Dynamics of skyrmions in chiral ferromagnets

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A ferromagnetic film

The magnetisation vector $\mathbf{M} = \mathbf{M}(x, y, t)$

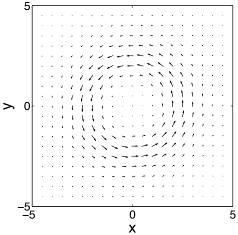
$\mathbf{M}^2(x, y, t) = M_s^2$, we typically normalise $\mathbf{m} = \mathbf{M}/M_s$, thus $\mathbf{m}^2 = 1$.

The skyrmion number

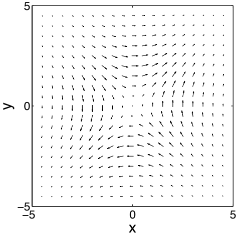
is a topological invariant and it counts the number of times that the magnetisation $\mathbf{m}$ covers the sphere $\mathbf{m}^2 = 1$:

$$Q = \frac{1}{4\pi} \int q \, d^2x, \quad q = \frac{1}{2} \epsilon_{\mu\nu} \mathbf{m} \cdot (\partial_\nu \mathbf{m} \times \partial_\mu \mathbf{m})$$ topological density

Skyrmion ($Q = 1$)



Antiskyrmion ($Q = -1$)



Antisymmetric exchange interaction: Dzyaloshinskii-Moriya (DM) materials

A typical and minimal energy functional for $\mathbf{m} = (m_1, m_2, m_3)$ is

$$W = W_{\rm ex} + W_{\rm a} + W_{\rm DM}.$$

- The usual symmetric exchange energy

$$W_{\rm ex} = \frac{1}{2} \int \partial_\mu \mathbf{m} \cdot \partial_\mu \mathbf{m} \, d^2x, \qquad \mu = 1, 2.$$

- An easy-axis anisotropy energy (with constant $\kappa > 0$)

$$W_{\rm a} = \frac{\kappa}{2} \int (m_1^2 + m_2^2) \, d^2x.$$

- An exchange of the Dzyaloshinskii-Moriya type ($\lambda = \pm 1$)

$$W_{\rm DM} = \lambda \int \mathbf{m} \cdot (\nabla \times \mathbf{m}) \, d^2x.$$

The Landau-Lifshitz (LL) equation

The conservative (Hamiltonian) LL equation associated with the energy is

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{f}, \qquad \mathbf{m}^2 = 1$$

$$\mathbf{f} \equiv -\frac{\delta W}{\delta \mathbf{m}} = \Delta \mathbf{m} + \kappa m_3 \hat{\mathbf{e}}_3 - 2\lambda \nabla \times \mathbf{m}.$$

Static solutions: $\mathbf{m} \times \mathbf{f} = 0$ - one dimension

A Bloch (domain) wall $\mathbf{m}(x) = (0, \sin\Theta(x), \cos\Theta(x))$, where

$$\tan\left(\frac{\Theta}{2}\right) = e^{\sqrt{\kappa}\,x}$$

has energy

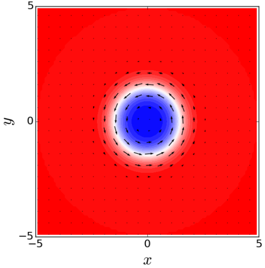
$$W = 2\sqrt{\kappa} - \pi\lambda.$$

Spiral state

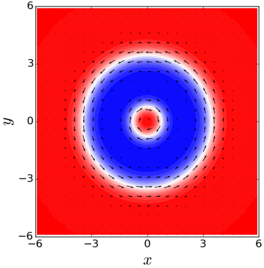
For $\kappa \to (\pi^2/4)\lambda^2$ the domain wall energy $W \to 0$. For $\kappa \geq (\pi^2/4)\lambda^2$ we have a proliferation of domain walls. A helical magnetisation configuration "a spiral" is the ground state of the system.

