# Liouville type properties for a class of weighted anisotropic elliptic equations

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**Abstract**

We establish Liouville type results for weighted anisotropic elliptic equations in divergence form in the strip $\mathbb{R}^{N-1} \times (-1, 1)$, $N \geq 2$. The weights depend on one variable and they include the case where they are powers of the distance functions to the boundary of the strip.

## 1 Introduction and main results

In this work our interest is to prove Liouville type results for the *anisotropic* elliptic operator

$$\mathcal{L}u = w_1 \Delta_{x'} u + \partial_\lambda (w_1 w_2 \partial_\lambda u), \tag{1.1}$$

where $x = (x', \lambda) \in S := \mathbb{R}^{N-1} \times (-1, 1)$, $N \geq 2$ and $w_i(\lambda) = w_i(|\lambda|)$ for $i = 1, 2$, are locally positive and bounded weight functions. That is, we look for conditions on $w_1$, $w_2$ under which the only bounded weak solutions of $\mathcal{L}u = 0$ are the constant solutions.

Let us recall the uniformly elliptic case

$$\sum_{i,j=1}^{N} \partial_i (a_{ij}(x) \partial_j u) = 0,$$

with

$$c_1 |\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \leq c_2 |\xi|^2, \qquad \forall \xi \in \mathbb{R}^N, \quad c_1, c_2 > 0.$$

The pioneering work of De Giorgi and Moser [DG, Mo1, Mo2], see also [HL], played a crucial role in establishing many properties of weak solutions such as Harnack inequality, Liouville type results, Holder continuity etc. Several extensions of these results were made by various authors in a number of directions, see e.g., [FKS, G, GSC].

To discuss the nonuniformly elliptic case we denote by $a(x)$ the matrix with entries $a_{ij}(x)$ and set

$$\lambda(x) := \inf_{\xi\in\mathbb{R}^N} \frac{\xi\cdot a(x)\xi}{|\xi|^2}, \qquad \mu(x) := \sup_{\xi\in\mathbb{R}^N} \frac{|a(x)\xi|^2}{\xi\cdot a(x)\xi}.$$

Assume that for $p, q \in (1, +\infty]$, $\mu \in L^p_{loc}(\mathbb{R}^N)$, $\lambda^{-1} \in L^q_{loc}(\mathbb{R}^N)$, and

$$\limsup_{R\to\infty} |B_R|^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \|\mu\|_{L^p(B_R)} \|\lambda^{-1}\|_{L^q(B_R)} < \infty.$$

Under essentially these assumptions, and provided that

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{N},$$

Trudinger [T], established Harnack inequality and Hölder continuity for nonnegative weak solutions, see also [MS]. Quite recently the same results have been proved by Bella and Schäffner [BS] under the weaker condition

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{N-1}.$$

As a consequence, every bounded weak solution is constant, cf Corollary 4.4 of [BS] for the precise result and the definition of weak solutions.

There is a recent interest in the study of anisotropic operators see e.g. [CL, HP, MNS1, MNS2]. Our motivation for studying (1.1) comes from the work of Caffarelli and Cordoba [CC] in phase transition analysis and is a continuation of [FMT2] and [M2]. In [FMT2] the aim was to establish various Sobolev type inequalities for anisotropic weighted operators whereas in [M2], Liouville type Theorems for (1.1) are presented, for particular choices of the weights.

We first consider the model *anisotropic* elliptic operator

$$\mathcal{L}_{\alpha,\nu} u = (1-|\lambda|)^{\alpha} \Delta_{x'} u + \partial_\lambda((1-|\lambda|)^{\alpha+\nu} \partial_\lambda u) \tag{1.2}$$

for $(x', \lambda) \in S := \mathbb{R}^{N-1} \times (-1, 1)$. We focus our attention only in the case $\alpha > -1$ and we state the results in three cases, the subcritical one, that is $\nu < 2$ and the critical or supercritical case corresponding to $\nu = 2$ and $\nu > 2$ respectively. Then, our first result reads

**Theorem 1.1 (Subcritical case)** *Let $\alpha > -1$.*

**(a)** *If $\nu < 1 - \alpha$ then the function*

$$u(\lambda) = \int_{-1}^{\lambda} (1-|t|)^{-\alpha-\nu} dt,$$

*is a nonnegative (and bounded) weak solution of $\mathcal{L}_{\alpha,\nu} u = 0$ in $S$.*

**(b)** *If $1 - \alpha \le \nu < 2$ then any nonnegative weak solution of $\mathcal{L}_{\alpha,\nu} u = 0$ in $S$ is constant.*

When $\nu \ge 2$ our result reads

**Theorem 1.2 (Critical and supercritical cases)** *Let $\alpha > -1$ and $\nu \ge 2$. Every bounded weak solutions of $\mathcal{L}_{\alpha,\nu} u = 0$ in $S$ is constant.*

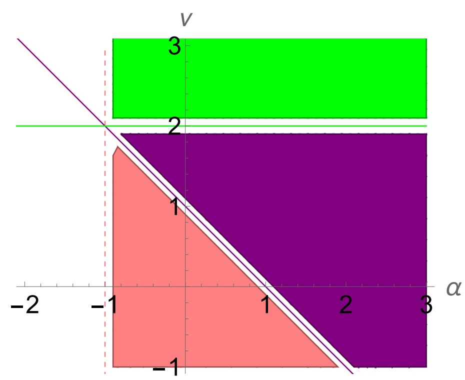


Figure 1: For $\alpha > -1$, the lines $\nu = 1 - \alpha$ and $\nu = 2$ define three regions in the plane $\alpha$–$\nu$. In the pink region (subcritical) there exist nonnegative non constant solutions. In the purple region (also subcritical) all nonnegative solutions are constants. Finally in the green region (supercritical) as well as in the case $\nu = 2$ (critical) all bounded solutions are constants.

The critical case $\nu = 2$ in the case $\alpha = 1$ was already treated in [M2]; in such a case the validity of a Liouville type result entails a positive answer to De Giorgi conjecture under the additional assumption that level sets are Lipschitz graphs, see also [BBG], [CC].

An operator like $\mathcal{L}_{\alpha,\nu}$ when $\nu = 2\alpha$ and $0 < \alpha \le 1$ (which corresponds to the subcritical and critical case in the present terminology) is naturally related to the phase transition analysis in [CC].

When $1 - \alpha \le \nu < 2$ our result is stronger than establishing that the only bounded weak solutions are the constant ones and is proved by means of a parabolic Harnack inequality up to the boundary.

We note that our results are outside the range of applicability of the ones by Bella and Schäffner [BS] mentioned above.

We next consider the more general elliptic operator (1.1). We assume that $w_i(\lambda) = w_i(|\lambda|)$, $i = 1, 2$, $-1 < \lambda < 1$, and $w_i \in L^\infty_{loc}(-1, 1)$. We only consider the case $w_1 \in L^1(0, 1)$ and we state the results in two cases, the subcritical one, which corresponds to the case $w_2^{-\frac{1}{2}} \in L^1(0, 1)$ and the critical or supercritical case which corresponds to the case $w_2^{-\frac{1}{2}} \notin L^1(0, 1)$. The results then are the following

**Theorem 1.3 (Subcritical case)** *Let $w_1 \in L^1(0, 1)$ and $(w_2)^{-\frac{1}{2}} \in L^1(0, 1)$.*

**(a)** *If $(w_1 w_2)^{-1} \in L^1(0, 1)$ then the function*

$$u(\lambda) = \int_{-1}^{\lambda} (w_1 w_2)^{-1}(t)dt$$

*is a nonnegative (and bounded) weak solution of $\mathcal{L}u = 0$ in $S$ .*

**(b)** *If $(w_1 w_2)^{-1} \notin L^1(0, 1)$ and there exists $\theta \ge 1$ and constants $c_1, c_2 > 0$ such that for any $\lambda \in (-1, 1)$ there holds*

$$c_1 \left( \int_{|\lambda|}^1 w_2^{-\frac{1}{2}}(t)dt \right)^\theta \le w_1(|\lambda|) w_2^{\frac{1}{2}}(|\lambda|) \le c_2 \left( \int_{|\lambda|}^1 w_2^{-\frac{1}{2}}(t)dt \right)^\theta , \tag{1.3}$$

