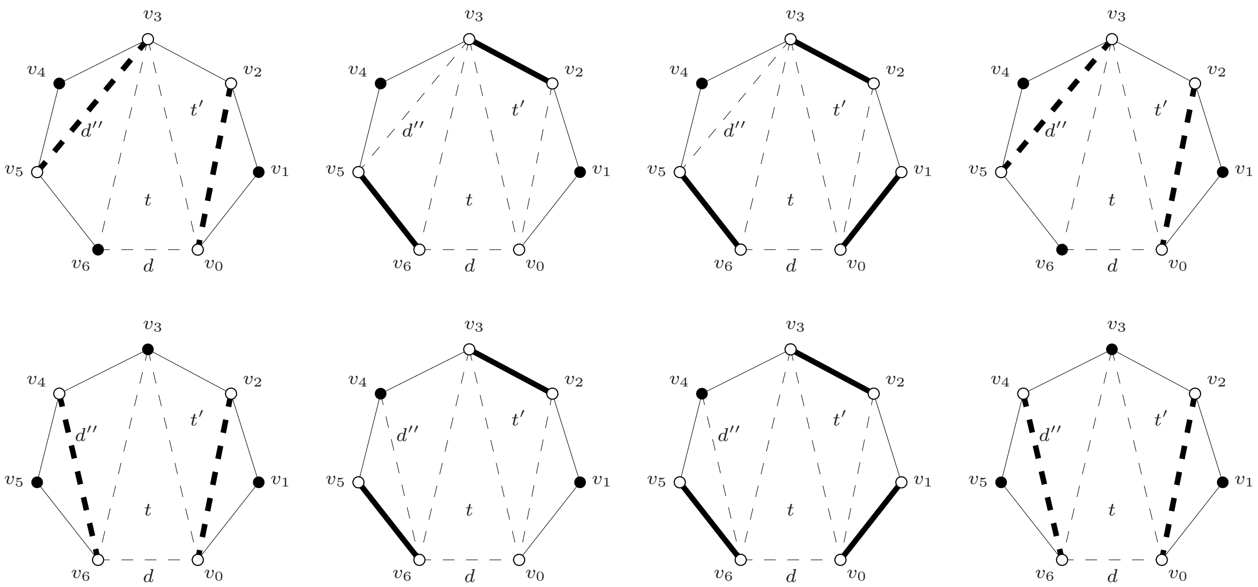


Figure 5: Proof of Theorem 3: the case $k = 6$ with $v' \equiv v_2$. Top row: $v'' \equiv v_5$. Bottom row: $v'' \equiv v_4$. Left column: $d_{02} \in D'$. Middle left column: $d_{02} \notin D'$ and $d_{36} \in D'$ and $v_0 \in D'$. Middle right column: $d_{02} \notin D'$ and $d_{36} \in D'$ and $v_0 \notin D'$. Right column: $d_{02}, d_{36} \notin D'$.

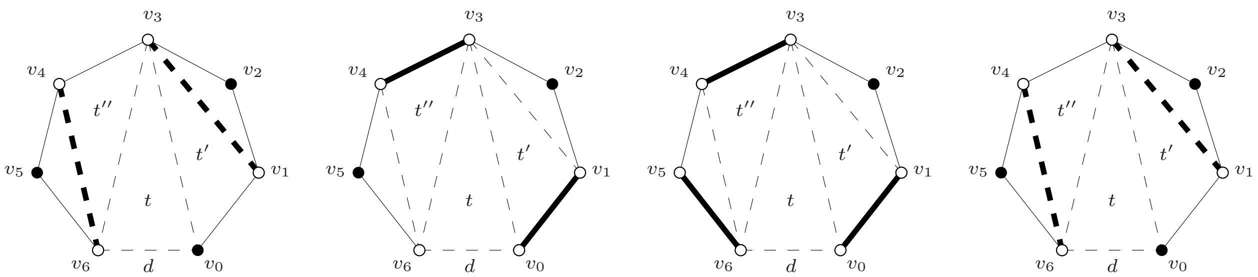


Figure 6: Proof of Theorem 3: the case $k = 6$ with $v' \equiv v_1$ and $v'' \equiv v_4$. Left: $d_{46} \in D''$. Middle left: $d_{46} \notin D''$ and $d_{03} \in D''$ and $v_6 \in D''$. Middle right: $d_{46} \notin D''$ and $d_{03} \in D''$ and $v_6 \notin D''$. Right: $d_{03}, d_{46} \notin D''$.

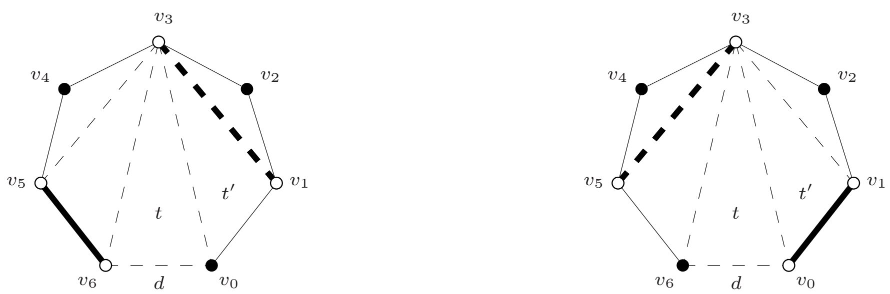


Figure 7: Proof of Theorem 3: the case $k = 6$ with $v' \equiv v_1$ and $v'' \equiv v_5$. Left: $d_{13} \in D'$; also $d_{13}, d_{03}, e_0 \notin D'$. Right: $d_{13} \notin D'$ and $d_{03} \in D'$; also $d_{13}, d_{03} \notin D'$ and $e_0 \in D'$.

$\boldsymbol{d_{13} \notin D'}$**.** We further distinguish between the following two cases:

$\boldsymbol{d_{03} \in D'}$**.** Set $D = (D' \setminus \{d_{03}\}) \cup \{e_0, d_{35}\}$.

$\boldsymbol{d_{03} \notin D'}$**.** If $e_0 \in D'$, set $D = D' \cup \{d_{35}\}$. Otherwise, i.e., if $e_0 \notin D'$, $v_1$ cannot be in $D'$. Since the triangle $t'$ is 2-dominated in $D'$, both $v_0$ and $v_3$ have to belong to $D'$. Since the diagonal $d_{03}$ does not belong to $D'$, the diagonal $d_{36}$ has to belong to $D'$ in order for $v_3$ to be in $D'$. Thus, set $D = (D' \setminus \{d_{36}\}) \cup \{d_{13}, e_5\}$.

Let us now turn our attention to establishing the lower bound. Consider the triangulation graphs $T_i$, $i = 1, 2, 3$, with $n = 3m + i - 1$ vertices, shown in Fig. 8, and let $D_i$ be the diagonal 2-dominating set of $T_i$. The central part of $T_i$ is triangulated arbitrarily. Notice that each subgraph of $T_i$, shown in either light or dark gray, requires at least one among its edges or diagonals to be in $D_i$ in order to be 2-dominated.

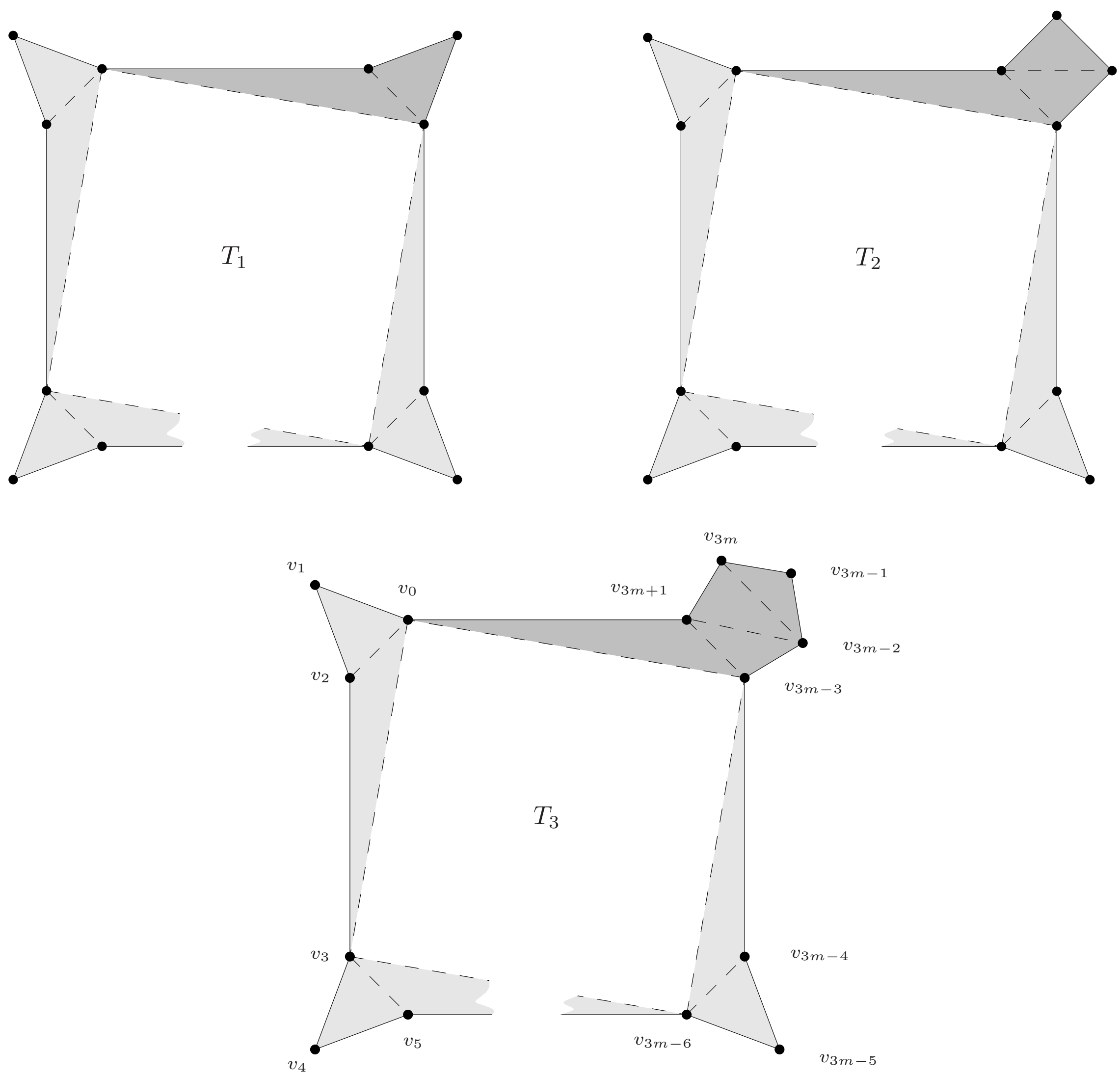


Figure 8: Three triangulation graphs $T_i$, $i = 1, 2, 3$, with $n = 3m + i - 1$ vertices, respectively (the central part of the graph is triangulated arbitrarily). All three triangulation graphs require at least $\lfloor \frac{n+1}{3} \rfloor$ diagonal guards in order to be 2-dominated.

Consider, for example, the quadrilateral $v_0v_1v_2v_3$ of $T_3$ (the situation for all other subgraphs shown in light gray is analogous, whereas the subgraphs shown in dark gray have at least as many vertices as those shown in light gray, and, thus, could not possibly be 2-dominated with fewer diagonal guards with respect to the subgraphs shown in light gray). Even if both $v_0$ and $v_3$ belong to $D_3$ due to edges or diagonals of the neighboring shaded subgraphs, or due to diagonals of the central part of $T_3$, the triangle $v_0v_1v_2$ is not 2-dominated unless either one of the edges $e_0$, $e_1$, $e_2$, or the diagonal $d_{02}$ belongs to $D_3$. This observation immediately establishes a lower bound of $\lfloor \frac{n}{3} \rfloor$.

Let us now assume that $|D_3| = \lfloor \frac{n}{3} \rfloor$. Notice that, under this assumption, each shaded subgraph in $T_3$ must have *exactly one* among its edges or diagonals in $D_3$. Moreover, none of the diagonals in the central part of $T_3$ (not shown in Fig. 8(bottom)) can belong to $D_3$, since then the size of $D_3$ would be greater than $\lfloor \frac{n}{3} \rfloor$. Consider the triangulated hexagon $H := v_0v_{3m-3}v_{3m-2}v_{3m-1}v_{3m}v_{3m+1}$. In order for $H$ to be 2-dominated with exactly one of its edges or diagonals, both $v_0$ and $v_{3m-3}$ have to be in $D_3$ due to edges or diagonals in the neighboring shaded subgraphs, while the unique edge or diagonal of $H$ in $D_3$ must be the diagonal $d_{3m-2,3m}$. Since we require that $v_{3m-3}$ must belong to $D_3$ via an edge or diagonal of the quadrilateral $v_{3m-6}v_{3m-5}v_{3m-4}v_{3m-3}$, and at the same time we require that exactly one of the edges or diagonals of $v_{3m-6}v_{3m-5}v_{3m-4}v_{3m-3}$ to be in $D_3$, the edge $e_{3m-4}$ must belong to $D_3$ and $v_{3m-6}$ must be in $D_3$ due to an edge or diagonal in the quadrilateral $v_{3m-9}v_{3m-8}v_{3m-7}v_{3m-6}$. Cascading this argument, we conclude that, since $v_3$ must belong to $D_3$ due to an edge or diagonal of the quadrilateral $v_0v_1v_2v_3$, and at the same time exactly one of the edges or diagonals of $v_0v_1v_2v_3$ must be in $D_3$, the edge $e_2$ must belong to $D_3$ and $v_0$ must belong to $D_3$ due to an edge or diagonal in the hexagon $H$. But this yields a contradiction, since the unique edge or diagonal of $H$ in $D_3$ is $d_{3m-2,3m}$, which is not incident to $v_0$. Hence $T_3$ requires $\lfloor \frac{n+1}{3} \rfloor$ diagonal guards in order to be monitored. □

### *2.1. Computing diagonal 2-dominating sets*

The proof of Theorem 3 can almost immediately be transformed into an $O(n)$ time and space algorithm. The triangulation graph $T_P$ of $P$ is assumed to be represented via a half-edge representation. Half-edges and vertices in our representation are assumed to have additional flags for indicating whether a half-edge is a boundary edge of the polygon, or whether a half-edge or a vertex of $T_P$ is marked as being in the diagonal 2-dominating set of $T_P$. Under these assumptions, adding or removing a half-edge or a vertex from the sought-for 2-dominating set, querying a half-edge or a vertex for membership in the 2-dominating set, as well as forming the triangulation graph for the recursive calls, all take $O(1)$ time.

Consider a diagonal $d$ that separates $T_P$ into two triangulation graphs $T_1$ and $T_2$, where $T_1$ contains $k = 4, 5$ or 6 edges of $P$; recall from the proof of Lemma 1 (for $\lambda = 4$) that the value of $k$ is minimal. Let $\Delta$ be the dual tree of $T_P$, $\Delta_1$ the dual tree of $T_1$ and $\Delta_1' = \Delta_1 \cup \{d'\}$, where $d'$ is the dual edge of $d$ in $\Delta$. $\Delta_1$ consists of a subtree of $\Delta$ with 2, 3 or 4 edges of $\Delta$, connected with the rest of $\Delta$ via a degree-2 or a

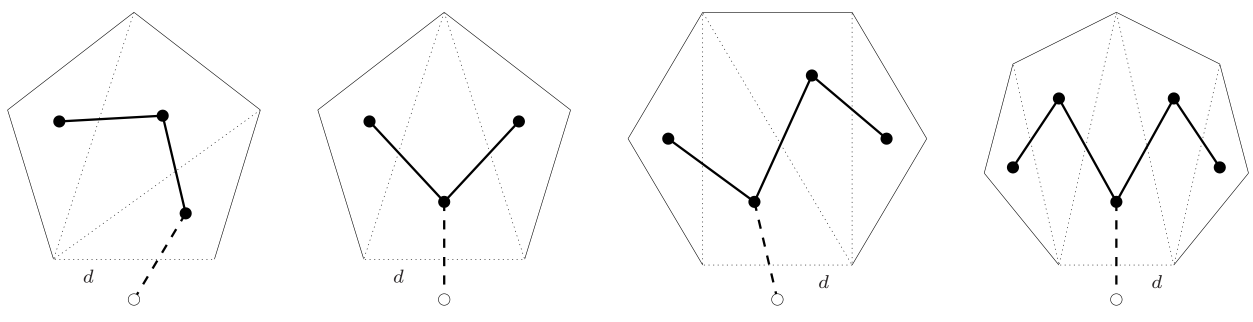


Figure 9: The four possible configurations for the dual trees $\Delta_1$ for $4 \leq k \leq 6$, shown as thick solid lines. The diagonal $d$ separates $T_1$ from $T_2$. The triangulations shown are indicative: all other triangulations yield isomorphic trees.

degree-3 node (see Fig. 9). Moreover, for $n \geq 13$, the subtrees $\Delta_1'$ corresponding to different diagonals $d$ of $T_P$ must be edge disjoint (otherwise the number of vertices of $P$ would be less than 13).

Having made these observations we can now describe the algorithm for computing the diagonal 2-dominating set $D$ for $T_P$. We first describe the initialization steps:

1. Initialize $D$ to be empty.
2. Create a queue $Q$, and initialize it to be empty. $Q$ will consist of diagonals of $T_P$.
3. For each diagonal $d$ of $T_P$ determine whether it separates off $k$ edges of $P$ in $T_P$, with $4 \leq k \leq 6$ and $k$ being minimal. In other words, determine if the dual edge $d'$ of $d$ in $\Delta$ is adjacent to subtrees of the form shown in Fig. 9. If so, put $d$ in $Q$.

The recursive part of the algorithm is as follows:

1. If the number of vertices of $T_P$ is less than 13, find a diagonal 2-dominating set $D$ and return.
2. If $Q$ is not empty:
   (a) Pop a diagonal $d$ out of $Q$.
   (b) If $T_2$ has less than 13 vertices, empty the queue $Q$ and find a 2-dominating set $D_2$ for $T_2$. Based on $D_2$, and according to the cases in the proof of Theorem 3, compute $D$ and return.
   (c) Using the cases in the proof of Theorem 3, determine the triangulation graph $\hat{T}$ for which we are supposed to find the 2-dominating set recursively, and let $\hat{\Delta}$ be the dual tree of $\hat{T}$. Let $V$ be the set of vertices in $\hat{\Delta} \cap \Delta_1'$. For any $v \in V$ determine if $v$ is a leaf-node to a subtree of $\hat{\Delta}$ like the subtrees in Fig. 9. If so, add the appropriate diagonal to $Q$. Neither one of the trees $\hat{\Delta}$ and $\hat{\Delta} \cap \Delta_1'$, nor the set $V$ are computed explicitly; the set $V$ is, in fact, evaluated using the cases in the proof of Theorem 3 without computing $\hat{\Delta} \cap \Delta_1'$.
   (d) Recursively, find a diagonal 2-dominating $\hat{D}$ for $\hat{T}$, using $Q$ as the queue.
   (e) Construct from $\hat{D}$ a diagonal 2-dominating set $D$ for $T_P$ and return.

The initialization part of our algorithm takes linear time, since Step 2 of the initialization takes constant time per diagonal. Let $T(n)$ be the time spent for the recursive part of our algorithm. Step 1 of the recursive part obviously takes constant time. Step 2 of the recursive part takes $T(n-3)+O(1)$ time. Let us be more precise. Popping a diagonal from $Q$ takes $O(1)$ time. Step 2(b) takes $O(1)$ time since we need to solve our problem for a constant value of $n$. Determining the case for $d$ takes $O(1)$ time. $V$ has constant size and can be computed in constant time, while checking for new diagonals to be added to the queue $Q$, as well as adding them to $Q$ also takes $O(1)$ time. Therefore, Step 2(c) costs $O(1)$ time. Step 2(d) is the recursive call, so it takes $T(n-3)$ time. Clearly, Step 2(e) takes $O(1)$ time, since constructing $D$ is a matter of updating some flags.

