Optimizing Improved Hardy Inequalities

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Abstract

Let Ω be a bounded domain in $I\!R^N$, $N \geq 3$, containing the origin. Motivated by a question of Brezis and Vázquez, we consider an Improved Hardy Inequality with best constant b, that we formally write as: $-\Delta \geq \left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2} + bV(x)$. We first give necessary conditions on the potential V, under which the previous inequality can or cannot be further improved. We show that the best constant b is never achieved in $H_0^1(\Omega)$, and in particular that the existence or not of further correction terms is not connected to the non achievement of b in $H_0^1(\Omega)$. Our analysis reveals that the original inequality can be repeatedly improved by adding in the right hand side specific potentials. This leads to an infinite series expansion of Hardy's inequality. The series obtained is in some sense optimal. In establishing these results we derive various sharp Improved Hardy-Sobolev inequalities.

1 Introduction

Throughout this work Ω is a bounded domain in $I\!R^N$, $N \geq 3$, containing the origin. The classical Hardy inequality asserts that for all $u \in H_0^1(\Omega)$:

$$\int_\Omega |\nabla u(x)|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2(x)}{|x|^2} dx. \qquad (1.1)$$

It is well known that $\left(\frac{N-2}{2}\right)^2$ is the best constant for inequality (1.1), and that this constant is not attained in $H_0^1(\Omega)$; see [OK] for a comprehensive account of Hardy inequalities and [D] for a recent review. The fact that the best constant is not attained suggests that one might look for an error term in (1.1). Indeed, Brezis and Vázquez [BV], have obtained the following Improved Hardy Inequalities valid for any $u \in H_0^1(\Omega)$:

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + \lambda_\Omega \int_\Omega u^2 dx, \tag{1.2}$$

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + K\|u\|^2_{L^p(\Omega)}. \tag{1.3}$$

In (1.3) we assume that $1 < p < 2N/(N-2)$. The constant λ_Ω in (1.2) is given by:

$$\lambda_\Omega = z_0^2 \ \omega_N^{\frac{2}{N}} \ |\Omega|^{-\frac{2}{N}}, \tag{1.4}$$

where ω_N and $|\Omega|$ denote the volume of the unit ball and Ω respectively, and $z_0 = 2.4048\ldots$ denotes the first zero of the Bessel function $J_0(z)$. The constant appearing in (1.4) is optimal when Ω is a ball, but again, it is not achieved in $H_0^1(\Omega)$.

Similar improved inequalities have been recently proved if instead of (1.1) one considers the Hardy inequality involving the distance from the boundary, or even the corresponding L^p Hardy inequalities. In all these cases a correction term is added in the right hand side; see, e.g, [BM], [BMS], [BFT], [FHT], [GGM], [VZ].

Hardy inequalities as well as their improved versions are used in many contexts. For instance, they have been useful in the study of the stability of solutions of semi-linear elliptic and parabolic equations (cf [BV], [CM1] [PV], [V]), in the existence and asymptotic behavior of the heat equation with singular potentials, (cf [CM2], [VZ]), as well as in the study of the stability of eigenvalues in elliptic problems (cf [D], [FHT]).

The motivation for the present work comes from the following question raised in [BV] (cf Problem 2, Section 8): In case Ω is a ball centered at zero, are the two terms in the right hand side of (1.2) just the first two terms of a series? Is there a further improvement of (1.3)?

We will address these questions in a more general setting. Thus, instead of (1.2) we will consider a more general Improved Hardy inequality:

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + b\int_\Omega V u^2 dx, \qquad \forall\ u \in H_0^1(\Omega). \tag{1.5}$$

We want the potential V to be a lower order potential compared to the Hardy potential $\frac{1}{|x|^2}$. For that reason we give the following definition of the admissible class $\mathcal{A}$ of potentials :

Definition 1.1 *We say that a potential V is an admissible potential, that is $V \in \mathcal{A}$, if V is not everywhere nonpositive, $V \in L_{loc}^{\frac{N}{2}}(\Omega \setminus \{0\})$, and there exists a positive constant C such that*

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + C\int_\Omega |V| u^2 dx, \qquad \forall\ u \in H_0^1(\Omega). \tag{1.6}$$

The presence of the absolute value in the right hand side of (1.6) ensures that the negative part of V is itself a lower order potential compared to the Hardy potential, and therefore the Hardy potential is truly present in (1.5).

