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Semiclassical nonlinear Schrödinger on the half line

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We are studying the semiclassical limit of the $1+1$ dimensional integrable nonlinear Schrödinger equation with defocusing cubic nonlinearity on the half line. Our analysis relies on the recent theory of Fokas *et al.*, which reduces boundary value problems for soliton equations to Riemann–Hilbert factorization problems. We employ the method of nonlinear steepest descent to asymptotically deform the given Riemann–Hilbert problem to an explicilty solvable one. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1624091]

I. AN INITIAL-BOUNDARY VALUE PROBLEM FOR THE NONLINEAR SCHRÖDINGER EQUATION

In recent years there has been a series of results by Fokas and others on *boundary value problems* for soliton equations (see Ref. 1 for a comprehensive review). The Fokas method goes beyond existence and uniqueness. In fact, it reduces such problems to Riemann–Hilbert factorization problems in the complex plane, thus generalizing the existing theory which reduces *initial value problems* to Riemann–Hilbert problems via the method of inverse scattering. One of the main advantages of the Riemann-Hilbert formulation is that one can use recent powerful results on the asymptotic behavior of solutions to these problems (as some parameter goes to infinity) to derive asymptotics for the solution of the associated soliton equation. Such methods were pioneered by Its and made rigorous and systematic by Deift and Zhou; the Deift–Zhou method is known as "nolinear steepest descent" in analogy with the linear steepest descent method which is applicable to asymptotic problems for Fourier-type integrals (see, e.g., Ref. 2). A generalization of the steepest descent method developed in Ref. 3 is able to give rigorous results for the so-called "semiclassical" or "zero dispersion" limit of the solution of the Cauchy problem for $1+1$ dimensional integrable evolution equations, in the case where the Lax operator is self-adjoint. The method has been further extended in Ref. 4 for the "non-self-adjoint" case, where in fact a "steepest descent" contour is, for the first time, introduced and its characterization and computation made systematic.

In this paper we consider the most basic example, that is the defocusing nonlinear Schrödinger (NLS) equation. (In a recent paper[5] we dealt with the simple problem of so-called linearizable data, for both the defocusing NLS and Korteweg–de Vries equations.) We make use of the recent results of Ref. 6 in order to study the so-called "semiclassical" limit of a particular initial-boundary value problem. More precisely we consider the $1+1$ dimensional, integrable, defocusing, nonlinear Schrödinger equation on the half-line

$$ihu_t(x,t)+h^2u_{xx}(x,t)-2|u(x,t)|^2u(x,t)=0,$$

$$u(x,0)=0, \qquad u(0,t)=f_0(t), \tag{1a}$$

$$x\geqslant 0, \quad t\geqslant 0,$$

where f_0 is assumed to be in the Schwartz space of the positive real line. We also assume that all

[a)]Electronic mail: spyros@mpim-bonn.mpg.de derivatives of $f_0(t)$ vanish at $t=0$. The special restriction of zero initial data is not essential but makes the computations and proofs somewhat easier. It is known[7] that the above problem is well-posed.

Our analysis is based on the results of Ref. 6, which considers the *a priori* overdetermined problem:

$$
\begin{gathered}
ihu_t(x,t)+h^2u_{xx}(x,t)-2|u(x,t)|^2u(x,t)=0,\\
u(x,0)=0, \qquad u(0,t)=f_0(t), \qquad u_x(0,t)=f_1(t),\\
x\geqslant 0, \quad t\geqslant 0.
\end{gathered}
\tag{1b}
$$

However, following Ref. 6, we will eventually impose a compatibility condition (the so-called "global relation") on the data f_0, f_1 which will ensure the existence (and uniqueness under such a condition) of a solution to (1b). It is also worthwhile noting (see Ref. 6) that given data $u(x,0)$ and $u(0,t)$ only, the global relation implicitly selects a function $f_1(t)=u_x(0,t)$ which complements the data f_0 and the initial data, and which will then ensure the existence of a solution to (1) and the validity of the Riemann–Hilbert formulation (see Theorem 1 below).

For the convenience of the reader we include an Appendix at the end of this paper containing a statement of some of the main results of Ref. 6.

It is well known that the above-mentioned equation admits a "Lax-pair" formulation. It arises as the compatibility condition for the equations $L\mu=0$ and $B\mu=0$ where the operators L, B are given by

$$
L=\begin{pmatrix} \partial_x-ik & iu \\ -i\overline{u} & \partial_x+ik \end{pmatrix},
$$

$$
B=\begin{pmatrix} ih\partial_t+4ik^2+i|u|^2 & -2ku-iu_t \\ -2k\overline{u}+i\overline{u_t} & ih\partial_t-i|u|^2 \end{pmatrix}.
$$

Here the bar denotes complex conjugation, k is the spectral variable, and $u=u(x,t)$ is the solution of (1a).

The traditional method of solving initial value problems for soliton equations that admit a similar Lax-pair formulation is to focus on the L operator and apply the theory of scattering and inverse scattering to that very operator. On the other hand, one of the main ideas of the Fokas method is that for initial-boundary value problems the two operators L and B should be on an equal footing. The scattering transform should be applied to both operators simultaneously, while a global relation has to be imposed on the data to ensure compatibility.

II. THE RIEMANN–HILBERT PROBLEM

As shown in Ref. 6, problem (1a) can be reduced to the following Riemann–Hilbert problem, under the special assumption that the so-called global relation holds (see relation (3.18) of Ref. 6; see also relation (5) below). One way to look at the global relation is as a way of selecting a solution of problem (1b). In fact it is known (see Ref. 7), using methods unrelated to inverse scattering theory, that (1b) has a unique solution. On the other hand, it has been shown in Ref. 6 that given data f_0 in (1b), there exists a function $f_1(t)$ such that the problem (1a) admits a solution, which furthermore can be explicitly characterized via inverse scattering and a particular Riemann–Hilbert factorization problem. Indeed, let Σ be the contour $\mathbb{R}\cup i\mathbb{R}$ with the following orientation:

(i) the real axis is oriented from left to right,
(ii) the positive imaginary axis is oriented from infinity toward zero,
(iii) the negative imaginary axis is oriented from infinity toward zero.

We use the following convention: the $+-$side of an oriented contour is always to its left, according to the given orientation.

Letting M_+ and M_- denote the limits of M on Σ from left and right, respectively, we define the Riemann–Hilbert factorization problem

$$M_+(x,t,k)=M_-(x,t,k)J(x,t,k),$$

where

$$\begin{aligned} J(x,t,k)=J_4^{-1}, &\qquad k\in\mathbb{R}^+,\\ J_1^{-1}, &\qquad k\in i\mathbb{R}^+,\\ J_3^{-1}, &\qquad k\in i\mathbb{R}^-,\\ J_2=J_3J_4^{-1}J_1, &\qquad k\in\mathbb{R}^-, \end{aligned} \tag{2}$$

with

$$J_1=\begin{pmatrix} 1 & 0\\ \Gamma(k)e^{2i\Theta} & 1\end{pmatrix},$$

$$J_4=\begin{pmatrix} 1 & -\gamma(k)e^{-2i\Theta}\\ \overline{\gamma}(k)e^{2i\Theta} & 1-|\gamma(k)|^2\end{pmatrix},$$

$$J_3=\begin{pmatrix} 1 & -\overline{\Gamma}(\overline{k})e^{-2i\Theta}\\ 0 & 1\end{pmatrix},$$

$$\Theta(x,t,k)=\frac{\theta}{h},$$

where

$$\theta=kx+2k^2t.$$

The functions γ,Γ are defined in terms of the spectral functions of the problem [see Appendix A, or Ref. 6, (2.28), (2.25)], with important analyticity properties [see Appendix A, or (2.21) and (2.22) of Ref. 6]. In particular note that

$$\Gamma(k)=\frac{1}{a(k)\left(a(k)\dfrac{\overline{A}(\overline{k})}{\overline{B}(\overline{k})}-b(k)\right)}, \tag{3}$$

where a, b are the spectral functions for the x problem and A, B are the spectral functions for the t problem. The functions a, b are analytic and bounded in the upper half-plane, while A, B are analytic and bounded in the first and third quadrants of the k plane. For our special choice of zero initial data, $b=0$, $a=1$. Hence $\gamma=0$ and $\Gamma(k)=\overline{B}(\overline{k})/\overline{A}(\overline{k})$. Note that a, b, A, B, Γ all depend on h.

