Gröbli solution for three magnetic vortices

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Fluid vortices

Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen.

(Von Herrn *H. Helmholtz.*)

J. Reine Angew. Math. **55**, 22 (1858)

N ist in diesem Falle die Potentialfunction unendlich langer Linien; diese selbst ist unendlich grofs, aber ihre Differentialquotienten sind endlich. Sind *a* und *b* die Coordinaten eines Wirbelfadens, dessen Querschnitt *da db* ist, so ist

$$-v = \frac{dN}{dx} = \frac{\zeta\, da\, db}{\pi} \cdot \frac{x-a}{r^2}, \qquad u = \frac{dN}{dy} = \frac{\zeta\, da\, db}{\pi} \cdot \frac{y-b}{r^2}.$$

Derivation of the equations of fluid motion in the presence of straight vortex lines (point vortices).

Kirchhoff's lectures

"Vorlesungen über mathematische Physik. Mechanik"

(Teubner, Leipzig, 1876)

Equations for N point vortices at positions $\mathbf{r}_\alpha = (x_\alpha, y_\alpha)$ in Hamiltonian form:

$$\Gamma_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha}, \qquad \Gamma_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \qquad \alpha = 1, \ldots, N$$

where Γ_α is the vortex circulation, and H is the Hamiltonian

$$H = -\frac{1}{4\pi} \sum_{\alpha,\beta}{}' \, \Gamma_\alpha \Gamma_\beta \, \ln |\mathbf{r}_\alpha - \mathbf{r}_\beta|.$$

Specielle Probleme über die Bewegung geradliniger paralleler Wirbelfäden.

Von

Dr. W. Gröbli.

Gröbli (1877) (Zürcher and Furrer, Zurich): Explicit reduction to quadratures of the three-vortex problem for arbitrary vortex circulations.

Poincaré (1893): Noted existence of three integrals in involution. Thus the three-vortex problem is completely integrable for arbitrary vortex circulations.

Synge (1949) (Can. J. Math. Phys.): Geometrical interpretation of Gröbli's solutions, through use of trilinear coordinates.

Aref (1979) (Phys. Fluids): Rederivation of Gröbli's solution, and use of trilinear coordinates to interpret the results.

Ferromagnets

Consider a 2D ferromagnetic material (e.g., a ferromagnetic film). Magnetization properties are described by the local magnetization vector $\mathbf{m} = \mathbf{m}(\mathbf{r}, t)$ with $\mathbf{m}^2 = 1$.

A magnetic vortex

a magnetization $\mathbf{m} = (m_1, m_2, m_3)$ configuration satisfying:

$$m_1 + i\, m_2 = e^{i\kappa(\phi - \phi_0)}, \quad m_3 = 0, \quad \text{as } |\mathbf{r}| \to \infty$$

$$m_3(\mathbf{r} = 0) = \lambda.$$

$\kappa = \pm 1, \dots$ is the winding number (a topological invariant)

$\lambda = \pm 1$ is the vortex polarity

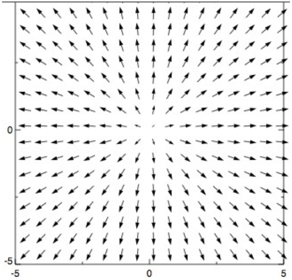
ϕ_0: the vortex phase (constant)

The skyrmion number s

is a further topological invariant and it counts the number of times $\mathbf{m}$ covers the sphere $\mathbf{m}^2 = 1$ (the degree of the mapping from the plane to the sphere). We have

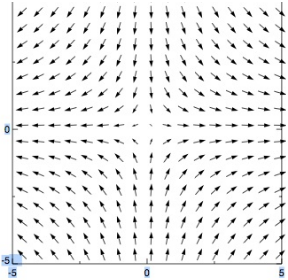
$$s = -\frac{1}{2}\kappa\lambda, \qquad \text{for simplicity}: \quad s = \kappa\lambda.$$

Vortex ($\kappa = 1,\ s = \pm 1$)



(m_1, m_2)

Antivortex ($\kappa = -1,\ s = \pm 1$)



(m_1, m_2)

Point magnetic vortices

In a collective-coordinate approximation the energy of N vortices is

$$H = -\sum_{\alpha<\beta} \kappa_\alpha \kappa_\beta \, \ln |\mathbf{r}_\alpha - \mathbf{r}_\beta|$$

$\mathbf{r}_\alpha = (x_\alpha, y_\alpha)$ are the vortex positions.