*then any nonnegative weak solution of $\mathcal{L}u = 0$ in $S$ is constant.*

Also,

**Theorem 1.4 (Critical and supercritical cases)** *Let $w_1 \in L^1(0,1)$ and $(w_2)^{-\frac{1}{2}} \notin L^1(0,1)$ and define*

$$\varphi(\lambda) = 1 + \int_0^{|\lambda|} (w_1 w_2)^{-1}(t)dt. \tag{1.4}$$

*We assume that there exists $m > 2$ such that $\varphi^{-\frac{1}{m}} w_2^{-\frac{1}{2}} \in L^1(0,1)$ and $\theta > 0$ such that for some constants $c_1, c_2 > 0$ and any $\lambda \in (-1,1)$ there holds*

$$c_1 \left( \int_{|\lambda|}^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t)dt \right)^\theta \leq w_1(|\lambda|) w_2^{\frac{1}{2}}(|\lambda|) \varphi^{\frac{1}{m}}(|\lambda|) \leq c_2 \left( \int_{|\lambda|}^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t)dt \right)^\theta . \tag{1.5}$$

*Then any bounded weak solution of $\mathcal{L}u = 0$ in $S$ is constant.*

Notice that if $w_1 \in L^1(0,1)$ and $(w_2)^{-\frac{1}{2}} \notin L^1(0,1)$ then necessarily $(w_1 w_2)^{-1} \notin L^1(0,1)$, as it follows easily from the decomposition $w_2^{-\frac{1}{2}} = w_1^{\frac{1}{2}} (w_1 w_2)^{-\frac{1}{2}}$, whence

$$\varphi(\lambda) \to +\infty \quad \text{as} \quad |\lambda| \to 1.$$

The result of Theorem 1.4 is weaker than the one in Theorem 1.3(b). This is not due to our approach, since if one considers the strip

$$S_+ = \mathbb{R}^{N-1} \times (0,1),$$

with $w_1$, $w_2$ as in Theorem 1.4 that is $w_1 \in L^1(0,1)$ and $(w_2)^{-\frac{1}{2}} \notin L^1(0,1)$, then the function

$$\varphi(\lambda) = 1 + \int_0^{\lambda} (w_1 w_2)^{-1}(t)dt,$$

is a nonnnegative weak solution of $\mathcal{L}u = 0$ in $S_+$ which is actually unbounded. Hence the requirement of boundedness of weak solutions in Theorem 1.4 cannot be replaced by the nonnegativity of weak solutions.

To prove Theorem 1.3(b) we establish a parabolic Harnack inequality up to the boundary for nonnegative weak solutions $u(x,t)$ of

$$\frac{\partial u}{\partial t} = \frac{1}{w_1} \mathcal{L}\, u \text{ in } C_R \times (0, R^2) \ . \tag{1.6}$$

with

$$C_R := \{|x'| < R, \quad |\lambda| < 1\}.$$

Parabolic Harnack inequality follows once one establishes Poincaré and Sobolev inequalities as well as a doubling volume growth condition as is shown in [FKS, CS]. See also [GSC, SC] for extensions on complete Riemannian manifolds. In the present work we follow an adaptation made in [FMT1], cf Theorem 2.11 there. In particular the proper energy space is now given by the following norm

$$||u||^2_{H^1_{w_1,w_2}(C_R)} := \int_{C_R} \left( u^2 + |\nabla_{x'} u|^2 + w_2 (\partial_\lambda u)^2 \right) w_1 dx' d\lambda.$$

This is done in Section 2.

To prove Theorem 1.4 we make use of the oscillation decrease method, cf section 4.3 of [HL], as adapted in Theorem 1.4 of [M2] to the anisotropic setting. This is done in Section 3.

In Section 4 we give the proofs of Theorems 1.1 and 1.2. We also discuss various extensions of our results.

## 2 Subcritical case: Proof of Theorem 1.3

In this section we give the proof of Theorem 1.3(b). This will be done by means of a parabolic Harnack inequality up to the boundary, using the Moser iteration scheme, as adapted to isotropic degenerate elliptic operators on bounded domains in [FMT1]. There is a difference in the cut off functions used here as compared to the ones used in [FMT1]. In this work our cut off functions take into account the geometry of the cylinder and they depend only on $x'$. We combine this with a density argument similar to [FMT1] that takes care of the $\lambda$ direction.

The three ingredients needed for the scheme to work are the doubling volume-growth condition, a local weighted Sobolev inequality as well as a local weighted Poincaré inequality.

The doubling property follows easily from the fact that

$$V(C_R) = \int_{C_R} w_1 dx' d\lambda = \left( \int_{B'_R} dx' \right) \int_{-1}^{1} w_1 d\lambda = C R^{N-1}, \tag{2.1}$$

for some uniform constant $C$ (independent from $R$) and any $R > 0$. Here we denote with $B'_R$ the Euclidean ball of radius $R$ in $\mathbb{R}^{N-1}$.

Concerning the local weighted Sobolev inequality we have

**Lemma 2.1 (local weighted Sobolev)** *Let $w_1 \in L^1(0,1)$, $(w_2)^{-\frac{1}{2}} \in L^1(0,1)$ and $(w_1 w_2)^{-1} \notin L^1(0,1)$. In addition we suppose that there exists $\theta > 0$ and some constants $c_1, c_2 > 0$ such that for any $\lambda \in (-1,1)$ there holds*

$$c_1 \left( \int_{|\lambda|}^{1} w_2^{-\frac{1}{2}}(t) dt \right)^{\theta} \leq w_1(|\lambda|) w_2^{\frac{1}{2}}(|\lambda|) \leq c_2 \left( \int_{|\lambda|}^{1} w_2^{-\frac{1}{2}}(t) dt \right)^{\theta}. \tag{2.2}$$

*Then for $q = \frac{2(N+\theta)}{N-2+\theta}$ there exists a positive constant $C_S$ such that for any $R \geq 1$ and for all $f \in C_0^{\infty}(\{|x'| < R\})$ there holds*

$$\left( \int_{C_R} |f|^q w_1 dx' d\lambda \right)^{\frac{2}{q}} \leq C_S R^2 (V(C_R))^{\frac{2}{q}-1} \int_{C_R} \left( |\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2 \right) w_1 dx' d\lambda. \tag{2.3}$$

*Proof:* It is clear that it is enough to prove the inequality in the half cylinder,

$$C_R^+ = C_R \cap \{\lambda > 0\} = \{|x'| < R,\ 0 < \lambda < 1\}.$$

Thus, we will prove that for any $f \in C_0^{\infty}(\{|x'| < R\})$

$$\left( \int_{C_R^+} |f|^q w_1 dx' d\lambda \right)^{\frac{2}{q}} \leq C_S R^2 V(C_R^+)^{\frac{2}{q}-1} \int_{C_R^+} \left( |\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2 \right) w_1 dx' d\lambda\ . \tag{2.4}$$

We change variables by defining

$$s = s(\lambda) = \left( \int_{\lambda}^{1} w_2^{-\frac{1}{2}}(t) dt \right) \left( \int_{0}^{1} w_2^{-\frac{1}{2}}(t) dt \right)^{-1}, \qquad g(x', s) = f(x', \lambda). \tag{2.5}$$

With this change of variables and taking into account (2.2), inequality (2.4) takes the following equivalent form

$$\left( \int_{\mathcal{C}_R^+} |g|^q s^{\theta} dx' ds \right)^{\frac{2}{q}} \leq C_S\, R^2 V^{\frac{2}{q}-1}(\mathcal{C}_R^+) \int_{\mathcal{C}_R^+} \left( |\nabla_{x'} g|^2 + (\partial_s g)^2 \right) s^{\theta} dx' ds, \tag{2.6}$$

with $\mathcal{C}_R^+ = \{|x'| < R,\, 0 < s < 1\}$ and $V(\mathcal{C}_R^+) = c_N R^{N-1}$ . For $R = 1$ the above inequality is written

$$\left( \int_{\mathcal{C}_1^+} |g|^q s^\theta dx' ds \right)^{\frac{2}{q}} \leq C_S \, V^{\frac{2}{q}-1}(\mathcal{C}_1^+) \int_{\mathcal{C}_1^+} \left( |\nabla_{x'} g|^2 + (\partial_s g)^2 \right) s^\theta dx' ds. \tag{2.7}$$

This is true by Proposition 2.1 of [FMT2] with $QB = 2A = \theta$ there. As a consequence $q = \frac{2(N+\theta)}{N-2+\theta}$.