The solution of (1a) can be recovered from the solution of (2) as follows:

$$u(x,t)=2i\ \ h\lim_{k\to\infty}(kM^{12}(x,t,k)), \tag{4}$$

where the index 12 here denotes the (12)-entry of a matrix.

The following "global relation" (see Appendix A for motivation and derivation) is imposed on the scattering data:

$$a(k)B(k)-b(k)A(k)=e^{4ik^2T}c(k),$$

where $c(k)$ is analytic and bounded for $\operatorname{Im} k>0$, and $c(k)=O(1/k)$ as $k\to\infty$. Here T is the time up to which we solve the initial boundary value problem for NLS. In general A, B are functions of T.

If $u_x(0,t)$ is denoted by $f_1(t)$ then there is a complicated relation between the data f_0 and f_1; the global relation is the expression of this in the spectral space.

In our particular case [problem (1a)] $T=\infty$ and the global relation becomes

$$a(k)B(k)-b(k)A(k)=0, \tag{5}$$

for $\arg(k)\in[0,\pi/2]$. For the special choice of zero initial data, since $b=0$, $a=1$, one has $B=0$ for $\arg(k)\in[0,\pi/2]$. In particular, $\Gamma(0)=0$.

The following is proved in Sec. V of Ref. 6.

Theorem 1: Given a Schwartz function f_0, there exists a unique f_1, also Schwartz, such that the above-given global relation is satisfied, and such that all derivatives of f_0,f_1 vanish at 0 (so that f_0,f_1 are compatible with NLS at $x=0,t=0$).

Using the theory developed in Ref. 6, we will then consider the (seemingly) overdetermined problem (1b) which in fact does have a unique solution, and which of course is the solution of problem (1a).

We note that in both the negative half-line and the positive half-line the jump matrix is of the same form. For positive k,

$$J=\begin{pmatrix} 1-|\gamma(k)|^2 & \gamma(k)e^{-2i\Theta} \\ -\bar{\gamma}(k)e^{2i\Theta} & 1 \end{pmatrix},$$

while for negative k,

$$J=\begin{pmatrix} 1-|\gamma(k)-\bar{\Gamma}|^2 & (\gamma(k)-\bar{\Gamma})e^{-2i\Theta} \\ -(\bar{\gamma}(k)-\Gamma)e^{2i\Theta} & 1 \end{pmatrix}.$$

In fact, let

$$r(k)=\gamma(k)-\bar{\Gamma}(k)=\frac{bA-aB}{\bar{a}A-\bar{b}B}, \qquad k<0,$$

$$r(k)=\gamma=\frac{b(k)}{\overline{a(k)}}, \quad k\geqslant 0.$$

Then, for all nonzero real k,

$$J=\begin{pmatrix} 1-|r(k)|^2 & r(k)e^{-2i\Theta} \\ -\bar{r}(k)e^{2i\Theta} & 1 \end{pmatrix}.$$

In the special case of zero initial data, the jump reduces to the identity for positive k, while for negative k, $r=-B/A$, so

$$J=\begin{pmatrix} 1-\left|\frac{B}{A}\right|^2 & \frac{-B}{A}e^{-2i\Theta} \\ \frac{\overline{B}}{\overline{A}}e^{2i\Theta} & 1 \end{pmatrix}. \tag{6}$$

III. DIRECT SCATTERING AS $h \to 0$

It is important to have some information about the "spectral" coefficent $r=-B/A$, for real values of k.

Theorem 2: For $k<0$, the spectral function $r(k,h)$ has the following asymptotic expression. There exist functions $\tilde{r},R_0$ of k alone, such that

$$r(k,h) \sim \tilde{r}(k)\exp\left(\frac{2iR_0(k)}{h}\right), \tag{7}$$

as $h\to 0$, where $\tilde{r}(k)$ is analytic and bounded as $k\to\infty$, and $R_0(k)$ is analytic. When $k \in i\mathbb{R}^+$, $\mathrm{Re}(iR_0)\leqslant 0$. Also $|\tilde{r}|\leqslant 1$.

Proof: The representation (7) follows from the standard Wentzel–Kramers–Brillouin (WKB) theory. Indeed, A, B admit representations of the form $s(k)\exp(iR(k)/h)$. Formula (7) thus follows. The analyticity of $\tilde{r}$, R_0 follows from the analyticity of A, B. The fact that $|A(k)|^2-|B(k)|^2=1$ (for real k) implies $|\tilde{r}|\leqslant 1$.

More detailed information about r can be recovered after a detailed analysis of the spectral problem for the second Lax operator. An easy calculation shows that the associated spectral problem reduces to a WKB problem of the type

$$h^2 y_{tt}=S(t,k)y,$$

where $S(t,k)$ is real. The spectral coefficients A, B can then be asymptotically estimated along the lines of Ref. 8 (Sec. 10.6). Eventually one is able to show the following.

Theorem 3: Let

$$f_2(t)=\frac{5}{8\cdot 2^{1/3}}[\mathrm{Re}(i\overline{f_1}(t)f_0(t))]^{4/3}+|f_0(t)|^4-|f_1(t)|^2.$$

Let $-f=\min\{f_2(t)\}$ over the interval $[0,\infty]$. Without loss of generality, we assume $-f<0$. [Otherwise the analysis becomes trivial; the coefficent $r(k,h)$ is everywhere small.] On the real line, the following holds.

For $-f<k<0$,

$$r(k,h)\sim -ie^{[2i\sigma(k)]/h},$$

where σ is smooth in $(-f,0)$ and takes real values. Also σ can be extended analytically in a small neighborhood of the segment $(-f,0)$.

For values of k away from the interval $(-f,0)$, $r(k,h)$ is either zero or uniformly exponentially small in h.

Furthermore one has an asymptotic formula for $1-|r(k)|^2$, $-f<k<0$. Indeed,

$$1-|r(k)|^2\sim\exp\left(\frac{-2\tau(k)}{h}\right),$$

where $\tau(k)$ is positive and can be extended analytically in a small neighborhood of $(-f,0)$.

Remark: An explicit integral formula for σ and τ in terms of the data can only be found assuming $u_x = f_1$ is known. In general, even though the existence of f_1 is guaranteed (given f_0) via the global relation, f_1 is not effectively computable.

However, in the special case of so-called ''linearizable data,'' f_1 and σ are indeed effectively computable (see Ref. 5).