The equations of motion can be written in Hamiltonian form:

$$s_\alpha \frac{dx_\alpha}{dt} = \frac{\partial H}{\partial y_\alpha}, \qquad s_\alpha \frac{dy_\alpha}{dt} = -\frac{\partial H}{\partial x_\alpha}\,; \quad \alpha = 1, 2, \ldots N$$

$$\Rightarrow \lambda_\alpha \frac{dx_\alpha}{dt} = -\sum_{\beta\neq\alpha} \kappa_\beta \frac{y_\alpha - y_\beta}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|^2}, \qquad \lambda_\alpha \frac{dy_\alpha}{dt} = \sum_{\beta\neq\alpha} \kappa_\beta \frac{x_\alpha - x_\beta}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|^2}$$

Compare to Helmholtz-Kirchhoff equations:

- presence of skyrmion numbers $s_\alpha = \pm 1$
- κ_α take only integer values ($\kappa_\alpha = \pm 1$)

Conservation laws

Energy

$$H = -\sum_{\alpha<\beta} \kappa_\alpha \kappa_\beta \ \ln |\mathbf{r}_\alpha - \mathbf{r}_\beta|$$

Linear momentum

$$P_x = -\sum_\alpha s_\alpha \ y_\alpha, \qquad P_y = \sum_\alpha s_\alpha \ x_\alpha,$$

Angular momentum

$$L = \frac{1}{2}\sum_\alpha s_\alpha \ (x_\alpha^2 + y_\alpha^2)$$

From the above we can construct three integrals in involution.

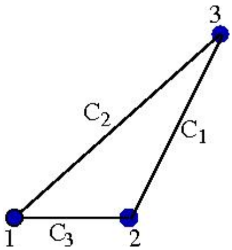
Therefore, the N-vortex problem for $N \leq 3$ is completely integrable.

Three magnetic vortices

Six equations of motion for (x_1, y_1), (x_2, y_2), (x_3, y_3).

Relative distances $[\mathbf{r}_\alpha = (x_\alpha, y_\alpha)]$:

$$C_1 = |\mathbf{r}_2 - \mathbf{r}_3|, \quad C_2 = |\mathbf{r}_3 - \mathbf{r}_1|, \quad C_3 = |\mathbf{r}_1 - \mathbf{r}_2|$$



The equations for the C_α's form the *closed system*

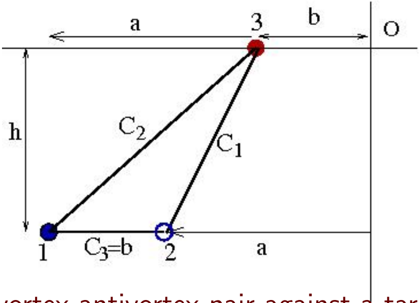
$$\frac{d}{dt}(C_1^2) = 4\kappa_1 A \left(\frac{1}{\lambda_3 C_2^2} - \frac{1}{\lambda_2 C_3^2} \right)$$
$$\frac{d}{dt}(C_2^2) = 4\kappa_2 A \left(\frac{1}{\lambda_1 C_3^2} - \frac{1}{\lambda_3 C_1^2} \right)$$
$$\frac{d}{dt}(C_3^2) = 4\kappa_3 A \left(\frac{1}{\lambda_2 C_1^2} - \frac{1}{\lambda_1 C_2^2} \right)$$

where A is the signed area of the vortex triangle.

A special three-vortex system

We focus on the specific case

$$(\kappa_1, \lambda_1) = (1, 1), \quad (\kappa_2, \lambda_2) = (-1, 1), \quad (\kappa_3, \lambda_3) = (1, -1)$$

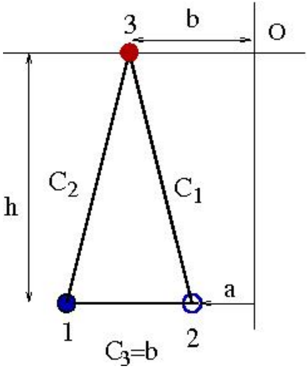


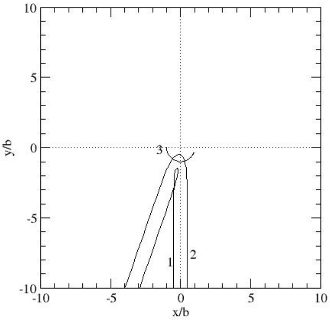
[scattering of a vortex-antivortex pair against a target vortex]