To establish (2.6), after a rescaling in the $x'$ variables the inequality takes the form

$$\left( \int_{\mathcal{C}_1^+} |g|^q s^\theta dx' ds \right)^{\frac{2}{q}} \leq C_S \, V^{\frac{2}{q}-1}(\mathcal{C}_1^+) \int_{\mathcal{C}_1^+} \left( |\nabla_{x'} g|^2 + R^2 \, (\partial_s g)^2 \right) s^\theta dx' ds.$$

This is true by (2.7) and the fact that $R \geq 1$. This completes the proof.

$\square$

We next consider the local weighted Poincare iequality. If

$$\bar{f} := \frac{1}{V(C_R)} \int_{C_R} f(x', \lambda) \; w_1 dx' d\lambda \; ,$$

we have

**Lemma 2.2 (local weighted Poincare)** *Let $w_1 \in L^1(0,1)$, $(w_2)^{-\frac{1}{2}} \in L^1(0,1)$ and $(w_1 w_2)^{-1} \notin L^1(0,1)$. In addition we suppose that there exists $\theta > 0$ and some constants $c_1, c_2 > 0$ such that for any $\lambda \in (-1,1)$ there holds*

$$c_1 \left( \int_{|\lambda|}^1 w_2^{-\frac{1}{2}}(t) dt \right)^\theta \leq w_1(|\lambda|) w_2^{\frac{1}{2}}(|\lambda|) \leq c_2 \left( \int_{|\lambda|}^1 w_2^{-\frac{1}{2}}(t) dt \right)^\theta . \tag{2.8}$$

*Then there exist positive constant $C_P$ such that for any $R \geq 1$ and for all $f \in C^1(\overline{C_R})$ there holds*

$$\int_{C_R} |f - \bar{f}|^2 w_1 dx' d\lambda \leq C_P R^2 \int_{C_R} \left( |\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2 \right) w_1 dx' d\lambda, \tag{2.9}$$

*Proof:* The result will follow once we establish that for any $f \in C^1(\overline{C_R})$ we have the following inequality in the upper half cylinder $C_R^+$,

$$\int_{C_R^+} |f - \xi|^2 w_1 dx' d\lambda \leq C_P R^2 \int_{C_R^+} \left( |\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2 \right) w_1 dx' d\lambda \tag{2.10}$$

for some positive constant $C_P$ (independent on $R$), with the choice

$$\xi = \frac{\int_{|x'|<R} f(x', 0) dx'}{\omega_{N-1} R^{N-1}}.$$

A similar inequality will hold in the lower half cylinder $C_R^-$ with the same choice of $\xi$. Then, since

$$\int_{C_R} |f - \bar{f}|^2 w_1 dx' d\lambda = \min_{\xi \in \mathbb{R}} \int_{C_R} |f - \xi|^2 w_1 dx' d\lambda,$$

the required inequality in $C_R$ will follow.

Making use of the change of variables (2.5) and taking into account (2.8) inequality (2.10) takes the following equivalent form (modulo absolute constants)

$$\int_{\{|x'|<R,\ 0<s<1\}} |g-\xi|^2 s^\theta dx' ds \le C_P R^2 \int_{\{|x'|<R,\ 0<s<1\}} \left(|\nabla_{x'} g|^2 + (\partial_s g)^2\right) s^\theta dx' ds. \tag{2.11}$$

We note that

$$\xi = \frac{\int_{\{|x'|<R\}} g(x',1) dx'}{\omega_{N-1} R^{N-1}}.$$

Once again it is enough to establish the result for $R=1$. The general case then follows by scaling in $x'$ and using the fact that $R \ge 1$, as it was done in the proof of (2.6).

For $s \in [0,1]$ we define

$$\bar{g}(s) = \frac{\int_{\{|x'|<1\}} g(x',s) dx'}{\omega_{N-1}},$$

and note that $\xi = \bar{g}(1)$. There holds

$$\begin{aligned}
&\int_{\{|x'|<1,\ 0<s<1\}} |g(x',s)-\xi|^2 s^\theta dx' ds \\
&\qquad \le \ 2\int_{\{|x'|<1,\ 0<s<1\}} |g(x',s)-\bar{g}(s)|^2 s^\theta dx' ds + 2\int_{\{|x'|<1,\ 0<s<1\}} |\bar{g}(s)-\bar{g}(1)|^2 s^\theta dx' ds\ .
\end{aligned} \tag{2.12}$$

We next consider the first integral on the right hand side. By Poincaré in the $x'$ variables we have

$$\begin{aligned}
\int_{\{|x'|<1,\ 0<s<1\}} |g(x',s)-\bar{g}(s)|^2 s^\theta dx' ds &= \int_0^1 s^\theta \left( \int_{|x'|<1} |g(x',s)-\bar{g}(s)|^2 dx' \right) ds \\
&\le C \int_{\{|x'|<1,\ 0<s<1\}} |\nabla_{x'} g(x',s)|^2 s^\theta \ dx' ds.
\end{aligned} \tag{2.13}$$

Concerning the second integral on the right hand side of (2.12) we have the following one dimensional Poincaré

$$\begin{aligned}
\int_0^1 |\bar{g}(s)-\bar{g}(1)|^2 s^\theta \, ds &= \int_0^1 |\bar{g}(s)-\bar{g}(1)|^2 \left(\frac{s^{\theta+1}}{\theta+1}\right)' ds = -\frac{2}{\theta+1}\int_0^1 (\bar{g}(s)-\bar{g}(1))\bar{g}'(s) s^{\theta+1} \, ds \\
&\le \frac{2}{\theta+1}\left(\int_0^1 (\bar{g}(s)-\bar{g}(1))^2 s^{\theta+2}\, ds\right)^{\frac12} \left(\int_0^1 \bar{g}'^2(s) s^\theta \, ds\right)^{\frac12} \\
&\le \frac{2}{\theta+1}\left(\int_0^1 (\bar{g}(s)-\bar{g}(1))^2 s^{\theta}\, ds\right)^{\frac12} \left(\int_0^1 \bar{g}'^2(s) s^\theta \, ds\right)^{\frac12},
\end{aligned}$$

whence,

$$\int_0^1 |\bar{g}(s)-\bar{g}(1)|^2 s^\theta \, ds \le \frac{4}{(\theta+1)^2} \int_0^1 \bar{g}'^2(s) s^\theta \, ds,$$

from which it follows that

$$\int_{\{|x'|<1,\ 0<s<1\}} |\bar{g}(s)-\bar{g}(1)|^2 s^\theta dx' ds \le \frac{4}{(\theta+1)^2} \int_{\{|x'|<1,\ 0<s<1\}} (\partial_s g)^2 s^\theta dx' ds. \tag{2.14}$$

Combining (2.12), (2.13) and (2.14) we obtain (2.11) with $R=1$ and this completes the proof.