Proof of Theorem 3: We simply note that the turning curve for the t-spectral problem is given by

$$S(t,k) = 4k^4 + k\,\mathrm{Re}(i\overline{f_1}(t) f_0(t)) + |f_0(t)|^4 - |f_1(t)|^2 = 0.$$

Indeed consider the t-problem

$$By = \begin{pmatrix} ih\partial_t + 4ik^2 + i|f_0|^2 & -2kf_0 - if_1 \\ -2k\overline{f_0} + i\overline{f_1} & ih\partial_t - i|f_0|^2 \end{pmatrix} \qquad y = 0.$$

Applying the operator

$$B^0 = \begin{pmatrix} ih\partial_t - i|f_0|^2 & 2kf_0 + if_1 \\ -2k\overline{f_0} - i\overline{f_1} & ih\partial_t + 4ik^2 + i|f_0|^2 \end{pmatrix}$$

we end up, up to errors of order $O(h)$, with

$$h^2 y_{tt} = S(t,k) y, \tag{8}$$

with $S(t,k)$ as above.

Note here that real k for which there exist L_2-solutions y of $By = 0$, are *a priori* excluded (see Ref. 6, p.16). So we do not need to concern ourselves with the possibility of real k for which the solutions to (8) are in L_2. We can then follow the WKB analysis of the semiclassical Schrödinger operator without essential changes (e.g., Ref. 8, Sec. 10.6). We can thus show that at all k such that $S(t,k) = k$ for some t, the reflection and transmission coefficients are given by the formulas above, while otherwise $r(k,h)$ is exponentially small (or zero). A short calculation shows that $S(t,k)$ as a function of k has only one local minimum, at $f_2(t)$, as defined in the statement of Theorem 3. The result follows immediately.

IV. REDUCING TO A PROBLEM ON THE REAL LINE

We next consider two Riemann–Hilbert problems with sole jumps given by

$$J_1^{-1}, \qquad k \in i\mathbb{R}^+,$$

$$J_3^{-1}, \qquad k \in i\mathbb{R}^-,$$

respectively.

We want U to be a function analytic in the complex plane except the upper imaginary axis, with normalization $\lim_{k\to\infty} U(x,t,k) = I$. The jump is prescribed by

$$U_+(x,t,k) = U_-(x,t,k) J_1^{-1}(x,t,k), \qquad k \in i\mathbb{R}^+,$$

with

$$J_1 = \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\Theta} & 1 \end{pmatrix},$$
$$\Theta(x,t,k) = \frac{\theta}{h}, \tag{9}$$

where

$$\theta = kx + 2k^2 t.$$

(The symbol U stands for "upper" since the jump is on the upper half-plane. We will immediately see however that U is a lower triangular matrix.)

Similarly, we want L to be a function analytic in the complex plane except the lower imaginary axis, with normalization $\lim_{k\to\infty} L(x,t,k) = I$. The jump is prescribed by

$$L_+(x,t,k) = L_-(x,t,k) J_3^{-1}(x,t,k), \qquad k \in i\mathbb{R}^-, \tag{10}$$

with

$$J_3 = \begin{pmatrix} 1 & -\overline{\Gamma}(\overline{k})e^{-2i\Theta} \\ 0 & 1 \end{pmatrix}.$$

(The symbol L stands for "lower" since the jump is on the upper half-plane. However L is an upper triangular matrix.)

The two Riemann–Hilbert problems above can be easily solved explicitly, since the jumps are triangular matrices. Indeed, direct calculations show that

$$U(x,t,k) = \begin{pmatrix} 1 & 0 \\ u(x,t,k) & 1 \end{pmatrix}, \tag{11}$$

where

$$u(x,t,k) = \frac{1}{2\pi i} \int_{i\mathbb{R}^+} \frac{\Gamma(s)e^{2i\Theta(x,t,s)}ds}{s-k}$$

satisifies (9). Similarly,

$$L(x,t,k) = \begin{pmatrix} 1 & l(x,t,k) \\ 0 & 1 \end{pmatrix}, \tag{12}$$

where

$$l(x,t,k) = \frac{1}{2\pi i} \int_{i\mathbb{R}^-} \frac{-\overline{\Gamma}(\overline{s})e^{-2i\Theta(x,t,s)}ds}{s-k}$$

satisfies (10). The direction of the integral contours is as prescribed in Sec. II, i.e., from infinity to zero.

Note that $u(k) = -\overline{l(\overline{k})}$. Note also that the integrals in (11) and (12) are not singular at $k = 0$, as $\Gamma(0) = 0$.

We next show that the Riemann–Hilbert problem (2) is equivalent to a problem on the real line. Indeed, let

$$N(x,t,k)=M(x,t,k)U^{-1}(x,t,k), \qquad \mathrm{Im}\, k>0,$$
$$N(x,t,k)=M(x,t,k)L^{-1}(x,t,k), \qquad \mathrm{Im}\, k<0. \tag{13}$$

Then $N(x,t,k)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$, with $\lim_{k\to\infty}N(x,t,k)=I$, and across $\mathbb{R}$ the jump is given by

$$N_+(x,t,k)=N_-(x,t,k)L(x,t,k)J(x,t,k)U^{-1}(x,t,k). \tag{14}$$

In fact, the new jump is given by

$$LJU^{-1}=\begin{pmatrix} 1-|l(k)+r(k)e^{-2i\Theta}|^2 & l(k)+r(k)e^{-2i\Theta} \\ -\overline{l}(k)-\overline{r}(k)e^{2i\Theta} & 1 \end{pmatrix}. \tag{15}$$

Note here that while $r(k)$ depends only on k but not on $x,t,l(k)$ depends on x,t,h via Θ by (12).

We have thus reduced the Riemann–Hilbert problem (2) to the problem (15) with only jump on the real line.

V. THE g-FUNCTION

We next show how the Riemann–Hilbert problem can be "deformed" to a problem that is explicitly solvable. We are essentially following the ideas of Ref. 3 (see also Ref. 4).

The first idea involves the so-called "g-function." We introduce a scalar function $g(k)$ which is to be analytic in $\mathbb{C}\backslash\mathbb{R}$ and decay like $O(1/k)$ at infinity. This function will be uniquely specified eventually.

Let

$$O(k)=N(k)\exp\left(\frac{ig(k)\sigma_3}{h}\right).$$

If N satisfies $N_+=N_-J, \quad k<0$, with J given by (6), then O solves a Riemann–Hilbert problem with jump matrix v_O, say, that is

$$O_+(x,t,k)=O_-(x,t,k)v_O(x,t,k), \tag{16}$$

$$v_O(k)=\begin{pmatrix} e^{[i(g_+-g_-)]/h}\left(1-\left|l+\frac{B}{A}e^{-2i\Theta}\right|^2\right) & \left(l-\frac{B}{A}e^{-2i\Theta}\right)e^{[-i(g_++g_-)]/h} \\ \left(-\overline{l}-\frac{\overline{B}}{\overline{A}}e^{2i\Theta}\right)e^{[i(g_++g_-)]/h} & e^{-(ig_+-ig_-)/h} \end{pmatrix}, \qquad k<0,$$

$$v_O(k)=\begin{pmatrix} e^{[i(g_+-g_-)]/h}(1-|l|^2) & l(k)e^{[-i(g_++g_-)]/h} \\ -\overline{l}(k)e^{[i(g_++g_-)]/h} & e^{[-(ig_+-ig_-)]/h} \end{pmatrix}, \qquad k>0,$$

$$\lim_{k\to\infty}O(k)=I.$$

Here g_+,g_- denote the limits of g from above and below the negative real axis, respectively.

