COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **13**, Number **6**, November **2014**

doi:10.3934/cpaa.2014.13.

pp. –

TRACE HARDY–SOBOLEV–MAZ'YA INEQUALITIES FOR THE HALF FRACTIONAL LAPLACIAN

Stathis Filippas

Department of Mathematics and Applied Mathematics
University of Crete, 70013 Heraklion, Greece

Luisa Moschini

Dipartimento di Scienze di Base ed Applicate per l'Ingegneria
University of Rome "La Sapienza", 00185 Rome, Italy

Achilles Tertikas

Department of Mathematics and Applied Mathematics
University of Crete, 70013 Heraklion, Greece

(Communicated by Xavier Cabre)

Abstract. In this work we establish trace Hardy-Sobolev-Maz'ya inequalities with best Hardy constants, for weakly mean convex domains. We accomplish this by obtaining a new weighted Hardy type estimate which is of independent inerest. We then produce Hardy-Sobolev-Maz'ya inequalities for the spectral half Laplacian. This covers a critical case left open in [9].

1. **Introduction.** The Hardy-Sobolev-Maz'ya (HSM) inequalities combine Sobolev and Hardy terms, the latter with best constant. For instance, for the regular (local) Laplacian and for a domain $\Omega \subset \mathbf{R^n}$ $n \geq 3$ it states that, if $d(x) = \mathrm{dist}(x, \partial\Omega)$, there exists a positive constant c such that

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{|u|^2}{d^2(x)} dx + c\left(\int_\Omega |u|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}, \quad u \in C_0^\infty(\Omega)\ . \tag{1.1}$$

Such an inequality was first proven in [12] in the special case where Ω is the half space. In [8] it was proven under the assumption that Ω is a weakly mean convex domain, that is, it satisfies in the distributional sense,

$$-\Delta d(x) \geq 0,\ in\ \Omega\ . \tag{1.2}$$

We note that in the case where $\partial\Omega \in C^2$ condition (1.2) is equivalent to the mean convexity of $\partial\Omega$, see [11, 13].

In [10] inequality (1.1) was established with a constant c independent of Ω, under the stronger hypothesis that Ω is convex. We note that mean convexity is equivalent

2000 *Mathematics Subject Classification.* Primary: 35J60, 42B20, 46E35; Secondary: 26D10, 35J15, 35P15, 47G30.

Key words and phrases. Hardy inequality, fractional Sobolev inequality, fractional Laplacian, critical exponent, best constant, trace inequality.

The authors were partially supported by the FP7-REGPOT project ACMAC: Archimedes Center for Modeling, Analysis and Computations of the University of Crete. AT was also partially supported by ELKE grant, University of Crete.

to convexity in $n = 2$ dimensions but it is a much weaker assumption for $n \geq 3$, cf [1].

Our interest in this work is in the fractional (non local) Laplacian in a bounded domain. Various fractional s–Laplacians ($0 < s < 1$) have been recently studied, see [2, 3, 7, 9] and references therein. In [9] the limiting case of obtaining Hardy–Sobolev–Maz'ya inequalities for the half Laplacian was left open in the case of a domain Ω. In fact, the half Laplacian is a border line case, since different behaviors are observed for $s < \frac{1}{2}$ and $s > \frac{1}{2}$. For instance, the fractional Laplacian considered in [6], satisfies Hardy inequality for $\frac{1}{2} < s < 1$ but not for $0 < s \leq \frac{1}{2}$, in the case of smooth bounded domains. Similarly a dichotomy appears, for a different fractional Laplacian this time, in the context of Γ–convergence of non local phase transitions, or in the context of non local surface diffusion, see [14, 4]. In our case certain aymptotics are different for $s > 1/2$ than for $s = 1/2$ and as a consequence the analysis in [9] fails for the limiting case of the half Laplacian.

As we have already mentioned, there are several fractional Laplacians, but in this work we will focus on the spectral fractional Lapacian that was recently considered in [3]. We will do this as in [9] via a suitable extension problem in the spirit of [5]. In our case the appropriate extension problem is the following:

$$\begin{aligned} -\Delta_{(x,y)} u &= 0, && \textit{in } \Omega \times (0,\infty)\ , && (1.3)\\ u &= 0, && \textit{on } \partial\Omega \times (0,\infty)\ , && (1.4)\\ u(x,0) &= f(x), && \textit{in } \Omega\ , && (1.5) \end{aligned}$$

the energy of which is given by

$$J[u] = \frac{1}{2}\int_0^{+\infty}\int_\Omega |\nabla_{(x,y)} u(x,y)|^2 dxdy\ .$$