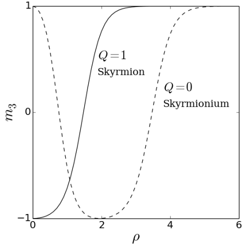
Static solutions: $\mathbf{m} \times \mathbf{f} = 0$ - two dimensions

Skyrmion ($Q = 1$)



Skyrmionium ($Q = 0$)





Stable excited states for $\kappa \geq (\pi^2/4)\lambda^2$

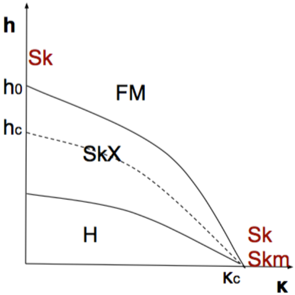
[A. N. Bogdanov and A. Hubert, JMMM (1999)]

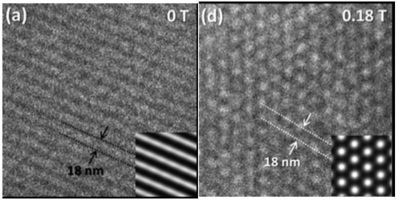
Skyrmionium-type configurations observed in (non-DM):

[Moutafis, et al, PRB (2007)]

[Finazzi, et al, PRL (2013)]

Phase diagram (sketch)





H: helix, **FM**: ferromagnetic state, **SkX**: skyrmion lattice (ground states)
Sk: skyrmion, **Skm**: skyrmionium (excited states)

$$h_c = \pi^2/16, \quad h_0 \approx 0.8, \qquad \kappa_c = \pi^2/4.$$

Dynamics of skyrmions

Fundamental relation for evolution of topological density [Papanicolaou, Tomaras, 1991]:

$$\dot{q} = -\epsilon_{\mu\nu}\partial_\mu(\mathbf{f}\cdot\partial_\nu\mathbf{m}) = \epsilon_{\mu\nu}\,\partial_\mu\partial_\lambda\sigma_{\nu\lambda}, \quad \mu,\nu,\lambda = 1,2$$

where $\mathbf{f}\cdot\partial_\mu\mathbf{m} = -\partial_\nu\sigma_{\mu\nu}$.

The tensor $\sigma_{\mu\nu}$ has components

$$\sigma_{11} = \frac{1}{2}\,(\partial_2\mathbf{m}\cdot\partial_2\mathbf{m} - \partial_1\mathbf{m}\cdot\partial_1\mathbf{m}) + \frac{\kappa}{2}(m_1^2 + m_2^2) + \lambda(m_1\partial_2 m_3 - m_3\partial_2 m_1$$

$$\sigma_{12} = -\,\partial_1\mathbf{m}\cdot\partial_2\mathbf{m} + \lambda(m_3\partial_1 m_1 - m_1\partial_1 m_3)$$

$$\sigma_{21} = -\,\partial_1\mathbf{m}\cdot\partial_2\mathbf{m} + \lambda(m_2\partial_2 m_3 - m_3\partial_2 m_2)$$

$$\sigma_{22} = \frac{1}{2}\,(\partial_1\mathbf{m}\cdot\partial_1\mathbf{m} - \partial_2\mathbf{m}\cdot\partial_2\mathbf{m}) + \frac{\kappa}{2}(m_1^2 + m_2^2) + \lambda(m_3\partial_1 m_2 - m_2\partial_1 m_3$$

Dynamics of skyrmions: I_μ

Define the moments of topological density q:

$$I_\mu = \int x_\mu q \, d^2x \qquad \mu = 1, 2.$$

Prove that they are conserved $\dot{I}_\mu = 0$ (by application of fundamental relation in previous page).

A rigid translation of spatial coordinates by a constant vector

$$x_\mu \to x_\mu + c_\mu \quad \Rightarrow \quad I_\mu \to I_\mu + 4\pi Q\, c_\mu$$

reveals difference between topological ($Q \neq 0$) and non-topological ($Q = 0$) magnetic solitons.

- For $Q \neq 0$, the (I_1, I_2) gives position of skyrmion and this is fixed.
- For $Q = 0$, skyrmions may propagate freely (solitary waves).

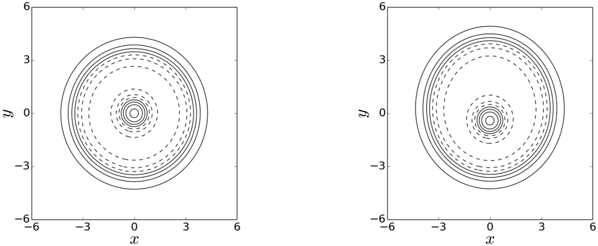
$Q = 0$ skyrmionium as a traveling wave

Assume propagating skyrmionium with velocity v (solitary wave). We make the traveling wave ansatz $\mathbf{m} = \mathbf{m}(x - vt, y)$ and this satisfies

$$v \frac{\partial \mathbf{m}}{\partial x} = \mathbf{m} \times \mathbf{f}.$$

We find numerically traveling solutions for $0 \leq v < v_c \approx 0.102$

$v = 0$ $\quad\quad$ $v = 0.07$

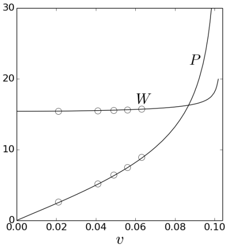


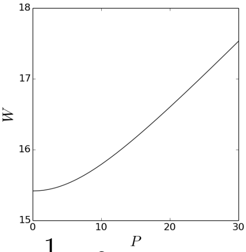
m_3 contour plots (solid lines: $m_3 > 0$, dashed lines: $m_3 < 0$)