Note that problem (16) is exactly (not just asymptotically) equivalent to the original Riemann–Hilbert problem (2). Formula (4) has to be replaced by

$$u(x,t)=2i\ \ h\lim_{k\to\infty}(kO^{12}(x,t,k))+2\partial_x\tilde{g}, \tag{17}$$

where $\tilde{g}$ is the residue of g at infinity.

VI. REDUCTION TO A SOLVABLE RIEMANN–HILBERT PROBLEM

Our first approximation involves getting rid of the functions l,u appearing in the jumps. The reason is simple. By formulas (11) and (12) u,l can be shown to be at worst $O(h)$ by use of the Laplace method for asymptotic evaluation of integrals, since the phase iR_0 of Theorem 2 has negative or zero real part. This suggests that l can be eventually erased from formula (16). In fact, we shall see right away that l *can* be neglected, not only because it is small, but also because of the precise factorization that follows.

Indeed, an easy calculation shows that the jump v_O of (16) can be written as

$$v_O=\begin{pmatrix} 1 & le^{2ig_+/h} \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} e^{[i(g_+-g_-)]/h}(1-|B/A|^2) & -\dfrac{B}{A}e^{-2i\Theta}e^{[-i(g_++g_-)]/h} \\ -\dfrac{\overline{B}}{\overline{A}}e^{2i\Theta}e^{[i(g_++g_-)]/h} & e^{[-(ig_+-ig_-)]/h} \end{pmatrix}$$

$$\cdot\begin{pmatrix} 1 & 0 \\ -\overline{l}e^{-2ig_-/h} & 1 \end{pmatrix}, \quad k<0, \tag{18}$$

$$v_O=\begin{pmatrix} 1 & le^{2ig_+/h} \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ -\overline{l}e^{-2ig_-/h} & 1 \end{pmatrix}, \qquad k>0.$$

Since g takes real values on $\mathbb{R}$ [this will be clear later, see formula (28)] and since $l=O(h)$, it follows that the triangular factors in (18) can be taken as the identity plus a resulting error of order at worst $O(h)$ in formula (18).

We have asymptotically reduced the Riemann–Hilbert problem (16) to a new Riemann–Hilbert problem for a matrix function $Q(z)$, say.

If Q is defined by

$$Q_+(x,t,k)=Q_-(x,t,k)v_Q(x,t,k),$$

$$v_Q(k)=\begin{pmatrix} e^{[i(g_+-g_-)]/h}(1-|B/A|^2) & -\dfrac{B}{A}e^{-2i\Theta}e^{[-i(g_++g_-)]/h} \\ \dfrac{\overline{B}}{\overline{A}}e^{2i\Theta}e^{[i(g_++g_-)]/h} & e^{[-(ig_+-ig_-)]/h} \end{pmatrix}, \qquad k<0, \tag{19}$$

$$\lim_{k\to\infty}Q(k)=I,$$

then Q is asymptotically equivalent to O in a neighborhood of ∞. In particular,

$$u(x,t)\sim 2ih\lim_{k\to\infty}(kO^{12}(x,t,k))+2\partial_x\tilde{g}. \tag{20}$$

The matrix v_Q can be written as

$$v_Q=\begin{pmatrix} e^{[i(g_+-g_-)]/h}(1-|r|^2) & r(k)e^{-2i\Theta}e^{[-i(g_++g_-)]/h} \\ -\overline{r}(k)e^{2i\Theta}e^{[i(g_++g_-)]/h} & e^{[-(ig_+-ig_-)]/h} \end{pmatrix}$$

$$=\begin{pmatrix} e^{[i(g_+-g_-)-2\tau]/h} & -ie^{[-2i\theta-i(g_++g_-)-2i\sigma]/h} \\ -ie^{[2i\theta+i(g_++g_-)+2i\sigma]/h} & e^{[-(ig_+-ig_-)]/h} \end{pmatrix},$$

$$\text{if } -f<k<0, \tag{21}$$

$$\sim\begin{pmatrix} e^{[i(g_+-g_-)]/h} & 0 \\ 0 & e^{[-(ig_+-ig_-)]/h} \end{pmatrix}, \quad \text{otherwise.}$$

We remind the reader that the functions σ,τ where introduced in the statement of Theorem 3.

At this point it becomes obvious that we should also impose

$$g_+-g_-=0, \qquad k>0, \quad or \qquad k<-f.$$

So g is to be analytic in $\mathbb{C}\setminus[-f,0]$.

Let

$$H=-g_+-g_--2\theta-2\sigma.$$

In the spirit of Ref. 3, we seek to reduce v_Q to a jump of one of the three following types:

$$\begin{pmatrix} 0 & -ie^{iH/h} \\ -ie^{-iH/h} & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -ie^{iH/h} \\ -ie^{-iH/h} & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -ie^{iH/h} \\ -ie^{-iH/h} & 0 \end{pmatrix}. \tag{22}$$

The motivation is the following. We expect that (22) will be deformable to a RH problem that can be explicitly solvable in terms of finite genus theta functions. Such a problem will have a "finite gap" structure. This means that the real line will be divided into a finite number of subintervals. In some of them the jump matrix has to look like

$$\begin{pmatrix} 0 & -ie^{iH/h} \\ -ie^{-iH/h} & 0 \end{pmatrix}$$

and in others it has to be the identity. We know however (through "lens"-type arguments, see Appendix B) that matrices of the first or third form in (22) can be reduced to the identity. Hence the ansatz (22).

To arrive from (21) to (22) we impose some seemingly artificial conditions on the functions g,H. For any given x,t, we will consider finite sequences of real numbers $-1\leqslant k_1<k_2\leqslant k_3<k_4\leqslant\cdots\leqslant k_{2G+1}<k_{2G+2}\leqslant 1$. We call the $G+1$ intervals $I_1=[k_1,k_2],\ldots,I_{G+1}=[k_{2G+1},k_{2G+2}]$ the "bands." Both the integer G and the real numbers $k_j, j=1,\ldots,2G+2$ are to be eventually determined for each x,t.

We make the ansatz that for each x,t the interval $[-f,0]$ can be subdivided into intervals by such k_j, such that on each of the arising intervals one of the three conditions holds:

$$-2i\tau=g_+-g_- \qquad \text{and} \qquad H'<0,$$

or

$$-2i\tau<g_+-g_-<0, \qquad \text{and} \qquad H'=0, \tag{23}$$

or

$$g_+-g_-=0, \qquad \text{and} \qquad H'>0.$$

H' denotes the derivative of H with respect to k. In particular, the intervals where $H'=0$ are to be the bands I_j, while on the intervals of which $[-f,0]\backslash\cup I_j$ consists, either the first or the third condition has to hold.

We will eventually see that conditions (23) amount to a scalar Riemann–Hilbert problem that can be solved explicitly, plus a set of algebraic conditions on the end points k_j defining the gap structure, plus a set of inequalities which essentially pick up the appropriate number of k_j's.