At this point we recall that the inner radius of a domain Ω is defined as $R_{in} := \sup_{x\in\Omega} d(x)$. We say that the domain Ω has finite inner radius whenever $R_{in} < \infty$. Our first result is the following Trace Hardy-Sobolev-Maz'ya inequality:

Theorem 1.1. *Let $n \geq 2$ and $\Omega \subset \mathbf{R^n}$ be a uniformly Lipschitz domain with finite inner radius which in addition satisfies*

$$-\Delta d(x) \geq 0, \ \ \textit{in } \Omega\ . \tag{1.6}$$

Then there exists a positive constant c such that for all $u \in C_0^\infty(\Omega \times \mathbf{R})$ there holds

$$\int_0^{+\infty}\int_\Omega |\nabla_{(x,y)} u(x,y)|^2 dxdy \geq \frac{2}{\pi}\int_\Omega \frac{u^2(x,0)}{d(x)}dx + c\left(\int_\Omega |u(x,0)|^{\frac{2n}{n-1}}dx\right)^{\frac{n-1}{n}}\ . \tag{1.7}$$

We note that the constant $\frac{2}{\pi}$ is the best constant for the corresponding trace Hardy inequality see Theorem 1 of [9] for the precise statement.

We will apply Theorem 1.1 to the spectral fractional Laplacian that is defined as follows. Let $\Omega \subset \mathbf{R^n}$ be a bounded domain, and λ_i and ϕ_i be the Dirichlet eigenvalues and orthonormal eigenfunctions of the Laplacian, i.e. $-\Delta\phi_i = \lambda_i\phi_i$ in Ω, with $\phi_i = 0$ on $\partial\Omega$. Then, for $f(x) = \sum c_i\phi_i(x)$ the s–fractional Laplacian is defined by

$$(-\Delta)^s f(x) = \sum_{i=1}^\infty c_i\lambda_i^s\phi_i(x), \ \ 0 < s < 1\ . \tag{1.8}$$

In the sequel we will be interested in the case $s = \frac{1}{2}$. More precisely, the Hardy-Sobolev-Maz'ya inequality for the spectral half Laplacian reads:

Theorem 1.2. *Let $n \geq 2$ and $\Omega \subset \mathbf{R^n}$ be a bounded Lipschitz domain which in addition satisfies*

$$-\Delta d(x) \geq 0, \ \text{ in } \Omega \ . \tag{1.9}$$

Then there exists a positive constant c such that for all $f \in C_0^\infty(\Omega)$ there holds

$$((-\Delta)^{\frac{1}{2}} f, f)_\Omega \geq \frac{2}{\pi} \int_\Omega \frac{f^2(x)}{d(x)} dx + c \left(\int_\Omega |f(x)|^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} . \tag{1.10}$$

Again, the constant $\frac{2}{\pi}$ is the best constant for the corresponding Hardy inequality see Theorem 3 of [9] for the precise statement.

We note that the proof is based on the following crucial estimate that was missing in [9].

Theorem 1.3. *Let $\Omega \subset \mathbf{R^n}$ be a domain with finite inner radius R_{in} such that*

$$-\Delta d(x) \geq 0, \ \text{ in } \Omega \ .$$

If in addition $A + 1 > 0$, then for all $u \in C_0^\infty(\mathbf{R^n} \times \mathbf{R})$ there holds

$$\begin{aligned} &\int_0^{+\infty} \int_\Omega \frac{y^{A+2} d^{A+2}}{(d^2+y^2)^{A+2}} |\nabla u|^2 dx dy \\ &+ \frac{(A+1)(4A+9)}{4(2A+5)^2} \int_0^{+\infty} \int_\Omega \frac{y^{A+1} d^{A+1} X}{(d^2+y^2)^{A+\frac{3}{2}}} (-\Delta d) u^2 dx dy \\ \geq &\frac{(A+1)^2}{8(2A+5)^2} \int_0^{+\infty} \int_\Omega \frac{y^A d^A X^2}{(d^2+y^2)^{A+1}} u^2 dx dy \ , \end{aligned} \tag{1.11}$$

where $X = X(\frac{d(x)}{R_{in}})$ and $X(t) = (1 - \ln t)^{-1}$, $0 < t \leq 1$.

For Theorem 1.3 it is important that the domain has a finite inner radius.

Using Theorem 1.3 and quite similar arguments to the ones leading to the proof of Theorems 1.1 and 1.2 one can establish HSM–inequalities for the Dirichlet half Laplacian defined in [9]. In particular Theorems 4, 5 and 12 of [9] are valid for the limiting case $s = 1/2$.

In Section 2 we give the proof of Theorem 1.3 after establishing a more general result, where weak mean convexity of the domain is not required. In the final Section 3 we give the proofs of Theorems 1.1 and 1.2.