□

For the Moser iteration scheme to work, we will also need the analogue of Theorem 2.11 of [FMT1]. We first introduce the following norm

$$||u||^2_{H^1_{w_1,w_2}(C_R)} := \int_{C_R} \left(u^2 + |\nabla_{x'} u|^2 + w_2(\partial_\lambda u)^2\right) w_1 dx' d\lambda.$$

We next define the following two Hilbert spaces: $H^1_{w_1,w_2}(C_R)$ is the completion of $C^\infty(C_R)$ under the above norm, whereas $H^1_{0,w_1,w_2}(C_R)$ is the completion under the same norm, of functions that have compact support in $\lambda \in (-1,1)$, that is

$$H^1_{0,w_1,w_2}(C_R) = \overline{C^\infty_0(\mathbb{R}^{N-1}\times(-1,1))}^{\,||\cdot||_{H^1_{w_1,w_2}(C_R)}}.$$

We then have

**Proposition 2.3 (Density)** *Let $w_1 \in L^1(0,1)$, $(w_2)^{-\frac{1}{2}} \in L^1(0,1)$ and $(w_1 w_2)^{-1} \notin L^1(0,1)$. In addition we suppose that there exists $\theta > 0$ and some constants $c_1, c_2 > 0$ such that for any $\lambda \in (-1,1)$ there holds*

$$c_1 \left(\int_{|\lambda|}^1 w_2^{-\frac{1}{2}}(t)dt\right)^\theta \le w_1(|\lambda|) w_2^{\frac{1}{2}}(|\lambda|) \le c_2 \left(\int_{|\lambda|}^1 w_2^{-\frac{1}{2}}(t)dt\right)^\theta . \tag{2.15}$$

*If $\theta \ge 1$, then*

$$H^1_{w_1,w_2}(C_R) = H^1_{0,w_1,w_2}(C_R).$$

*Proof:* Once again we change variables by (2.5) and work in the half cylinder

$$\mathcal{C}^+_R = \{|x'| < R,\ 0 < s < 1\}.$$

We recall that $\lambda = 1$ corresponds to $s = 0$. The norm now takes the form

$$||u||^2_{H^1(\mathcal{C}^+_R, s^\theta dx' ds)} := \int_{\mathcal{C}^+_R} \left(u^2 + |\nabla_{x'} u|^2 + (\partial_s u)^2\right) s^\theta dx' ds,$$

and the corresponding function spaces are now $H^1(\mathcal{C}^+_R, s^\theta dx'ds)$ and $H^1_0(\mathcal{C}^+_R, s^\theta dx'ds)$. We need to prove that any function in $H^1(\mathcal{C}^+_R, s^\theta dx'ds)$ can be approximated by functions in $H^1_0(\mathcal{C}^+_R, s^\theta dx'ds)$.

By Theorem 7.2 of [K] it is known that the set $C^\infty(\overline{\mathcal{C}^+_R})$ is dense in $H^1(\mathcal{C}^+_R, s^\theta dx'ds)$. Hence for any $v \in H^1(\mathcal{C}^+_R, s^\theta dx'ds)$ and any $\varepsilon > 0$, there exists $w \in C^\infty(\overline{\mathcal{C}^+_R})$ such that $||v - w||_{H^1} \le \epsilon$. We then define the function

$$\varphi_k(\lambda) = \begin{cases} 0 & \text{if } \ s \le \frac{1}{k^2}\ , \\ 1 + \frac{\ln(ks)}{\ln(k)} & \text{if } \ \frac{1}{k^2} < s < \frac{1}{k}\ , \\ 1 & \text{if } \ s \ge \frac{1}{k}\ , \end{cases}$$

and set

$$w_k := w\varphi_k \in C^{0,1}_0(\{s > 0\})\Big|_{\mathcal{C}^+_R} .$$

Then,

$$\begin{aligned} ||w - w_k||^2_{H^1} &= ||w(1-\varphi_k)||^2_{H^1} \\ &\le 2\int_{\mathcal{C}^+_R} (w^2 + |\nabla_{x'} w|^2 + (\partial_s w)^2)(1-\varphi_k)^2 s^\theta dx' ds + 2\int_{\mathcal{C}^+_R} w^2 (\partial_s \varphi_k)^2 s^\theta dx' ds \\ &\le 2\int_{\{|x'|<R,\ 0<s<\frac{1}{k}\}} (w^2 + |\nabla_{x'} w|^2 + (\partial_s w)^2)\ s^\theta dx' ds + C R^{N-1} ||w||^2_{L^\infty(\mathcal{C}^+_R)} \int_{\frac{1}{k^2}<s<\frac{1}{k}} \frac{1}{s^2(\ln(k))^2}\ s^\theta ds. \end{aligned}$$

For $\theta > 1$ there holds

$$\int_{\frac{1}{k^2}<s<\frac{1}{k}} \frac{1}{s^2(\ln(k))^2}\ s^\theta ds \le \frac{1}{\theta-1}\left(\frac{1}{k}\right)^{\theta-1}\frac{1}{(\ln(k))^2}\ ,$$

whereas for $\theta = 1$,

$$\int_{\frac{1}{k^2}<s<\frac{1}{k}} \frac{1}{s^2(\ln(k))^2}\ s^\theta ds = \frac{1}{(\ln(k))^2}\int_{\frac{1}{k^2}<s<\frac{1}{k}} \frac{1}{s}ds \le \frac{1}{\ln(k)}.$$

Thus, for $\theta \ge 1$ and $k$ large enough we have $||v - w_k||_{H^1} \le 2\epsilon$ and the result follows.

□

We are now ready to study positive weak solutions $u(x', \lambda, t)$ of the parabolic problem

$$\frac{\partial u}{\partial t} = \frac{1}{w_1}\mathcal{L}\, u \text{ in } C_R \times (0, R^2)\ . \tag{2.16}$$

To this end we first have

**Definition 2.4** *A weak solution $u(x', \lambda, t)$ of (2.16) is a function*

$$u \in C^1((0,R^2); L^2(C_R, w_1 dx'd\lambda) \cap C^0((0,R^2); H^1_{w_1,w_2}(C_R))$$

*such that for any $\Phi \in C^0((0,R^2); C_0^\infty(C_R))$ and any $0 < t_1 < t_2 < R^2$ we have*

$$\int_{t_1}^{t_2}\int_{C_R}\left\{\frac{\partial u}{\partial t}\ \Phi + < \nabla_{x'}u, \nabla_{x'}\Phi > + \partial_\lambda u\ \partial_\lambda \Phi\ w_2\right\} w_1\ dx'd\lambda\ dt = 0\ .$$

Thus, the Moser iteration scheme entails the following result

**Theorem 2.5 (Parabolic Harnack inequality up to the boundary)**. *Let $N \ge 2$, $w_1 \in L^1(0,1)$, $(w_2)^{-\frac{1}{2}} \in L^1(0,1)$ and $(w_1 w_2)^{-1} \notin L^1(0,1)$. In addition we suppose that there exists $\theta \ge 1$ and some constants $c_1, c_2 > 0$ such that for any $\lambda \in (-1,1)$ there holds*

$$c_1\left(\int_{|\lambda|}^1 w_2^{-\frac{1}{2}}(t)dt\right)^\theta \le w_1(|\lambda|)w_2^{\frac{1}{2}}(|\lambda|) \le c_2\left(\int_{|\lambda|}^1 w_2^{-\frac{1}{2}}(t)dt\right)^\theta . \tag{2.17}$$

*Then, there exists a positive constant $C_H$ such that for any $R \ge 1$ and any positive weak solution $u(x', \lambda, t)$ of (2.16), the following estimate holds true*

$$ess\ sup_{(x',\lambda,t)\in C_{\frac{R}{2}}\times(\frac{R^2}{4},\frac{R^2}{2})}\ u(x',\lambda,t) \le C_H\ ess\ inf_{(x',\lambda,t)\in C_{\frac{R}{2}}\times(\frac{3}{4}R^2,R^2)}\ u(x',\lambda,t)\ .$$

As a consequence of Theorem 2.5, the only nonnegative stationary weak solutions of the parabolic problem (2.16) are the constants and this proves Theorem 1.3(b).