Now differentiating (with respect to k) the scalar Riemann–Hilbert problem given by the equalities in (23) and solving for g' leads to

$$g'(k)=(p(k))^{1/2}\left(\int_{\cup I_j}\frac{2\sigma'(\mu)-2\theta'(\mu)}{(p(\mu))_+^{1/2}(\mu-k)}\frac{d\mu}{2\pi i}+\int_{(-f,0)\backslash\cup I_j}\frac{-2i\tau'(\mu)}{(p(\mu))_+^{1/2}(\mu-k)}\frac{d\mu}{2\pi i}\right),$$

where

$$p(k)=\Pi_{j=1}^{G+1}(k-k_{2j-1})(k-k_{2j}). \tag{24}$$

We have imposed the condition $g(k)=O(k^{-1}),\ \ as\ \ k\to\infty$. Easy calculations then show that g' has to satisfy the moment conditions

$$\int_{\cup I_j}\frac{\sigma'(k)-\theta'(k)}{(p(k))_+^{1/2}}k^l dk+\int_{(-f,0)\backslash\cup I_j}\frac{-i\tau'(k)}{(p(k))_+^{1/2}}k^l dk=0, \tag{25}$$

$$l=0,1,2,\ldots,G.$$

Also, integrating g' around I_j and requiring $H'<0$, we obtain

$$\int_{I_j}(g'_+-g'_-)d\lambda=-2i(\tau(k_{2j-1})-\tau(k_{2j})),\quad j=1,\ldots,G+1. \tag{26}$$

Conditions (25) and (26) form a set of $2G+2$ equations for $2G+2$ unknowns. They enable us to solve for k_j.

At this point, we note that H is smooth in $[-f,0]$. We also note that it admits analytic continuations in (possibly small) lens-like domains of the complex plane, not including the points k_j.

In fact, conditions (25) and (26) together with the inequalities in (23) reduce to the Euler–Lagrange conditions of a variational problem. This is virtually the same variational problem introduced by Lax and Levermore and the existence and uniqueness of its solution is guaranteed by the theory of variational problems of logarithmic potentials (see Ref. 9 for a discussion). Since a complete written proof has not appeared anywhere so far, we will simply state a hypothesis.

Hypothesis: Assume that the data f_0 are real analytic and rapidly decaying (say Schwartz). Then for each x,t there is a finite non-negative integer G for which both equalities and inequalities in (23) have a solution. In other words, the ‘‘finite genus ansatz’’ can be eventually justified.

Once the existence of an appropriate ‘‘g-function’’ is guaranteed, it is straightforward to reduce our Riemann–Hilbert problem to its final form.

At the end of this procedure, and because of conditions (23), the jump contour consists of the bands $I_j, j=1,\ldots,G+1$ and on each band, the jump matrix is given by

$$w_j=\begin{pmatrix}0 & -ie^{iH/h}\\ -ie^{-iH/h} & 0\end{pmatrix}.$$

Furthermore, H is constant on each band. We actually have

$$w_j=\begin{pmatrix} 0 & -ie^{i\Omega_j/h} \\ -ie^{-i\Omega_j/h} & 0 \end{pmatrix}, \tag{27}$$

where the Ω_j are real constants.

The Ω_j can be computed explicitly. But first, let us note that the Riemann–Hilbert problem with jumps along the intervals I_j given by (27) can be explicitly solved in terms of theta functions. To appropriately define those functions we first need to introduce an underlying Riemann surface, together with some associated holomorphic differentials.

Let X be the two-sheeted Riemann surface of genus G associated with $(p(k))^{1/2}$, obtained by adjoining two copies of the slit plane $\mathbb{C}\setminus\cup_k I_k$. On the "top" sheet $(p(k))^{1/2}\sim k^{G+1}$ and on the "bottom" sheet $(p(k))^{1/2}\sim -k^{G+1}$. The branch points of the surface will be the end points of the "bands," that is, $k_1,\ldots,k_{2G+2}$. The homology cycles are defined in a standard way as follows. The cycles A_k lie on the top sheet and encircle slits I_k. The cycles B_k pass from the top sheet through the slit I_1 to the bottom sheet and back again through I_k.

The basis $\omega=(\omega_1,\ldots,\omega_G)$ of holomorphic differentials on X is determined by the normalization

$$\int_{A_i}\omega_j=\delta_{ij},1\leqslant i,\quad j\leqslant G.$$

The Riemann-matrix of periods is

$$\tau=(\tau_{ij})=\left(\int_{B_i}\omega_j\right)_{1\leqslant i,j\leqslant G}.$$

By the Riemann bilinear relations, τ is symmetric and $i\tau$ is negative definite. We can thus define the associated theta function

$$\theta_G(s)=\Sigma_{m\in\mathbb{Z}^G}\exp(2\pi i(m,s)+\pi i(m,\tau m)),\qquad s\in\mathbb{C}^G,$$

where (.,.) is the real scalar product. Note that θ_N is an even function.

Now, solving the scalar Riemann–Hilbert for g (not its derivative) we get

$$g(k)=(p(k))^{1/2}\left(\int_{\cup I_j}\frac{2\sigma(\mu)-2\theta(\mu)-\Omega_j}{(p(\mu))_+^{1/2}(\mu-k)}\frac{d\mu}{2\pi i}+\int_{(-f,0)\setminus\cup I_j}\frac{-2i\tau(\mu)}{(p(\mu))_+^{1/2}(\mu-k)}\frac{d\mu}{2\pi i}\right). \tag{28}$$

Applying the condition that $g(k)=O(k^{-1}),\ \ as\ \ k\to\infty$ once more, we get the conditions

$$\int_{\cup I_j}(2\sigma(k)-2\theta(k)-\Omega_j)\omega_l+\int_{(-f,0)\setminus\cup I_j}(-2i\tau(k))\omega_l=0,$$

$$l=1,2,\ldots,G.$$

Recalling the definition of the normalized differentials ω_l, we immediately get the following:

$$\Omega_l=\int_{\cup I_j}(2\sigma(k)-2\theta(k))\omega_l+\int_{(-f,0)\setminus\cup I_j}(-2i\tau(k))\omega_l,$$

$$l=1,2,\ldots,G. \tag{29}$$

We also define the following function:

$$\zeta(k)=\left[\Pi_{i=1}^{G+1}\frac{k-k_{2i}}{k-k_{2i-1}}\right]^{1/4},$$

where ζ is meant to be analytic off the union of the "gaps," i.e., the intervals between the bands, and $\zeta(k)\sim 1$, as $k\to\infty$. The function ζ has the important property that $\zeta\pm\zeta^{-1}$ has $G+1$ roots $(\zeta_j^{\pm})_{j=1}^{G+1}$, lying in the bands I_j, one root in each band. Note also that $\zeta_+=i\zeta_-$ across the gaps.

We next define the "Abel map" integral, for k on the top sheet of the Riemann surface X. Let

$$u(k)=\int_0^k\omega,$$

where the integral is taken along any path on the top sheet. Note that it is well-defined modulo $\mathbb{Z}^G$. Also define the constant vector

$$d=-K-\sum_{j=1}^{G}\int_0^{P_2(\zeta_j)}\omega,$$

where K is the vector of Riemann constants and $P_2(z)$ denotes the preimage of a point $z\in X$ in the "bottom" sheet. We can now state the following theorem.