2. **The proof of Theorem 1.3.**

In this section we will prove Theorem 1.3. In fact we will prove a more general result that does not require any sign assumption on the measure $-\Delta d(x)$.

Theorem 2.1. *Let $\Omega \subset \mathbf{R^n}$ be a domain with finite inner radius R_{in}. If $A+1 > 0$, then for all $u \in C_0^\infty(\mathbf{R^n} \times \mathbf{R})$ there holds*

$$\begin{aligned} &\int_0^{+\infty} \int_\Omega \frac{y^{A+2} d^{A+2}}{(d^2+y^2)^{A+2}} |\nabla u|^2 dx dy \\ &+ \frac{A+1}{4(2A+5)^2} \int_0^{+\infty} \int_\Omega \left[\frac{y^{A+2} d^{A+1} X^2}{(d^2+y^2)^{A+2}} + 4(A+2) \frac{y^{A+2} d^{A+3} X}{(d^2+y^2)^{A+3}} \right] (-\Delta d) u^2 dx dy \\ \geq &\frac{(A+1)^2}{8(2A+5)^2} \int_0^{+\infty} \int_\Omega \frac{y^A d^A X^2}{(d^2+y^2)^{A+1}} u^2 dx dy \ , \end{aligned} \tag{2.1}$$

where $X = X(\frac{d(x)}{R_{in}})$ *and* $X(t) = (1 - \ln t)^{-1}$, $0 < t \leq 1$.

From this estimate we have:

Proof of Theorem 1.3. The result follows from Theorem 2.1 using the sign assumption $-\Delta d(x) \geq 0$ in Ω. □

The rest of this section is devoted to the proof of Theorem 2.1. We first present some auxilliary Lemmas. Our first Lemma covers a limiting case of Lemma 8 of [9].

Lemma 2.2. *Let* $\Omega \subset \mathbf{R^n}$ *be a domain with finite inner radius* R_{in}*. If* A *and* B *are constants such that* $A + 1 > 0$ *and* $B + 1 > 0$ *then for all* $u \in C_0^\infty(\mathbf{R^n} \times \mathbf{R})$ *there holds*

$$\begin{aligned}
&\int_0^{+\infty} \int_\Omega \frac{y^A d^B X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}} |u| dx dy \\
\leq & \frac{A+B+2}{A+1} \int_0^{+\infty} \int_\Omega \frac{y^{A+2} d^B X}{(d^2+y^2)^{\frac{A+B+4}{2}}} (-d\Delta d) |u| dx dy \\
&+ \frac{A+B+3}{A+1} \int_0^{+\infty} \int_\Omega \frac{y^{A+1} d^B X}{(d^2+y^2)^{\frac{A+B+2}{2}}} |\nabla u| dx dy \ , \qquad (2.2)
\end{aligned}$$

where $X = X(\frac{d(x)}{R_{in}})$ *and* $X(t) = (1 - \ln t)^{-1}$, $0 < t \leq 1$.

Proof. Integrating by parts in the y-variable we compute

$$\begin{aligned}
&(A+1) \int_0^{+\infty} \int_\Omega \frac{y^A d^B X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}} |u| dx dy \\
\leq & (A+B+2) \int_0^{+\infty} \int_\Omega \frac{y^{A+2} d^B X^2}{(d^2+y^2)^{\frac{A+B+4}{2}}} |u| dx dy \\
&+ \int_0^{+\infty} \int_\Omega \frac{y^{A+1} d^B X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}} |u_y| dx dy. \qquad (2.3)
\end{aligned}$$

In the previous calculation there is no boundary term due to our assumptions. To continue we will estimate the first term in the right hand side above. To this end we define the vector field $\vec{F} : \Omega \times (0, +\infty) \to \mathbf{R}^n \times \mathbf{R}$ by

$$\vec{F}(x,y) := \left(\frac{y^{A+2} d^{B+1} X \nabla d}{(d^2+y^2)^{\frac{A+B+4}{2}}}, \ \frac{y^{A+3} d^B X}{(d^2+y^2)^{\frac{A+B+4}{2}}} \right). \qquad (2.4)$$

We then have

$$\int_0^{+\infty} \int_\Omega \mathrm{div} \vec{F} |u| dx dy = - \int_0^{+\infty} \int_\Omega \vec{F} \cdot \nabla |u| dx dy \leq \int_0^{+\infty} \int_\Omega |\vec{F}| |\nabla u| dx dy. \qquad (2.5)$$