# 3 Critical and supercritical cases: Proof of Theorem 1.4

To obtain the result we make use of the oscillation decrease method, cf section 4.3 of [HL], as adapted in Theorem 1.4 of [M2] to the anisotropic setting; we recall its statement for the convenience of the reader.

We first define a nondecreasing positive function $\gamma = \gamma(R)$, $R \ge 1$, that satisfies $\lim_{R\to\infty}\gamma(R) = +\infty$ and in addition has the following property (level growth) :

For $\tau > 0$, there exist two positive functions $l(\tau)$ and $L(\tau)$ such that

$$l(\tau) \leq \frac{\gamma(\tau R)}{\gamma(R)} \leq L(\tau) \tag{3.1}$$

for all $R \geq 1$ and $\lim_{\tau \to 0^+} L(\tau) = 0$ .

For a positive and locally Lipschitz function $\varphi = \varphi(x', \lambda)$ we define

$$L_R := L_{R,\gamma(R)} = \{(x', \lambda) \in S : |x'| < R,\ \varphi < \gamma(R)\} \ .$$

We assume that $\varphi$ has the following two additional properties

**(i)** $\varphi$ is an almost-supersolution, that is, there exist constants $C_{as} \geq 0$, $\gamma_0 \geq 0$ and $\beta > 0$, such that

$$\frac{1}{w_1} \mathcal{L}\varphi \leq C_{as} \frac{\gamma(R)}{R^{2+\beta}} \quad \text{in} \ \ L_R \setminus L_{R,\gamma_0} = \{|x'| < R,\ \gamma_0 \leq \varphi < \gamma(R)\},$$

for every $R > 1$ such that $\gamma(R) > \gamma_0$.

**(ii)** (balancing property) As $R \to +\infty$,

$$K(R) := \frac{R}{\gamma(R)} V^{-\frac{1}{2}}(L_R) \left( \int_{L_R} \left( |\nabla_{x'}\varphi|^2 + w_2 (\partial_\lambda \varphi)^2 \right) w_1 dx' d\lambda \right)^{\frac{1}{2}} \to 0 \ .$$

In addition we ask for the following properties.

**(iii)** (volume doubling) There exists a positive constant $C_D$ (independent of $R$) such that

$$V(L_{2R}) \leq C_D V(L_R) \ ,$$

for every $R > 1$, where $V(D) := \int_D w_1 dx' d\lambda$.

**(iv)** (local weighted Sobolev) For some $q > 2$ there exists a positive constant $C_S$ (independent of $R$) such that,

$$\left( \frac{1}{V(L_R)} \int_{L_R} |f|^q w_1 dx' d\lambda \right)^{\frac{2}{q}} \leq C_S R^2 \frac{1}{V(L_R)} \int_{L_R} \left( |\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2 \right) w_1 dx' d\lambda,$$

for every $R > 1$ and $f \in C_0^\infty(L_R)$.

**(v)** (local weighted Poincaré) There exists a positive constant $C_P$ (independent of $R$) such that, for every $R > 1$ and every $f \in C^1(\overline{L_R})$ satisfying $f = 0$ on $\{|x'| \leq R,\ \varphi = \gamma(R)\}$ and

$$V\left( \{(x', \lambda) \in L_R : f(x', \lambda) = 0\} \right) \geq \frac{1}{2} V(L_R),$$

there holds

$$\int_{L_R} f^2(x', \lambda) w_1 dx' d\lambda \leq C_P R^2 \int_{L_R} \left( |\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2 \right) w_1 dx' d\lambda.$$

Then, by Theorem 1.4 of [M2] it follows that if $\mathrm{osc}_{L_R} u := \sup_{L_R} u - \inf_{L_R} u$, then

**(a)** (Density theorem) Let $u \in H^1_{w_1,w_2}(L_{2R})$, $u \geq 0$, $-\mathcal{L}\ u \geq 0$ (weakly) in $L_{2R}$, $\mathrm{osc}_{L_{2R}} u \leq 2$ and

$$V\left( \{(x', \lambda) \in L_R : u \geq 1\} \right) \geq \frac{1}{2} V(L_R)$$

for $R$ big enough. Then we have $\inf_{L_{\frac{R}{2}}} u \geq \delta$ , for some $\delta > 0$ independent from $R$.

**(b)** (Oscillation decrease) Let $u \in H^1_{w_1,w_2}(L_{2R})$, $\mathcal{L}\ u = 0$ (weakly) in $L_{2R}$ for $R$ big enough then for some $\delta' > 0$ independent from $R$,

$$\mathrm{osc}_{L_{\frac{R}{2}}} u \leq [1-\delta']\mathrm{osc}_{L_{2R}} u.$$

**(c)** (Liouville theorem) Any bounded (weak) solution of $\mathcal{L}\ u = 0$ in $S$ is constant.

In the sequel we take $\gamma(R) = R^m$ for $m > 2$ and the function $\varphi(\lambda)$ as defined in (1.4). Assuming that $w_1$, $w_2$ satisfy the hypotheses of Theorem 1.4, we will verify properties (i)–(v). It is easily seen that $\gamma$ satisfies the level growth estimate (3.1).

Function $\varphi(\lambda)$ is in fact a solution of $\mathcal{L}\ u = 0$, in $S$ away from $\lambda = 0$; moreover, since $(w_1 w_2)^{-1} \notin L^1(0,1)$, it is unbounded as $|\lambda| \to 1$. Thus, the almost-supersolution property (i) is satisfied with $C_{as} = 0$.

Recall that

$$L_R = \{|x'| < R,\ \varphi < R^m\}.$$

Since $w_1 \in L^1(0,1)$ for $R$ large enough we have

$$d_1 R^{N-1} \leq V(L_R) := \int_{L_R} w_1 dx' d\lambda \leq d_2 R^{N-1},$$

for suitable positive constants $d_1$, $d_2$, therefore the volume doubling property (iii) is satisfied too.

Concerning the balancing property (ii), we compute

$$\begin{aligned}\int_{L_R} (|\nabla_{x'}\varphi|^2 + w_2(\partial_\lambda\varphi)^2) w_1 dx' d\lambda &= CR^{N-1}\int_{\{\varphi<R^m\}} w_1 w_2 (\partial_\lambda \varphi)^2\, d\lambda \\ &\leq CR^{N-1}\int_{\{\varphi<R^m\}} (w_1 w_2)^{-1}\, d\lambda \leq CR^{N-1+m}.\end{aligned}$$

It follows that

$$K(R) \leq CR^{1-m} R^{-\frac{N-1}{2}} R^{\frac{N-1}{2}} R^{\frac{m}{2}} = CR^{1-\frac{m}{2}},$$

which tends to zero as $R$ tends to $+\infty$ since $m > 2$. Hence the balancing property (ii) is also satisfied.

It only remains to prove the local weighted Sobolev and Poincaré inequalities, which we believe are of independent interest.