From the analysis above we conclude that the cost $T(n)$ for the recursive part of our algorithm satisfies the recursive relation

$$T(n) = \begin{cases} T(n-3) + O(1), & n \geq 13 \\ O(1), & 3 \leq n \leq 12 \end{cases}$$

which yields $T(n) = O(n)$. Since initialization takes linear time, and our space requirements are obviously linear in the size of $P$ (we do not duplicate parts of $T_P$ for the recursive calls, but rather set appropriately the boundary flags for some half-edges), we arrive at the following theorem.

**Theorem 4.** *Given the triangulation graph $T_P$ of a polygon $P$ with $n \geq 3$ vertices, we can compute a diagonal 2-dominating set for $T_P$ of size at most $\lfloor \frac{n+1}{3} \rfloor$ in $O(n)$ time and space.*

## 3. 2-dominance of triangulation graphs: edge guards

Let $T_P$ be a triangulation graph of a polygon $P$, and let $u$ and $v$ be two nodes of $T_P$ connected via an edge $e$. The *contraction* of $e$ is a transformation that removes the nodes $u$ and $v$ and replaces them with a new node $x$, that is adjacent to every node that $u$ and $v$ was adjacent to. The contraction transformation can be used to prove the following lemma, which is the analogue of Lemma 3.2 in [29] in the context of 2-dominance.

**Lemma 5.** *Suppose $f(n)$ diagonal (resp., edge) guards are always sufficient to 2-dominate an $n$-node triangulation graph. If $T_P$ is an arbitrary triangulation graph of a polygon $P$, $v$ any vertex of $P$ and $e$ any of the two incident edges of $v$, then $T_P$ can be 2-dominated with $f(n-1)$ diagonal (resp., edge) guards, plus a vertex guard at $v$. Moreover, $e$, if specified, does not belong to the 2-dominating set of $T_P$.*

*Proof.* Let $u$ be the chosen vertex at which the guard is to be placed. If the edge $e$ is specified, let $v$ be the node adjacent to $u$ across $e$; otherwise, let $e$ be any of two the edges of $P$ incident to $u$, and $v$ the node adjacent to $u$ across $e$. Let $t_e$ be the triangle of $T_P$ adjacent to $e$ and let $w$ be the third vertex of $t_e$, besides $u$ and $v$. Edge contract $T_P$ across $e$, producing the triangulation graph $T'_P$ of $n-1$ nodes. Since $T'_P$ is a triangulation graph of a polygon (cf. Lemma 3.1 in [29]), it can be 2-dominated by $f(n-1)$ diagonal (resp., edge) guards.

Let $x$ be the node of $T'_P$ that replaced $u$ and $v$, and let $D'$ be the 2-dominating set of $T'_P$ consisting of $f(n-1)$ diagonal (resp., edge) guards. Suppose that no guard is placed at $x$, that is $x$ is not an endpoint of a edge or diagonal (resp., edge) in $D'$. Then $D = D' \cup \{u\}$ is a dominating set for $T_P$, since the guard at $u$ dominates $t_e$, whereas the remaining triangles of $T_P$ are dominated by edges or diagonals (resp. edges) in $D'$. Moreover, every triangle in $T_P$, except the triangles adjacent to $u$ or $v$, has two of its vertices in $D'$, and thus in $D$. Since $x$ is not in $D'$, all the vertices of $T'_P$ adjacent to $x$ have to be in $D'$. Hence, all triangles adjacent to $u$ or $v$, except $t_e$ have two of their vertices in $D'$ and thus in $D$. Finally, $t_e$ has also two vertices in $D$, namely $u$ and $w$. Thus, $D$ is a 2-dominating set for $T_P$.

Suppose now that a guard is used at $x$ in $D'$. If $xw$ is an edge or diagonal guard in $D'$, assign $xw$ to $vw$. Every other edge or diagonal guard $g$ in $D'$ incident to $x$, if any, becomes an edge or diagonal guard in $D$, incident to either $u$ or $v$, depending on whether $g$ is incident to $u$ or $v$ in $T_P$. As in the previous case, every triangle in $T_P$ is dominated and has at least two of its vertices in $D$. More precisely, every triangle in $T'_P$ not containing $x$ has two of its vertices in $D'$ and thus in $D$. Every triangle $t'$ in $T'_P$ containing $x$ is now a triangle in $T_P$ containing either $u$ or $v$ or both (this is the case for $t_e$). Therefore every triangle in $T_P$, except $t_e$, that contains $u$ or $v$ has one vertex in $D'$ plus either $u$ or $v$. Clearly, $t_e$ has both $u$ and $v$ in $D$. ☐

Before proceeding with the first main result of this section, let us state and prove an intermediate lemma concerning edge 2-dominating sets for small values of $n$.

**Lemma 6.** *Every triangulation graph $T_P$ with $3 \leq n \leq 9$ vertices, corresponding to a polygon $P$, can be 2-dominated by $\lfloor \frac{2n+1}{5} \rfloor$ edge guards, except for $n = 4$, where one additional guard is required.*

*Proof.* Let $v_i$, $1 \leq i \leq n$ be the vertices of $T_P$, and let $e_i$ be the edge $v_i v_{i+1}$. For each value of $n$ we are going to define an edge 2-dominating set $D$ of size $\lfloor \frac{2n+1}{5} \rfloor$.

$\boldsymbol{n \in \{3, 4, 5, 7\}}$. Set $D$ to be the set of edges of $P$ with odd index.

$\boldsymbol{n = 6}$. See proof of Lemma 2.

$\boldsymbol{n = 8}$. Let $t_1$ and $t_2$ be two ears of $T_P$ and consider their relative positions as shown in Fig. 10. In each case define the set $D$ as shown in Fig. 10.

$\boldsymbol{n = 9}$. Since $n \geq 6$, by means of Lemma 1 with $\lambda = 3$, there exists diagonal $d$ that partitions $T_P$ into two triangulation graphs $T_1$ and $T_2$, where $T_1$ contains $k$ boundary edges of $T_P$, $3 \leq k \leq 4$. Let $d \equiv d_{0k}$ be the common edge of $T_1$ and $T_2$, where $d_{ij}$ denotes the diagonal $v_i v_j$. Consider each of the two values of $k$ separately (see also Fig. 11):

$\boldsymbol{k = 3}$. Let $t$ be the triangle adjacent to the diagonal $d_{03}$ in $T_2$ and let $v$ be its apex. The cases $v \equiv v_4$, $v \equiv v_8$ and $v \equiv v_5$, $v \equiv v_7$ are symmetric, so we only need to consider the cases $v \in \{v_4, v_5, v_6\}$:

$\boldsymbol{v \equiv v_4}$. Let $t'$ be the triangle incident to $d_{04}$ in the hexagon $v_0 v_4 v_5 v_6 v_7 v_8$, and let $v'$ be its apex. Consider the subcases:

$\boldsymbol{v' \equiv v_5}$. Set $D = \{e_2, e_5, e_8\}$.

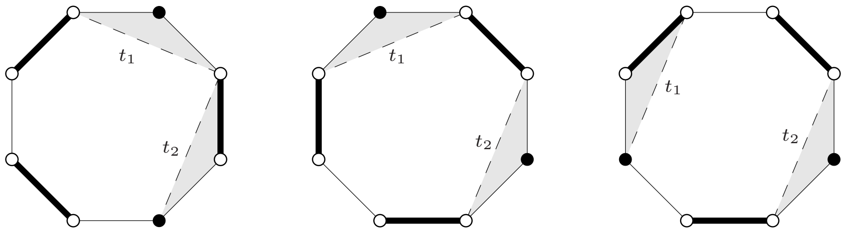


Figure 10: Proof of Lemma 6 for $n = 8$. The shaded triangles $t_1$ and $t_2$ are two ears of $T_P$. The subfigures correspond to the three possible relative positions of $t_1$ and $t_2$ in $T_P$.

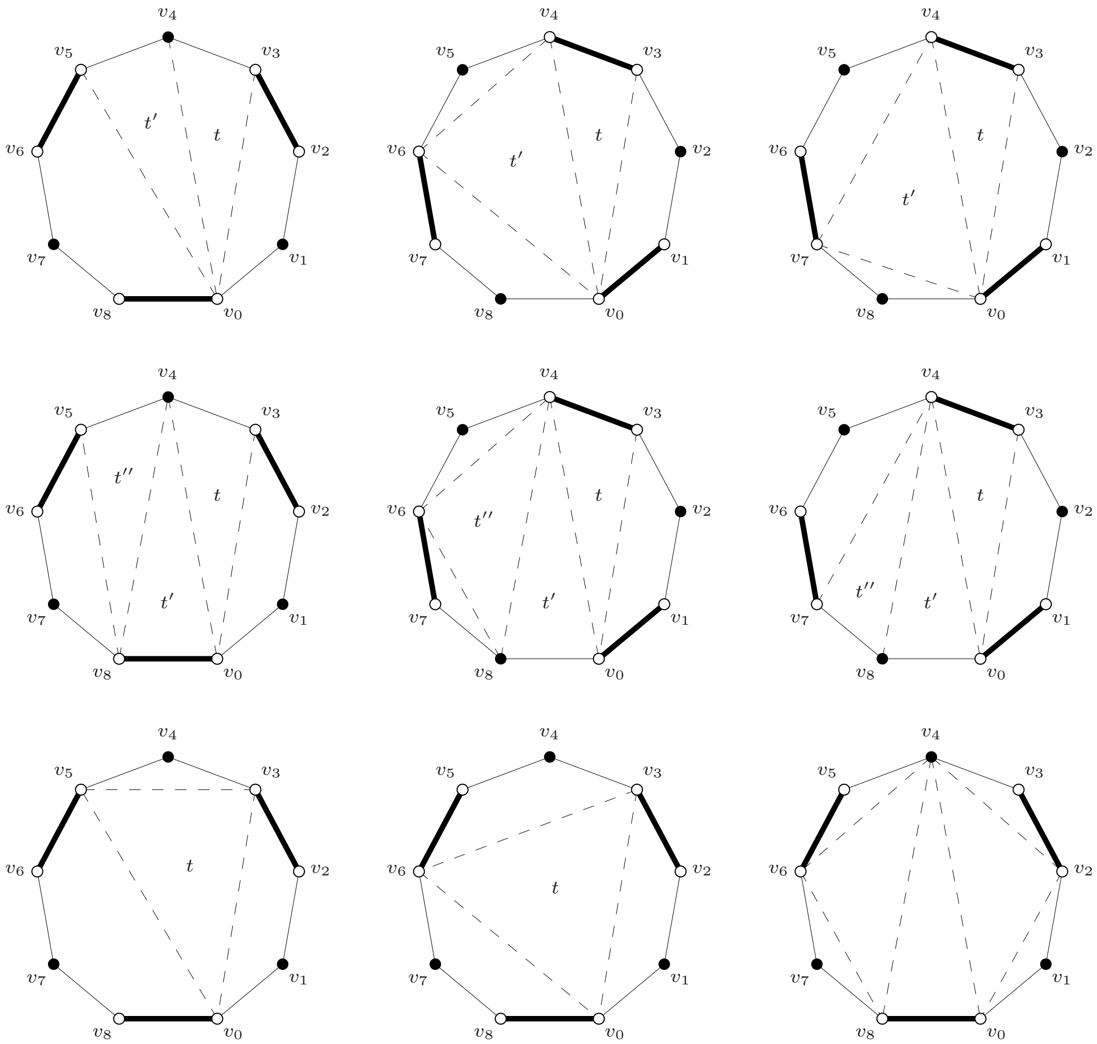


Figure 11: Proof of Lemma 6 for $n = 9$. Top row (left to right): $k = 3$, $v \equiv v_4$ and $v' \equiv v_5$; $k = 3$, $v \equiv v_4$ and $v' \equiv v_6$; $k = 3$, $v \equiv v_4$ and $v' \equiv v_7$. Middle row (left to right): $k = 3$, $v \equiv v_4$, $v' \equiv v_8$ and $v'' \equiv v_5$; $k = 3$, $v \equiv v_4$, $v' \equiv v_8$ and $v'' \equiv v_6$; $k = 3$, $v \equiv v_4$, $v' \equiv v_8$ and $v'' \equiv v_7$. Bottom row (left to right): $k = 3$ and $v \equiv v_5$; $k = 3$ and $v \equiv v_6$; $k = 4$.

$\boldsymbol{v' \in \{v_6, v_7\}}$. Set $D = \{e_0, e_3, e_6\}$.

$\boldsymbol{v' \equiv v_8}$. Let $t'' \neq t'$ be the triangle supported by $d_{48}$ and let $v''$ be its apex. If $v'' \equiv v_5$, set $D = \{e_2, e_5, e_8\}$. Otherwise, if $v'' \in \{v_6, v_7\}$, set $D = \{e_0, e_3, e_6\}$.

$\boldsymbol{v \in \{v_5, v_6\}}$. Set $D = \{e_2, e_5, e_8\}$.

$\boldsymbol{k = 4}$. By the minimality of $k$, the apex of the triangle supported by $d_{04}$ in $T_1$ must be $v_2$. Again, by the minimality of $k$, the diagonals $d_{47}$, $d_{58}$ and $d_{06}$ cannot exist. This implies that either $d_{48}$ or $d_{05}$ must belong to $T_P$. The two cases are symmetric, so we can assume, without loss of generality, that $d_{48} \in T_P$. Again, by the minimality of $k$, the diagonals $d_{46}$ and $d_{68}$ must be in $T_P$. In this case set $D = \{e_2, e_5, e_8\}$. □

In the next two theorems we state and prove the first two main results of this section concerning worst-case upper and lower bounds on the number of edge guards required in order to 2-dominate a triangulation graph.

**Theorem 7.** *Let $P$ be a polygon with $n \geq 3$ vertices and $T_P$ its triangulation graph. $\lfloor \frac{2n+1}{5} \rfloor$ edge guards are always sufficient in order to 2-dominate $T_P$, except for $n = 4$, where one additional guard is required.*

*Proof.* In Lemma 6, we have shown the result for $3 \leq n \leq 9$. Let us now assume that $n \geq 10$ and that the theorem holds for all $n'$ such that $5 \leq n' < n$. By means of Lemma 1 with $\lambda = 5$, there exists diagonal $d$ that partitions $T_P$ into two triangulation graphs $T_1$ and $T_2$, where $T_1$ contains $k$ boundary edges of $T_P$, $5 \leq k \leq 8$. Let $v_0, \ldots, v_k$ be the $k+1$ vertices of $T_1$, as we encounter them while traversing $P$ counterclockwise, and let $v_0 v_k$ be the common edge of $T_1$ and $T_2$. For each value of $k$ we are going to define an edge 2-dominating set $D$ for $T_P$ of size $\lfloor \frac{2n+1}{5} \rfloor$. In what follows $d_{ij}$ denotes the diagonal $v_i v_j$, whereas $e_i$ denotes the edge $v_i v_{i+1}$. Consider each of the four values of $k$ separately:

$\boldsymbol{k = 5}$. Let $t$ be the triangle supported by $d$ in $T_1$, and let $v$ be the apex of this triangle. $|T_2| = n - 4$, and by Lemma 5 there exists a 2-dominating set $D_0$ (resp., $D_5$) for $T_2$, consisting of $f(n-5)$ edge guards plus $v_0$ (resp., $v_5$), such that $d \notin D_0$ (resp., $d \notin D_5$). If $v \in \{v_3, v_4\}$, set $D = D_0 \cup \{e_0, e_3\}$. If $v \in \{v_1, v_2\}$, set $D = D_5 \cup \{e_1, e_4\}$ (see Fig. 12).

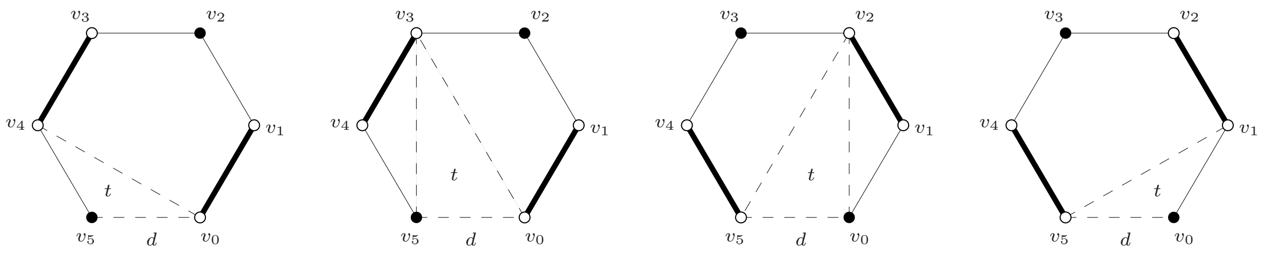

Figure 12: Proof of Theorem 7: the case $k = 5$. Left two: $v \in \{v_3, v_4\}$. Right two: $v \in \{v_1, v_2\}$.

$\boldsymbol{k = 6}$. The presence of diagonals $d_{05}$ or $d_{16}$ would violate the minimality of $k$. Let $t$ be the triangle supported by $d$ in $T_1$. The apex $v$ of this triangle should be $v_2$, $v_3$ or $v_4$. The cases $v \equiv v_2$ and $v \equiv v_4$ are symmetric, so we only consider the cases $v \equiv v_2$ and $v \equiv v_3$. Since $T_2$ has $n-5$ vertices, by our induction hypothesis we have that $T_2$ can be dominated with $f(n-5) = \lfloor \frac{2n+1}{5} \rfloor - 2$ edge guards. Let $D_2$ be the edge 2-dominating set for $T_2$. Consider the following cases (see also Fig. 13):

$\boldsymbol{d_{06} \in D_2}$. Set $D = (D_2 \setminus \{d_{06}\}) \cup \{e_0, e_2, e_5\}$.

$\boldsymbol{d_{06} \notin D_2}$. Since $D_2$ is a 2-dominating set for $T_2$, either $v_0$ or $v_6$ belongs to $D_2$. If $v_0 \in D_2$, set $D = D_2 \cup \{e_2, e_4\}$. Otherwise, $v_6 \in D_2$, in which case set $D = D_2 \cup \{e_1, e_3\}$.

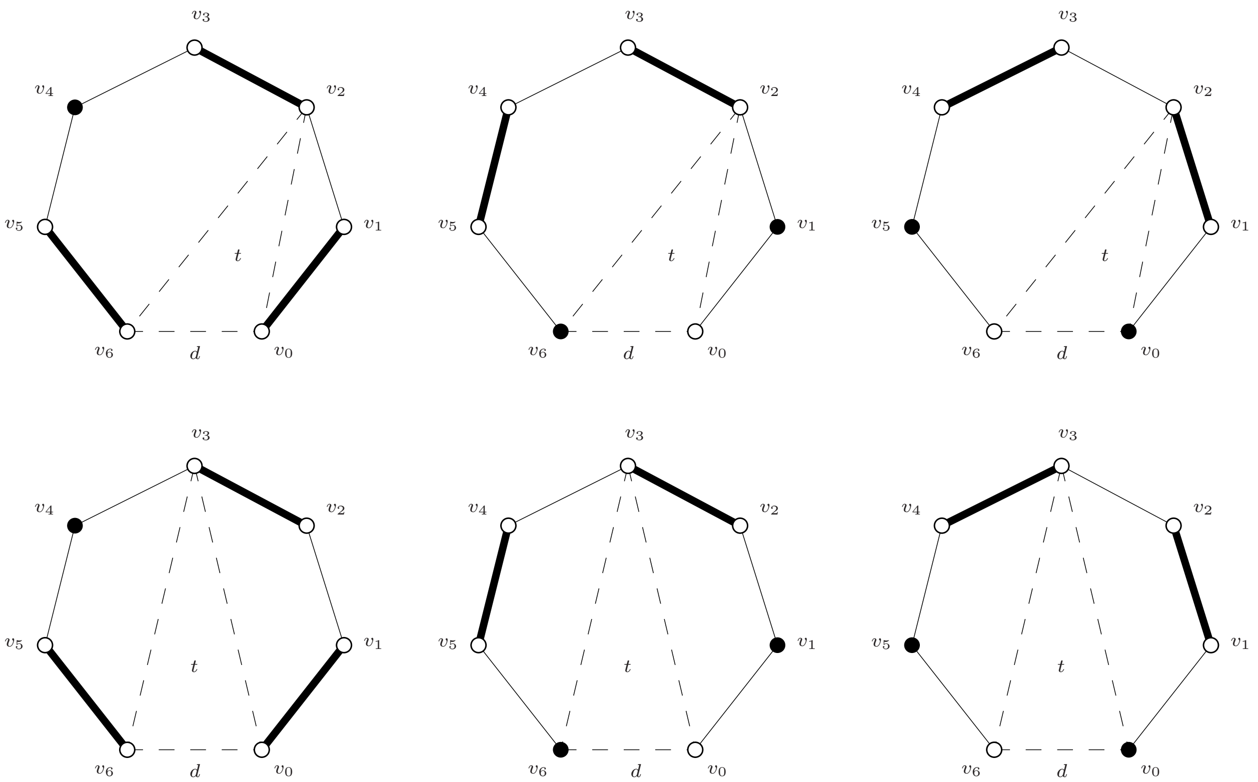


Figure 13: Proof of Theorem 7: the case $k = 6$. Top row: the apex of $t$ is $v_2$. Bottom row: the apex of $t$ is $v_3$. Left column: $d_{06} \in D_2$. Middle column: $d_{06} \notin D_2, v_0 \in D_2$. Right column: $d_{06} \notin D_2, v_6 \in D_2$.