We note that because of our assumptions $A + 1 > 0$ and $B + 1 > 0$, there are no boundary terms in (2.5). After a straightforward calculation (in the sense of distributions), taking also into account that $|\nabla d| = 1$ a.e., we end up with,

$$\mathrm{div} \vec{F} = \frac{y^{A+2} d^B X (d\Delta d)}{(d^2+y^2)^{\frac{A+B+4}{2}}} + \frac{y^{A+2} d^B X^2}{(d^2+y^2)^{\frac{A+B+4}{2}}}, \qquad (2.6)$$

and

$$|\vec{F}| = \frac{y^{A+2} d^B X}{(d^2+y^2)^{\frac{A+B+3}{2}}} \leq \frac{y^{A+1} d^B X}{(d^2+y^2)^{\frac{A+B+2}{2}}}. \qquad (2.7)$$

From (2.5)–(2.7) we get

$$\int_0^{+\infty}\int_\Omega \frac{y^{A+2}d^B X^2}{(d^2+y^2)^{\frac{A+B+4}{2}}}|u|dxdy \le \int_0^{+\infty}\int_\Omega \frac{y^{A+2}d^B X(-d\Delta d)}{(d^2+y^2)^{\frac{A+B+4}{2}}}|u|dxdy + \int_0^{+\infty}\int_\Omega \frac{y^{A+1}d^B X}{(d^2+y^2)^{\frac{A+B+2}{2}}}|\nabla u|dxdy \ . \tag{2.8}$$

Combining (2.3) and (2.8) the result follows. □

We next obtain the L^2–analogue of Lemma 2.2:

Lemma 2.3. *Let $\Omega\subset\mathbf{R^n}$ be a domain with finite inner radius R_{in}. If A and B are constants such that $A+1>0$ and $B+1>0$ then for all $u\in C_0^\infty(\mathbf{R^n}\times\mathbf{R})$ there holds*

$$\begin{aligned}&\int_0^{+\infty}\int_\Omega \frac{y^{A}d^B X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}}u^2dxdy\\ \le&\frac{2(A+B+2)}{A+1}\int_0^{+\infty}\int_\Omega \frac{y^{A+2}d^B X}{(d^2+y^2)^{\frac{A+B+4}{2}}}(-d\Delta d)u^2dxdy\\ &+\frac{4(A+B+3)^2}{(A+1)^2}\int_0^{+\infty}\int_\Omega \frac{y^{A+2}d^B}{(d^2+y^2)^{\frac{A+B+2}{2}}}|\nabla u|^2dxdy \ ,\end{aligned} \tag{2.9}$$

where $X=X(\frac{d(x)}{R_{in}})$ and $X(t)=(1-\ln t)^{-1}$, $0<t\le 1$.

Proof. We apply Lemma 2.2 to u^2. We then use Young's inequality in the last term of the right hand side:

$$2y^{A+1}X|u||\nabla u|\le \varepsilon y^A X^2u^2+\frac{1}{\varepsilon}y^{A+2}|\nabla u|^2,$$

with

$$\varepsilon=\frac{A+1}{2(A+B+3)} \ .$$

We omit the details. □

The following is a variation of Lemma 6 of [9], in the sense that no assumption on the sign of $(-\Delta d)$ is required.

Lemma 2.4. *Suppose that $\Omega\subset\mathbf{R^n}$ has finite inner radius. If A, B are constants such that $A+1>0$, $B+1>0$, then for all $u\in C_0^\infty(\mathbf{R^n}\times\mathbf{R})$ there holds*

$$\begin{aligned}&(B+1)\int_0^{+\infty}\int_\Omega \frac{y^{A}d^B X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}}|u|dxdy\\ \le&(A+B+3)\int_0^{+\infty}\int_\Omega \frac{y^{A}d^{B+1} X}{(d^2+y^2)^{\frac{A+B+2}{2}}}|\nabla u|dxdy\\ &+\int_0^{+\infty}\int_\Omega \left[\frac{y^{A}d^{B+1} X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}}+(A+B+2)\frac{y^{A}d^{B+3} X}{(d^2+y^2)^{\frac{A+B+4}{2}}}\right](-\Delta d)|u|dxdy \ ,\end{aligned} \tag{2.10}$$

where $X=X(\frac{d(x)}{R_{in}})$ and $X(t)=(1-\ln t)^{-1}$, $0<t\le 1$.

Proof. Integrating by parts in the x-variables we compute

$$
\begin{aligned}
&(B+1)\int_0^{+\infty}\int_\Omega \frac{y^A d^B X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}}|u|dxdy + 2\int_0^{+\infty}\int_\Omega \frac{y^A d^B X^3}{(d^2+y^2)^{\frac{A+B+2}{2}}}|u|dxdy \\
&\leq \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1} X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}}|\nabla u|dxdy + \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1} X^2(-\Delta d)}{(d^2+y^2)^{\frac{A+B+2}{2}}}|u|dxdy \\
&\quad + (A+B+2)\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+2} X^2}{(d^2+y^2)^{\frac{A+B+4}{2}}}|u|dxdy. \qquad (2.11)
\end{aligned}
$$