**Lemma 3.1 (local weighted Sobolev)** *Let $w_1 \in L^1(0,1)$ and $(w_2)^{-\frac{1}{2}} \notin L^1(0,1)$ and define*

$$\varphi(\lambda) = 1 + \int_0^{|\lambda|} (w_1 w_2)^{-1}(t) dt. \tag{3.2}$$

*We assume that there exists $m > 2$ such that $\varphi^{-\frac{1}{m}} w_2^{-\frac{1}{2}} \in L^1(0,1)$ and $\theta > 0$ such that for some constants $c_1, c_2 > 0$ and any $\lambda \in (-1,1)$ there holds*

$$c_1 \left(\int_{|\lambda|}^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t) dt\right)^\theta \leq w_1(|\lambda|) w_2^{\frac{1}{2}}(|\lambda|) \varphi^{\frac{1}{m}}(|\lambda|) \leq c_2 \left(\int_{|\lambda|}^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t) dt\right)^\theta. \tag{3.3}$$

*Then for $q = \frac{2(N+\theta)}{N-2+\theta}$ there exists a positive constant $C_S$ such that for any $R \geq 1$ and for all $f \in C_0^\infty(L_R)$ there holds*

$$\left(\int_{L_R} |f|^q w_1 dx' d\lambda\right)^{\frac{2}{q}} \leq C_S R^2 (V(L_R))^{\frac{2}{q}-1} \int_{L_R} \left(|\nabla_{x'} f|^2 + w_2(\partial_\lambda f)^2\right) w_1 dx' d\lambda. \tag{3.4}$$

*Proof:* The proof is similar to the proof of Lemma 2.1, we therefore sketch it. By scaling in the $x'$–variables

$$x' = Ry', \qquad g(y',\lambda) = f(y'R,\lambda) \in C_0^\infty(|y'|<1, \varphi < R^m),$$

estimate (3.4) takes the following equivalent form

$$\left(\int_{\{|y'|<1,\ \varphi<R^m\}} |g|^q w_1 dy'd\lambda\right)^{\frac{2}{q}} \le C_S \int_{\{|y'|<1,\ \varphi<R^m\}} \left(|\nabla_{y'} g|^2 + R^2 w_2 (\partial_\lambda g)^2\right) w_1 dy'd\lambda;$$

Since $\varphi(\lambda) < R^m$ is equivalent to $\varphi^{\frac{2}{m}}(\lambda) < R^2$, it is enough to establish

$$\left(\int_{\{|y'|<1,\ \varphi<R^m\}} |g|^q w_1 dy'd\lambda\right)^{\frac{2}{q}} \le C_S \int_{\{|y'|<1,\ \varphi<R^m\}} \left(|\nabla_{y'} g|^2 + \varphi^{\frac{2}{m}} w_2 (\partial_\lambda g)^2\right) w_1 dy'd\lambda;$$

In fact we will prove a stronger inequality, namely for $g \in C_0^\infty(|y'|<1,\ |\lambda|<1)$.

$$\left(\int_{\{|y'|<1,\ |\lambda|<1\}} |g|^q w_1 dy'd\lambda\right)^{\frac{2}{q}} \le C_S \int_{\{|y'|<1,\ |\lambda|<1\}} \left(|\nabla_{y'} g|^2 + \varphi^{\frac{2}{m}} w_2 (\partial_\lambda g)^2\right) w_1 dy'd\lambda.$$

It is enough to prove the result in the upper half cylinder, that is for $g \in C_0^\infty(|y'|<1,\ |\lambda|<1)$,

$$\left(\int_{\{|y'|<1,\ 0<\lambda<1\}} |g|^q w_1 dy'd\lambda\right)^{\frac{2}{q}} \le C_S \int_{\{|y'|<1,\ 0<\lambda<1\}} \left(|\nabla_{y'} g|^2 + \varphi^{\frac{2}{m}} w_2 (\partial_\lambda g)^2\right) w_1 dy'd\lambda. \tag{3.5}$$

To do this we change variables by

$$s = s(\lambda) = \left(\int_\lambda^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t)dt\right)\left(\int_0^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t)dt\right)^{-1}, \quad h(y',s) = g(y',\lambda), \quad \lambda \in (0,1). \tag{3.6}$$

Taking into account (3.3), inequality (3.5) takes the form

$$\left(\int_{\{|y'|<1,\ 0<s<1\}} |h|^q s^\theta dy'ds\right)^{\frac{2}{q}} \le C_S \int_{\{|y'|<1,\ 0<s<1\}} \left(|\nabla_{y'} h|^2 + (\partial_s h)^2\right) s^\theta dy'ds,$$

with $h \in C_0^\infty(|y'|<1)$. Since $\theta > 0$, this is true because of (2.7).

□

Next, after recalling that

$$\bar{f} := \frac{1}{V(L_R)} \int_{L_R} f(x',\lambda)\ w_1 dx'd\lambda\ ,$$

we have

**Lemma 3.2 (local weighted Poincaré)** *Let $w_1 \in L^1(0,1)$ and $(w_2)^{-\frac{1}{2}} \notin L^1(0,1)$ and define*

$$\varphi(\lambda) = 1 + \int_0^{|\lambda|} (w_1 w_2)^{-1}(t)dt. \tag{3.7}$$

*We assume that there exists $m > 2$ such that $\varphi^{-\frac{1}{m}} w_2^{-\frac{1}{2}} \in L^1(0,1)$ and $\theta > 0$ such that for some constants $c_1, c_2 > 0$ and any $\lambda \in (-1,1)$ there holds*

$$c_1 \left(\int_{|\lambda|}^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t)dt\right)^\theta \le w_1(|\lambda|) w_2^{\frac{1}{2}}(|\lambda|) \varphi^{\frac{1}{m}}(|\lambda|) \le c_2 \left(\int_{|\lambda|}^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t)dt\right)^\theta. \tag{3.8}$$

*Then there exists a positive constant $C_P$ (independent of $R$) such that, for every $f \in C^1(\overline{L_R})$ satisfying $f = 0$ on $\{|x'| \leq R,\ \varphi = R^m\}$, there holds*

$$\int_{L_R} |f - \bar{f}|^2 w_1 dx' d\lambda \leq C_P R^2 \int_{L_R} \left(|\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2\right) w_1 dx' d\lambda \tag{3.9}$$

*for every $R > 1$. Moreover, if in addition $f$ is such that*

$$V\left(\{(x', \lambda) \in L_R : f(x', \lambda) = 0\}\right) \geq \frac{1}{2} V(L_R),$$

*then, for every $R > 1$, we also have*

$$\int_{L_R} f^2(x', \lambda) w_1 dx' d\lambda \leq C_P R^2 \int_{L_R} \left(|\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2\right) w_1 dx' d\lambda. \tag{3.10}$$

*Proof:* Assuming that (3.9) has been established, we first show that it implies (3.10). To this end we show that if $f$ satisfies $V(\{f = 0\} \cap L_R) \geq \frac{1}{2} V(L_R)$, we then have

$$\int_{L_R} f^2 w_1 dx' d\lambda \leq 2 \int_{L_R} |f - \bar{f}|^2 w_1 dx' d\lambda. \tag{3.11}$$

Indeed (3.11) follows easily from the following computation:

$$\begin{aligned}
\int_{L_R} |f - \bar{f}|^2 w_1 dx' d\lambda &= \int_{L_R} f^2 w_1 dx' d\lambda - \frac{(\int_{L_R} f w_1 dx' d\lambda)^2}{V(L_R)} \\
&= \int_{L_R} f^2 w_1 dx' d\lambda - \frac{(\int_{\{f \neq 0\} \cap L_R} f w_1 dx' d\lambda)^2}{V(L_R)} \\
&\geq \int_{L_R} f^2 w_1 dx' d\lambda - \frac{\left(\int_{\{f \neq 0\} \cap L_R} f^2 w_1 dx' d\lambda\right) V(\{f \neq 0\} \cap L_R)}{V(L_R)} \\
&= \int_{L_R} f^2 w_1 dx' d\lambda - \frac{\left(\int_{L_R} f^2 w_1 dx' d\lambda\right) V(\{f \neq 0\} \cap L_R)}{V(L_R)} \\
&\geq \frac{1}{2} \int_{L_R} f^2 w_1 dx' d\lambda,
\end{aligned}$$

and (3.11) follows. In the sequel we will give the proof of (3.9).