Energy – Momentum relation

The linear momentum $\mathbf{P} = (P_1, P_2)$ is defined from

$$P_\mu = \epsilon_{\mu\nu} I_\nu.$$





$$(P_1 =)\ P = mv, \qquad W = W_0 + \frac{1}{2} m v^2, \qquad v \ll v_c$$

We may associate a mass (m) to the skyrmionium

At low momenta $W = W_0 + \frac{P^2}{2m}$ (Newtonian)

At high momenta $W \approx v_c P$ (relativistic).

Force and acceleration on a $Q = 0$ skyrmionium

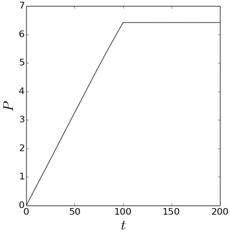
Apply an external non-homogeneous magnetic field, e.g.,

$$\mathbf{h} = (0, 0, h), \qquad h = g\,x.$$

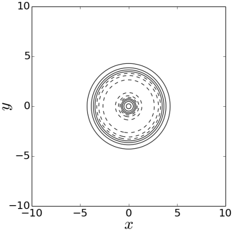
The force changes the linear momentum

$$\dot{P}_x = -\int \partial_x h (1 - m_3)\, d^2x, \quad \dot{P}_y = 0.$$

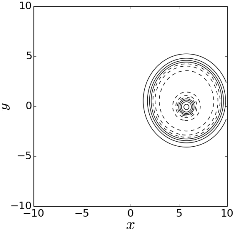
Force for $t \leq 100$



$t = 0$



$t = 160$



Skyrmion dynamics for $Q = 0$

When forced, it accelerates. Propagates freely in the absence of force.

Force on $Q = 1$ skyrmions

Apply a magnetic field gradient

$$\mathbf{h} = (0, 0, h), \qquad h = g\,x.$$

Skew deflection of magnetic bubbles in field-gradient

[Malozemoff, Slonczewski, "Magnetic Domain Walls in Bubble Materials", 1979]

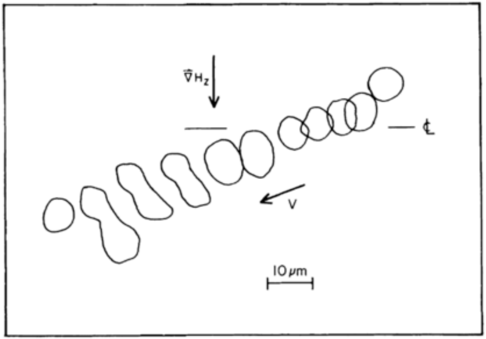


Fig. 13.2. Initial and final normal photographs and nine intermediate superimposed high-speed photographs of a hard bubble at the end of each of a sequence of nine gradient pulses of length 2 μsec and strength $H_g = |r\nabla H_z| = 4.5$ Oe oriented as indicated in a EuGaYIG film. The overall direction of the bubble motion illustrates the skew deflection of hard bubbles and the elliptical transient shape suggests a bunching effect. The horizontal lines indicate the center line of the gradient (after Patterson *et al.*[357]).

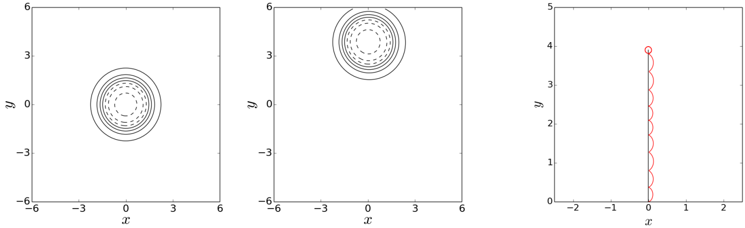
Hall motion of $Q = 1$ skyrmion

We follow the skyrmion guiding center $\mathbf{R} = (R_1, R_2)$:

$$R_\mu = \frac{I_\mu}{4\pi Q} = \frac{1}{4\pi Q} \int x_\mu q \, d^2x.$$

The evolution equations are calculated as

$$\dot{R}_x = 0, \quad \dot{R}_y = -\frac{1}{4\pi Q} \int \partial_x h \, (1 - m_3) \, d^2x.$$



Skyrmion dynamics for $Q \neq 0$

When forced, propagates with constant velocity.