It follows from (1.3), by means of Holder's inequality that if V is not everywhere nonpositive and $V \in L^p(\Omega)$ with $p > N/2$, then $V \in \mathcal{A}$. As a matter of fact $\mathcal{A}$ contains potentials which are not in $L^p(\Omega)$ with $p > N/2$. This will follow from the following Improved Hardy-Sobolev inequality with critical exponent. We set

$$X(t) = (-\log t)^{-1}. \tag{1.7}$$

We then have:

Theorem A (**Improved Hardy-Sobolev Inequality**) *Let $D \geq \sup_{x\in\Omega}|x|$. Then, there exists $c > 0$ such that for all $u \in H_0^1(\Omega)$:*

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + c\left(\int_\Omega |u|^{\frac{2N}{N-2}} X^{1+\frac{N}{N-2}}\left(\frac{|x|}{D}\right) dx\right)^{\frac{N-2}{N}}. \tag{1.8}$$

We note that estimate (1.8) is sharp in the sense that $X^{1+\frac{N}{N-2}}$ cannot be replaced by a smaller power of X. This is in contrast with the Hardy-Sobolev inequalities derived by Maz'ja ([M], Corollary 3, p. 97) where however distance is not taken from a point but from a hyperplane; see also [BFT], [VZ], [BL] for related results.

As a consequence of Theorem A, the class $\mathcal{A}$ contains all non everywhere nonpositive potentials V such that $\int_\Omega |V|^{\frac{N}{2}} X^{1-N} dx < \infty$.

We now return to inequality (1.5) where $V \in \mathcal{A}$ and $b > 0$ is the best constant, and we pose our main question: Can we further improve (1.5)? That is, we ask whether there is a potential $W \in \mathcal{A}$, and a positive constant b_1 such that:

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + b\int_\Omega V u^2 dx + b_1 \int_\Omega W u^2 dx, \qquad \forall\, u \in H_0^1(\Omega). \tag{1.9}$$

To answer the question the following quantity plays an important role:

$$\mathcal{C}^0 := \lim_{r\downarrow 0} C_r, \qquad C_r = \inf_{\substack{u\,\in\, C_0^\infty(B_r) \\ \int_{B_r} V u^2 dx > 0}} \frac{\int_{B_r} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{B_r} \frac{u^2}{|x|^2} dx}{\int_{B_r} V u^2 dx}. \tag{1.10}$$

If in (1.10) there is no $u \in C_0^\infty(B_r)$ such that $\int_{B_r} V u^2 dx > 0$ for some $r > 0$, we set $C_r = \infty$. We may think of $\mathcal{C}^0$ as the the local best constant of (1.5) near zero.

It is evident that $b \leq \mathcal{C}^0$. We then prove:

Theorem B *Let $V \in \mathcal{A}$. If*

$$b < \mathcal{C}^0,$$

then, we cannot improve (1.5) by adding a nonnegative potential $W \in \mathcal{A}$.

We note however that if we allow W to change sign then improvement of (1.5) is possible under some extra condition on W; see Proposition 3.8 for the precise statement.

A consequence of Theorems A and B is the following (cf Corollary 3.7):

Corollary 1.2 *Let $D > \sup_{x\in\Omega}|x|$. Suppose V is not everywhere nonpositive, and such that $\int_\Omega |V|^{\frac{N}{2}} X^{1-N}(|x|/D) dx < \infty$. Then, there is no improvement of (1.5) with nonnegative $W \in \mathcal{A}$.*

We next address the question of whether the best constants in Hardy type inequalities, such as (1.5) or (1.9) are achieved or not in $H_0^1(\Omega)$. In this direction we establish a more general result which is of independent interest. In order to state this result, let us first consider the following problem:

$$\begin{aligned}\Delta u+\left(\frac{N-2}{2}\right)^2\frac{u}{|x|^2}+V(x)u &= 0, \qquad \text{in }\ \Omega,\\ u>0, \quad \text{in} \quad \Omega\setminus\{0\}, \qquad u &= 0, \qquad \text{on }\ \partial\Omega. \end{aligned} \tag{1.11}$$

We denote by V_+ and V_- the positive and negative part of V. That is $V_+=\max\{0,V\}$ and $V_-=\max\{0,-V\}$. We then have:

Theorem C *Let $V\in C^{0,\alpha}_{loc}(\Omega\backslash\{0\})$, for some $\alpha\in(0,1)$. We also assume that $V_+\in L^{\frac{N}{2},\infty}(\Omega)$ and $V_-\in L^q(\Omega)$ with $q>\frac{N}{2}$. Then problem (1.11) has no $H^1_0(\Omega)$ solutions.*

As a consequence of this, the best constants in the aforementioned Hardy type inequalities are not achieved in $H^1_0(\Omega)$. In particular, the existence or not of further correction terms in these inequalities does not follow from the non-achievement of the best constants in $H^1_0(\Omega)$. For instance, by Theorem C the best constant λ_Ω in (1.2) is not achieved in $H^1_0(\Omega)$, yet, by Corollary 1.2 it cannot be further improved by adding a nonnegative potential in the right hand side. By theorem B, a necessary condition for further improvement, is the equality of the global and local best constants.