***k* = 7.** The presence of diagonals $d_{06}$, $d_{05}$, $d_{17}$ or $d_{27}$ would violate the minimality of $k$. Let $t$ be the triangle supported by $d$ in $T_1$. The apex $v$ of this triangle is either $v_3$ or $v_4$. The two cases are symmetric, so we can assume without loss of generality that the apex of $t$ is $v_3$ (see Fig. 14). Consider the triangulation graph $T' = T_2 \cup \{t\}$. It has $n - 5$ vertices and, by our induction hypothesis, it can be 2-dominated with $f(n-5) = \lfloor \frac{2n+1}{5} \rfloor - 2$ edge guards. Let $D'$ be the 2-dominating set of $T'$. Consider the following two cases:

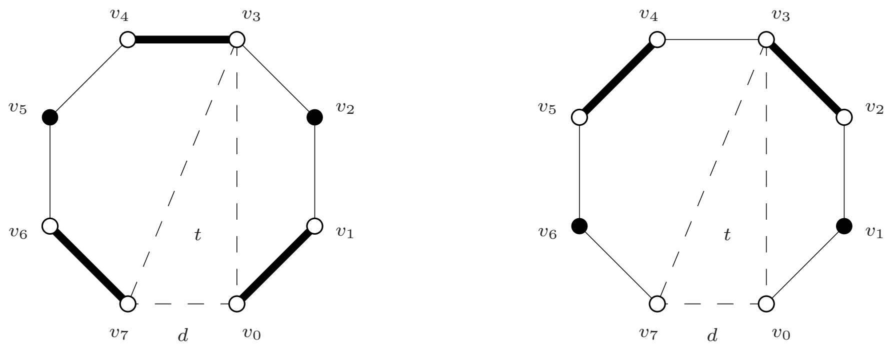


Figure 14: Proof of Theorem 7: the case $k = 7$. Left: $|D' \cap \{d_{03}, d_{37}\}| \geq 1$. Right: $d_{03}, d_{37} \notin D'$.

$|\boldsymbol{D'} \cap \{\boldsymbol{d_{03}}, \boldsymbol{d_{37}}\}| \geq \mathbf{1}.$ Set $D = (D' \setminus \{d_{03}, d_{37}\}) \cup \{e_0, e_3, e_6\}$.

$\boldsymbol{d_{03}}, \boldsymbol{d_{37}} \notin \boldsymbol{D'}$. In this case $v_3$ cannot be in $D'$, since either $d_{03}$ or $d_{37}$ would have to be in $D'$. This

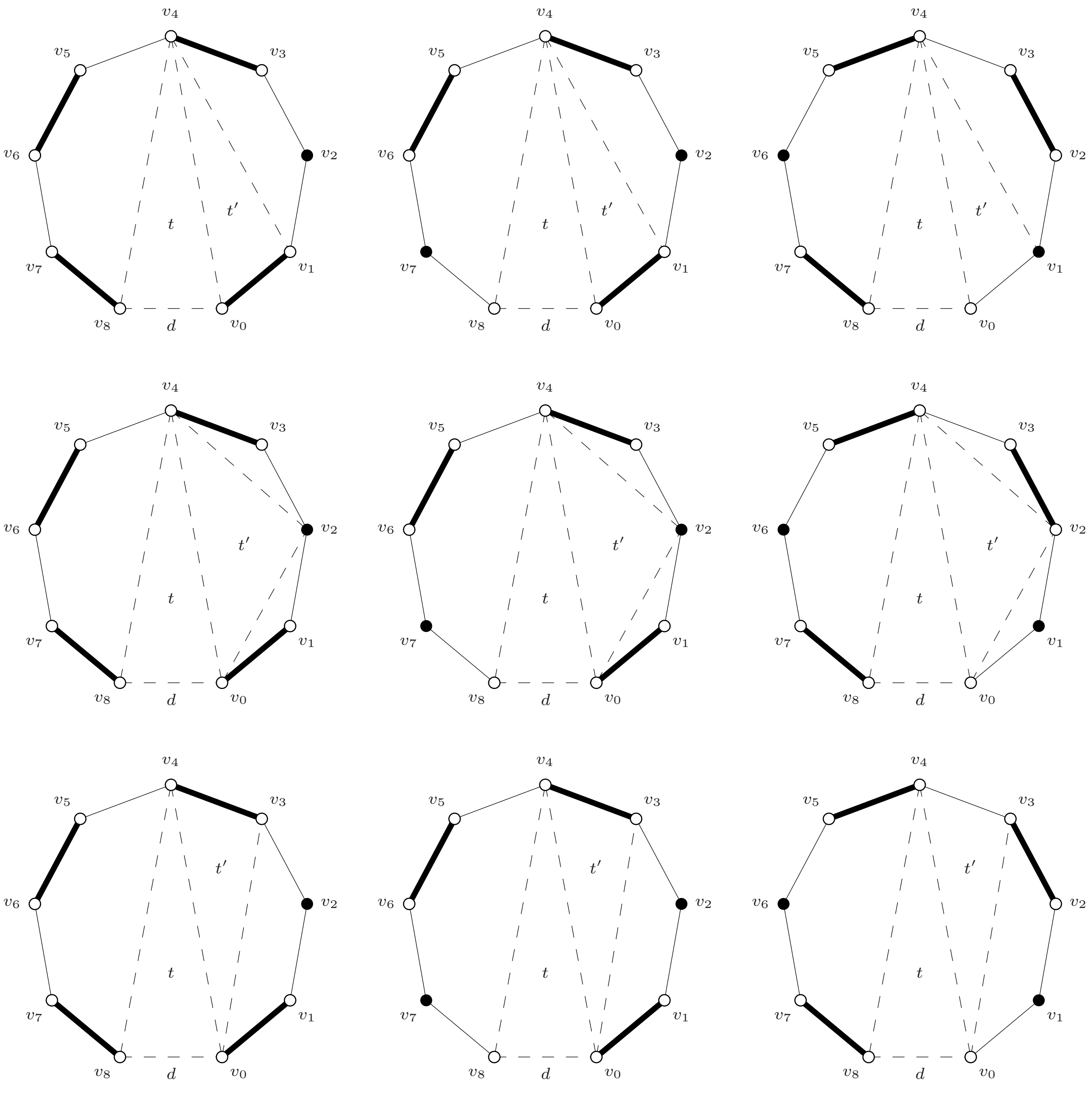


Figure 15: Proof of Theorem 7: the case $k = 8$. Rows (top to bottom): $v' \equiv v_1$; $v' \equiv v_2$; $v' \equiv v_3$. Top row (left to right): $d_{14}, d_{48} \in D'$; $d_{14} \in D', d_{48} \notin D'$, $v_8 \in D'$, and also $d_{14}, d_{48} \notin D'$; $d_{14} \in D', d_{48} \notin D'$, $v_0 \in D'$, and also $d_{14} \notin D', d_{48} \in D'$. Middle row (left to right): $|\{d_{02}, d_{24}, d_{48}\} \cap D'| \geq 2$; $d_{02} \in D', d_{24}, d_{48} \notin D'$, and also $d_{24} \in D', d_{02}, d_{48} \notin D'$, $v_8 \in D'$; $d_{24} \in D', d_{02}, d_{48} \notin D'$, $v_0 \in D'$, and also $d_{48} \in D', d_{02}, d_{24} \notin D'$. Bottom row (left to right): $d_{03}, d_{48} \in D'$, and also $d_{03} \in D', d_{48} \notin D'$, $e_3 \in D'$, as well as $d_{03} \notin D', d_{48} \in D'$, $e_3 \in D'$; $d_{03} \in D', d_{48} \notin D'$, $e_3 \notin D'$, and also $d_{03}, d_{48} \notin D'$, $v_8 \in D'$; $d_{03} \notin D', d_{48} \in D'$, $e_3 \notin D'$, and also $d_{03}, d_{48} \notin D'$, $v_0 \in D'$.

implies that both $v_0$ and $v_7$ have to be in $D'$ (2-dominance of $t$). Set $D = D' \cup \{e_2, e_4\}$.

$\boldsymbol{k = 8}$**.** The presence of diagonals $d_{07}$, $d_{06}$, $d_{05}$, $d_{18}$, $d_{28}$ or $d_{38}$ would violate the minimality of $k$. Thus, the apex of the triangle $t$ in $T_1$ that is supported by $d$ is $v_4$. Let $t' \neq t$ be the triangle incident to $d_{04}$, and let $v'$ be its vertex opposite $d_{04}$. Clearly, $v' \in \{v_1, v_2, v_3\}$. Consider the triangulation graph $T' = T_2 \cup \{t, t'\}$. It has $n-5$ vertices and, by our induction hypothesis, it can be 2-dominated with $f(n-5) = \lfloor \frac{2n+1}{5} \rfloor - 2$ edge guards. Let $D'$ be the 2-dominating set of $T'$. Consider the following cases (see also Fig. 15):

$\boldsymbol{v' \equiv v_1}$**.** Consider the following subcases:

$\boldsymbol{d_{14}, d_{48} \in D'}$**.** Set $D = (D' \setminus \{d_{14}, d_{48}\}) \cup \{e_0, e_3, e_5, e_7\}$.

$\boldsymbol{d_{14} \in D', d_{48} \notin D'}$**.** If $v_8 \in D'$, set $D = (D' \setminus \{d_{14}\}) \cup \{e_0, e_3, e_5\}$. Otherwise, $v_0 \in D'$ (2-dominance of $t$), in which case set $D = (D' \setminus \{d_{14}\}) \cup \{e_2, e_4, e_7\}$.

$\boldsymbol{d_{14} \notin D', d_{48} \in D'}$**.** In this case either $v_0$ or $v_1$ belongs to $D'$ (2-dominance of $t'$). Since $d_{14} \notin D'$, we must have that either $v_0 \in D'$ or $e_0 \in D'$, which implies, in either case, that $v_0 \in D'$. Hence, set $D = (D' \setminus \{d_{48}\}) \cup \{e_2, e_4, e_7\}$.

$\boldsymbol{d_{14}, d_{48} \notin D'}$**.** In this case $v_4 \notin D'$, which implies that $v_0, v_1, v_8 \in D'$. But then $e_0 \in D'$. Therefore, set $D = D' \cup \{e_3, e_5\}$.

$\boldsymbol{v' \equiv v_2}$**.** Notice that in this case it is not possible that $d_{02}, d_{24}, d_{48} \notin D'$, since then $v_2, v_4 \notin D'$, which contradicts the 2-dominance of $t'$ by $D'$ in $T'$. Consider the remaining subcases:

$\boldsymbol{|\{d_{02}, d_{24}, d_{48}\} \cap D'| \geq 2}$**.** Set $D = (D' \setminus \{d_{02}, d_{24}, d_{48}\}) \cup \{e_0, e_3, e_5, e_7\}$.

$\boldsymbol{d_{02} \in D', d_{24}, d_{48} \notin D'}$**.** Then $v_4 \notin D'$, which implies that $v_8 \in D'$ (2-dominance of $t$). Set $D = (D' \setminus \{d_{02}\}) \cup \{e_0, e_3, e_5\}$.

$\boldsymbol{d_{24} \in D', d_{02}, d_{48} \notin D'}$**.** $v_0$ or $v_8$ belongs to $D'$ (2-dominance of $t$). If $v_0 \in D'$, set $D = (D' \setminus \{d_{24}\}) \cup \{e_2, e_4, e_7\}$. Otherwise, if $v_8 \in D'$, set $D = (D' \setminus \{d_{24}\}) \cup \{e_0, e_3, e_5\}$.

$\boldsymbol{d_{48} \in D', d_{02}, d_{24} \notin D'}$**.** Then $v_2 \notin D'$, which implies that $v_0 \in D'$ (2-dominance of $t'$). Set $D = (D' \setminus \{d_{48}\}) \cup \{e_2, e_4, e_7\}$.

$\boldsymbol{v' \equiv v_4}$**.** Consider the following subcases:

$\boldsymbol{d_{03}, d_{48} \in D'}$**.** Set $D = (D' \setminus \{d_{03}, d_{48}\}) \cup \{e_0, e_3, e_5, e_7\}$.

$\boldsymbol{d_{03} \in D', d_{48} \notin D'}$**.** If $e_3 \in D'$, set $D = (D' \setminus \{d_{03}\}) \cup \{e_0, e_5, e_7\}$. Otherwise, $v_4 \notin D'$, i.e., both $v_0$ and $v_8$ belong to $D'$. Set $D = (D' \setminus \{d_{03}\}) \cup \{e_0, e_3, e_5\}$.

$\boldsymbol{d_{03} \notin D', d_{48} \in D'}$**.** If $e_3 \in D'$, set $D = (D' \setminus \{d_{48}\}) \cup \{e_0, e_5, e_7\}$. Otherwise, $v_3 \notin D'$, i.e., $v_0$ belongs to $D'$ (2-dominance of $t'$). Set $D = (D' \setminus \{d_{48}\}) \cup \{e_2, e_4, e_7\}$.

$\boldsymbol{d_{03}, d_{48} \notin D'}$**.** Since $d_{03}, d_{48} \notin D'$, $t'$ can be 2-dominated in $D'$ only if $e_3 \in D'$. Now, if $v_8 \in D'$, set $D = D' \cup \{e_0, e_5\}$; otherwise, i.e., if $v_8 \notin D'$, $v_0$ has to be in $D'$, in which case set $D = (D' \setminus \{e_3\}) \cup \{e_2, e_4, e_7\}$. □

**Theorem 8.** *There exists a family of triangulation graphs with $n \geq 3$ vertices any edge 2-dominating set of which has cardinality at least $\lfloor \frac{2n+1}{5} \rfloor$, except for $n = 4$, where any edge 2-dominating set has cardinality at least 2.*

*Proof.* Our claim is trivial for $n \in \{3, 4\}$. We are first going to prove the lower bound for all $n = 5m + k$, where $m \geq 1$ and $k \in \{0, 1, 3, 4\}$. The case $n = 5m + 2$, for $m \geq 1$, is a bit more complicated and is dealt with separately.

Consider the triangulation graphs $\Gamma_{5m}$, $\Gamma_{5m+1}$, $\Gamma_{5m+3}$ and $\Gamma_{5m+4}$, $m \geq 1$, shown in Fig. 16. The central part of these graphs is triangulated arbitrarily. $\Gamma_{5m+i}$, $i = 0, 1, 3, 4$, consists of $n = 5m + i$ vertices, and requires a minimum of two edge guards per hexagon shown in light gray (this is true even if the two vertices of these hexagons that also belong to the neighboring shaded polygons are in the 2-dominating set due to edges of these polygons). Moreover, $\Gamma_{5m}$ and $\Gamma_{5m+1}$ require two more edge guards for the hexagon and heptagon, respectively, shown in dark gray, whereas $\Gamma_{5m+3}$ and $\Gamma_{5m+4}$ require three more edge guards for

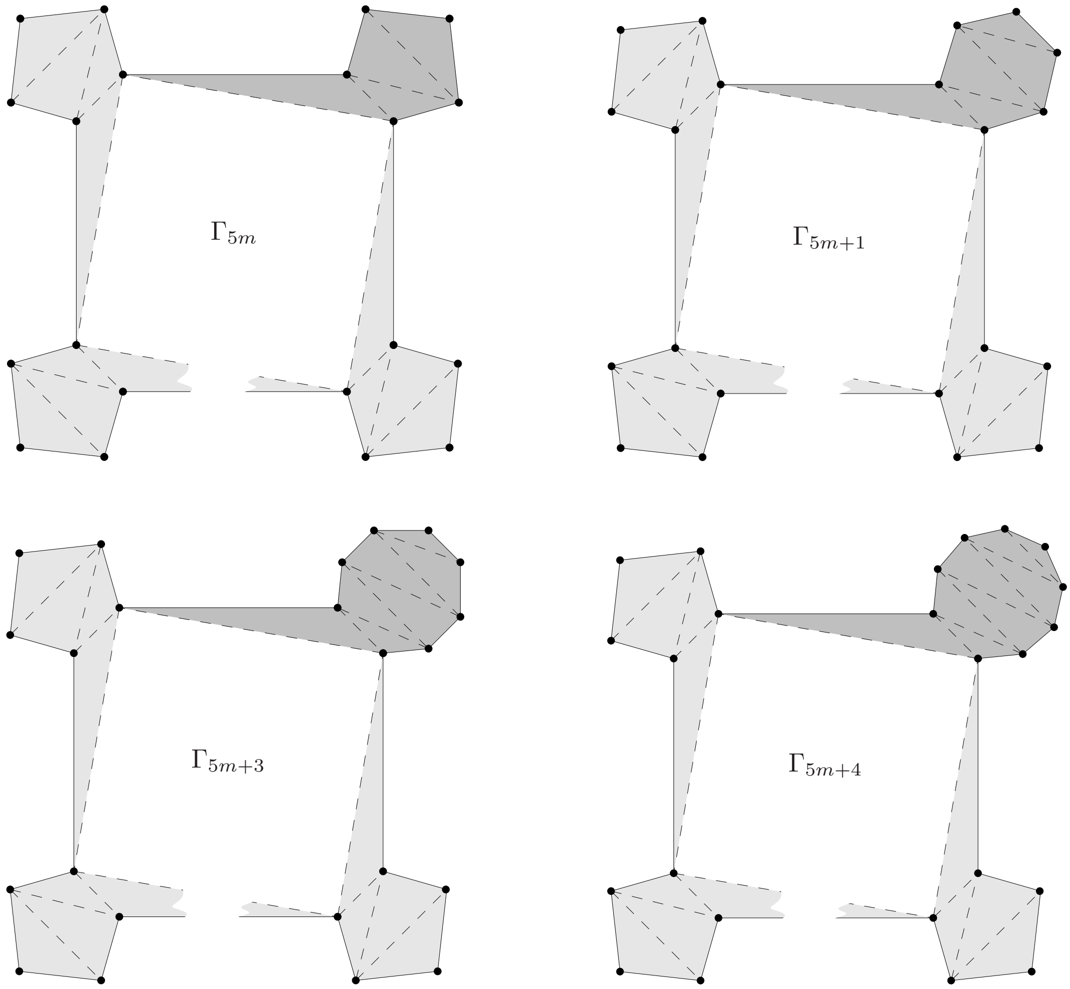


Figure 16: The triangulation graphs $\Gamma_{5m+i}$, $i = 0, 1, 3, 4$, with $n = 5m + i$ vertices, respectively (the central parts of the graphs are triangulated arbitrarily). All four triangulation graphs require at least $\lfloor \frac{2n+1}{5} \rfloor$ edge guards in order to be 2-dominated.

the enneagon and decagon shown in dark gray (this is true even if the two vertices of these polygons that also belong to the neighboring shaded polygons are in the 2-dominating set due to edges of these polygons). Hence, $\Gamma_{5m}$, $\Gamma_{5m+1}$, $\Gamma_{5m+3}$ and $\Gamma_{5m+4}$ require $\lfloor \frac{2n+1}{5} \rfloor$ edge guards in order to be 2-dominated.