Since

$$\int_{L_R} |f - \bar{f}|^2 w_1 dy' d\lambda = \min_{\xi \in \mathbb{R}} \int_{L_R} |f - \xi|^2 w_1 dy' d\lambda,$$

it is enough to prove that for every $f \in C^1(\overline{L_R})$ satisfying $f = 0$ on $\{|x'| \leq R,\ \varphi = R^m\}$, the following inequality holds

$$\int_{L_R} |f - \xi|^2 w_1 dx' d\lambda \leq C_P R^2 \int_{L_R} \left(|\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2\right) w_1 dx' d\lambda, \tag{3.12}$$

for a particular choice of the constant $\xi$ that we will specify later.

Once again we rescale by

$$x' = R y', \qquad g(y', \lambda) = f(y' R, \lambda),$$

and (3.12) takes the following equivalent form

$$\int_{\{|y'| < 1,\ \varphi < R^m\}} |g - \xi|^2 w_1 dy' d\lambda \leq C_P \int_{\{|y'| < 1,\ \varphi < R^m\}} \left(|\nabla_{y'} g|^2 + R^2 w_2 (\partial_\lambda g)^2\right) w_1 dy' d\lambda$$

for any $g \in C^1(\overline{\{|y'| < 1,\ \varphi < R^m\}})$ such that $g = 0$ on $\{|y'| \le 1,\ \varphi = R^m\}$. This inequality will follow after establishing

$$\int_{\{|y'|<1,\ |\lambda|<1\}} |g - \xi|^2 w_1 dy' d\lambda \le C_P \int_{\{|y'|<1,\ |\lambda|<1\}} \left( |\nabla_{y'} g|^2 + \varphi^{\frac{2}{m}} w_2 (\partial_\lambda g)^2 \right) w_1 dy' d\lambda \tag{3.13}$$

for any $g \in C^1(\overline{\{|y'| < 1,\ |\lambda| < 1\}})$.

To prove (3.13), once again we work in the upper half cylinder and choose

$$\xi = \frac{\int_{|y'|<1} g(y', 0) dy'}{\omega_{N-1}} .$$

We will show that for every $g \in C^1(\overline{\{|y'| < 1, 0 < \lambda < 1\}})$ there holds

$$\int_{\{|y'|<1,\ 0<\lambda<1\}} |g - \xi|^2 w_1 dy' d\lambda \le C_P \int_{\{|y'|<1,\ 0<\lambda<1\}} \left( |\nabla_{y'} g|^2 + \varphi^{\frac{2}{m}} w_2 (\partial_\lambda g)^2 \right) w_1 dy' d\lambda, \tag{3.14}$$

Using (3.8) and making the change of variables (3.6) we are lead to prove that for $\xi = \frac{\int_{|y'|<1} h(y',1)dy'}{\omega_{N-1}}$ and $\theta > 0$ there holds,

$$\int_{\{|y'|<1,\ 0<s<1\}} |h - \xi|^2 s^\theta dy' ds \le C_P \int_{\{|y'|<1,\ 0<s<1\}} \left( |\nabla_{y'} h|^2 + (\partial_s h)^2 \right) s^\theta dy' ds,$$

for any $h \in C^1(\overline{\{|y'| < 1,\ 0 < s < 1\}})$. This inequality follows from (2.11) with $R = 1$.

□

# 4 The distance function weight and final remarks

In this section we first make specific choices of the weights $w_1$, $w_2$ and give the proof of Theorems 1.1 and 1.2. We next present some extensions of our results.

We make the following choices

$$w_1(\lambda) = (1 - |\lambda|)^\alpha \quad \text{and} \quad w_2(\lambda) = (1 - |\lambda|)^\nu .$$

*Proof of Theorem 1.1:* It is a consequence of Theorem 1.3. For part (a) we note that $\alpha > -1$ is equivalent to $w_1 \in L^1(0, 1)$ and $\nu < 1 - \alpha$ is equivalent to $(w_1 w_2)^{-1} \in L^1(0, 1)$. Since $\alpha > -1$ and $\nu < 1 - \alpha$ it follows that $\nu < 2$, which is equivalent to $w_2^{-\frac{1}{2}} \in L^1(0, 1)$. Similarly, for part (b) when $1 - \alpha \le \nu < 2$ then $(w_1 w_2)^{-1} \notin L^1(0, 1)$, and (1.3) is satisfied by choosing $\theta = \frac{\alpha + \frac{\nu}{2}}{1 - \frac{\nu}{2}} \ge 1$.

□

We next have

*Proof of Theorem 1.2:* It is a consequence of Theorem 1.4. As we have seen, $\alpha > -1$ is equivalent to $w_1 \in L^1(0, 1)$. We next note that $(w_2)^{-\frac{1}{2}} \notin L^1(0, 1)$ corresponds to $\nu \ge 2$. Then,

$$\varphi(\lambda) = 1 + \int_0^{|\lambda|} (w_1 w_2)^{-1}(t) dt = 1 + \int_0^{|\lambda|} (1 - |t|)^{-\alpha-\nu} dt \sim (1 - |\lambda|)^{-(\alpha+\nu-1)},$$

for $|\lambda| \sim 1$. When $\frac{\alpha+\nu-1}{m} - \frac{\nu}{2} + 1 > 0 \Leftrightarrow m(\nu - 2) < 2(\alpha + \nu - 1)$, then

$$\varphi^{-\frac{1}{m}}(\lambda) w_2^{-\frac{1}{2}}(\lambda) \sim (1 - |\lambda|)^{\frac{\alpha+\nu-1}{m} - \frac{\nu}{2}} \in L^1(0, 1).$$

Moreover in this case

$$\int_{|\lambda|}^{1} \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t) dt \sim (1-|\lambda|)^{\frac{\alpha+\nu-1}{m}-\frac{\nu}{2}+1} .$$

It follows that (1.5) holds if we choose $\theta$ such that

$$\theta\left(\frac{\alpha+\nu-1}{m}-\frac{\nu}{2}+1\right)=\alpha+\frac{\nu}{2}-\frac{\alpha+\nu-1}{m}. \tag{4.1}$$

If $m$ is such that

$$\frac{\alpha+\nu-1}{2}>\frac{\alpha+\nu-1}{m}>\frac{\nu}{2}-1,$$

then, since $\alpha>-1$ and $\nu\geq 2$ we have that $\frac{\alpha+\nu-1}{\alpha+\frac{\nu}{2}}<2$ and it follows from (4.1) the positivity of $\theta$. Hence, all hypothesis of Theorem 1.4 are satisfied and the result follows.

□

**Remark** (i) Our Liouville type results in the supercritical case, that is $\alpha>-1$ and $\nu>2$ of Theorem 1.2, can be transformed to Liouville type results for the isotropic equation of the form

$$div((1+|s|)^{\tau}\nabla v(x',s))=0, \quad \text{in } \mathbb{R}^N \quad \text{for} \quad \tau=\frac{2\alpha+\nu}{2-\nu}; \tag{4.2}$$

here $\tau$ can be any number in the interval $(-\infty,-1)$. This can be done via the change of variables

$$s=\int_0^{\lambda}(1-|t|)^{-\frac{\nu}{2}}\,dt, \qquad v(x',s)=u(x,\lambda).$$

It follows that every bounded weak solution of (4.2) is constant.
(ii) In the critical case, that is $\nu=2$, using the same change of variables, one can obtain Liouville type results for the isotropic equation of the form

$$div(e^{\tau|s|}\nabla v(x',s))=0 \text{ in } \mathbb{R}^N, \qquad \tau=-(1+\alpha),$$

here $\tau$ can be any number in the interval $(-\infty,0)$.