In connection with this let us make the following observation. In the plain Hardy inequality (1.1) it is well known that for r small:

$$\inf_{u\,\in\, C^\infty_0(\Omega)}\frac{\int_\Omega|\nabla u|^2dx}{\int_\Omega\frac{u^2}{|x|^2}dx}=\inf_{u\,\in\, C^\infty_0(B_r)}\frac{\int_{B_r}|\nabla u|^2dx}{\int_{B_r}\frac{u^2}{|x|^2}dx}=\left(\frac{N-2}{2}\right)^2.$$

Thus, the global and local best constants are equal and improvement of (1.1) is possible.

We then look for potentials $V\in\mathcal{A}$ for which (1.5) holds true and at the same time $b=\mathcal{C}^0$. It turns out that such potentials do exist for which further improvement of (1.5) is possible. The next natural question is whether we can repeat this process, of successively improving (1.1), thereby obtaining some sort of "series expansion" for Hardy inequality. It turns out that this is possible. Before stating our result let us first introduce some notation.

For $t\in(0,1]$ we define the following functions:

$$X_1(t)=(1-\log t)^{-1},\qquad X_k(t)=X_1(X_{k-1}(t)),\qquad k=2,3,\dots.$$

We then have:

Theorem D (Series expansion of Hardy's Inequality) *Let $D\geq\sup_{x\in\Omega}|x|$. Then, the following inequality holds for any $u\in H^1_0(\Omega)$:*

$$\begin{aligned}\int_\Omega|\nabla u(x)|^2dx \;\geq\; &\left(\frac{N-2}{2}\right)^2\int_\Omega\frac{u^2(x)}{|x|^2}dx\\ &+\;\frac14\sum_{i=1}^\infty\int_\Omega\frac{1}{|x|^2}X_1^2\Big(\frac{|x|}{D}\Big)X_2^2\Big(\frac{|x|}{D}\Big)\dots X_i^2\Big(\frac{|x|}{D}\Big)u^2(x)dx.\end{aligned}\tag{1.12}$$

Moreover, for each $k=1,2,\dots$ the constant $1/4$ is the best constant for the corresponding k- Improved Hardy inequality, that is

$$\frac14=\inf_{u\in H^1_0(\Omega)}\frac{\int_\Omega|\nabla u|^2dx-\left(\frac{N-2}{2}\right)^2\int_\Omega\frac{u^2}{|x|^2}dx-\frac14\sum_{i=1}^{k-1}\int_\Omega\frac{1}{|x|^2}X_1^2X_2^2\dots X_i^2u^2dx}{\int_\Omega\frac{1}{|x|^2}X_1^2X_2^2\dots X_k^2u^2dx}.$$

If we cut the above series at the k step, we then obtain the k-Improved Hardy inequality. Let us introduce the notation:

$$I_k[u] = \int_\Omega |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx - \frac{1}{4}\sum_{i=1}^{k} \int_\Omega \frac{1}{|x|^2} X_1^2 X_2^2 \dots X_i^2 u^2 dx. \quad (1.13)$$

Then, the k-Improved Hardy inequality can be written as $I_k[u] \geq 0$, for $k = 1, 2, \dots$. The particular choice of the potentials we add in the right hand side of (1.1) at each step, is suggested by Theorem B. Thus, the first potential $V_0 = |x|^{-2} X_1^2$ is such that $b = \mathcal{C}^0 = 1/4$. The same logic underlies the choice of the other potentials. More precisely, suppose that at the k step we ask whether there are potentials V_k for which the following inequality holds:

$$I_k[u] \geq b_k \int_\Omega V_k u^2 dx. \quad (1.14)$$

As before, we want V_k to be a lower order potential compared to the ones appearing in $I_k[u]$. We then define the admissible class $\mathcal{A}_k$ in analogy with $\mathcal{A}$:

Definition 1.3 *We say that a potential V_k is a k- admissible potential, that is $V_k \in \mathcal{A}_k$, if V_k is not everywhere nonpositive, $V_k \in L^{\frac{N}{2}}_{loc}(\Omega \setminus \{0\})$, and there exists a positive constant C such that*

$$I_k[u] \geq C \int_\Omega |V_k| u^2 dx, \qquad \forall\, u \in H_0^1(\Omega). \quad (1.15)$$

The corresponding k-Improved Hardy-Sobolev inequality becomes:

Theorem A' *(**k-Improved Hardy-Sobolev Inequality**) Let $D \geq \sup_{x\in\Omega} |x|$. Then, there exists $c > 0$ such that for all $u \in H_0^1(\Omega)$:*

$$I_k[u] \geq c \left(\int_\Omega |u|^{\frac{2N}{N-2}} \left(\Pi_{i=1}^{k+1} X_i\left(\frac{|x|}{D}\right) \right)^{1+\frac{N}{N-2}} dx \right)^{\frac{N-2}{N}}. \quad (1.16)$$