In the previous calculation there are no boundary terms due to our assumptions. To continue we will estimate the middle term in the right hand side above. To this end we use (2.5) with the following choice of the vector field $\vec{F} : \Omega \times (0, +\infty) \to \mathbf{R}^n \times \mathbf{R}$

$$
\vec{F}(x,y) := \left(\frac{y^A d^{B+3} X \nabla d}{(d^2+y^2)^{\frac{A+B+4}{2}}}, \ \frac{y^{A+1} d^{B+2} X}{(d^2+y^2)^{\frac{A+B+4}{2}}} \right).
$$

Straightforward calculations show that

$$
\mathrm{div}\vec{F} = \frac{y^A d^{B+3} X(\Delta d)}{(d^2+y^2)^{\frac{A+B+4}{2}}} + \frac{y^A d^{B+2} X^2}{(d^2+y^2)^{\frac{A+B+4}{2}}},
$$

and

$$
|\vec{F}| = \frac{y^A d^{B+2} X}{(d^2+y^2)^{\frac{A+B+3}{2}}} \leq \frac{y^A d^{B+1} X}{(d^2+y^2)^{\frac{A+B+2}{2}}}.
$$

We then have that

$$
\begin{aligned}
&\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+2} X^2}{(d^2+y^2)^{\frac{A+B+4}{2}}}|u|dxdy \\
&\leq \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+3} X}{(d^2+y^2)^{\frac{A+B+4}{2}}}(-\Delta d)|u|dxdy + \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1} X}{(d^2+y^2)^{\frac{A+B+2}{2}}}|\nabla u|dxdy.
\end{aligned}
$$

Combining the above with (2.11) and the fact that $X \leq 1$ we conclude the proof. ☐

We next obtain the L^2–analogue of Lemma 2.4:

Lemma 2.5. *Suppose that $\Omega \subset \mathbf{R^n}$ has finite inner radius. If A, B are constants such that $A+1>0$, $B+1>0$, then for all $u \in C_0^\infty(\mathbf{R^n} \times \mathbf{R})$ there holds*

$$
\begin{aligned}
&\int_0^{+\infty}\int_\Omega \frac{y^A d^B X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}} u^2 dxdy \\
&\leq \frac{4(A+B+3)^2}{(B+1)^2}\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+2}}{(d^2+y^2)^{\frac{A+B+2}{2}}}|\nabla u|^2 dxdy \qquad (2.12) \\
&\quad + \frac{2}{B+1}\int_0^{+\infty}\int_\Omega \left[\frac{y^A d^{B+1} X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}} + (A+B+2)\frac{y^A d^{B+3} X}{(d^2+y^2)^{\frac{A+B+4}{2}}} \right](-\Delta d) u^2 dxdy \ ,
\end{aligned}
$$

where $X = X(\frac{d(x)}{R_{in}})$ and $X(t) = (1-\ln t)^{-1}$, $0 < t \leq 1$.

Proof. We apply Lemma 2.4 to u^2 We then use Young's inequality in the first term of the right hand side:

$$
2d^{B+1} X|u||\nabla u| \leq \varepsilon d^B X^2 u^2 + \frac{1}{\varepsilon} d^{B+2}|\nabla u|^2,
$$

with

$$\varepsilon = \frac{B+1}{2(A+B+3)} \ .$$

We omit the details. □

We are now ready to give the proof of Theorem 2.1:

Proof of Theorem 2.1. We first apply Lemma 2.3 with $B = A+2$ to get:

$$\int_0^{+\infty}\int_\Omega \frac{y^A d^{A+2} X^2}{(d^2+y^2)^{A+2}} u^2 dxdyl = \frac{4(2A+5)^2}{(A+1)^2}\int_0^{+\infty}\int_\Omega \frac{y^{A+2}d^{A+2}}{(d^2+y^2)^{A+2}}|\nabla u|^2 dxdy + \frac{4(A+2)}{A+1}\int_0^{+\infty}\int_\Omega \frac{y^{A+2}d^{A+3}X}{(d^2+y^2)^{A+3}}(-\Delta d)u^2 dxdy \ . \quad (2.13)$$

We next use Lemma 2.5 with $A = B+2$, to obtain:

$$\int_0^{+\infty}\int_\Omega \frac{y^{B+2}d^B X^2}{(d^2+y^2)^{B+2}}u^2 dxdy \le \frac{4(2B+5)^2}{(B+1)^2}\int_0^{+\infty}\int_\Omega \frac{y^{B+2}d^{B+2}}{(d^2+y^2)^{B+2}}|\nabla u|^2 dxdy + \frac{2}{B+1}\int_0^{+\infty}\int_\Omega \left[\frac{y^{B+2}d^{B+1}X^2}{(d^2+y^2)^{B+2}} + 2(B+2)\frac{y^{B+2}d^{B+3}X}{(d^2+y^2)^{B+3}}\right](-\Delta d)u^2 dxdy \ , \quad (2.14)$$

Replacing B by A in (2.14) and adding it to (2.13) we conclude the proof. □

3. **The proofs of Theorems 1.1 and 1.2.**

We first establish the following Hardy–Sobolev estimate, that will be used in an essential way in the proof of Theorem 1.1. For the definition of the " uniformly Lipschitz domain" see for instance [9].