To prove the lower bound for all remaining $n \geq 7$, we are going to inductively construct a family of triangulation graphs $\Gamma_{5m+2}$, $m \geq 1$, as follows. The triangulation graph $\Gamma_7$ is shown in Fig. 17(top left). $\Gamma_{12}$ is constructed by *gluing* two copies $\Gamma_7'$ and $\Gamma_7''$ of $\Gamma_7$ along the edge $e_0$ of $\Gamma_7'$ and the edge $e_6$ of $\Gamma_7''$, such that the vertex $v_0$ (resp., $v_1$) of $\Gamma_7'$ is identified with the vertex $v_0$ (resp., $v_6$) of $\Gamma_7''$ (see Fig. 17(top right)). In $\Gamma_{12}$, $v_0$ is the vertex that used to be $v_0$ in both $\Gamma_7'$ and $\Gamma_7''$, while all other vertices are numbered in the counterclockwise sense. $\Gamma_{5m+7}$, $m \geq 2$, is constructed by gluing $\Gamma_{5m+2}$ with $\Gamma_7$ along the edge $e_0$ of $\Gamma_{5m+2}$ and the edge $e_6$ of $\Gamma_7$, such that the vertex $v_0$ (resp., $v_1$) of $\Gamma_{5m+2}$ is identified with the vertex $v_0$ (resp., $v_6$) of $\Gamma_7$ (see Fig. 17(bottom row) for $\Gamma_{17}$ and $\Gamma_{22}$). In $\Gamma_{5m+7}$, $v_0$ is the vertex that used to be $v_0$ in both $\Gamma_{5m+2}$ and $\Gamma_7$, while all other vertices are numbered in the counterclockwise sense.

We are now ready to proceed with our proof of the lower bound for the triangulation graphs $\Gamma_{5m+2}$, $m \geq 1$. More precisely, we will show, by induction on $m$, that every edge 2-dominating set of the triangulation

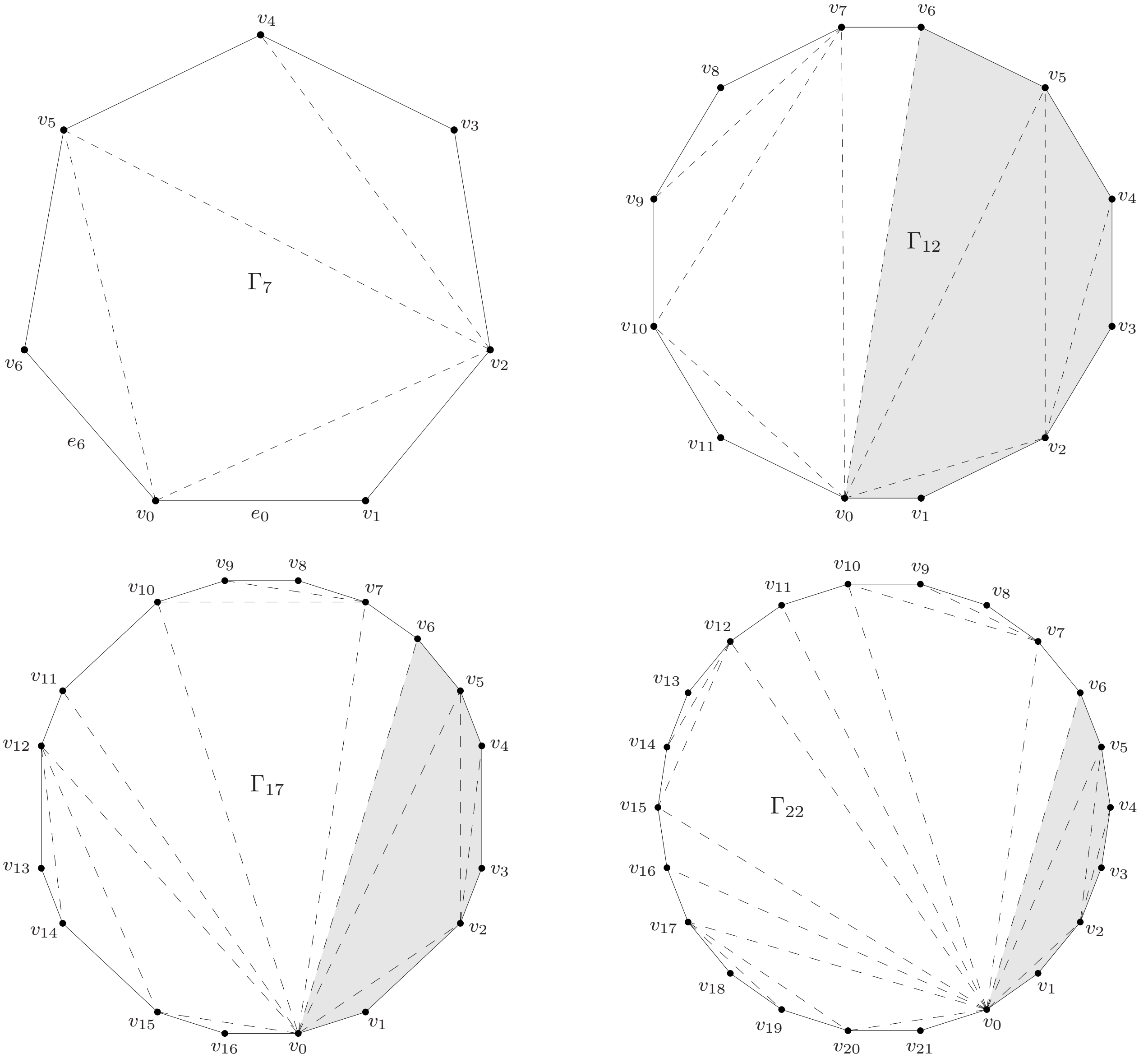


Figure 17: The triangulation graphs $\Gamma_7$, $\Gamma_{12}$, $\Gamma_{17}$ and $\Gamma_{22}$, with $n = 7, 12, 17$ and 22 vertices, respectively. Each of these graphs requires $\lfloor \frac{2n+1}{5} \rfloor$ edge guards in order to be 2-dominated. The shaded part of the graph $\Gamma_n$, $n = 12, 17, 22$, corresponds to the graph $\Gamma_7$ that is glued to $\Gamma_{n-5}$ in order to construct $\Gamma_n$.

graph $\Gamma_{5m+2}$ has size at least $2m + 1$. We start by the base case, i.e., $m = 1$. $\Gamma_7$ cannot be 2-dominated by less than three edges, since then we would be able to find an edge $e$ of $\Gamma_7$ such that its two endpoints are not in the edge 2-dominating set of $\Gamma_7$, and thus the triangle of $\Gamma_7$ incident to $e$ would not be 2-dominated. Let us now assume that our claim holds true for some $m \geq 1$, i.e., every edge 2-dominating set of $\Gamma_{5m+2}$ has size at least $2m + 1$.

Consider the triangulation graph $\Gamma_{5m+7}$. Let $D$ be an edge 2-dominating set for $\Gamma_{5m+7}$, and let us assume that $|D| < 2(m + 1) + 1$, i.e., $|D| \leq 2m + 2$. Let $T_1$ and $T_2$ be the triangulation graphs that we get by *cutting* $\Gamma_{5m+7}$ along the diagonal $d_{06}$, with $T_2$ being the one containing the vertex $v_1$ (see Fig. 18(left)), and, moreover, let $T_3$ and $T_4$ be the triangulation graphs that we get by cutting $\Gamma_{5m+7}$ along

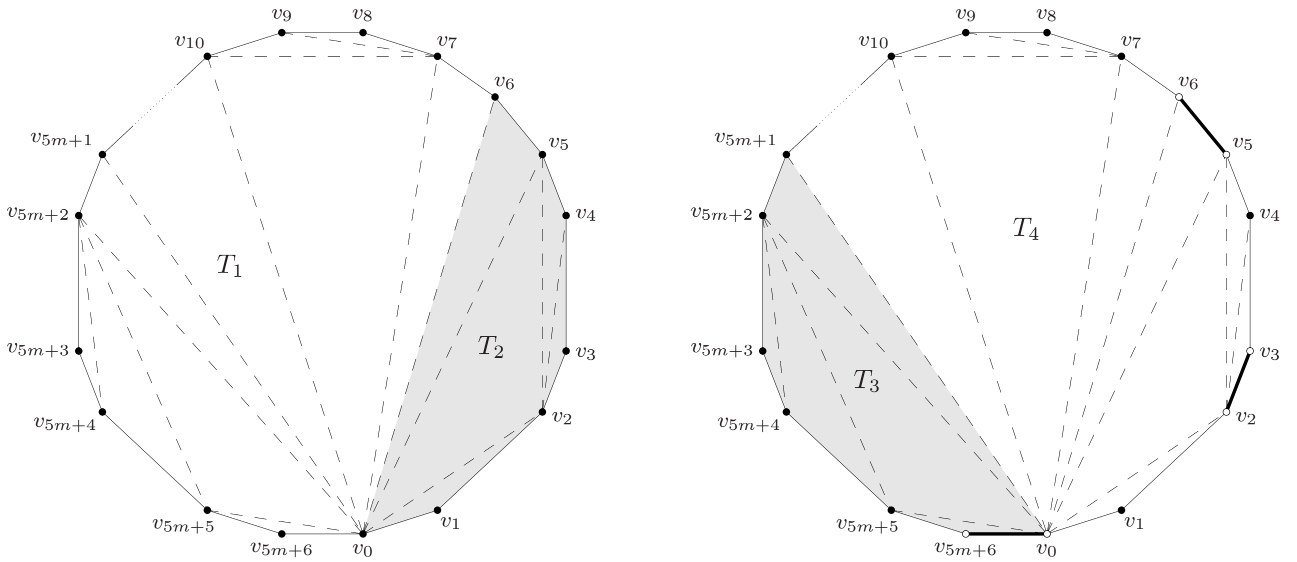


Figure 18: The triangulation graph $\Gamma_{5m+7}$. The shaded subgraphs of $\Gamma_{5m+7}$ are the triangulation graphs $T_2$ and $T_3$, while the non-shaded subgraphs of $\Gamma_{5m+7}$ are the triangulation graph $T_1$ and $T_4$, respectively. In the right subfigure we also depict some of the edges in $D$ when $D_2 = \{e_2, e_5\}$.

the diagonal $d_{0,5m+1}$, with $T_4$ being the one containing the vertex $v_1$ (see Fig. 18(right)). Notice that $T_1$ and $T_4$ (resp., $T_2$ and $T_3$) are isomorphic to $\Gamma_{5m+2}$ (resp., $\Gamma_7$). Let $D_1$ (resp., $D_2$) be the subset of $D$ containing the edges of $D$ in $T_1$ (resp., $T_2$), and define $D_3$ and $D_4$ analogously. Finally, notice that the sets $D_1' = D_1 \cup \{d_{06}\}$ and $D_4' = D_4 \cup \{d_{0,5m+1}\}$ are edge 2-dominating sets of $T_1$ and $T_4$, respectively. It is easy to verify that $|D_2| \geq 2$ (resp., $|D_3| \geq 2$), since otherwise we would be able to find an edge in $\{e_1, e_2, e_3, e_4\}$ (resp., $\{e_{5m+2}, e_{5m+3}, e_{5m+4}, e_{5m+5}\}$) such that its two endpoints are not endpoints of edges in $D$; notice that this is true even if both $v_0$ and $v_6$ (resp., $v_0$ and $v_{5m+1}$) belong to $D$ due to edges in $D_1$ (resp., $D_4$). Consider the following cases:

$\boldsymbol{|D_2| \geq 3.}$ In this case we have $|D_1| = |D| - |D_2| \leq (2m+2) - 3 = 2m - 1$, which further implies that $|D_1'| = |D_1| + 1 \leq 2m$. This contradicts our inductive assumption, since $D_1'$ is an edge 2-dominating set of $T_1$, and thus of $\Gamma_{5m+2}$.

$\boldsymbol{|D_2| = 2.}$ In this case $|D_1| = |D| - |D_2| \leq (2m+2) - 2 = 2m < 2m+1$. Observe that $D_2$ can only be one of the following four subsets of $\{e_0, e_1, \ldots, e_5\}$ of size two: $\{e_1, e_3\}$, $\{e_1, e_4\}$, $\{e_2, e_4\}$ and $\{e_2, e_5\}$. All other subsets of size two of $\{e_0, e_1, \ldots, e_5\}$, except $\{e_0, e_3\}$, are such that there exists an edge in $\{e_1, e_2, e_3, e_4\}$ with the property that its two endpoints are not endpoints of edges in $D$. Lastly, if $D_2$ was equal to $\{e_0, e_3\}$, the triangle $v_0v_2v_5$ would not be 2-dominated by $D$. Consider the following subcases:

$\boldsymbol{D_2 \in \{\{e_1, e_3\}, \{e_1, e_4\}, \{e_2, e_4\}\}.}$ Refer to Fig. 18(left). Notice that none of the vertices of edges in $D_2$ is a vertex of a triangle in $T_1$, i.e., the vertices of edges in $D_2$ do not contribute to the 2-domination of triangles in $T_1$. This further implies that the triangles in $T_1$ are essentially 2-dominated by the edges in $D_1$, which suggests the existence of an edge 2-dominating set for $\Gamma_{5m+2}$ of size $|D_1| = 2m < 2m+1$, a contradiction with respect to our inductive hypothesis.

$\boldsymbol{D_2 = \{e_2, e_5\}.}$ Refer to Fig. 18(right). In order for the triangle $v_0v_1v_2$ to be 2-dominated we must have that $e_{5m+6} \in D_1$, and, more importantly, that $e_{5m+6} \in D_3$. Recall that $|D_3| \geq 2$; we argue that in this case $|D_3| \geq 3$. To verify that, suppose that $|D_3| = 2$. Then the unique edge in $D_3 \setminus \{e_{5m+6}\}$ cannot be one of $e_{5m+1}$, $e_{5m+2}$, $e_{5m+4}$ or $e_{5m+5}$, since then we would be able to find an edge in $\{e_{5m+2}, e_{5m+3}, e_{5m+4}, e_{5m+5}\}$, such that its two endpoints are not endpoints of edges in $D$; moreover, if the unique edge in $D_3 \setminus \{e_{5m+6}\}$ is $e_{5m+3}$, the triangle $v_0v_{5m+2}v_{5m+5}$ is not 2-dominated by $D$. Since $|D_3| \geq 3$, we get that the size of $D_4$ has to be $|D_4| = |D| - |D_3| \leq (2m+2) - 3 = 2m - 1$, which gives that $|D_4'| = |D_4| + 1 \leq 2m$. As for the case $|D_2| \geq 3$ above, the bound on the size of $|D_4'|$ contradicts our inductive assumption, since $D_4'$ is an edge 2-dominating set of $T_4$, and thus of $\Gamma_{5m+2}$. □

### *3.1. Computing edge 2-dominating sets in linear time*

Unlike the case of diagonal 2-dominating sets, the proof of Theorem 7 uses edge contractions, which yields an $O(n^2)$ time and $O(n)$ space algorithm. A linear time and space algorithm is, however, feasible by relaxing the requirement on the size of the edge 2-dominating set. More precisely, we prove in this subsection that we can 2-dominate a triangulation graph with $\lfloor \frac{3n}{7} \rfloor$ edge guards. Although this result is weaker with respect to the result of Theorem 7, the proof technique is analogous to the technique in the proof of Theorem 3, i.e., it does not use edge contractions. Consequently, in analogy to the considerations of Section 2.1, we can devise a linear time and space algorithm for computing an edge 2-dominating set of size at most $\lfloor \frac{3n}{7} \rfloor$.

**Theorem 9.** *Every triangulation graph $T_P$ of a polygon $P$ with $n \geq 3$ vertices can be 2-dominated by $\lfloor \frac{3n}{7} \rfloor$ edge guards, except for $n = 4$, where one additional guard is required.*

*Proof.* By Theorem 7, and since $\lfloor \frac{2n+1}{5} \rfloor = \lfloor \frac{3n}{7} \rfloor$ for all $3 \leq n \leq 11$, we conclude that our theorem holds true for all $n$, with $3 \leq n \leq 11$.

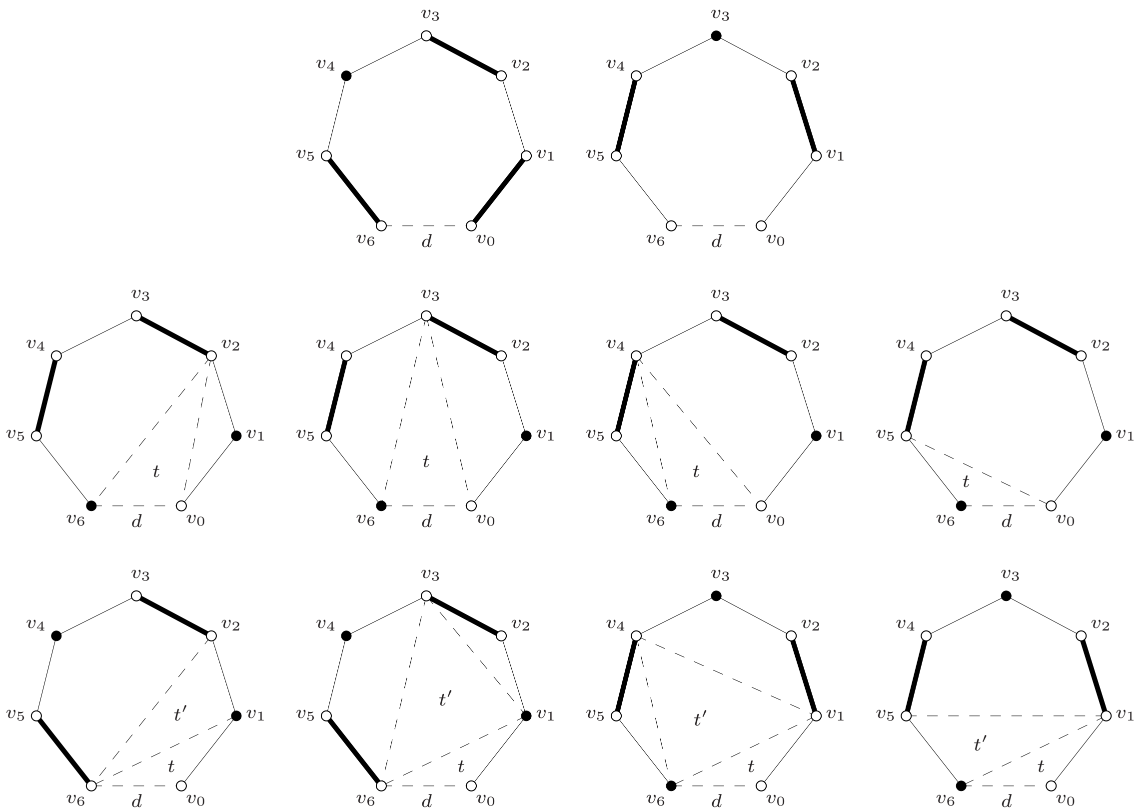


Figure 19: Proof of Theorem 9: the case $k = 6$. Top row (left to right): $d_{06} \in D_2$; $d_{06} \notin D_2$, $v_0, v_6 \in D_2$. Middle row (left to right): $d_{06} \notin D_2$, $v_0 \in D_2, v_6 \notin D_2$ and $v \in \{v_2, v_3, v_4, v_5\}$. Bottom row (left to right): $d_{06} \notin D_2$, $v_0 \in D_2, v_6 \notin D_2$, $v \equiv v_1$ and $v' \in \{v_2, v_3, v_4, v_5\}$.