Theorem 4: The function Q defined by problem (19) is asymptotically equivalent, as $h\to 0$, to

$$\mathrm{diag}\left(\frac{\theta_G(u(\infty)+d)}{\theta_G\left(u(\infty)+\dfrac{\Theta_G}{2\pi h}+d\right)},\frac{\theta_G(-u(\infty)+d)}{\theta_G\left(-u(\infty)+\dfrac{\Theta_G}{2\pi h}-d\right)}\right)$$
$$\cdot\begin{pmatrix}\dfrac{\zeta+\zeta^{-1}}{2}\dfrac{\theta_G\left(u(k)+\dfrac{\Theta_G}{2\pi h}+d\right)}{\theta_G(u(k)+d)} & e^{(-iH_{G+1})/h}\dfrac{\zeta-\zeta^{-1}}{2i}\dfrac{\theta_G\left(u(k)+\dfrac{\Theta_G}{2\pi h}-d\right)}{\theta_G(u(k)-d)}\\ e^{(iH_{G+1})/h}\dfrac{\zeta-\zeta^{-1}}{-2i}\dfrac{\theta_G\left(-u(k)+\dfrac{\Theta_G}{2\pi h}+d\right)}{\theta_G(-u(k)+d)} & \dfrac{\zeta+\zeta^{-1}}{2}\dfrac{\theta_G\left(-u(k)+\dfrac{\Theta_G}{2\pi h}-d\right)}{\theta_G(-u(k)+d)}\end{pmatrix},\tag{30}$$

where $\Theta_G=(\Omega_1,\ldots,\Omega_G)^T$, the Ω_j being given by (29). The asymptotics is uniform in compact subsets of the Riemann sphere with the bands I_j deleted.

The proof consists of a straightforward check of the jump relations. The important fact is that (because of our choice of d) the zeros of $\zeta\pm\zeta^{-1}$ exactly cancel the poles of the theta function quotients.

The semiclassical asymptotics for the solution of (1a) follows from (30) and (4).

Theorem 5: The asymptotics for $u(x,t,h)$, the solution of (1a), as $h\to 0$, is given by

$$u(x,t,h)\sim[\Sigma_{j=1}^{2G+2}k_j]e^{(-iH_{G+1})/h}\cdot\frac{\theta_G(u(\infty)+d)\,\theta_G(u(\infty)+\Theta_G/(2\pi h)-d)}{\theta_G(u(\infty)-d)\,\theta_G(u(\infty)+\Theta_G/(2\pi\hbar)+d)}.\tag{31}$$

Formula (31) expresses a slowly modulated wave.

VII. CONCLUSION

Since we have been able to reduce our Riemann–Hilbert problem to one that arises in the full-line problem, the results of Refs. 10, 11, and 3 on the phenomenology of the solution as $h\to 0$ apply.

Semiclassically, the half-plane $x,t\geqslant 0$ can be divided in two regions. In the first ("smooth") region the strong semiclassical limit exists and satisfies the formallly limiting Euler system. In the second ("turbulent") region fast oscillations appear that can be described in terms of slowly modulating finite-gap solutions. Weak limits of an infinite number of densities including $|u|^2$, $ih(u\overline{u_x}-u_x\overline{u})$ exist.

We also note that the Whitham equations theory is still relevant. The functions $k_j(x,t)$ are in fact the Riemann invariants of the Whitham equations. The equations themselves can be derived by differentiating (25) and (26) (see, e.g., Ref. 4).

Let us also note that, even though the assumption that the initial data are equal to zero makes the analysis somewhat easier, it is not essential. In particular the above qualitative discussion of the semiclassical limit is still valid.

Finally, let us speculate on the long time asymptotics of the semiclassical limit.

There are two ways of computing the long time semiclassical limit of the defocusing NLS on the full line (see Ref. 12 or 13). One is to use the existing theory for times of order 1 (as in Ref. 11) and take the limit $t\to\infty$.

Alternatively, one should in principle be able to look at the long time behavior of the problem with fixed ϵ and *then* take $\epsilon\to 0$. This is by no means obvious *a priori*, but it turns out that this idea gives the right results. See, for example, Ref. 12, where the author has computed the long time semiclassical limit of the defocusing NLS on the full line.

On the half-line, it is already known what the long time of the problem with fixed ϵ is. As in the full-line case, any initial data degenerate into a sequence of finitely many separated solitons (see Refs. 14 and 6).

It then should follow, in exact analogy with the full line case,[12] that the long time asymptotics of the semiclassical limit in the half line case can be described by a sequence of solitons (in the turbulent region). The number of solitons is finite but increasing like $O(1/\epsilon)$ as $\epsilon\to 0$. Their width is $O(\epsilon)$ and they are separated by a distance of order $O(\epsilon t)$. In the smooth region, the solution simply dies out.

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APPENDIX A: THE SCATTERING DATA FOR THE PROBLEM ON THE HALF LINE

In this appendix we quote freely from the paper of Fokas, Its and Sung (Ref. 6). We introduce the quantities a,b,A,B referred to in Sec. II and we state the analytic properties of these quantities.

We consider the NLS equation

$$iu_t+u_{xx}-2|u|^2u=0. \tag{A1}$$

Here we have set $h=1$. For general positive h one can reduce Eq. (1a) to Eq. (A1) through the obvious rescaling $x\to x/h$, $t\to t/h$. Equation (A1) admits the Lax pair

$$\mu_x+ik[\sigma_3,\mu]=Q(x,t)\mu,$$

$$\mu_t+2ik^2[\sigma_3,\mu]=\tilde{Q}(x,t,k)\mu,$$

where $\sigma_3=\mathrm{diag}(1,-1)$, and

$$Q(x,t)=\begin{pmatrix} 0 & u(x,t) \\ \overline{u(x,t)} & 0 \end{pmatrix}, \qquad \tilde{Q}(x,t,k)=2kQ-iQ_x\sigma_3-i\lambda|u|^2\sigma_3.$$

Let $\hat{\sigma}_3$ denote the commutator with respect to σ_3, then $(\exp\hat{\sigma}_3)A$ can be computed easily:

$$\hat{\sigma}_3A=[\sigma_3,A], \quad e^{\hat{\sigma}_3}A=e^{\sigma_3}Ae^{-\sigma_3},$$

where A is a 2×2 matrix.

The Lax pair can be rewritten as

$$d(e^{i(kx+2k^2t)\hat{\sigma}_3}\mu(x,t,k))=W(x,t,k), \tag{A2}$$

where the exact one-form W is defined by

$$W=e^{i(kx+2k^2t)\hat{\sigma}_3}(Q\mu\,\mathrm{d}x+\tilde{Q}\mu\,\mathrm{d}t).$$

Let Eq. (A1) be valid for

$$0<x<\infty, \quad 0<t<T,$$

where $T\leqslant\infty$ is a given positive constant. Assume that there exists a function $u(x,t)$ with sufficient smoothness and decay. A solution of Eq. (A2) is given by

$$\mu_*(x,t,k)=I+\int_{(x_*,t_*)}^{(x,t)} e^{-i(kx+2k^2t)\hat{\sigma}_3}W(\xi,\tau,k), \tag{A3}$$

where I is the 2×2 identity matrix, (x_*,t_*) is an arbitrary point in the domain $0<\xi<\infty$, $0<\tau<T$, and the integral is over a (piecewise) smooth curve from (x_*,t_*) to (x,t). Since the one-form W is exact, μ_* is independent of the path of integration. The analyticity properties of μ_* with respect to k depend on the choice of (x_*,t_*). It was shown in Ref. 15 that for a polygonal domain there exists a canonical way of choosing the points (x_*,t_*), namely, they are the corners of the associated polygon. Thus we define three different solutions μ_1, μ_2, μ_3, corresponding to $(0,T)$, $(0,0)$, (∞,t). Also we choose the particular contours as follows: The first contour consists of the oriented linear segments $(0,T)$ to $(0,t)$ and $(0,t)$ to (x,t). The second contour consists of the oriented linear segments from $(0,0)$ to $(0,t)$ and from $(0,t)$ to (x,t). The third contour is parallel to the x axis and is oriented from $(0,+\infty)$ to (x,t).