Theorem 3.1. *Let $n \ge 2$ and $\Omega \subset \mathbf{R^n}$ be a uniformly Lipschitz domain with finite inner radius that in addition satisfies*

$$-\Delta d(x) \ge 0, \ \ in \ \Omega \ . \quad (3.1)$$

Then there exists a positive constant c such that for all $u \in C_0^\infty(\Omega \times \mathbf{R})$ there holds

$$\int_0^{+\infty}\int_\Omega |\nabla_{(x,y)}u(x,y)|^2 dxdy \ge \frac{2}{\pi}\int_\Omega \frac{u^2(x,0)}{d(x)}dx + c\left(\int_0^{+\infty}\int_\Omega |u(x,y)|^{\frac{2(n+1)}{n-1}}dxdy\right)^{\frac{n-1}{n+1}} \ . \quad (3.2)$$

Proof of Theorem 3.1. We first recall inequality (2.11), from [9], with $s = 1/2$ (and $a = 0$), that is

$$\int_0^{+\infty}\int_\Omega |\nabla u|^2 dxdy \ge \frac{2}{\pi}\int_\Omega \frac{u^2(x,0)}{d(x)}dx + \int_0^{+\infty}\int_\Omega |\nabla u - \frac{\nabla\phi}{\phi}u|^2 dxdy - \int_0^{+\infty}\int_\Omega \frac{\Delta\phi}{\phi}u^2 dxdy \ , \quad (3.3)$$

where ϕ is given by

$$\phi(x,y) = A\left(\frac{y}{d}\right), \ y > 0, \ x \in \Omega \ , \quad (3.4)$$

and A solves (2.3), (2.4) in [9], that is

$$A(t) = 1 - \frac{2}{\pi}\arctan t, \qquad t \ge 0. \quad (3.5)$$

The result will follow after establishing the following inequality:

$$\begin{aligned}\int_0^{+\infty}\int_\Omega |\nabla u-\frac{\nabla\phi}{\phi}u|^2dxdy-\int_0^{+\infty}\int_\Omega \frac{\Delta\phi}{\phi}u^2dxdy\\ \geq c\left(\int_0^{+\infty}\int_\Omega |u(x,y)|^{\frac{2(n+1)}{n-1}}dxdy\right)^{\frac{n-1}{n+1}} .\end{aligned} \tag{3.6}$$

To this end we start with the Sobolev inequality,

$$\int_0^{+\infty}\int_\Omega |\nabla u|dxdy\geq c\left(\int_0^{+\infty}\int_\Omega |u(x,y)|^{\frac{n+1}{n}}dxdy\right)^{\frac{n}{n+1}} \ ,\ u\in C_0^\infty(\Omega\times\mathbf{R})\ ,$$

with the choice $u=\phi^{\frac{2n}{n-1}}v$. Hence we obtain

$$\begin{aligned}\int_0^{+\infty}\int_\Omega \phi^{\frac{2n}{n-1}}|\nabla v|dxdy+\frac{2n}{n-1}\int_0^{+\infty}\int_\Omega \phi^{\frac{n+1}{n-1}}|\nabla\phi||v|dxdy\\ \geq c\left(\int_0^{+\infty}\int_\Omega |\phi^{\frac{2n}{n-1}}v|^{\frac{n+1}{n}}dxdy\right)^{\frac{n}{n+1}} .\end{aligned} \tag{3.7}$$

Next we will control the second term of the LHS using Lemma 7 of [9]. To this end we recall that we have the following asymptotics (cf Lemma 2 of [9]),

$$\phi^{\frac{2n}{n-1}}\sim\frac{d^{\frac{2n}{n-1}}}{(d^2+y^2)^{\frac{n}{n-1}}}\ ,\qquad \phi^{\frac{n+1}{n-1}}|\nabla\phi|\sim\frac{d^{\frac{n+1}{n-1}}}{(d^2+y^2)^{\frac{n}{n-1}}}\ . \tag{3.8}$$