Let us now assume that $n \geq 12$ and that the theorem holds for all $n'$ such that $5 \leq n' < n$. By Lemma 1 with $\lambda = 6$, there exists a diagonal $d$ that partitions $T_P$ into two triangulation graphs $T_1$ and $T_2$, where $T_1$ contains $k$ boundary edges of $T_P$ with $6 \leq k \leq 10$. Let $v_i$, $0 \leq i \leq k$, be the $k+1$ vertices of $T_1$, as we encounter them while traversing $P$ counterclockwise, and let $v_0 v_k$ be the common edge of $T_1$ and $T_2$. For each value of $k$ we are going to define an edge 2-dominating set $D$ for $T_P$ of size $\lfloor \frac{3n}{7} \rfloor$. In what follows $d_{ij}$ denotes the diagonal $v_i v_j$, whereas $e_i$ denotes the edge $v_i v_{i+1}$. Consider each value of $k$ separately.

$\boldsymbol{k = 6.}$ In this case $T_2$ contains $n-5$ vertices. By our induction hypothesis we can dominate $T_2$ with $f(n-5) \leq \lfloor \frac{3n}{7} \rfloor - 2$ edge guards. Let $D_2$ be the edge 2-dominating set for $T_2$. Consider the following cases: (see Fig. 19):

$\boldsymbol{d_{06} \in D_2.}$ Set $D = (D_2 \setminus \{d_{06}\}) \cup \{e_0, e_2, e_5\}$.

$\boldsymbol{d_{06} \notin D_2.}$ Since $T_2$ is 2-dominated by $D_2$, at least one of the vertices $v_0$ and $v_6$ belongs to $D_2$. We distinguish between the following subcases:

$\boldsymbol{v_0, v_6 \in D_2.}$ Set $D = D_2 \cup \{e_1, e_4\}$.

$\boldsymbol{v_0 \in D_2, v_6 \notin D_2.}$ Let $t$ be the triangle supported by $d$ in $T_1$ and let $v$ be its vertex opposite to $d$. If $v \in \{v_2, v_3, v_4, v_5\}$, set $D = D_2 \cup \{e_2, e_4\}$. If $v \equiv v_1$, let $t'$ be the second triangle supported by $d_{16}$ beyond the triangle $t$, and let $v'$ be its vertex opposite to $d_{16}$. If $v' \in \{v_2, v_3\}$, set $D = D_2 \cup \{e_2, e_5\}$. Otherwise, i.e., if $v' \in \{v_4, v_5\}$, set $D = D_2 \cup \{e_1, e_4\}$.

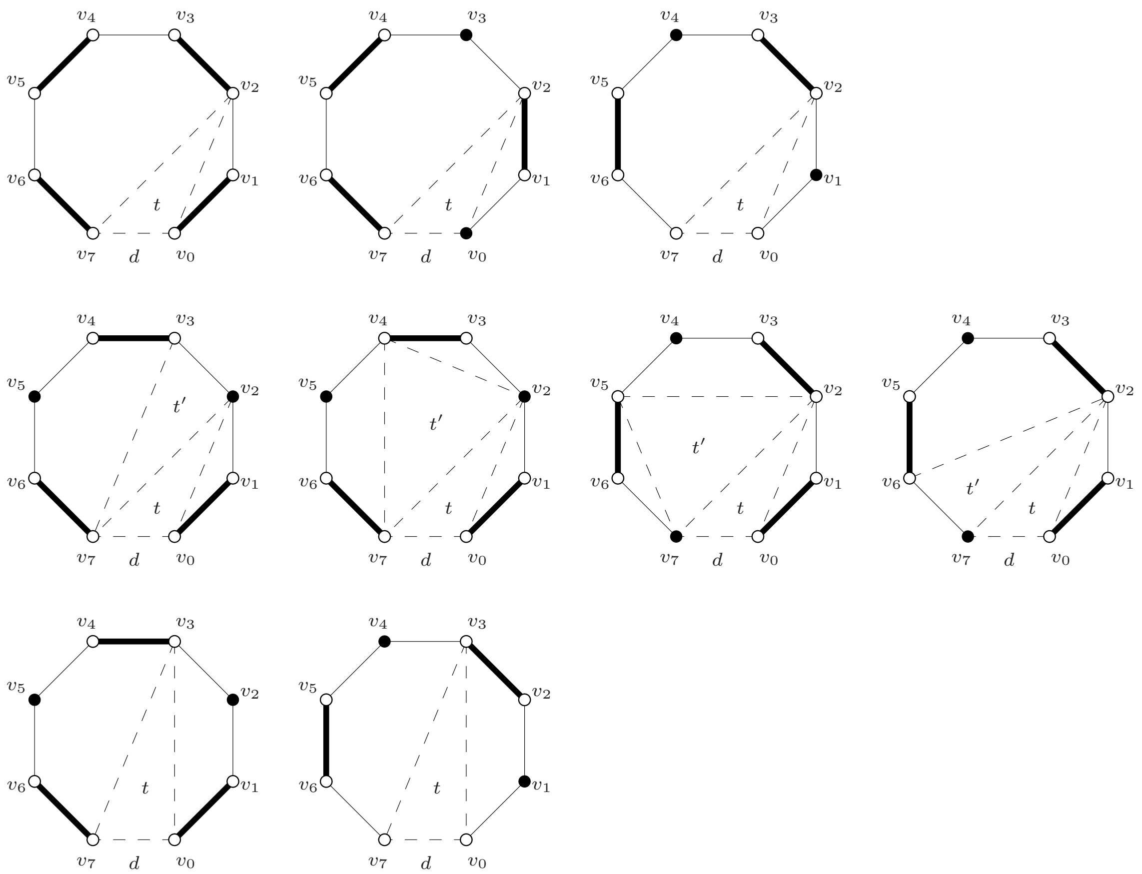


Figure 20: Proof of Theorem 9: the case $k = 7$. Top and middle rows: $v \equiv v_2$. Bottom row: $v \equiv v_3$. Top row (left to right): $d_{02}, d_{27} \in D'$; $d_{02} \notin D', d_{27} \in D'$; $d_{02}, d_{27} \notin D'$. Middle row (left to right): $d_{02} \in D', d_{27} \notin D'$ and $v' \in \{v_3, v_4, v_5, v_6\}$. Bottom row (left to right): $d_{02} \in D'$ or $d_{27} \in D'$; $d_{02}, d_{27} \notin D'$.

**$v_0 \notin D_2, v_6 \in D_2$.** This case is symmetric to the previous one. Let $t$ be the triangle supported by $d$ in $T_1$ and let $v$ be its vertex opposite to $d$. If $v \in \{v_1, v_2, v_3, v_4\}$, set $D = D_2 \cup \{e_1, e_3\}$. If $v \equiv v_5$, let $t'$ be the second triangle supported by $d_{05}$ beyond the triangle $t$, and let $v'$ be its vertex opposite to $d_{05}$. If $v' \in \{v_1, v_2\}$, set $D = D_2 \cup \{e_1, e_4\}$. Otherwise, i.e., if $v' \in \{v_3, v_4\}$, set $D = D_2 \cup \{e_0, e_3\}$.

**$k = 7$.** The presence of diagonals $d_{06}$ or $d_{17}$ would violate the minimality of $k$. Let $t$ be the triangle supported by $d$ in $T_1$ and let $v$ its vertex opposite to $d$. Consider the triangulation graph $T' = T_2 \cup \{t\}$. It has $n-5$ vertices, hence, by our induction hypothesis, it can be 2-dominated with $f(n-5) \leq \lfloor \frac{3n}{7} \rfloor - 2$ edge guards. Let $D'$ be the 2-dominating set for $T'$. Clearly, $v \in \{v_2, v_3, v_4, v_5\}$; furthermore notice that the cases $v \equiv v_2$ and $v \equiv v_5$, and $v \equiv v_3$ and $v \equiv v_4$ are symmetric. We, therefore, consider only the cases $v \equiv v_2$ and $v \equiv v_3$ (see Fig. 20):

**$v \equiv v_2$.** We distinguish between the following subcases:

**$d_{02}, d_{27} \in D'$.** Set $D = (D' \setminus \{d_{02}, d_{27}\}) \cup \{e_0, e_2, e_4, e_6\}$.

**$d_{02} \in D', d_{27} \notin D'$.** Let $t' \neq t$ be the triangle supported by $d_{27}$, and let $v'$ be its vertex opposite to $d_{27}$. If $v' \in \{v_3, v_4\}$, set $D = (D' \setminus \{d_{02}\}) \cup \{e_0, e_3, e_6\}$. Otherwise, if $v' \in \{v_5, v_6\}$, set $D = (D' \setminus \{d_{02}\}) \cup \{e_0, e_2, e_5\}$.

**$d_{02} \notin D', d_{27} \in D'$.** Set $D = (D' \setminus \{d_{27}\}) \cup \{e_1, e_4, e_6\}$.

**$d_{02}, d_{27} \notin D'$.** In this case $v_2$ cannot belong to $D'$. Hence in order for $t$ to be 2-dominated we must have that $v_0, v_7 \in D'$. Hence, set $D = D' \cup \{e_2, e_5\}$.

**$v \equiv v_3$.** Consider the following subcases:

**$d_{02}$ *or* $d_{27} \in D'$.** Set $D = (D_2 \setminus \{d_{02}, d_{27}\}) \cup \{e_0, e_3, e_6\}$.

**$d_{02}, d_{27} \notin D'$.** In this case $v_3$ cannot belong to $D'$. Hence in order for $t$ to be 2-dominated we must have that $v_0, v_7 \in D'$. Hence, set $D = D' \cup \{e_2, e_5\}$.

**$k = 8$.** The presence of diagonals $d_{07}$, $d_{06}$, $d_{18}$ or $d_{28}$ would violate the minimality of $k$. Let $t$ be the triangle supported by $d$ in $T_1$ and let $v$ its vertex opposite to $d$. In this case $T_2$ contains $n-7$ vertices, hence, it can be 2-dominated with $f(n-7) = \lfloor \frac{3n}{7} \rfloor - 3$ edge guards. Let $D_2$ be the 2-dominating set for $T_2$. Clearly, $v' \in \{v_3, v_4, v_5\}$; furthermore notice that the cases $v \equiv v_3$ and $v \equiv v_5$ are symmetric. We, therefore, consider only the cases $v \equiv v_3$ and $v \equiv v_4$. In fact, both cases can be treated jointly. Consider the following subcases (see Fig. 21):

**$d_{08} \in D_2$.** Set $D = (D_2 \setminus \{d_{08}\}) \cup \{e_0, e_3, e_5, e_7\}$.

**$d_{08} \notin D_2$.** Then either $v_0$ or $v_8$ belongs to $D_2$.

**$v_0 \in D_2$.** Set $D = D_2 \cup \{e_2, e_4, e_7\}$.

**$v_8 \in D_2$.** Set $D = D_2 \cup \{e_0, e_3, e_5\}$.

**$k = 9$.** The presence of diagonals $d_{08}$, $d_{07}$, $d_{06}$, $d_{19}$, $d_{29}$ or $d_{39}$ would violate the minimality of $k$. Let $t$ be the triangle supported by $d$ in $T_1$ and let $v$ its vertex opposite to $d$. Consider the triangulation graph $T' = T_2 \cup \{t\}$, and let $D'$ be its edge 2-dominating set. $T'$ has $n-7$ vertices, hence, by our induction hypothesis, $D'$ consists of $f(n-7) = \lfloor \frac{3n}{7} \rfloor - 3$ edge guards. Clearly, $v \in \{v_4, v_5\}$. The two cases are symmetric, so we only need to consider the case $v \equiv v_4$. Consider the following subcases (see Fig. 22):

**$d_{04}$ *or* $d_{49} \in D'$.** Set $D = (D_2 \setminus \{d_{04}, d_{49}\}) \cup \{e_0, e_3, e_5, e_8\}$.

**$d_{04}, d_{49} \notin D'$.** In this case $v_4$ cannot belong to $D'$. Hence in order for $t$ to be 2-dominated we must have that $v_0, v_9 \in D'$. Hence, set $D = D' \cup \{e_2, e_4, e_6\}$.

**$k = 10$.** The presence of diagonals $d_{09}$, $d_{08}$, $d_{07}$, $d_{06}$, $d_{1,10}$, $d_{2,10}$ $d_{3,10}$ or $d_{4,10}$ would violate the minimality of $k$. Let $t$ be the triangle supported by $d$ in $T_1$. Clearly, the vertex of $t$ opposite to $d$ is $v_5$. Let $t' \neq t$ be the triangle in $T_1$ supported by $d_{05}$, and let $v'$ be its vertex opposite to $d_{05}$. Consider the triangulation

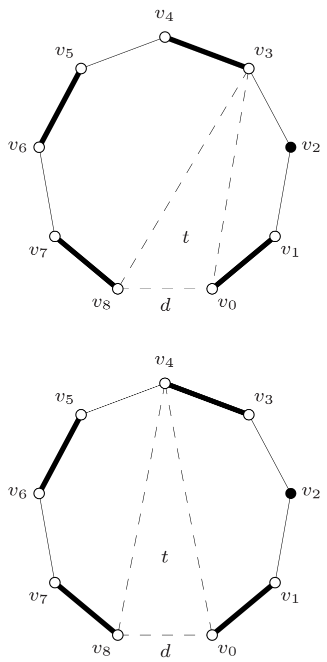


Figure 21: Proof of Theorem 9: the case $k = 8$. Top row: $v \equiv v_3$. Bottom row: $v \equiv v_4$. Left column: $d_{08} \in D_2$. Middle column: $d_{08} \notin D_2$ and $v_0 \in D_2$. Right column: $d_{08} \notin D_2$ and $v_8 \in D_2$.

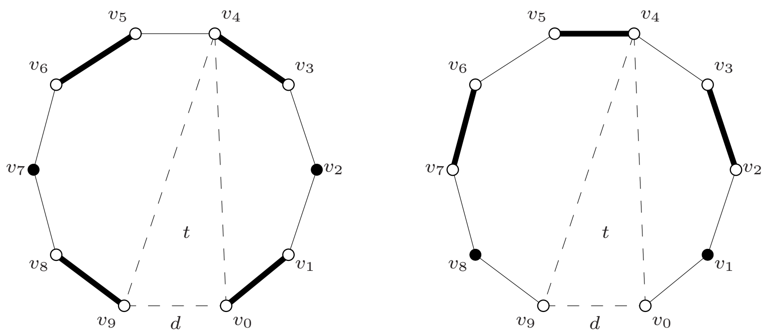


Figure 22: Proof of Theorem 9: the case $k = 9$. Left: $d_{04}$ or $d_{49} \in D'$. Right: $d_{04}, d_{49} \notin D'$.

graph $T' = T_2 \cup \{t, t'\}$, and let $D'$ be its edge 2-dominating set. $T'$ has $n - 7$ vertices, hence, by our induction hypothesis, $D'$ contains $f(n-7) = \lfloor \frac{3n}{7} \rfloor - 3$ edge guards. Clearly, $v' \in \{v_1, v_2, v_3, v_4\}$. Consider each of the following three cases for $v'$ (see Fig. 23):

$\boldsymbol{v' \equiv v_1}$**.** We distinguish between the following subcases:

- $\boldsymbol{d_{15}}$ ***or*** $\boldsymbol{d_{5,10} \in D'}$**.** Set $D = (D' \setminus \{d_{15}, d_{5,10}\}) \cup \{e_1, e_4, e_6, e_9\}$.
- $\boldsymbol{d_{15}, d_{5,10} \notin D'}$**.** In this case $v_5$ cannot belong to $D'$. Hence in order for $t$ and $t'$ to be 2-dominated we must have that $v_0, v_1, v_{10} \in D'$. Since $d_{15} \notin D'$, we must have that $e_0 \in D'$, in order for $v_1$ to be in $D'$. Hence, given that $e_0, v_{10} \in D'$, set $D = D' \cup \{e_3, e_5, e_7\}$.

$\boldsymbol{v' \in \{v_2, v_3\}}$**.** Let $d'$ be the diagonal $v_0 v'$ and $d''$ the diagonal $v' v_5$. Notice that at least one of $d'$, $d''$ and $d_{5,10}$ must belong to $D'$, since otherwise both $v'$ and $v_5$ would not belong to $D'$ (both their incident edges in $T'$ would not belong to $D'$), which implies that the triangle $t'$ would not

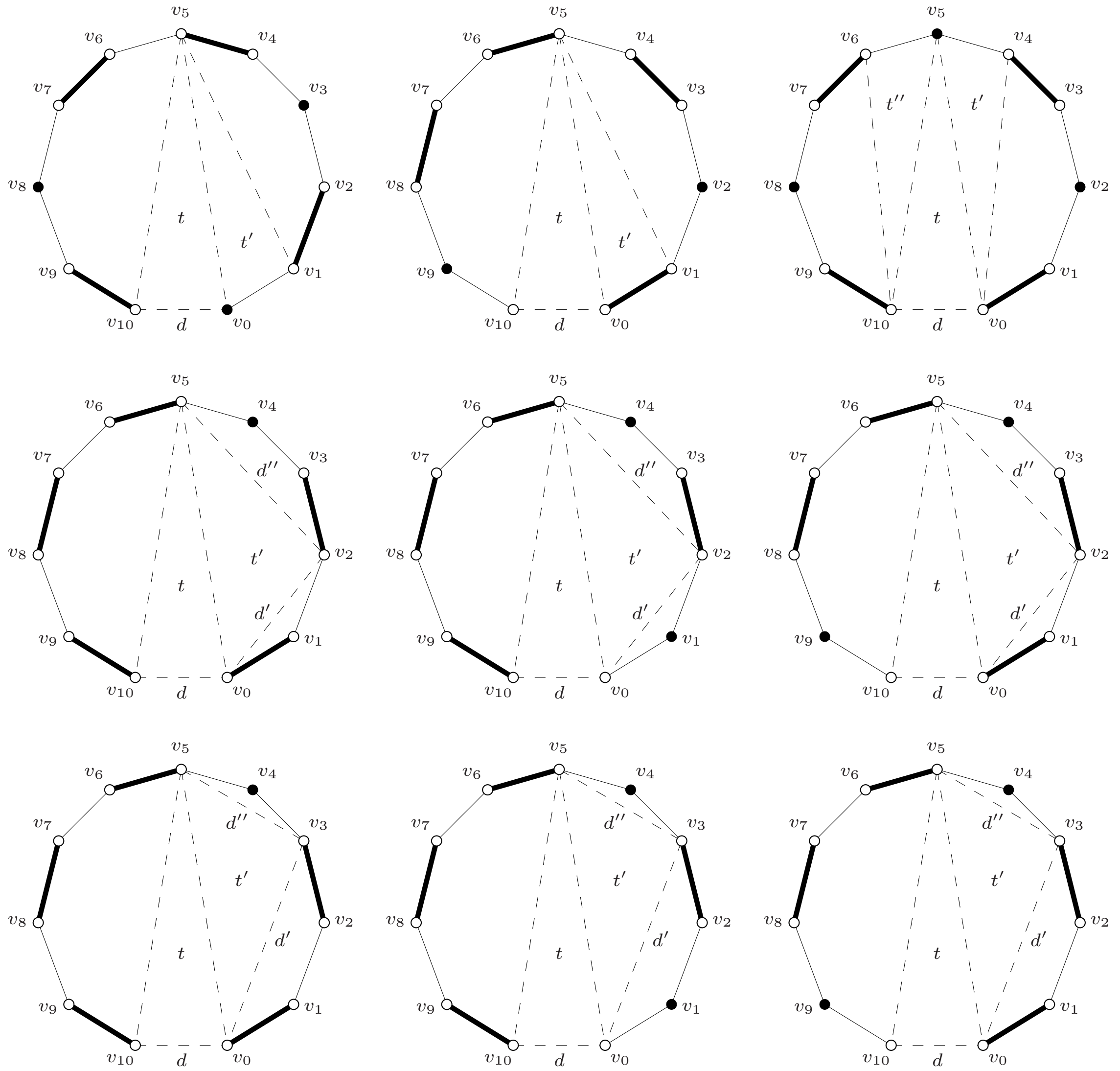


Figure 23: Proof of Theorem 9: the case $k = 10$. Top row (left to right): $v' \equiv v_1$, and $d_{15}$ or $d_{5,10} \in D'$; $v' \equiv v_1$ and $d_{15}, d_{5,10} \notin D'$; $v' \equiv v_4$. Middle row: $v' \equiv v_2$. Bottom row: $v' \equiv v_3$. Middle and bottom rows (left to right): $D' \cap \{d', d'', d_{5,10}\}| \geq 2$; $|D' \cap \{d', d'', d_{5,10}\}| = 1$ and $v_0 \in D' \setminus \{d'\}$; $|D' \cap \{d', d'', d_{5,10}\}| = 1$ and $v_0 \notin D' \setminus \{d'\}$.

be 2-dominated by $D'$. Given this fact, we distinguish between the following cases:

$\boldsymbol{|D' \cap \{d', d'', d_{5,10}\}| \geq 2,}$ i.e., at least two among $d'$, $d''$ and $d_{5,10}$ belong to $D'$. Set $D = (D' \setminus \{d', d'', d_{5,10}\}) \cup \{e_0, e_2, e_5, e_7, e_9\}$.