The existence of nontrivial potentials $V_k \in \mathcal{A}_k$, follows from Theorem A'. Consider (1.14) with $V_k \in \mathcal{A}_k$ and b_k its best constant. We now define the local best constant as:

$$\mathcal{C}_k^0 := \lim_{r\downarrow 0} C_{k,r}, \qquad C_{k,r} = \inf_{\substack{u \in C_0^\infty(B_r) \\ \int_{B_r} V_k u^2 dx > 0}} \frac{I_k[u]}{\int_{B_r} V_k u^2 dx}. \quad (1.17)$$

The analogue of Theorem B reads:

Theorem B' *Let $V_k \in \mathcal{A}_k$. If*

$$b_k < \mathcal{C}_k^0,$$

then, we cannot improve (1.14) by adding a nonnegative potential $W_k \in \mathcal{A}_k$.

The choice then of potentials in Theorem D is such that at each step $b_k = \mathcal{C}_k^0$ $(= \frac{1}{4})$.

We finally discuss some of the ideas underlying the proofs. The following change of variables

$$w(x) = u(x)|x|^{\frac{N-2}{2}}, \qquad x \in \Omega, \quad (1.18)$$

already introduced in [BV], plays an important role in our approach. By means of (1.18) we can reformulate inequality (1.5) in terms of w. If b is the best constant in (1.5) we first show that $b = B$, where

$$B = \inf_{\substack{w \in C_0^\infty(B_r) \\ \int_{B_r} |x|^{-(N-2)} V w^2 dx > 0}} Q[w], \qquad Q[w] := \frac{\int_\Omega |x|^{-(N-2)} |\nabla w|^2 dx}{\int_\Omega |x|^{-(N-2)} V w^2 dx}. \tag{1.19}$$

The natural space to study this functional is a suitable Hilbert space that we denote by $W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$. It then turns out that if $b < \mathcal{C}^0$, then b is achieved in $W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$. This is the crucial ingredient in the proof of Theorem B. Similar ideas are used in the proof of Theorem B' To prove Theorem D we use a change of variables similar to (1.18) and various identities. For Theorem C after taking the spherical average of the terms appearing in (1.11) we reduce the problem to a suitable ODE and then use an argument by contradiction. Once again the change of variables (1.18) is used.

The rest of the paper is organized as follows. In Section 2 we introduce the space $W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$ and establish some preliminary estimates. In particular we prove Theorem A. In Section 3 we prove Theorem B and other related results, whereas in Section 4 we give the proof of Theorem C. In Section 5, as an application of the techniques of Section 3, we consider the special case $V = 1$, that is inequality (1.2), and we obtain some information about the best constant λ_Ω. The last two Sections are then dedicated to the infinite improvement of Hardy's inequality, and Theorems D, A' and B' are proved.

After this work was completed we learned that related results have been obtained in [ACR, AS] by different methods.

2 Preliminaries

In this Section we will introduce the space $W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$ and we will establish some preliminary results.

Clearly, the best constant b in (1.5) is given by:

$$b = \inf_{\substack{u \in H_0^1(\Omega) \\ \int_\Omega V u^2 dx > 0}} R[u], \tag{2.1}$$

where:

$$R[u] = \frac{\int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx}{\int_\Omega V u^2 dx}.$$

Let $u \in H_0^1(\Omega)$ and set $w(x) = |x|^{\frac{N-2}{2}} u(x)$. We easily check that $\nabla(|x|^{-(N-2)}) \nabla w^2 \in L^1(\Omega)$ and

$$\begin{aligned} I[u] &:= \int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx \\ &= \int_\Omega |x|^{-(N-2)} |\nabla w|^2 dx + \frac{1}{2} \int_\Omega \nabla(|x|^{-(N-2)}) \nabla w^2 dx. \end{aligned} \tag{2.2}$$

We next show that the last integral above is equal to zero. Let $B_\varepsilon = \{x : \ |x| < \varepsilon\}$ and $S_\varepsilon = \{x : \ |x| = \varepsilon\}$. We then write:

$$\int_\Omega \nabla(|x|^{-(N-2)})\nabla w^2 dx = \int_{B_\varepsilon} \nabla(|x|^{-(N-2)})\nabla w^2 dx + \int_{\Omega - B_\varepsilon} \nabla(|x|^{-(N-2)})\nabla w^2 dx.$$

The integrand in the above integrals is easily checked to be an L^1 function and therefore the first integral in the right hand side tends to zero as $\varepsilon \to 0$. Concerning the second integral, integrating by parts and using the fact that $\Delta(|x|^{-(N-2)}) = 0$ we end up with:

$$\int_{\Omega - B_\varepsilon} \nabla(|x|^{-(N-2)})\nabla w^2 dx = (N-2)\varepsilon^{-N+1} \int_{S_\varepsilon} w^2 dS = \frac{N-2}{\varepsilon} \int_{S_\varepsilon} u^2 dS.$$