We then use Lemma 7 of [9] with the choice $A=0$, $B=\frac{n+1}{n-1}$ and $\Gamma=\frac{n}{n-1}$ taking into account that $A+B+2-2\Gamma=1>0$, to obtain the estimate

$$\begin{aligned}\int_0^{+\infty}\int_\Omega \frac{d^{\frac{n+1}{n-1}}}{(d^2+y^2)^{\frac{n}{n-1}}}|v|dxdy\leq c_1\int_0^{+\infty}\int_\Omega \frac{d^{\frac{2n}{n-1}}}{(d^2+y^2)^{\frac{n}{n-1}}}|\nabla v|dxdy\\ +c_2\int_0^{+\infty}\int_\Omega \frac{d^{\frac{2n}{n-1}}}{(d^2+y^2)^{\frac{n}{n-1}}}|v|dxdy\ .\end{aligned} \tag{3.9}$$

From this and (3.7) we have that

$$\int_0^{+\infty}\int_\Omega \phi^{\frac{2n}{n-1}}|\nabla v|dxdy+\int_0^{+\infty}\int_\Omega \phi^{\frac{2n}{n-1}}|v|dxdy\geq c\left(\int_0^{+\infty}\int_\Omega |\phi^{\frac{2n}{n-1}}v|^{\frac{n+1}{n}}dxdy\right)^{\frac{n}{n+1}} .$$

To continue we next set $v=|w|^{\frac{2n}{n-1}}$ and apply Schwartz inequality in the LHS. After a simplification we arrive at:

$$\int_0^{+\infty}\int_\Omega \phi^2|\nabla w|^2dxdy+\int_0^{+\infty}\int_\Omega \phi^2w^2dxdy\geq c\left(\int_0^{+\infty}\int_\Omega |\phi w|^{\frac{2(n+1)}{n-1}}\right)^{\frac{n-1}{n+1}} . \tag{3.10}$$

To conclude the proof of the Theorem we need the following estimate:

$$c\int_0^{+\infty}\int_\Omega \phi^2w^2dxdy\leq\int_0^{+\infty}\int_\Omega \phi^2|\nabla w|^2dxdy-\int_0^{+\infty}\int_\Omega (\Delta\phi)\phi w^2dxdy\ . \tag{3.11}$$

It is worth noticing that the estimates of [9] that work for $\frac{1}{2}<s<1$ fail to give (3.11) for the limiting value $s=\frac{1}{2}$. It is at this point that we will use the more refined estimate of Theorem 1.3 with $A=0$, that is,

$$\int_0^{+\infty}\int_\Omega \frac{X^2 w^2}{d^2+y^2}dxdy \le 200\int_0^{+\infty}\int_\Omega \frac{y^2d^2|\nabla w|^2}{(d^2+y^2)^2}dxdy + 18\int_0^{+\infty}\int_\Omega \frac{ydX(-\Delta d)w^2}{(d^2+y^2)^{\frac{3}{2}}}dxdy\ , \tag{3.12}$$

here $X=X\left(\frac{d}{R_{in}}\right)$. This implies

$$\frac{1}{R_{in}^2}\int_0^{+\infty}\int_\Omega \frac{d^2 w^2}{d^2+y^2}dxdy \le 200\int_0^{+\infty}\int_\Omega \frac{d^2|\nabla w|^2}{d^2+y^2}dxdy + 18\int_0^{+\infty}\int_\Omega \frac{yd(-\Delta d)w^2}{(d^2+y^2)^{\frac{3}{2}}}dxdy\ . \tag{3.13}$$

Taking into account the asymptotics of ϕ, cf (3.8), as well as the fact that

$$-\Delta\phi = \frac{2}{\pi}\frac{y}{d^2+y^2}(-\Delta d)\ ,$$

estimate (3.13) leads to (3.11). We omit further details. $\square$

We next give the proof of Theorem 1.1.

Proof of Theorem 1.1. We will use (3.3) where ϕ is given, as before, by (3.4), (3.5). The result then will follow once we establish:

$$\int_0^{+\infty}\int_\Omega |\nabla u - \frac{\nabla\phi}{\phi}u|^2 dxdy - \int_0^{+\infty}\int_\Omega \frac{\Delta\phi}{\phi}u^2 dxdy \ge c\left(\int_\Omega |u(x,0)|^{\frac{2n}{n-1}}dx\right)^{\frac{n-1}{n}}. \tag{3.14}$$

To this end we start with the trace inequality,

$$\int_0^{+\infty}\int_\Omega |\nabla u| dxdy \ge c\int_\Omega |u(x,0)|dx\ ,$$

valid for $u\in C_0^\infty(\Omega\times\mathbf{R})$. We apply this to $u=\phi^{\frac{2n}{n-1}}v$. Hence we obtain

$$\int_0^{+\infty}\int_\Omega \phi^{\frac{2n}{n-1}}|\nabla v|dxdy + \frac{2n}{n-1}\int_0^{+\infty}\int_\Omega \phi^{\frac{n+1}{n-1}}|\nabla\phi||v|dxdy \ge c\int_\Omega |\phi^{\frac{2n}{n-1}}v|dx \tag{3.15}$$