$\boldsymbol{|D' \cap \{d', d'', d_{5,10}\}| = 1,}$ i.e., exactly one among $d'$, $d''$ and $d_{5,10}$ belongs to $D'$. Consider the two cases:

- $\boldsymbol{v_0 \in D' \setminus \{d'\}.}$ Set $D = (D' \setminus \{d', d'', d_{5,10}\}) \cup \{e_2, e_5, e_7, e_9\}$.
- $\boldsymbol{v_0 \notin D' \setminus \{d'\}.}$ In order for $t$ to be 2-dominated by $D'$, we must have that $v_{10} \in D'$. Hence, set $D = (D' \setminus \{d', d'', d_{5,10}\}) \cup \{e_0, e_2, e_5, e_7\}$.

$\boldsymbol{v' \equiv v_4.}$ Let $t'' \neq t$ be the triangle in $T_1$ supported by $d_{5,10}$, and let $v''$ be its vertex opposite $d_{5,10}$. If

$v'' \not\equiv v_6$, we have a configuration that is symmetric to one of the cases $v' \equiv v_1$, $v' \equiv v_2$ or $v' \equiv v_3$, treated above. Hence, we only need to consider the case $v'' \equiv v_6$. We distinguish between the following cases:

$\boldsymbol{d_{04}}$ ***or*** $\boldsymbol{d_{5,10} \in D'}$. Set $D = (D' \setminus \{d_{04}, d_{5,10}\}) \cup \{e_0, e_3, e_6, e_9\}$.

$\boldsymbol{d_{04}, d_{5,10} \notin D'}$. In order for $t'$ to be 2-dominated by $D'$, either $v_4$ or $v_5$ has to belong to $D'$. Since both $d_{04}$ and $d_{5,10}$ do not belong to $D'$, we conclude that $e_4$ must belong to $D'$. Hence, set $D = (D' \setminus \{e_4\}) \cup \{e_0, e_3, e_6, e_9\}$. □

In a manner analogous to the case of diagonal 2-dominating sets, the proof of Theorem 9 can almost

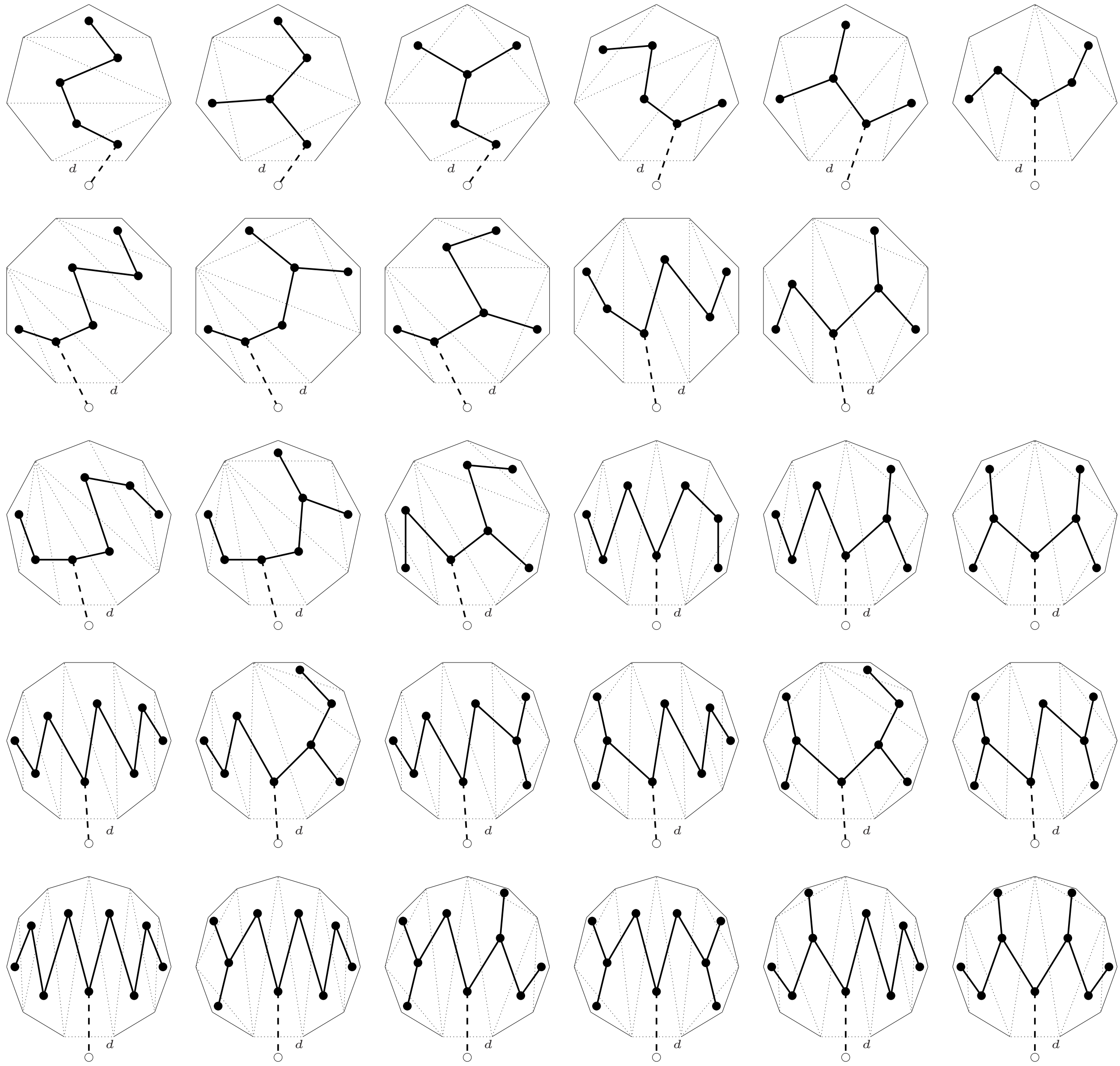

Figure 24: The 29 possible configurations for the dual trees $\Delta_1$ for $6 \le k \le 10$, shown as thick solid lines. The diagonal $d$ separates $T_1$ from $T_2$. The triangulations shown are indicative: all other triangulations yield isomorphic trees.

immediately be transformed into an $O(n)$ time and space algorithm. The algorithm is, in fact, almost identical to the algorithm presented in Section 2.1 for computing diagonal 2-dominating sets. The differences, which by no means alter the spirit of the algorithm, are related to how the proof of Theorem 9 is incorporated. More precisely, the values of $k$ are $6, 7, 8, 9$ and 10, instead of $4, 5$ and 6, whereas the dual trees $\Delta$ are those in Fig. 24, instead of those in Fig. 9. Finally, the cut-off value for the recursion is 21 (instead of 13): for $n \geq 21$, the subtrees $\Delta_1'$ corresponding to different diagonals $d$ of $T_P$ must be edge disjoint (otherwise the number of vertices of $P$ would be less than 21).

The analysis of the edge 2-dominance linear time algorithm, sketched above, is entirely analogous to the analysis of the algorithm for computing diagonal 2-dominating sets. Initialization takes linear time and space, whereas the recursive part of the algorithm requires linear space, and its time requirements satisfy the recursive relation

$$T(n) \leq \begin{cases} T(n-5) + O(1), & n \geq 21 \\ O(1), & 3 \leq n \leq 20 \end{cases}$$

which, clearly, yields $T(n) = O(n)$. Hence, we arrive at the following theorem.

**Theorem 10.** *Given the triangulation graph $T_P$ of a polygon $P$ with $n \geq 3$ vertices, we can compute an edge 2-dominating set for $T_P$ of size at most $\lfloor \frac{3n}{7} \rfloor$ (except for $n = 4$, where one additional edge guard is required) in $O(n)$ time and space.*

## 4. Piecewise-convex polygons

Let $v_1, \ldots, v_n$, $n \geq 2$, be a sequence of points and $a_1, \ldots, a_n$ a set of curvilinear arcs, such that $a_i$ has as endpoints the points $v_i$ and $v_{i+1}$. We will assume that the arcs $a_i$ and $a_j$, $i \neq j$, do not intersect, except when $j = i - 1$ or $j = i + 1$, in which case they intersect only at the points $v_i$ and $v_{i+1}$, respectively. We define a *curvilinear polygon* $P$ to be the closed region of the plane delimited by the arcs $a_i$. The points $v_i$ are called the vertices of $P$. An arc $a_i$ is a *convex arc* if every line on the plane intersects $a_i$ at at most two points or along a line segment. A polygon $P$ is called a *locally convex polygon*, if for every point $p$ on the boundary of $P$, with the possible exception of $P$'s vertices, there exists a disk centered at $p$, say $D_p$, such that $P \cap D_p$ is convex (see Fig. 25(left)). A polygon $P$ is called a *piecewise-convex polygon*, if it is locally convex, and the portion of the boundary between every two consecutive vertices is a convex arc (see Fig. 25(right)).

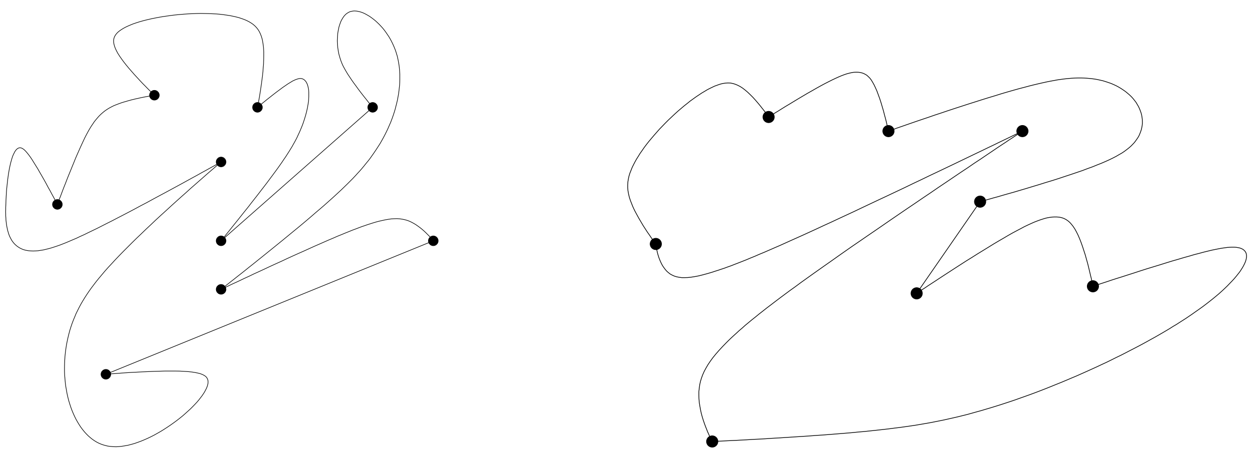

Figure 25: Left: A locally convex polygon. Right: A piecewise-convex polygon.

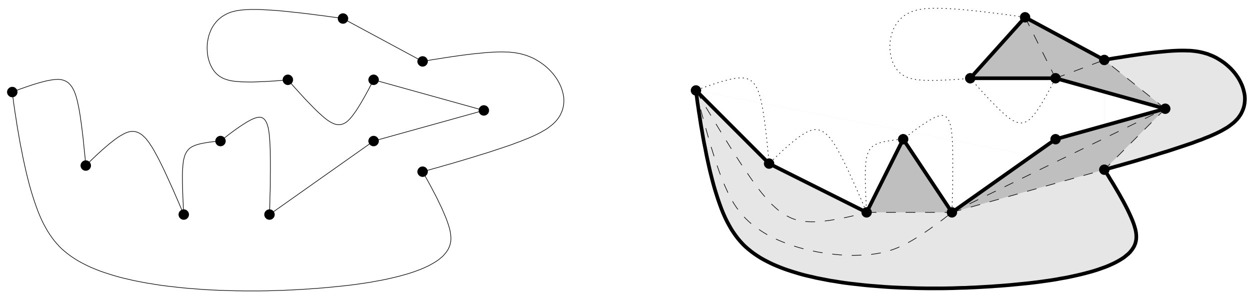

Figure 26: Left: A piecewise-convex polygon $P$. Right: The triangulation graph $T_P$ of $P$. The boundary edges of $T_P$ are shown as thick solid lines. The two crescents of $P$ are shown in light gray, whereas the three stars of $P$ are shown in dark gray.

Let $a_i$ be an edge of a piecewise-convex polygon $P$ with endpoints $v_i$ and $v_{i+1}$. We call the convex region $r_i$ delimited by $a_i$ and $\overline{v_i v_{i+1}}$ a *room*, where $\overline{xy}$ denotes the line segment from $x$ to $y$. A room is called degenerate if the arc $a_i$ is a line segment. For $p, q \in a_i$, $\overline{pq}$ is called a *chord* of $a_i$; the chord of $r_i$ is $\overline{v_i v_{i+1}}$. An *empty room* is a non-degenerate room that does not contain any vertex of $P$ in the interior of $r_i$ or in the interior of $\overline{v_i v_{i+1}}$. A *non-empty room* is a non-degenerate room that contains at least one vertex of $P$ in the interior of $r_i$ or in the interior of $\overline{v_i v_{i+1}}$.

We say that a point $p$ in the interior of a piecewise-convex polygon $P$ is visible from a point $q$ if $\overline{pq}$ lies in the closure of $P$. We say that $P$ is *monitored* by a *guard set* $G$ if every point in $P$ is visible from at least one point belonging to some guard in $G$. A *diagonal* of a piecewise-convex polygon $P$ is a straight-line segment in the closure of $P$ the endpoints of which are vertices of $P$. An *edge (resp., mobile) guard* is an edge (resp., edge or diagonal) of $P$ belonging to a guard set $G$ of $P$. An *edge (resp., mobile) guard set* is a guard set that consists of only edge (resp., mobile) guards.

Let $P$ be a piecewise-convex polygon with $n \geq 3$ vertices. Consider a convex arc $a_i$ of $P$, with endpoints $v_i$ and $v_{i+1}$, and let $r_i$ be the corresponding room. If $r_i$ is a non-empty room, let $X_i$ be the set of vertices of $P$ that lie in the interior of $\overline{v_i v_{i+1}}$, and let $R_i$ be the set of vertices of $P$ in the interior of $r_i$ or in $X_i$. If $R_i \neq X_i$, let $C_i$ be the set of vertices in the convex hull of the vertex set $(R_i \setminus X_i) \cup \{v_i, v_{i+1}\}$; if $R_i = X_i$, let $C_i = X_i \cup \{v_i, v_{i+1}\}$. Finally, let $C_i^* = C_i \setminus \{v_i, v_{i+1}\}$. If $r_i$ is an empty room, let $C_i = \{v_i, v_{i+1}\}$ and $C_i^* = \emptyset$.

We are now going to construct a constrained triangulation graph $T_P$ of $P$. The vertex set of $T_P$ is the set of vertices of $P$. The edges and diagonals of $T_P$, as well as their embedding, are defined as follows (see also Fig. 26):

- If $a_i$ is a line segment or $r_i$ is an empty room, the edge $(v_i, v_{i+1})$ is an edge in $T_P$, and is embedded as $\overline{v_i v_{i+1}}$.
- If $r_i$ is a non-empty room, the following edges or diagonals belong to $T_P$:
  1. $(v_i, v_{i+1})$,
  2. $(c_{i,j}, c_{i,j+1})$, for $1 \leq j \leq |C_i| - 1$, where $c_{i,1} \equiv v_i$ and $c_{i,|C_i|} \equiv v_{i+1}$. The remaining $c_i$'s are the vertices of $P$ in $C_i^*$ as we encounter them when walking inside $r_i$ and on the convex hull of the point set $C_i$ from $v_i$ to $v_{i+1}$, and
  3. $(v_i, c_{i,j})$, for $3 \leq j \leq |C_i| - 1$, provided that $|C_i| \geq 4$. We call these diagonals *weak diagonals*.

  The diagonals $(c_{i,j}, c_{i,j+1})$, $1 \leq j \leq |C_i| - 1$ are embedded as $\overline{c_{i,j}, c_{i,j+1}}$, whereas the diagonals $(v_i, c_{i,j})$, $3 \leq j \leq |C_i| - 1$, are embedded as curvilinear segments. Finally, the edges $(v_i, v_{i+1})$ are embedded as curvilinear segments, namely, the arcs $a_i$.

The edges $(v_i, v_{i+1})$, along with the diagonals $(c_{i,j}, c_{i,j+1})$, $1 \leq j \leq |C_i| - 1$, partition $P$ into subpolygons of two types: (1) subpolygons that lie entirely inside a non-empty room, called *crescents*, and (2) subpolygons delimited by edges of the polygon $P$, as well as diagonals of the type $(c_{i,j}, c_{i,j+1})$, called *stars*. In general, a piecewise-convex polygon may only have crescents, or only stars, or both. The crescents are triangulated by means of the diagonals $(v_i, c_{i,j})$, $3 \leq j \leq |C_i| - 1$. To finish the definition of the triangulation graph $T_P$, we simply need to triangulate all stars inside $P$. Since the delimiting edges of stars are embedded as line segments, i.e., stars are linear polygons, any polygon triangulation algorithm may be used to triangulate them.

In direct analogy to the types of subpolygons we can have inside $P$, we have two possible types of triangles in $T_P$: (1) triangles inside stars, called *star triangles*, and (2) triangles inside a crescent, called *crescent triangles*. Crescent triangles have at least one edge that is a weak diagonal, except when the number of vertices of $P$ in the interior of the corresponding room $r$ is exactly one, in which case none of the three edges of the unique crescent triangle in $r$ is a weak diagonal. A crescent triangle that has at least one weak diagonal among its edges is called a *weak triangle*.

### *4.1. Mobile guards*

Let $G_{T_P}$ be a diagonal 2-dominating set of $T_P$. Based on $G_{T_P}$ we define a set $G$ of edges or straight-line diagonals of $P$ as follows (see also Fig. 27): (1) for every edge in $G_{T_P}$, add to $G$ the corresponding convex arc of $P$, (2) add to $G$ every non-weak diagonal of $G_{T_P}$, and (3) for every weak diagonal in $G_{T_P}$, add to $G$ the edge of $P$ delimiting the crescent that contains the weak diagonal. Clearly, $|G| \leq |G_{T_P}|$.