This choice implies the following inequalities:

$$\mu_1:\ \xi-x\leqslant 0, \quad \tau-t\geqslant 0,$$

$$\mu_2:\ \xi-x\leqslant 0, \quad \tau-t\leqslant 0,$$

$$\mu_3:\ \xi-x\geqslant 0.$$

The second column of the matrix equation (A3) involves $\exp[i(k(\xi-x)+2k^2(\tau-t))]$. Using the above-mentioned inequalities it follows that this exponential is bounded in the following regions of the complex k plane:

$$\mu_1:\{\Im k\leqslant 0\cap\Im k^2\geqslant 0\},$$

$$\mu_2:\{\Im k\leqslant 0\cap\Im k^2\leqslant 0\},$$

$$\mu_3:\{\Im k\geqslant 0\}.$$

Thus the second column vectors of μ_1, μ_2 and μ_3 are bounded and analytic for $\arg k$ in $(\pi,3\pi/2)$, $(3\pi/2,2\pi)$ and $(0,\pi)$, respectively. We will denote these vectors with superscripts (3), (4), and (12) to indicate that they are bounded and analytic in the third quadrant, fourth quadrant, and the upper half of the complex k plane, respectively. Similar conditions are valid for the first column vectors, thus

$$\mu_1(x,t,k)=(\mu_1^{(2)},\mu_1^{(3)}), \quad \mu_2(x,t,k)=(\mu_2^{(1)},\mu_2^{(4)}), \quad \mu_3(x,t,k)=(\mu_3^{(34)},\mu_3^{(12)}). \tag{A4}$$

We note that the functions μ_1 and μ_2 are entire functions of k. Equation (A4) together with the estimate

$$\mu_j(x,t,k)=I+O\left(\frac{1}{k}\right), \quad k\to\infty, \quad j=1,2,3, \tag{A5}$$

imply that the functions μ_j are the fundamental eigenfunctions needed for the formulation of a RH problem in the complex k plane. The jump matrix of this RH problem is uniquely defined in terms of the 2×2-matrix valued functions

$$s(k)=\mu_3(0,0,k), \quad S(k)=[e^{2ik^2T\hat{\sigma}_3}\mu_2(0,T,k)]^{-1}. \tag{A6}$$

This is a direct consequence of the fact that (in the domain where μ_3 is defined) any two solutions of (A3) are simply related,

$$\begin{gathered}\mu_3(x,t,k)=\mu_2(x,t,k)e^{-i(kx+2k^2t))\hat{\sigma}_3}\mu_3(0,0,k),\\ \mu_1(x,t,k)=\mu_2(x,t,k)e^{-i(kx+2k^2t))\hat{\sigma}_3}[e^{2ik^2T\hat{\sigma}_3}\mu_2(0,T,k)]^{-1}.\end{gathered} \tag{A7}$$

The functions $s(k)$ and $S(k)$ follow from the evaluations at $x=0$ and $t=T$, respectively, of the function $\mu_3(x,0,k)$ and of $\mu_2(0,t,k)$ which satisfy the following linear integral equations:

$$\begin{gathered}\mu_3(x,0,k)=I+\int_\infty^x e^{ik(\xi-x)\hat{\sigma}_3}(Q\mu_3)(\xi,0,k)\,d\xi,\\ \mu_2(0,t,k)=I+\int_0^t e^{2ik^2(\tau-t)\hat{\sigma}_3}(\tilde{Q}\mu_2)(0,\tau,k)\,d\tau.\end{gathered} \tag{A8}$$

The fact that Q and $\tilde{Q}$ are traceless together with (A5) imply $\det \mu_j(x,t,k)=1$ for $j=1,2,3$. Thus

$$\det s(k)=\det S(k)=1.$$

From the symmetry properties of Q and $\tilde{Q}$ it follows that

$$(\mu(x,t,k))_{11}=\overline{(\mu(x,t,\bar{k})_{22}}, \quad (\mu(x,t,k))_{21}=\overline{\mu(x,t,\bar{k})_{12}},$$

and thus

$$s_{11}(k)=\overline{s_{22}(\bar{k})}, \quad s_{21}(k)=\overline{s_{12}(\bar{k})}, \quad S_{11}(k)=\overline{S_{22}(\bar{k})}, \quad S_{21}(k)=\overline{S_{12}(\bar{k})}.$$

We will use the following notation for s and S:

$$s(k)=\begin{pmatrix}\overline{a(\bar{k})} & b(k)\\ \overline{b(\bar{k})} & a(k)\end{pmatrix}, \quad S(k)=\begin{pmatrix}\overline{A(\bar{k})} & B(k)\\ \overline{B(\bar{k})} & A(k)\end{pmatrix}.$$

The definitions of $\mu_j(0,t,k)$, $j=1,2$, and of $\mu_2(x,0,k)$ imply that these functions have larger domains of boundedness,

$$\mu_1(0,t,k)=(\mu_1^{(24)}(0,t,k),\mu_1^{(13)}(0,t,k)),$$

$$\mu_2(0,t,k)=(\mu_2^{(13)}(0,t,k),\mu_2^{(24)}(0,t,k)),$$

$$\mu_2(x,0,k)=(\mu_2^{(12)}(x,0,k),\mu_2^{(34)}(x,0,k)).$$

The definitions of $s(k)$, $S(k)$ imply

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \mu_3^{(12)}(0,0,k), \quad \begin{pmatrix} -e^{-4ik^2T}B(k) \\ \overline{A(\bar{k})} \end{pmatrix} = \mu_2^{(24)}(0,T,k),$$

where the vectors $\mu_3^{(12)}(x,0,k)$ and $\mu_2^{(24)}(0,t,k)$ satisfy the following ODEs:

$$\partial_x \mu_3^{(12)}(x,0,k) + 2ik \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mu_3^{(12)}(x,0,k) = Q(x,0)\mu_3^{(12)}(x,0,k), \quad 0 \leqslant \arg k \leqslant \pi,\ 0 < x < \infty,$$

$$\lim_{x\to\infty} \mu_3^{(12)}(x,0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and

$$\partial_t \mu_2^{(24)}(0,t,k) + 4ik^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mu_2^{(24)}(0,t,k)$$

$$= \tilde{Q}(0,t,k)\mu_2^{(24)}(0,t,k), \quad \arg k \in [\pi/2,\pi] \cup [3\pi/2, 2\pi], \quad 0 < t < T,$$

$$\mu_2^{(24)}(0,0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The above definitions imply the following properties:

$$a(k), b(k) \text{ are defined and analytic for } \arg k \in (0,\pi).$$

$$|a(k)|^2 - |b(k)|^2 = 1, \qquad k \in \mathbb{R}.$$

$$a(k) = 1 + O\left(\frac{1}{k}\right), \quad b(k) = O\left(\frac{1}{k}\right), \quad k \to \infty.$$

Also $A(k), B(k)$ are entire functions bounded for $\arg k \in [0,\pi/2] \cup [\pi, 3\pi/2]$. If $T = \infty$, the functions $A(k)$ and $B(k)$ are defined and analytic in the quadrants $\arg k \in (0,\pi/2) \cup (\pi, 3\pi/2)$.

$$A(k)\overline{A(\bar{k})} - B(k)\overline{B(\bar{k})} = 1, \quad k \in \mathbb{C}\ (k \in \mathbb{R} \cup i\mathbb{R}, \text{ if } T = \infty),$$

$$A(k) = 1 + O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right), \quad B(k) = O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right), \quad k \to \infty.$$

All of the above properties, except for the property that $B(k)$ is bounded for $\arg k \in [0,\pi/2] \cup [\pi, 3\pi/2]$, follow from the analyticity and boundedness of $\mu_3(x,0,k)$, $\mu_2(0,t,k)$, from the conditions of unit determinant, and from the large k asymptotics of these eigenfunctions. Regarding $B(k)$ we note that $B(k) = B(T,k)$, where $B(t,k) = -\exp(4ik^2t)(\mu_2^{(24)}(0,t,k))_1$. The above ODEs imply a linear Volterra integral equation for the vector $\exp(4ik^2t)\mu_2^{(24)}(0,t,k)$, from which it immediately follows that $B(t,k)$ is an entire function of k bounded for $\arg k \in [0,\pi/2] \cup [\pi, 3\pi/2]$.