Since $u \in H_0^1(\Omega)$, a simple limiting argument shows that along a sequence $\{\varepsilon_j\}$

$$\frac{N-2}{\varepsilon_j} \int_{S_{\varepsilon_j}} u^2 dS \to 0, \qquad \text{as } \varepsilon_j \to 0.$$

It then follows that the last term in (2.3) is zero, and the following identity holds:

$$I[u] = \int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx = \int_\Omega |x|^{-(N-2)} |\nabla w|^2 dx. \tag{2.3}$$

Using (2.3), we easily obtain:

$$R[u] = \frac{\int_\Omega |x|^{-(N-2)} |\nabla w|^2 dx}{\int_\Omega |x|^{-(N-2)} V w^2 dx} =: Q[w].$$

To study the functional $Q[w]$ we introduce an appropriate function space. We denote by $W_0^{1,2}(\Omega; \ |x|^{-(N-2)})$ the completion of $\mathcal{C}_0^\infty(\Omega)$ under the norm $\int_\Omega |x|^{-(N-2)} w^2 dx$ $+ \int_\Omega |x|^{-(N-2)} |\nabla w|^2 dx$. This is easily seen to be a Hilbert space with inner product $< f, \ g >= \int_\Omega |x|^{-(N-2)} fg dx + \int_\Omega |x|^{-(N-2)} \nabla f \cdot \nabla g dx$. Moreover, we have:

Lemma 2.1 *(i) If* $u \in H_0^1(\Omega)$ *then* $|x|^{\frac{N-2}{2}} u \in W_0^{1,2}(\Omega; \ |x|^{-(N-2)})$.
(ii) If $w \in W_0^{1,2}(\Omega; \ |x|^{-(N-2)})$ *then* $|x|^{-a} w \in H_0^1(\Omega)$ *for all* $a < \frac{N-2}{2}$.
(iii) $\left(\int_\Omega |x|^{-(N-2)} |\nabla w|^2 dx\right)^{1/2}$ *is an equivalent norm for the space* $W_0^{1,2}(\Omega; \ |x|^{-(N-2)})$.

Proof: (i) Let $u \in H_0^1(\Omega)$. A simple calculation shows that:

$$\begin{aligned} \int_\Omega |x|^{-(N-2)} |\nabla(|x|^{\frac{N-2}{2}} u)|^2 dx &= \int_\Omega |x|^{-(N-2)} \left| \frac{N-2}{2} |x|^{\frac{N-6}{2}} u x + |x|^{\frac{N-2}{2}} \nabla u \right|^2 dx \\ &\le 2\left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + 2\int_\Omega |\nabla u|^2 dx \le C\|u\|_{H_0^1(\Omega)} < +\infty, \end{aligned}$$

where in the last line we used the classical Hardy inequality.

(ii) Concerning the second statement let $w \in \mathcal{C}_0^\infty(\Omega)$. If $v = |x|^{-a} w$, then:

$$\int_\Omega |\nabla v|^2 dx \le a^2 \int_\Omega |x|^{-2a-2} w^2 dx + 2\int_\Omega |x|^{-2a} |\nabla w|^2 dx \tag{2.4}$$

The classical Hardy inequality, when applied to $v = |x|^{-a} w$ yields:

$$\left(a - \frac{N-2}{2}\right)^2 \int_\Omega |x|^{-2a-2} w^2 dx \le \int_\Omega |x|^{-2a} |\nabla w|^2 dx. \tag{2.5}$$

¿From this and (2.4) we get for some constant C_a depending only on a:

$$\|v\|^2_{H^1_0(\Omega)} \leq C_a \int_\Omega |x|^{-2a}|\nabla w|^2 \leq C_a \int_\Omega |x|^{-(N-2)}|\nabla w|^2 dx < +\infty.$$

The result then follows by a standard density argument.

(iii) This follows easily from (2.5) with $a = \frac{N-2}{2} - 1$. •

We will next give the proof of Theorem A. We first present an auxiliary lemma.

Lemma 2.2 *Let $X(t) = (-\log t)^{-1}$. For any $q \geq 2$, there exists a $c > 0$ such that*

$$\int_0^1 |v'|^2 r dr \geq c \left(\int_0^1 |v|^q r^{-1} X^{1+q/2}(r) dr \right)^{2/q}, \tag{2.6}$$

for any $v \in C_0^\infty(0,1)$.