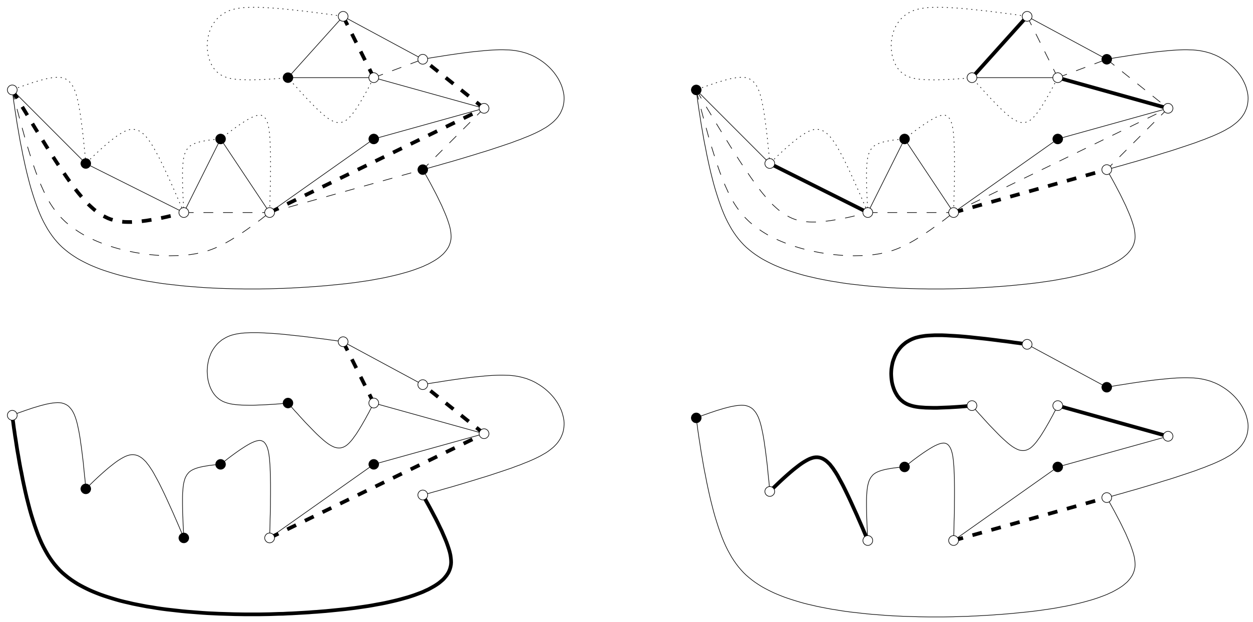

Figure 27: Top row: two diagonal 2-dominating sets for the triangulation graph $T_P$ of $P$ from Fig. 26. Bottom row: the corresponding mobile guard sets for $P$.

**Lemma 11.** *Let $P$ be a piecewise-convex polygon with $n \geq 3$ vertices, $T_P$ its constrained triangulation graph, and $G_{T_P}$ a diagonal 2-dominating set of $T_P$. The set $G$ of mobile guards, defined by mapping every edge of $G_{T_P}$ to the corresponding convex arc of $P$, every non-weak diagonal of $G_{T_P}$ to itself, and every weak diagonal $d$ of $G_{T_P}$ to the convex arc of $P$ delimiting the crescent that contains $d$, is a mobile guard set for $P$.*

*Proof.* Let $q$ be a point in the interior of $P$. $q$ is either inside: (1) an empty room $r_i$ of $P$, (2) a star triangle $t_s$ of $T_P$, (3) a non-weak crescent triangle $t_{nw}$ of $T_P$, or (4) a weak crescent triangle $t_w$ of $T_P$. In any of the four cases, $q$ is visible from at least two vertices $u_1$ and $u_2$ of $T_P$ that are connected via an edge or a diagonal in $T_P$. In the first case, $q$ is visible from the two endpoints $v_i$ and $v_{i+1}$ of $a_i$. In the second case, $q$ is visible from all three vertices of $t_s$. The third case arises when $q$ is inside a non-empty room $r_j$ with $|C_j^*| = 1$ ($t_{nw}$ is the unique crescent triangle in $r_j$), in which case $q$ is visible from at least two of the three vertices $v_j$, $v_{j+1}$ and $c_{j,1}$. Finally, in the fourth case, $q$ has to lie inside the crescent of a non-empty room $r_j$ with $|C_j^*| \geq 2$, and is visible from at least two consecutive vertices $c_{j,k}$ and $c_{j,k+1}$ of $C_j$.

Since $G$ is a diagonal 2-dominating set for $T_P$, and $(u_1, u_2) \in T_P$, at least one of $u_1$ and $u_2$ belongs to $G_{T_P}$. Without loss of generality, let us assume that $u_1 \in G_{T_P}$. If $u_1 \in G$, $q$ is monitored by $u_1$. If $u_1 \notin G$, $u_1$ has to be an endpoint of a weak diagonal $d_w$ in $G_{T_P}$. Let $r_\ell$ be the room, inside the crescent of which lies $d_w$. Since $d_w \in G_{T_P}$, we have that $a_\ell \in G$. If $q$ lies inside the closure of the crescent of the room $r_\ell$ (this can happen in case (4) above), $q$ is visible from $a_\ell$, and thus monitored by $a_\ell$. Otherwise, $u_1$ cannot be an endpoint of $a_\ell$ ($a_\ell \in G$, whereas $u_1 \notin G$), which implies that $u_1 \in C_\ell^*$, i.e., $u_1 \equiv c_{\ell,m}$, with $2 \leq m \leq |C_\ell| - 1$. But then $q$ lies inside the cone with apex $c_{\ell,m}$, delimited by the rays $c_{\ell,m}c_{\ell,m-1}$ and $c_{\ell,m}c_{\ell,m+1}$, and containing at least one of $v_\ell$ and $v_{\ell+1}$ in its interior. Since, $q$ is visible from the intersection point of the line $qu_1$ with $a_\ell$, $q$ is monitored by $a_\ell$. □

Our approach for computing the mobile guard set $G$ of $P$ consists of three major steps:

1. Construct the constrained triangulation $T_P$ of $P$.
2. Compute a diagonal 2-dominating set $G_{T_P}$ for the triangulation graph $T_P$.
3. Map $G_{T_P}$ to $G$.

The sets $C_i^*$, needed in order to construct the constrained triangulation $T_P$ of $P$ can be computed in $O(n \log n)$ time and $O(n)$ space (cf. [21]). Once we have the sets $C_i^*$, the constrained triangulation $T_P$ of $P$ can be constructed in linear time and space. By Theorem 4, computing $G_{T_P}$ takes linear time; furthermore $|G_{T_P}| \leq \lfloor \frac{n+1}{3} \rfloor$, which implies that $|G| \leq \lfloor \frac{n+1}{3} \rfloor$. Finally, the construction of $G$ from $G_{T_P}$ takes $O(n)$ time and space: for every edge in $G_{T_P}$ we need to add to $G$ the corresponding convex arc of $P$, while for every diagonal $d$ in $G_{T_P}$ we need to determine if it is a weak diagonal, in which case we need to add to $G$ the edge of $P$ delimiting the crescent in which $d$ lies, otherwise we simply add $d$ to $G$; by appropriate bookkeeping at the time of construction of $T_P$ these operations can take $O(1)$ per edge or diagonal. Summarizing, by Theorem 3, Lemma 11 and our analysis above, we arrive at the following theorem. The case $n = 2$ can be trivially established.

**Theorem 12.** *Let $P$ be a piecewise-convex polygon with $n \geq 2$ vertices. We can compute a mobile guard set for $P$ of size at most $\lfloor \frac{n+1}{3} \rfloor$ in $O(n \log n)$ time and $O(n)$ space.*

### *4.2. Edge guards*

We start by proving that an edge 2-dominating set for $T_P$ is also an edge guard set for $P$ (see also Fig. 28).

**Lemma 13.** *Let $P$ be a piecewise-convex polygon with $n \geq 3$ vertices, $T_P$ its constrained triangulation graph, and $G_{T_P}$ an edge 2-dominating set of $T_P$. The set $G$ of edge guards, defined by mapping every edge in $G_{T_P}$ to the corresponding convex arc of $P$, is an edge guard set for $P$.*

*Proof.* Let $q$ be a point in the interior of $P$. Recall the four cases for $q$ from the proof of Lemma 11. $q$ is either inside: (1) an empty room of $P$, (2) a star triangle of $T_P$, (3) a non-weak crescent triangle of $T_P$, or (4) a weak crescent triangle of $T_P$. In any of the four cases, $q$ is visible from at least two vertices $u_1$ and $u_2$ of $T_P$, such that the edge or diagonal $(u_1, u_2)$ belongs to $T_P$. Let $t$ be a triangle supported by $(u_1, u_2)$ in $T_P$. At least two of the vertices of $t$ belong to $G_{T_P}$, which implies that at least one of $u_1$ and $u_2$, belongs to $G_{T_P}$. Since the set of vertices that are endpoints of edges in $G_{T_P}$ is the same as the set of vertices that are endpoints of edges in $G$, we conclude that $q$ is monitored by a vertex that is an endpoint of an edge in $G$. □

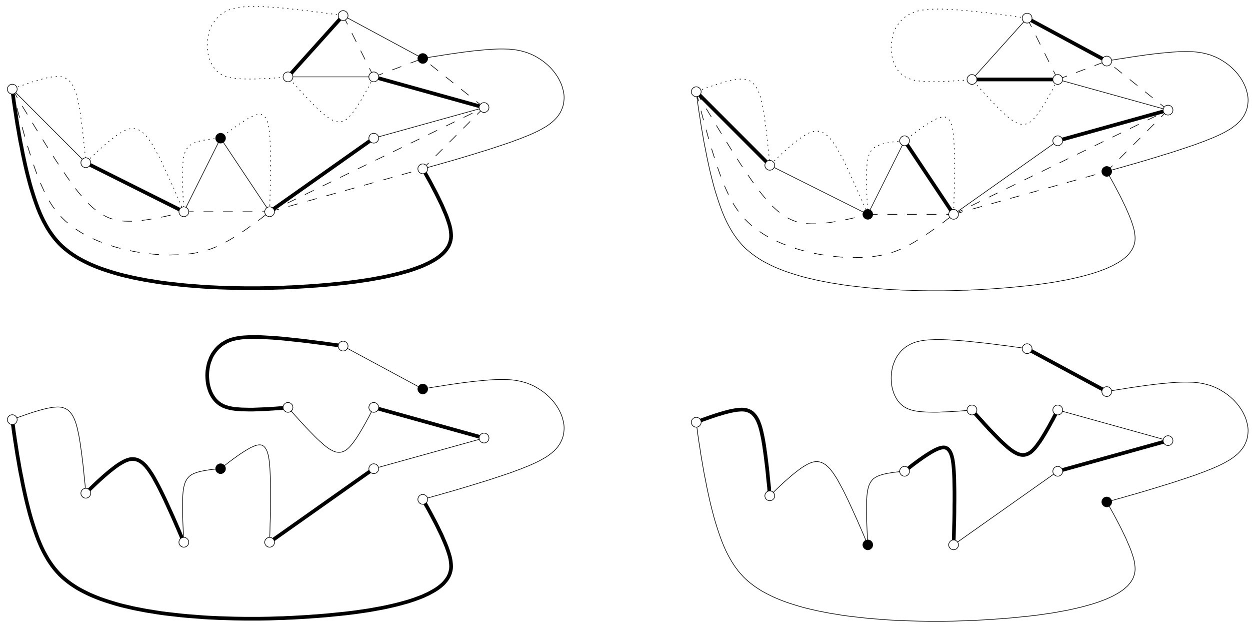

Figure 28: Top row: two edge 2-dominating sets for the triangulation graph $T_P$ of $P$ from Fig. 26. Bottom row: the corresponding edge guard sets for $P$.

By Theorems 7 and 10, we can either compute an edge 2-dominating set $G_{T_P}$ of size $\lfloor \frac{2n+1}{5} \rfloor$ in $O(n^2)$ time and $O(n)$ space, or an edge 2-dominating set $G_{T_P}$ of size $\lfloor \frac{3n}{7} \rfloor$ in linear time and space (except for $n = 4$ where one additional edge is needed in both cases). As in the case of mobile guards, the constrained triangulation graph $T_P$ of $P$ can be computed in $O(n \log n)$ time and $O(n)$ space. Since $|G| = |G_{T_P}|$, we arrive at the following theorem. The case $n = 2$ is trivial, since in this case any of the two edges of $P$ is an edge guard set for $P$.

**Theorem 14.** *Let $P$ be a piecewise-convex polygon with $n \geq 2$ vertices. We can either: (1) compute an edge guard set for $P$ of size $\lfloor \frac{2n+1}{5} \rfloor$ (except for $n = 4$, where one additional edge guard is required) in $O(n^2)$ time and $O(n)$ space, or (2) compute an edge guard set for $P$ of size $\lfloor \frac{3n}{7} \rfloor$ (except for $n = 2, 4$, where one additional edge guard is required) in $O(n \log n)$ time and $O(n)$ space.*

### *4.3. Lower bound constructions*

Consider the piecewise-convex polygon $P$ of Fig. 29. Each spike consists of three edges, namely, two line segments and a convex arc. In order for points in the non-empty room of the convex arc to be monitored, either one of the three edges of the spike, or a diagonal at least one endpoint of which is an endpoint of the convex arc, has to be in any guard set of $P$: the chosen edge or diagonal in a spike cannot monitor the non-empty room inside another spike of $P$. Since $P$ consists of $k$ spikes, yielding $n = 3k$ vertices, we need at least $k$ mobile guards to monitor $P$. We, thus, conclude that $P$ requires at least $\lfloor \frac{n}{3} \rfloor$ mobile guards in order to be monitored.

**Theorem 15.** *There exists a family of piecewise-convex polygons with $n \geq 3$ vertices any mobile guard set of which has cardinality at least $\lfloor \frac{n}{3} \rfloor$.*

Our lower bound for edge guards is slightly better than for mobile guards. Consider the fan-like $n$-vertex piecewise-convex polygon $F$ of Fig. 30. $F$ is constructed from a regular $n$-gon by replacing each edge of the $n$-gon by a highly tilted spike. The spike $s$, bounded by the edge $e_s$ of $F$, can only be monitored by the points of $e_s$, or some of the points of the two neighboring edges of $e_s$. This immediately implies that in

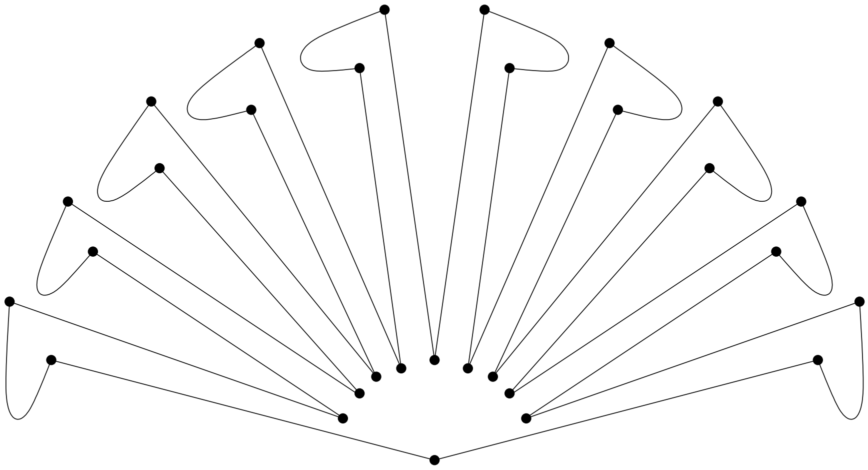

Figure 29: The lower bound construction for mobile guards: the polygon shown contains $n = 3k$ vertices, and requires $k = \lfloor \frac{n}{3} \rfloor$ mobile guards in order to be monitored.

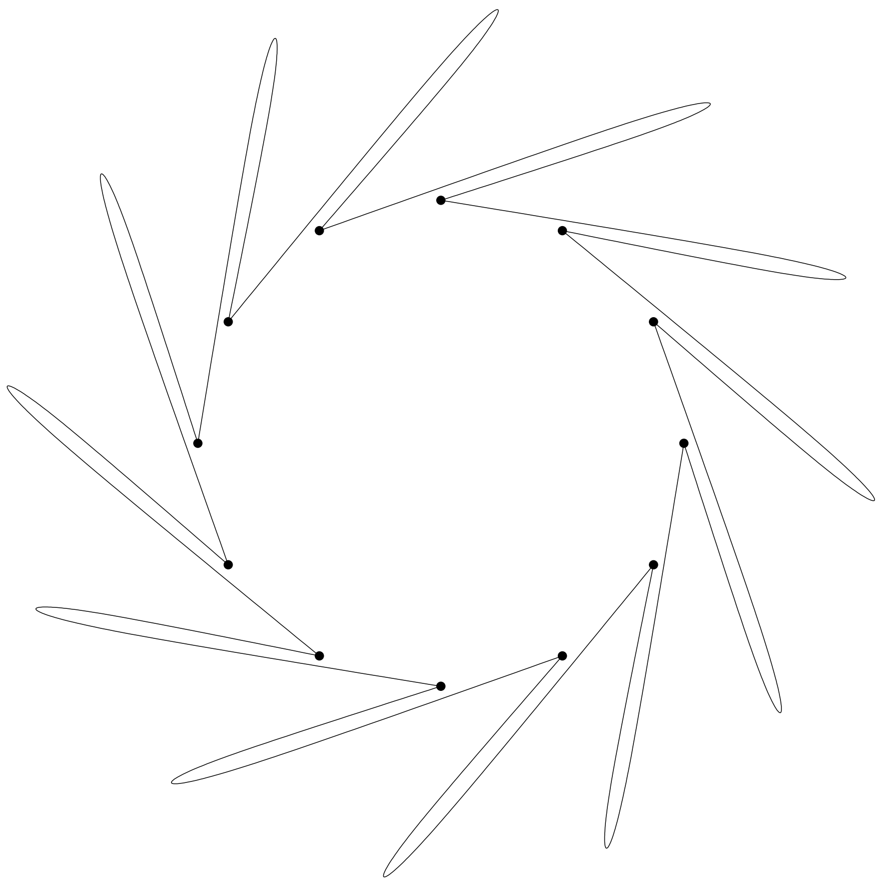

Figure 30: The lower bound construction for edge guards: the polygon shown contains $n$ vertices, and requires $k = \lceil \frac{n}{3} \rceil$ edge guards in order to be monitored.

order to monitor $F$ we need a minimum of $\lceil \frac{n}{3} \rceil$ edge guards. To see this, assume that there exists an edge guard set $G$ for $F$ of size $|G| < \lceil \frac{n}{3} \rceil$. Then we would be able to find three consecutive edges $e_1$, $e_2$, $e_3$ of $F$ that do not belong to $G$, which implies that the spike bounded by $e_2$ is not monitored by $G$, a contradiction.

**Theorem 16.** *There exists a family of piecewise-convex polygons with $n \geq 3$ vertices any edge guard set of which has cardinality at least $\lceil \frac{n}{3} \rceil$.*

## 5. Monotone piecewise-convex polygons

In this section we consider the special case of *monotone piecewise-convex polygons*. We start by restating the definition of monotonicity: a piecewise-convex polygon $P$ is called *monotone* if there exists a line $L$, such that every line $L^{\perp}$ perpendicular to $L$ intersects $P$ at at most two points or line segments. Without loss of generality we may assume that the line $L$, with respect to which $P$ is monotone, is the $x$-axis. Let $u_j$, $1 \leq j \leq n$, be the vertex of $P$ with the $j$-th largest $x$-coordinate — ties are broken lexicographically (also refer to Fig. 31). Let $u_0$ (resp., $u_{n+1}$) be the point of $P$ of minimal (resp., maximal) $x$-coordinate. Let $\ell_j$, $0 \leq j \leq n+1$, be the line passing through $u_j$, perpendicular to $L$. The collection $\mathcal{L} = \{\ell_0, \ell_1, \ldots, \ell_{n+1}\}$ of lines decompose the interior of $P$ into $n+1$ (possibly empty) convex regions $\kappa_j$, $0 \leq j \leq n$, that are free of vertices or edges of $P$. Each region $\kappa_j$, $0 \leq j \leq n$, has on its boundary both $u_j$ and $u_{j+1}$. Let $e_j^{\ell}$ (resp., $e_j^r$), $1 \leq j \leq n$, be the edge of $P$ that has $u_j$ as its right (resp., left) endpoint, i.e., $e_j^{\ell}$ (resp., $e_j^r$) lies to left (resp., right) of $u_j$. We define $e_0^r$ (resp., $e_{n+1}^{\ell}$) to be the edge containing $u_0$ (resp., $u_{n+1}$). For a vertex $u_j$, $1 \leq j \leq n$, let $e_j^{opp}$ be edge of $P$ opposite to $u_j$, i.e., the edge intersected by $\ell_j$ on the monotone chain on $P$ not containing $u_j$. Finally, for each $u_j$, $0 \leq j \leq n+1$, define its index $\sigma_j$ to be equal to 0 if $u_j$ lies on both the upper and monotone chain of $P$ (this is the case for $u_0$ and $u_{n+1}$), $+1$ if $u_j$ lies on the upper but not the lower monotone chain of $P$, and $-1$ if $u_j$ lies on the lower but not the upper monotone chain of $P$.