We note that the above results do not follow by the ones by Bella and Schäffner [BS] but they do follow from Theorem 2.1 in [M1].
(iii) Similarly, in the subcritical case, that is $1-\alpha\leq\nu<2$, one obtains Liouville type results as in Theorem 1.1(b), for the equation

$$div((1-|s|)^{\tau}\nabla v(x',s))=0, \quad \text{in } \mathbb{R}^{N-1}\times(-1,1), \quad \tau=\frac{2\alpha+\nu}{2-\nu};$$

here $\tau$ can be any number in the interval $[1,\infty)$.

In an other direction, we note that the same results with Theorems 1.1 and 1.2 hold for more general operators that can be thought of as perturbations of the operators we considered so far. More precisely, let

$$\begin{aligned}\mathcal{L}'_{\alpha,\nu}\,u \;&:=\; div(B_{\alpha,\nu}(x',\lambda)\nabla u)\\ &=\; \sum_{i,j=1}^{N-1}\frac{\partial}{\partial x_i}\left(A_{i,j}(1-|\lambda|)^{\alpha}\frac{\partial u}{\partial x_j}\right)+\frac{\partial}{\partial\lambda}\left(A_{N,N}(1-|\lambda|)^{\alpha+\nu}\frac{\partial u}{\partial\lambda}\right)\\ &\quad+\sum_{j=1}^{N-1}\frac{\partial}{\partial\lambda}\left(A_{N,j}(1-|\lambda|)^{\alpha+\frac{\nu}{2}}\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{N-1}\frac{\partial}{\partial x_i}\left(A_{i,N}(1-|\lambda|)^{\alpha+\frac{\nu}{2}}\frac{\partial u}{\partial\lambda}\right),\end{aligned} \tag{4.3}$$

in

$$S = \mathbb{R}^{N-1} \times (-1,1)\ , \qquad N \geq 2,$$

where the $N \times N$ matrix $A = (A_{i,j})$ has bounded and measurable entries $A_{i,j} = A_{i,j}(x', \lambda)$ for $(x', \lambda) \in S$, and it is symmetric and uniformly elliptic, that is, for some constants $0 < c_0 \leq C_0$ the following inequalities hold true for any $\xi = (\xi', \xi_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and $(x', \lambda) \in S$:

$$c_0|\xi|^2 \leq \sum_{i,j=1}^{N} A_{i,j}(x', \lambda)\xi_i\xi_j \leq C_0|\xi|^2.$$

Equivalently, we have

$$c_0(1-|\lambda|)^{\alpha}(|\xi'|^2 + (1-|\lambda|)^{\nu}|\xi_N|^2) \leq \sum_{i,j=1}^{N} (B_{\alpha,\nu})_{i,j}(x', \lambda)\xi_i\xi_j \leq C_0(1-|\lambda|)^{\alpha}(|\xi'|^2 + (1-|\lambda|)^{\nu}|\xi_N|^2).$$

Clearly, the model operator $\mathcal{L}_{\alpha,\nu}$, defined in (1.2), follows from $\mathcal{L}'_{\alpha,\nu}$ in the special case where $A_{i,j} = \delta_{i,j}$ , that is, when $A$ is the identity matrix. By quite similar arguments one can prove

**Theorem 4.1** *Let $\alpha > -1$.*

**(a)** *If in addition $1 - \alpha \leq \nu < 2$, then any nonnegative weak solution of $\mathcal{L}'_{\alpha,\nu}u = 0$ in $S$ is constant.*

**(b)** *If in addition $\nu \geq 2$ and*

$$\begin{aligned} &A_{i,N} \text{ do not depend on } x_i, \quad i=1,2,\dots,\ N\text{-}1, \\ &A_{N,N} \text{ does not depend on } \lambda, \end{aligned}$$

*every bounded weak solutions of $\mathcal{L}'_{\alpha,\nu}u = 0$ in $S$ is constant.*

We can also consider the more general operator

$$\begin{aligned} \mathcal{L}'\ u \ &:= \ div(B(x', \lambda)\nabla u) \\ &= \ \sum_{i,j=1}^{N-1} \frac{\partial}{\partial x_i}\left(A_{i,j}w_1\frac{\partial u}{\partial x_j}\right) + \frac{\partial}{\partial \lambda}\left(A_{N,N}w_1w_2\frac{\partial u}{\partial \lambda}\right) \\ &\quad + \sum_{j=1}^{N-1} \frac{\partial}{\partial \lambda}\left(A_{N,j}w_1\sqrt{w_2}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{N-1} \frac{\partial}{\partial x_i}\left(A_{i,N}w_1\sqrt{w_2}\frac{\partial u}{\partial \lambda}\right), \end{aligned} \tag{4.4}$$

in

$$S = \mathbb{R}^{N-1} \times (-1,1)\ , \quad N \geq 2$$

where the $N \times N$ matrix $A = (A_{i,j})$ has bounded and measurable entries $A_{i,j} = A_{i,j}(x', \lambda)$ for $(x', \lambda) \in S$, and it is symmetric and uniformly elliptic, that is, for some constants $0 < c_0 \leq C_0$ the following inequalities hold true for any $\xi = (\xi', \xi_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and for any $(x', \lambda) \in S$:

$$c_0|\xi|^2 \leq \sum_{i,j=1}^{N} A_{i,j}(x', \lambda)\xi_i\xi_j \leq C_0|\xi|^2.$$

Equivalently

$$c_0w_1(|\xi'|^2 + w_2|\xi_N|^2) \leq \sum_{i,j=1}^{N} B_{i,j}(x', \lambda)\xi_i\xi_j \leq C_0w_1(|\xi'|^2 + w_2|\xi_N|^2).$$

We recall that $w_i = w_i(|\lambda|)$, $i = 1, 2$. The model operator $\mathcal{L}$ in (1.1) follows from $\mathcal{L}'$ in the special case where $A$ is the $N \times N$ identity matrix. By quite similar arguments we have

**Theorem 4.2** *Let $w_1 \in L^1(0,1)$.*

**(a)** *If $(w_2)^{-\frac{1}{2}} \in L^1(0,1)$, $(w_1 w_2)^{-1} \notin L^1(0,1)$ and there exists $\theta \geq 1$ and constants $c_1, c_2 > 0$ such that for any $\lambda \in (-1,1)$ there holds*

$$c_1 \left( \int_{|\lambda|}^{1} w_2^{-\frac{1}{2}}(t)dt \right)^{\theta} \leq w_1(|\lambda|) w_2^{\frac{1}{2}}(|\lambda|) \leq c_2 \left( \int_{|\lambda|}^{1} w_2^{-\frac{1}{2}}(t)dt \right)^{\theta},$$

*then any nonnegative weak solution of $\mathcal{L}'u = 0$ in $S$, is constant.*

**(b)** *Let $(w_2)^{-\frac{1}{2}} \notin L^1(0,1)$. We define*

$$\varphi(\lambda) = 1 + \int_0^{|\lambda|} (w_1 w_2)^{-1}(t)dt.$$

*We assume that there exists $m > 2$ such that $\varphi^{-\frac{1}{m}} w_2^{-\frac{1}{2}} \in L^1(0,1)$ and $\theta > 0$ such that for some constants $c_1, c_2 > 0$ and any $\lambda \in (-1,1)$ there holds*

$$c_1 \left( \int_{|\lambda|}^{1} \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t)dt \right)^{\theta} \leq w_1(|\lambda|) w_2^{\frac{1}{2}}(|\lambda|) \varphi^{\frac{1}{m}}(|\lambda|) \leq c_2 \left( \int_{|\lambda|}^{1} \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t)dt \right)^{\theta}.$$

*In addition,*

$$\begin{aligned} &A_{i,N} \text{ do not depend on } x_i, \quad i=1,2,\ldots, N-1,\\ &A_{N,N} \text{ does not depend on } \lambda. \end{aligned}$$

*Then any bounded weak solution of $\mathcal{L}'u = 0$ in $S$ is constant.*

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