Next we will control the second term of the LHS exactly as we did in (3.9) in the proof of Theorem 3.1, to arrive at

$$\int_0^{+\infty}\int_\Omega \phi^{\frac{2n}{n-1}}|\nabla v|dxdy + \int_0^{+\infty}\int_\Omega \phi^{\frac{2n}{n-1}}|v|dxdy \ge c\int_\Omega |\phi^{\frac{2n}{n-1}}v(x,0)|dx\ .$$

To continue we next set $v=|w|^{\frac{2n}{n-1}}$ and apply Schwartz inequality in the LHS to get after elementary manipulations that

$$\left(\int_0^{+\infty}\int_\Omega |\phi w|^{\frac{2(n+1)}{n-1}}dxdy\right)\left[\int_0^{+\infty}\int_\Omega \phi^2|\nabla w|^2 dxdy + \int_0^{+\infty}\int_\Omega \phi^2 w^2 dxdy\right] \ge c\left(\int_\Omega |\phi w(x,0)|^{\frac{2n}{n-1}}dx\right)^2. \tag{3.16}$$

At this point we use (3.10) and then inequality (3.11) to arrive at

$$\int_0^{+\infty}\int_\Omega \phi^2|\nabla w|^2 dxdy - \int_0^{+\infty}\int_\Omega (\Delta\phi)\phi w^2 dxdy \ge c\left(\int_\Omega |\phi w(x,0)|^{\frac{2n}{n-1}}dx\right)^{\frac{n-1}{n}}\ ,$$

which is equivalent to (3.14) after the substitution $u = \phi|w|$. We omit further details. $\square$

Proof of Theorem 1.2. The result follows from Theorem 1.1 since the harmonic extension $u(x, y)$ in $\Omega \times [0, \infty)$ of f, that is, the solution of (1.3)–(1.5), has energy that satisfies

$$\int_0^{+\infty} \int_\Omega |\nabla u|^2 dxdy = ((-\Delta)^{\frac{1}{2}} f, f)_\Omega \ , \tag{3.17}$$

see (8.5) of [9]. $\square$

REFERENCES

[1] D. H. Armitage and U. Kuran, The convexity and the superharmonicity of the signed distance function, *Proc. Amer. Math. Soc.*, **93** (1985), 598–600.

[2] K. Bogdan, K. Burdzy and Z-Q. Chen, Censored stable processes, *Prob. theory related fields*, **127** (2003), 89–152.

[3] X. Cabré and J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, *Adv. Math.*, **224** (2010), 2052–2093.

[4] L. Caffarelli, J. M. Roquejoffre and O. Savin, Non local minimal surfaces, *Comm. Pure Appl. Math.*, **63** (2010), 1111–1144.

[5] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. P.D.E.* **32** (2007), 1245–1260.

[6] B. Dyda, A fractional order Hardy inequality, *Illinois J. Math.*, **48** (2004), 575–588.

[7] B. Dyda and R. L. Frank, Fractional Hardy-Sobolev-Maz'ya inequality for domains, *Studia Math.*, **208** (2012), 151–166.

[8] S. Filippas, V. Maz'ya and A. Tertikas, Critical Hardy-Sobolev inequalities, *J. Math. Pures Appl.*, **87** (2007), 37–56.

[9] S. Filippas, L. Moschini and A. Tertikas, Sharp Trace Hardy-Sobolev-Maz'ya inequalities and the fractional Laplacian, *Arch. Ration. Mech. Anal.*, **208** (2013), 109–161.

[10] R. L. Frank and M. Loss, Hardy-Sobolev-Maz'ya inequalities for arbitrary domains, *J. Math. Pures Appl.*, **97** (2012), 39–54.

[11] R. T. Lewis, J. Li and Y. Y. Li, A geometric characterization of a sharp Hardy inequality, *J. Funct. Anal.*, **262** (2012), 3159–3185.

[12] V. Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, Second, revised and augmented edition. Springer, Heidelberg, 2011.

[13] G. Psaradakis, L^1 Hardy inequalities with weights, *J. Geom. Anal.*, **23** (2013), 1703–1728.

[14] O. Savin and E. Valdinoci, Γ-convergence for nonlocal phase transitions, *Ann. Inst. H. Poincare Anal. Non Linire*, **29** (2012), 479–500.

Received September 2013; revised April 2014.

E-mail address: filippas@tem.uoc.gr
E-mail address: luisa.moschini@sbai.uniroma1.it
E-mail address: tertikas@math.uoc.gr