Proof: It follows from [M], Theorem 3, p. 44, with $d\mu = r^{-1}X^{1+q/2}\chi_{[0,1]}dr$ and $d\nu = r\chi_{[0,1]}dr$. •

We then have:

Theorem 2.3 *Let $D \geq \sup_{x\in\Omega}|x|$ and $u \in C_0^\infty(\Omega)$. Then, there exists $c > 0$ such that:*

$$I[u] = \int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2}dx \geq c\left(\int_\Omega |u|^{\frac{2N}{N-2}} X^{1+\frac{N}{N-2}}(\frac{|x|}{D})dx\right)^{\frac{N-2}{N}}. \tag{2.7}$$

Proof: Suppose first that Ω is the unit ball B. Following [VZ] we decompose u into spherical harmonics to get

$$u = \sum_{m=0}^\infty u_m(r) f_m(\sigma), \tag{2.8}$$

where the $f_m(\sigma)$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with corresponding eigenvalues $c_m = m(N+m-2)$, $m \geq 0$. In particular $u_0(r)$ is the radial part of u and $f_0(\sigma) = 1$. Observing that

$$\int_B |\nabla u|^2 dx = \sum_{m=0}^\infty \int_B \left(|\nabla u_m|^2 + c_m \frac{u_m^2}{|x|^2}\right) dx,$$

we calculate

$$I[u] = I[u_0] + \sum_{m=1}^\infty \int_B \left(|\nabla u_m|^2 - \left(\frac{(N-2)^2}{4} - c_m\right)\frac{u_m^2}{|x|^2}\right) dx. \tag{2.9}$$

We next estimate the nonradial part using the inequality

$$\int_B \left(|\nabla u_m|^2 - \left(\frac{(N-2)^2}{4} - c_m\right)\frac{u_m^2}{|x|^2}\right) dx \geq \frac{c_m}{c_m + \frac{(N-2)^2}{4}} \int_B \left(|\nabla u_m|^2 + c_m \frac{u_m^2}{|x|^2}\right) dx.$$

Taking into account that $c_m \geq N-1$, for $m \geq 1$, we estimate the infinite sum in (2.9) from below by $C_N \int_B |\nabla(u-u_0)|^2 dx$, $C_N = 4(N-1)/N^2$. Hence, we arrive at

$$I[u] \geq I[u_0] + C_N \int_B |\nabla(u-u_0)|^2 dx. \tag{2.10}$$

We now estimate $I[u_0]$. Setting $w_0(r) = r^{\frac{N-2}{2}} u_0(r)$ we calculate:

$$\begin{aligned} I[u_0] &= N\omega_N \int_0^1 w_0'^2(r) r dr \\ &\geq c \left(\int_0^1 |w_0|^{\frac{2N}{N-2}} r^{-1} X^{1+\frac{N}{N-2}} dr \right)^{(N-2)/N} \\ &= c \left(\int_B |u_0|^{\frac{2N}{N-2}} X^{1+\frac{N}{N-2}} dr \right)^{(N-2)/N}, \end{aligned}$$

where we also used (2.6) with $q = 2N/(N-2)$.

To estimate the nonradial part in (2.10) we use the Sobolev embedding and the fact that X is bounded to obtain:

$$\begin{aligned} \int_B |\nabla(u-u_0)|^2 dx &\geq c \left(\int_B |u-u_0|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \\ &\geq c \left(\int_B |u-u_0|^{\frac{2N}{N-2}} X^{1+\frac{N}{N-2}} dx \right)^{\frac{N-2}{N}}. \end{aligned}$$

It then follows from (2.10) that for any $u \in C_0^\infty(B)$

$$I[u] \geq c \left(\int_B |u|^{\frac{2N}{N-2}} X^{1+\frac{N}{N-2}} dx \right)^{(N-2)/N}. \tag{2.11}$$

It is clear that the same argument works for B_R, a ball of radius $R > 0$.

Consider now the case where Ω is a bounded domain. Then, for some $R > 0$ we have that $\Omega \subset B_R$. Since (2.11) is true for any $u \in C_0^\infty(B_R)$ it is true in particular for every $u \in C_0^\infty(\Omega)$. •

3 Existence of minimizers in $W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$

In this Section we will give the proof of Theorem B and related results. The main idea is to reformulate inequality (1.5) in terms of w in $W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$. Throughout this Section we assume that $V \in \mathcal{A}$. In particular V satisfies (1.6). We next show that (1.6) is equivalent to the following inequality:

$$\int_\Omega |x|^{-(N-2)} |\nabla w|^2 dx \geq C \int_\Omega |x|^{-(N-2)} |V| w^2 dx, \qquad \forall\, w \in W_0^{1,2}(\Omega;\ |x|^{-(N-2)}). \tag{3.1}$$

More precisely we have:

Lemma 3.1 *The best constants of inequalities (1.6) and (3.1) are equal.*

Proof: We denote by C_1 and C_2 the best constant of (1.6) and (3.1) respectively. Let $u \in H_0^1(\Omega)$. By Lemma 2.1, $w = |x|^{\frac{N-2}{2}} u \in W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$. We then have:

$$\frac{\int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx}{\int_\Omega |V| u^2 dx} = \frac{\int_\Omega |x|^{-(N-2)} |\nabla w|^2 dx}{\int_\Omega |x|^{-(N-2)} |V| w^2 dx} \geq C_2.$$

Taking the infimum over $u \in H_0^1(\Omega)$, we conclude that $C_1 \geq C_2$.