We are now ready to derive the so-called global relation. We present the discussion Sec. (2.4) of Ref. 6. We in fact show that the spectral functions are not independent but satisfy an important relation. Indeed, the integral of the one-form $W(x,t,k)$ around the boundary of the domain $\{(\xi,\tau): 0 < \xi < \infty, 0 < \tau < t\}$ vanishes. Let W be defined by (A2) with $\mu = \mu_3$. Then

$$\int_{\infty}^{0} e^{ik\xi\sigma_3}(Q\mu_3)(\xi,0,k)\,d\xi+\int_0^t e^{2ik^2\tau\hat{\sigma}_3}(\tilde{Q}\mu_3)(0,\tau,k)\,d\tau+e^{2ik^2t\hat{\sigma}_3}\int_0^{\infty} e^{ik\xi\hat{\sigma}_3}(Q\mu_3)(\xi,t,k)\,d\xi$$
$$=\lim_{x\to\infty} e^{ikx\hat{\sigma}_3}\int_0^t e^{2ik^2\tau\hat{\sigma}_3}(\tilde{Q}\mu_3)(x,\tau,k)\,d\tau. \tag{A9}$$

Using the definition of $s(k)$ above and (A8) it follows that the first term of this equation equals $s(k)-I$. Equation (A7) evaluated at $x=0$ gives

$$\mu_3(0,\tau,k)=\mu_2(0,\tau,k)e^{-2ik^2\tau\hat{\sigma}_3}s(k),$$

thus

$$e^{2ik^2\tau\hat{\sigma}_3}(\tilde{Q}\mu_3)(0,\tau,k)=[e^{2ik^2\tau\hat{\sigma}_3}(\tilde{Q}\mu_2)(0,\tau,k)]s(k);$$

this equation together with (A8) imply that the second term of (A9) equals

$$[e^{2ik^2t\hat{\sigma}_3}\mu_2(0,t,k)-I]s(k).$$

Hence assuming that u has sufficient decay as $x\to\infty$ Eq. (A9) becomes

$$-I+S(t,k)^{-1}s(k)+e^{2ik^2t\hat{\sigma}_3}\int_0^{\infty} e^{ik\xi\hat{\sigma}_3}(Q\mu_3)(\xi,t,k)\,d\xi=0, \tag{A10}$$

where the first and second columns of this equation are valid for $\arg k$ in the lower and the upper half of the complex k-plane, respectively, and $S(t,k)$ is defined by

$$S(t,k)=[e^{2ik^2t\hat{\sigma}_3}\mu_2(0,t,k)]^{-1}.$$

Letting $t=T$ and noting that $S(k)=S(T,k)$, Eq. (A10) becomes

$$-I+S(k)^{-1}s(k)+e^{2ik^2T\hat{\sigma}_3}\int_0^{\infty} e^{ik\xi\hat{\sigma}_3}(Q\mu_3)(\xi,T,k)\,d\xi=0.$$

The (12) component of this equation is

$$B(k)a(k)-A(k)b(k)=e^{4ik^2T}c^+(k), \quad \arg k\in[0,\pi],$$

$$c^+(k)=\int_0^{\infty} e^{ik\xi}(Q\mu_3)_{12}(\xi,T,k)\,dk.$$

This is the global relation, for finite T. For $T=\infty$ and assuming that f_0 is Schwartz, c^+ has to be set equal to zero.

APPENDIX B: THE "LENS" ARGUMENT

Suppose we have the following Riemann–Hilbert problem. We are seeking a matrix L, analytic in the complex plane except for a jump along the real interval $[\alpha,\beta]$, oriented from left to right. The normalization at infinity is to be $\lim_{k\to\infty}L=I$, and the jump across $[\alpha,\beta]$ is

$$L_+=L_-\begin{pmatrix} 0 & -ie^{iH/h} \\ -ie^{-iH/h} & 1 \end{pmatrix},$$

where $\mathrm{d}H/\mathrm{d}k<0$. We also assume that H is real on $[\alpha,\beta]$ and admits an analytic continuation in a small "lens"-like domain bounded by two analytic arcs C_u, C_l joining the points α,β (in that direction) and lying entirely in the upper and lower half-planes, respectively. We note the following factorization of the jump matrix:

$$\begin{pmatrix} 0 & -ie^{iH/h} \\ -ie^{-iH/h} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -ie^{iH/h} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -ie^{-iH/h} & 1 \end{pmatrix}.$$

This suggests the following definition. Let

$$L' = L, \quad \text{outside the domain bounded by} \quad C_u \cup C_l,$$

$$L' = L \cdot \begin{pmatrix} 1 & 0 \\ ie^{-iH/h} & 1 \end{pmatrix}, \text{ between } [\alpha,\beta] \text{ and } C_u.$$

$$L' = \begin{pmatrix} 1 & -ie^{iH/h} \\ 0 & 1 \end{pmatrix} \cdot L, \text{ between } [\alpha,\beta] \text{ and } C_l.$$

The Riemann–Hilbert problem for L' is as follows:

$$L'_+ = L'_- \cdot \begin{pmatrix} 1 & 0 \\ ie^{-iH/h} & 1 \end{pmatrix}, \quad k \in \quad C_u,$$

$$L'_+ = L'_- \cdot \begin{pmatrix} 1 & -ie^{iH/h} \\ 0 & 1 \end{pmatrix}, \quad k \in \quad C_l.$$

Now, since $\mathrm{d}\,\mathrm{Re}\,H/\mathrm{d}k<0$ on the interval $[\alpha,\beta]$, by the Cauchy–Riemann relations $\mathrm{dIm}\,H/\mathrm{d}k<0$ across the interval $[\alpha,\beta]$, in the positive imaginary direction. This means that $\mathrm{Im}\,H<0 \quad on \quad C_u$ if C_u is chosen to be close enough to $[\alpha,\beta]$, except at the end points α,β. Similarly $\mathrm{Im}\,H>0 \quad on \quad C_l$ if C_l is chosen to be close enough to $[\alpha,\beta]$, except at the end points α,β. Hence,

$$\mathrm{Re}(-iH)<0, \quad k \in C_u,$$

$$\mathrm{Re}(-iH)>0, \quad k \in C_l,$$

except at the end points α,β. In other words the jump matrix for L' is the identity plus an exponentially small quantity, at least away from the end points α,β. This implies that the contour $C_u \cup C_l$ can be erased, at least away from the the end points α,β.

Near the end points one can use a parametrix argument, which we omit (see, e.g., Ref. 4 for details).

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