It is spontaneously pinned in the absence of force.

Skyrmion dynamics under spin-transfer torque (and damping)

LL equation of motion

$$\frac{\partial \mathbf{m}}{\partial t} = -\,\mathbf{m} \times \mathbf{g}$$

$$\mathbf{g} = \frac{1}{1+\alpha^2}\left[\mathbf{f} + \alpha\,\mathbf{m} \times \mathbf{f} - (\beta-\alpha)u\,\partial_1\mathbf{m} - \alpha(\beta-\alpha)u\,\mathbf{m} \times \partial_1\mathbf{m}\right].$$

The time derivative of the topological density

$$\dot{q} = -\epsilon_{\mu\nu}\,\partial_\mu(\mathbf{g}\cdot\partial_\nu\mathbf{m})$$

The moments I_μ are no longer conserved

Integral relations which should be satisfied by all solutions of the LL eqn:

$$(1+\alpha^2)\dot{I}_1 = -(\beta-\alpha)u\,d_{12} + \alpha\,D_2 + (1+\alpha\beta)u\,(4\pi Q)$$

$$(1+\alpha^2)\dot{I}_2 = (\beta-\alpha)u\,d_{11} - \alpha\,D_1$$

where

$$d_{\mu\nu} = \int (\partial_\mu\mathbf{m}\cdot\partial_\nu\mathbf{m})\,d^2x, \qquad D_\mu = \int (\mathbf{m}\times\mathbf{f})\cdot\partial_\mu\mathbf{m}\,d^2x.$$

A traveling $Q = 0$ skyrmionium

We apply the integral relations for $Q = 0$ where the momentum is $(P_1, P_2) = (I_2, -I_1)$

$$(1+\alpha^2)\dot{P}_1 = (\beta - \alpha) u \, d_{11} - \alpha \, D_1$$
$$(1+\alpha^2)\dot{P}_2 = (\beta - \alpha) u \, d_{12} - \alpha \, D_2.$$

Stead-state propagation: virial relations

Spin torque + damping $\rightarrow$ steady-state

Assume a traveling wave

$$\mathbf{m}(x_1, x_2, t) = \mathbf{m}_0(\xi_1, \xi_2; v_1, v_2), \qquad \xi_1 \equiv x_1 - v_1 t, \quad \xi_2 = x_2 - v_2 t.$$

The LL equation reduces to

$$u\,\partial_1 \mathbf{m} - v_\nu \partial_\nu \mathbf{m} = -\mathbf{m} \times \mathbf{f} + \mathbf{m} \times (\beta\, u\, \partial_1 \mathbf{m} - \alpha v_\nu\, \partial_\nu \mathbf{m})\,.$$

Virial relations for steady states with velocity (v_1, v_2)

Take cross product of both sides with $\partial_\mu \mathbf{m}$ and then contract with $\mathbf{m}$:

$$(-4\pi Q + \alpha d_{21}) v_1 + \alpha d_{22} v_2 = \beta u\, d_{21} - u\,(4\pi Q)$$
$$\alpha d_{11} v_1 + (4\pi Q + \alpha d_{12}) v_2 = \beta u\, d_{11}.$$

Steady-state propagation for a $Q = 0$ skyrmionium

If the skyrmionium eventually reaches a traveling steady state then

$$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} \alpha v_1 - \beta u \\ \alpha v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} v_1 & = \frac{\beta}{\alpha} u \\ v_2 & = 0 \end{cases}$$

Concluding remarks

- The Dzyaloshinskii-Moriya interaction in ferromagnetic materials supports stable non-trivial magnetic patterns (domain wall, skyrmion, skyrmionium, etc).
- A topological $Q \neq 0$ skyrmion is pinned in a ferromagnetic film. It moves perpendicular to an applied force. The dynamics is analogous to the motion of an electron in a perpendicular magnetic field.
- A non-topological $Q = 0$ skyrmionium may move freely as a solitary wave. It responds as a Newtonian particle to forces.
- Integral relations are derived and used to test the results.