We next prove the reverse inequality. Given any $\varepsilon > 0$ there exists a $w_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$ such that

$$\frac{\int_\Omega |x|^{-(N-2)}|\nabla w_\varepsilon|^2 dx}{\int_\Omega |x|^{-(N-2)}|V|w_\varepsilon^2 dx} \le C_2 + \varepsilon.$$

Let $0 < a < \frac{N-2}{2}$. By Lemma 2.1 we have that $v_{a,\varepsilon} = |x|^{-a}w_\varepsilon \in H_0^1(\Omega)$. A straightforward calculation shows that:

$$\begin{aligned} C_1 &\le \frac{\int_\Omega |\nabla v_{a,\varepsilon}|^2 dx - \frac{(N-2)^2}{4}\int_\Omega \frac{v_{a,\varepsilon}^2}{|x|^2} dx}{\int_\Omega |V| v_{a,\varepsilon}^2 dx} \\ &= \frac{\int_\Omega |x|^{-2a}|\nabla w_\varepsilon|^2 dx - \left(a - \frac{N-2}{2}\right)^2 \int_\Omega |x|^{-2a-2} w_\varepsilon^2 dx}{\int_\Omega |x|^{-2a}|V|w_\varepsilon^2 dx}. \end{aligned}$$

We will take the limit as $a \to \frac{N-2}{2}$ (for ε fixed). To this end we first calculate:

$$\begin{aligned} &\left(a - \frac{N-2}{2}\right)^2 \int_\Omega |x|^{-2a-2} w_\varepsilon^2 dx \le \|w_\varepsilon\|^2_{L^\infty(\Omega)} \left(a - \frac{N-2}{2}\right)^2 \int_\Omega |x|^{-2a-2} dx \\ &\le C\|w_\varepsilon\|^2_{L^\infty(\Omega)} \left(a - \frac{N-2}{2}\right)^2 \frac{1}{N-2-2a} \longrightarrow 0, \quad \text{as} \quad a \to \frac{N-2}{2}, \end{aligned}$$

for some positive constant C. Passing to the limit $a \to \frac{N-2}{2}$ we conclude that $C_1 \le C_2 + \varepsilon$, and the result follows. •

By the same argument the Hardy-Sobolev inequality takes the following form:

Lemma 3.2 *Let $D \ge \sup_{x\in\Omega}|x|$. Then, there exists $c > 0$ such that for all $w \in W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$ there holds:*

$$\int_\Omega |x|^{-(N-2)}|\nabla w|^2 dx \ge c\left(\int_\Omega |x|^{-N}|w|^{\frac{2N}{N-2}} X^{1+\frac{N}{N-2}}(\frac{|x|}{D})dx\right)^{(N-2)/N}. \qquad (3.2)$$

We now consider inequality (1.5) with best constant b and $V \in \mathcal{A}$. We set

$$Q[w] = \frac{\int_\Omega |x|^{-(N-2)}|\nabla w|^2 dx}{\int_\Omega |x|^{-(N-2)} V w^2 dx},$$

and define

$$B = \inf_{\substack{w \in \mathcal{C}_0^\infty(\Omega) \\ \int_\Omega |x|^{-(N-2)} V w^2 dx > 0}} Q[w] = \inf_{\substack{w \in W_0^{1,2}(\Omega;\ |x|^{-(N-2)}) \\ \int_\Omega |x|^{-(N-2)} V w^2 dx > 0}} Q[w]. \qquad (3.3)$$

By practically the same argument as in Lemma 3.1 we have that:

Proposition 3.3 *There holds: $B = b$.*

The local best constant of inequality (1.5) near zero (cf (1.10)), can be written as:

$$\mathcal{C}^0 = \lim_{r\downarrow 0} C_r, \qquad C_r = \inf_{\substack{w \in C_0^\infty(B_r) \\ \int_{B_r} |x|^{-(N-2)} V w^2 dx > 0}} \frac{\int_{B_r} |x|^{-(N-2)}|\nabla w|^2 dx}{\int_{B_r} |x|^{-(N-2)} V w^2 dx}. \qquad (3.4)$$

If in (3.4) there is no $w \in C_0^\infty(B_r)$ such that $\int_\Omega |x|^{-(N-2)} V w^2 dx > 0$ for some $r > 0$, we set $C_r = \infty$. It is evident that $B \le \mathcal{C}^0$.