We are going to compute an edge set $G$ for $P$ of size at most $\lceil \frac{n+1}{4} \rceil$ as will be described below. The idea behind computing $G$ is to split $P$ into subpieces consisting of (at most) four convex regions $\kappa_j$ and for each such four-tuple of convex pieces choose an edge of $P$ that monitors them. The procedure for computing $G$ is as follows. For $j > n$, set $\kappa_j = \emptyset$, and initialize $G$ to be empty. Let

$$K_i = \bigcup_{j=4i-4}^{4i-1} \kappa_j, \quad 1 \leq i \leq \left\lceil \frac{n+1}{4} \right\rceil.$$

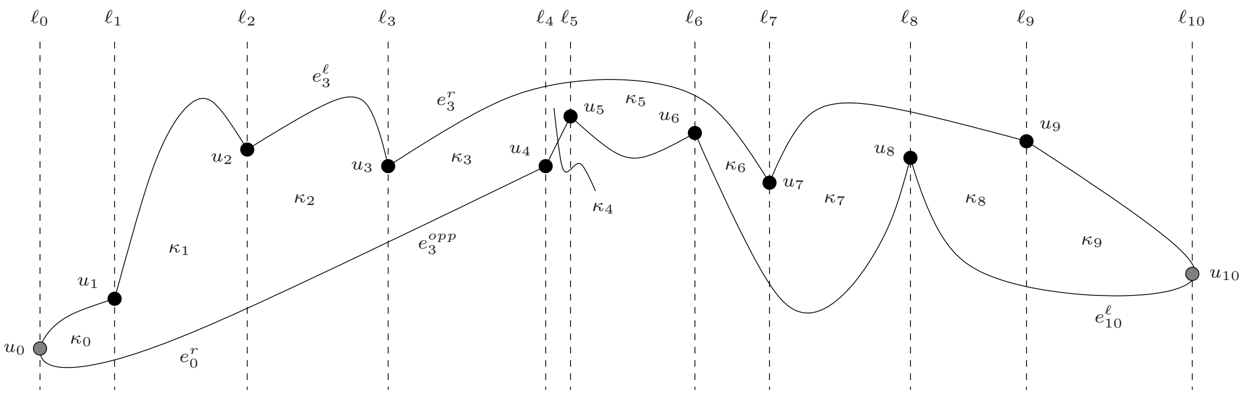


Figure 31: A monotone piecewise-convex polygon $P$ with 9 vertices. The decomposition of $P$ into the convex regions $\kappa_j$, $0 \leq j \leq 9$ is shown. The edges $e_3^{\ell}$ and $e_3^r$ are the edges of $P$ having $u_3$ to their left and right, respectively. The edge $e_3^{opp}$ is the edge of $P$ opposite to $u_3$ (i.e., the edge of $P$ intersected by $\ell_3$ lying on the monotone chain of $P$ not containing $u_3$). The indices of the vertices of $P$ are as follows: $\sigma_0 = \sigma_{10} = 0$; $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_7 = \sigma_9 = +1$; $\sigma_4 = \sigma_5 = \sigma_6 = \sigma_8 = -1$.

For each $K_i$, $1 \leq i < \lceil \frac{n+1}{4} \rceil$, we are going to add one edge of $P$ to $G$ according to the following procedure.

1. If $\sigma_{4i+1} \neq \sigma_{4i+2}$, add $e^r_{4i+1}$ to $G$.
2. Otherwise, if $\sigma_{4i+2} \neq \sigma_{4i+3}$, add $e^\ell_{4i+3}$ to $G$.
3. Otherwise, if $\sigma_{4i} \neq \sigma_{4i+1}$, add $e^r_{4i}$ to $G$.
4. Otherwise, if $\sigma_{4i+3} \neq \sigma_{4i+4}$, add $e^\ell_{4i+4}$ to $G$.
5. Otherwise, add $e^{opp}_{4i+2}$ to $G$.

The procedure for adding an edge of $P$ for $K_{\lceil \frac{n+1}{4} \rceil}$ is analogous or simpler, since we only need to account for four or less consecutive convex regions.

**Lemma 17.** *The edge set $G$ defined via the procedure above is an edge guard set for $P$.*

*Proof.* We are going to show that the set $K_i$, $1 \leq i < \lceil \frac{n+1}{4} \rceil$ is monitored by the corresponding edge added to $G$. The argument for $K_{\lceil \frac{n+1}{4} \rceil}$ is analogous or simpler and is omitted.

Given a point $p \in P$, let $\ell^\perp(p)$ be the line passing through $p$ that is perpendicular to $L$.

Suppose that $\sigma_{4i+1} \neq \sigma_{4i+2}$. The edge $e^r_{4i+1}$ has as right endpoint a vertex $u_\lambda$ with $\lambda \geq 4i+3$. Clearly, $\kappa_{4i}$ and $\kappa_{4i+1}$ are monitored by $u_{4i+1} \in e^r_{4i+1}$. If $\lambda = 4i+3$, then $\kappa_{4i+2}$ and $\kappa_{4i+3}$ are monitored by $u_{4i+3} \in e^r_{4i+1}$. Otherwise, $\lambda \geq 4i+4$, in which case for every point $p \in \kappa_{4i+2} \cup \kappa_{4i+3}$ the line $\ell^\perp(p)$ intersects $e^r_{4i+1}$. The argument is symmetric if $\sigma_{4i+1} = \sigma_{4i+2}$, but $\sigma_{4i+2} \neq \sigma_{4i+3}$.

Otherwise, consider the case $\sigma_{4i} \neq \sigma_{4i+1}$. The edge $e^r_{4i}$ has a right endpoint a vertex $u_\lambda$ of $P$, with $\lambda \geq 4i+3$. If $\lambda = 4i+3$, both $\kappa_{4i+2}$ and $\kappa_{4i+3}$ are monitored by $u_{4i+3}$. $\kappa_{4i}$ is monitored by $u_{4i}$, whereas for every point $p \in \kappa_{4i+1}$, the line $\ell^\perp(p)$ intersects $e^r_{4i}$. If $\lambda > 4i+3$, i.e., $\lambda \geq 4i+4$, then for every point $p \in K_i$, the line $\ell^\perp(p)$ intersects $e^r_{4i}$. The argument is symmetric if $\sigma_{4i} = \sigma_{4i+1} = \sigma_{4i+2} = \sigma_{4i+3}$, but $\sigma_{4i+3} \neq \sigma_{4i+4}$.

Finally, consider the case $\sigma_{4i} = \sigma_{4i+1} = \sigma_{4i+2} = \sigma_{4i+3} = \sigma_{4i+4}$. In this case for every point $p \in K_i$, the line $\ell^\perp(p)$ intersects $e^{opp}_{4i+2}$. ☐

Given Lemma 17 we can now state and prove the main result of this section.

**Theorem 18.** *Given a monotone piecewise-convex polygon $P$ with $n \geq 2$, $\lceil \frac{n+1}{4} \rceil$ edge or mobile guards are always sufficient and sometimes necessary in order to monitor $P$. We can compute such an edge guard set in $O(n)$ time and $O(n)$ space.*

*Proof.* Lemma 17 gives us the upper bound, since an edge guard set is also a mobile guard set. The time and space complexities are a result of the fact that determining whether a piecewise-convex polygon is monotone can be determined in linear time [32], and the fact that the procedure for computing an edge guard set described above takes linear time and space.

Let us now concentrate on proving the lower bound. It suffices to present the proof for the case of mobile guards. Our claim is trivial for $n \in \{2, 3\}$. Consider the monotone piecewise-convex polygons $M_1$ (top) and $M_2$ (bottom) of Fig. 32. $M_1$ consists of $n_1 = 2m_1 + 5$, $m_1 \geq 0$, vertices, whereas $M_2$ consists of $n_2 = 2m_2 + 4$, $m_2 \geq 0$, vertices (in our example $m_1 = m_2 = 4$). The rationale behind the construction of $M_i$, $i = 1, 2$, lies in the properties of the shaded regions $s_j$, $0 \leq j \leq n_i$, shown in Fig. 32. Each region $s_j$, $1 \leq j \leq n_i - 1$, is only visible by the two vertices $u_j$ and $u_{j+1}$ of $M_i$, some or all points on the edges $e^r_j$ and $e^\ell_{j+1}$, as well as points on diagonals of $M_i$ that have either $u_j$ or $u_{j+1}$ as one of their endpoints. Finally, the shaded region $s_0$ (resp., $s_{n_i}$) is only visible by $u_0$, all points on $e^r_0$ or the diagonals $d_{12}$, $d_{13}$ and $d_{23}$ (resp., by $u_{n_i}$, all points on $e^\ell_{n_i+1}$ or the diagonals $d_{n_i-2,n_i}$, $d_{n_i-1,n_i}$ and $d_{n_i-2,n_i-1}$).

Let $G_i$ be the mobile guard set for $M_i$, $i = 1, 2$. Suppose that we can monitor $M_i$ with less than $\lceil \frac{n+1}{4} \rceil$ mobile guards. This implies that the number of vertices of $M_i$ in $G_i$ is less than $\lceil \frac{n+1}{2} \rceil$, which further implies that either: (1) there exist two consecutive vertices of $M_i$ that do not belong to $G_i$, or: (2) $u_1$ or $u_{n_i}$ is not incident to an edge or diagonal of $M_i$ in $G_i$. In the former case, let $u_k$ and $u_{k+1}$ be the two consecutive vertices of $M_i$ that are not incident to edges that belong to $G_i$. This implies, in particular, that neither $e^r_k$ nor $e^\ell_{k+1}$, nor any diagonal of $M_i$ incident to $u_k$ or $u_{k+1}$, belongs to $G_i$ and therefore the shaded region $s_k$ is not monitored by the edges or diagonals in $G_i$, a contradiction. In the latter case, $e_0$, $d_{12}$, $d_{13}$ or $d_{23}$ (resp., $e_{n_i}$, $d_{n_i-2,n_i}$, $d_{n_i-1,n_i}$ or $d_{n_i-2,n_i-1}$) cannot belong to $G_i$, which implies that $s_0$ (resp., $s_{n_i}$) is not

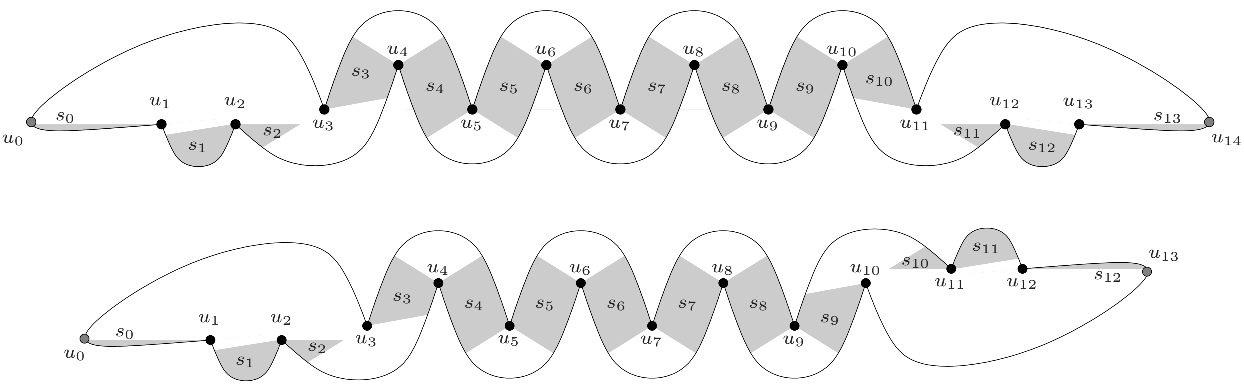


Figure 32: The lower bound construction for monotone piecewise-convex polygons. The polygon $M_1$ (top) consists of $n_1 = 13$ vertices, whereas $M_2$ (bottom) consists of $n_2 = 12$ vertices. Each region $s_j$, $1 \leq j \leq n_i - 1$, is only visible by $u_j$, $u_{j+1}$, some or all points on $e_j^r$ and $e_{j+1}^\ell$, or points on diagonals of $M_i$ that have either $u_j$ or $u_{j+1}$ as one of their endpoints. The shaded region $s_0$ (resp., $s_{n_i}$) is only visible by $u_0$, all points on $e_0^r$ or the diagonals $d_{12}$, $d_{13}$, $d_{23}$ (resp., by $u_{n_i}$, all points on $e_{n_i+1}^\ell$ or the diagonals $d_{n_i-2,n_i}$, $d_{n_i-1,n_i}$, $d_{n_i-2,n_i-1}$).

monitored by any of the edges or diagonals in $G_i$, again a contradiction. Hence our assumption that $M_1$ or $M_2$ can be monitored with less that $\lceil \frac{n+1}{4} \rceil$ edge guards is false. $\square$

**Remark 1.** *The results presented in this section for monotone piecewise-convex polygons are also valid for monotone locally convex polygons, i.e., curvilinear polygons that are locally convex except possibly at their vertices. The proof technique for producing the upper bound is identical to the case of monotone piecewise-convex polygons. Since monotone piecewise-convex polygons is a subclass of locally convex polygons, the lower bound construction presented in Theorem 18 still applies.*

## 6. Discussion and open problems

In this paper we have dealt with the problem of monitoring piecewise-convex polygons with edge or mobile guards. Our proof technique first transforms the problem of monitoring the piecewise-convex polygon to the problem of 2-dominating a constrained triangulation graph. For the problem of 2-dominance of triangulation graphs, we have shown that $\lfloor \frac{n+1}{3} \rfloor$ diagonal guards are always sufficient and sometimes necessary, while such a 2-dominating set can be computed in $O(n)$ time and space. When edge guards are to be used in the context of 2-dominance, $\lfloor \frac{2n+1}{5} \rfloor$ guards are always sufficient and sometimes necessary. We have not yet found a way to compute an edge 2-dominating set of size at most $\lfloor \frac{2n+1}{5} \rfloor$ in $o(n^2)$ time, whereas we have shown that it is possible to compute an edge 2-dominating set of size at most $\lfloor \frac{3n}{7} \rfloor$ in linear time and space. It, thus, remains an open problem how to compute an edge 2-dominating set of size at most $\lfloor \frac{2n+1}{5} \rfloor$ in $o(n^2)$ time and linear space.

Once a 2-dominating set $D$ has been found for the constrained triangulation graph, we either prove that $D$ is also a guard set for the piecewise-convex polygon (this is the case for edge guards) or we map $D$ to a mobile guard set for the piecewise-convex polygon. In the case of edge guards, the piecewise-convex polygon is actually monitored by the endpoints of the edges in the guard set. In the case of mobile guards, interior points of edges may also be needed in order to monitor the interior of the polygon. The latter observation should be contrasted against the corresponding results for the class of linear polygons, where, for both edge and mobile guards, the polygon is essentially monitored by the endpoints of these guards (cf. [29]). Based on our results on 2-dominance of triangulation graphs, we show that a mobile guard set of size at most $\lfloor \frac{n+1}{3} \rfloor$ can be computed in $O(n \log n)$ time and $O(n)$ space. As far as edge guards are concerned, we can either compute an edge guard set of size at most $\lfloor \frac{2n+1}{5} \rfloor$ in $O(n^2)$ time and $O(n)$ space, or an edge guard set of size at most $\lfloor \frac{3n}{7} \rfloor$ in $O(n \log n)$ time and $O(n)$ space. Finally, we have presented families of piecewise-convex polygons that require a minimum of $\lfloor \frac{n}{3} \rfloor$ mobile or $\lceil \frac{n}{3} \rceil$ edge guards in order to be monitored. An important remark, due to the lower bound of Theorem 8, is that the proof technique of this paper cannot possibly yield better results for the edge guarding problem. If we are to close the gap between the upper and lower bounds, a fundamentally different technique will have to be used.

When restricted to the subclass of monotone piecewise-convex polygons, we were able to derive better bounds on the number of edge or mobile guards that are sufficient in order to monitor these polygons. In particular, we can monitor monotone piecewise-convex polygons with $\lceil \frac{n+1}{4} \rceil$ edge or mobile guards, and this bound is tight for both types of guards. The same results apply to monotone locally convex polygons.

Thus far we have limited our attention to the class of piecewise-convex polygons. It would be interesting to attain similar results for locally concave polygons (i.e., curvilinear polygons that are locally concave except possibly at the vertices), for piecewise-concave polygons (i.e., locally concave polygons the edges of which are convex arcs), or for curvilinear polygons with holes.

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# Appendix

## A. 2-dominance with diagonal guards: alternative proof

The proof that follows is an alternative, much simpler proof for Theorem 3. Its disadvantage is that it makes use of edge contractions (cf. Lemma 5), thus yielding an $O(n^2)$ time and $O(n)$ space algorithm instead of a linear time and space algorithm, like the one provided in Section 2.

*Proof.* By Lemma 2 the theorem holds true for $3 \leq n \leq 7$. Let us now assume that $n \geq 8$ and that the theorem holds for all $n'$ such that $3 \leq n' < n$. By means of Lemma 1 with $\lambda = 3$, there exists a diagonal $d$ that partitions $T_P$ into two triangulation graphs $T_1$ and $T_2$, where $T_1$ contains $k$ boundary edges of $T_P$ with $3 \leq k \leq 4$. Let $v_i$, $0 \leq i \leq k$, be the $k+1$ vertices of $T_1$, as we encounter them while traversing $P$ counterclockwise, and let $v_0v_k$ be the common edge of $T_1$ and $T_2$. In what follows $d_{ij}$ denotes the diagonal $v_iv_j$, whereas $e_i$ denotes the edge $v_iv_{i+1}$. Consider each value of $k$ separately (see also Fig. A.33):

***k* = 3.** Without loss of generality let $d_{02}$ be the diagonal of the quadrilateral $T_1$. $T_2$ contains $n-2$ vertices. By Lemma 5 and our induction hypothesis, we can 2-dominate $T_2$ with $f(n-3)$ diagonal guards and $v_0$. $T_P$ can be 2-dominated by the $f(n-3)$ diagonal guards of $T_2$ plus the diagonal $d_{02}$.

***k* = 4.** In this case $T_2$ contains $n-3$ vertices. Let $t$ be the triangle in $T_1$ supported by $d$, and let $v$ be the third vertex of $t$ besides $v_0$ and $v_4$. The presence of diagonals $d_{03}$ or $d_{14}$ would violate the minimality of $k$, which implies that $v$ is actually $v_2$. By our induction hypothesis, we can 2-dominate $T_2$ with $f(n-3) = \lfloor \frac{n+1}{3} \rfloor - 1$ diagonal guards. Let $D_2$ be the diagonal 2-dominating set of $T_2$. Notice that at least one of $v_0$ and $v_4$ has to be in $D_2$. Let us assume, without loss of generality, that $v_0$ is in $D_2$. Then the set $D = D_2 \cup \{d_{24}\}$ is a diagonal 2-dominating set for $T_P$ of size $f(n-3)+1 = \lfloor \frac{n+1}{3} \rfloor$. $\square$

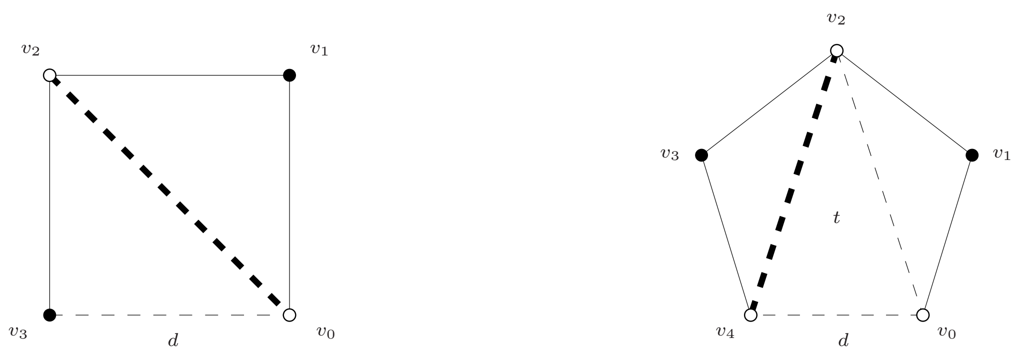


Figure A.33: Proof of Theorem 3. Left: $k = 3$. Right: $k = 4$.