- *b*: is the vortex-antivortex (12) separation
- *h*: is the distance of the VA pair from the target vortex (3)
- *a*: is the impact parameter
- Origin has been placed so that linear momentum vanishes

Symmetrically placed vortex-antivortex pair

Choose impact parameter $a = -b/2$

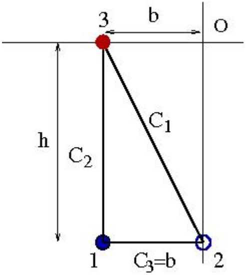


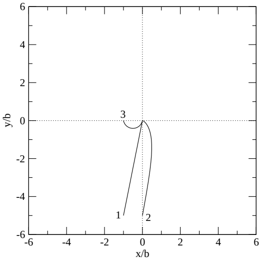


Scattering angle: $$\Delta\theta = \frac{2\pi}{\sqrt{5}}$$

Head-on collision

Choose $b = 1$ and impact parameter $a = 0$ (which gives angular momentum $L = 0$).





Solution

$$\frac{C_1}{B} - \arctan\left(\frac{C_1}{B}\right) = \frac{t_0 - t}{B^2}, \qquad B = \frac{b}{h}\sqrt{h^2 + b^2}$$

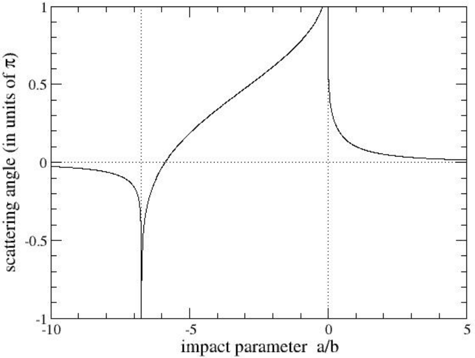
where t_0 (depends on initial condition) is the instance at which C_1, C_2, C_3 vanish simultaneously, or, the vortex triangle collapses to a point.

the scattering angle before collision is calculated as

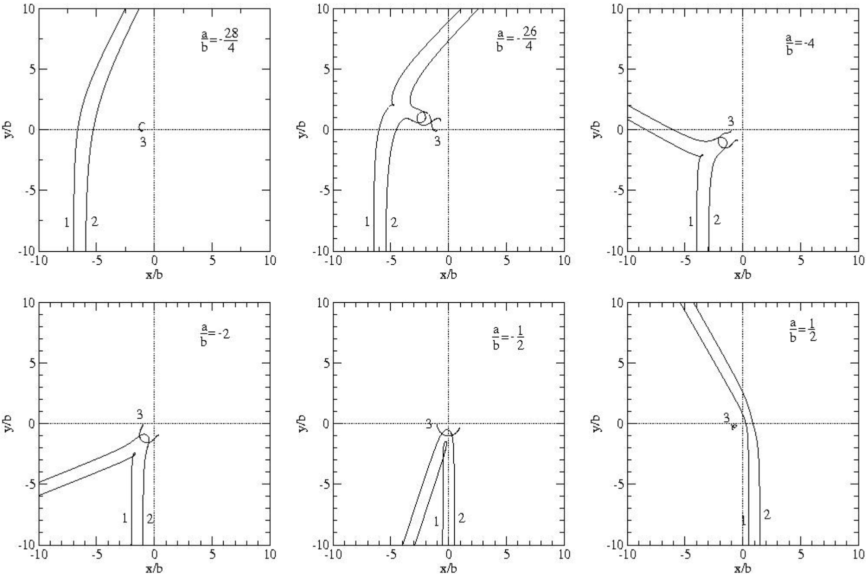
$$\arctan\left(\frac{h}{b}\right) \rightarrow \frac{\pi}{2}, \quad \text{for} \;\; h \rightarrow \infty.$$

That is, the total scattering angle is $2(\pi/2) = \pi$, and agrees with the picture of bouncing back for a particle after a head-on collsion.

Scattering angle as a function of impact parameter



Panel for three-vortex scattering



Experiment: Switching of vortex polarity

- An ac current generates an alternating magnetic field (250 MHz, 0.1 mT).
- Add a "burst" of 1.5 mT, for one period.
- Check that you obtained switching of vortex polarity!