Our next result is the crucial step towards proving Theorem B. We have

Proposition 3.4 *Suppose that $V \in \mathcal{A}$. Let B and $\mathcal{C}^0$ be as defined in (3.3) and (3.4) respectively. If*

$$B < \mathcal{C}^0, \tag{3.5}$$

then, every bounded in $W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$ minimizing sequence of (3.3) has a strongly in $W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$ convergent subsequence. In particular B is achieved by some $w_0 \in W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$.

Proof of Proposition 3.4: Let $\{w_k\}$ be a minimizing sequence for (3.3). We may normalize it so that

$$\int_\Omega |x|^{-(N-2)} V w_k^2 dx = 1. \tag{3.6}$$

It then follows that as $k \to \infty$:

$$\int_\Omega |x|^{-(N-2)} |\nabla w_k|^2 dx \to B. \tag{3.7}$$

In particular $\int_\Omega |x|^{-(N-2)} |\nabla w_k|^2 dx$ is bounded and therefore there exists a subsequence, still denoted by $\{w_k\}$, and a $w_0 \in W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$ such that as $k \to \infty$

$$w_k \rightharpoonup w_0, \qquad \text{weakly in } \ W_0^{1,2}(\Omega;\ |x|^{-(N-2)}), \tag{3.8}$$

and

$$w_k \to w_0, \qquad \text{strongly in } \ L^2(\Omega \setminus B_\rho), \quad \forall\ \rho > 0, \tag{3.9}$$

where B_ρ denotes a ball of radius ρ centered at zero. We set $v_k = w_k - w_0$. It then follows from (3.1), (3.6) and (3.8) that as $k \to \infty$

$$1 = \int_\Omega |x|^{-(N-2)} V v_k^2 dx + \int_\Omega |x|^{-(N-2)} V w_0^2 dx + o(1). \tag{3.10}$$

We similarly calculate that

$$B = \int_\Omega |x|^{-(N-2)} |\nabla v_k|^2 dx + \int_\Omega |x|^{-(N-2)} |\nabla w_0|^2 dx + o(1).$$

This has as a consequence the following two inequalities. The first one is (taking into account (3.3)):

$$B \geq \int_\Omega |x|^{-(N-2)} |\nabla v_k|^2 dx + B \int_\Omega |x|^{-(N-2)} V w_0^2 dx + o(1); \tag{3.11}$$

and the second one is:

$$B \geq \int_\Omega |x|^{-(N-2)} |\nabla w_0|^2 dx \tag{3.12}$$

¿From (3.5) we have that for ρ sufficiently small there holds:

$$B < C_\rho = \inf_{\substack{w \in C_0^\infty(B_\rho) \\ \int_\Omega |x|^{-(N-2)} V w^2 dx > 0}} \frac{\int_{B_\rho} |x|^{-(N-2)} |\nabla w|^2 dx}{\int_{B_\rho} |x|^{-(N-2)} V w^2 dx}. \tag{3.13}$$

Let $\phi \in C_0^\infty(B_\rho)$ be a smooth cutoff function, such that $0 \le \phi \le 1$ and $\phi = 1$ in $B_{\rho/2}$. We write $v_k = \phi v_k + (1-\phi)v_k$. Taking into account (3.10), we calculate as $k \to \infty$

$$\begin{aligned}\int_\Omega |x|^{-(N-2)}|\nabla v_k|^2 dx &= \int_\Omega |x|^{-(N-2)}|\nabla(\phi v_k)|^2 dx + o(1) + \\ &\quad + \int_\Omega |x|^{-(N-2)}|\nabla((1-\phi)v_k)|^2 dx + \\ &\quad +2\int_\Omega |x|^{-(N-2)}\phi(1-\phi)|\nabla v_k|^2 dx \\ &\ge \int_\Omega |x|^{-(N-2)}|\nabla(\phi v_k)|^2 dx + o(1). \end{aligned} \tag{3.14}$$

¿From (3.13) and the fact that $\phi v_k \in C_0^\infty(B_\rho)$ we obtain:

$$\int_\Omega |x|^{-(N-2)}|\nabla(\phi v_k)|^2 dx \ge C_\rho \int_\Omega |x|^{-(N-2)} V(\phi v_k)^2 dx. \tag{3.15}$$

Since $V \in L^{\frac{N}{2}}_{loc}(\Omega \setminus \{0\})$ it is standard (see e.g., [T], Corollary 3.6) that:

$$\int_{\Omega\setminus B_{\rho/2}} |x|^{-(N-2)} V v_k^2 dx \to 0, \qquad \text{as } k \to \infty.$$

In view of this, (3.14) and (3.15) we write:

$$\int_\Omega |x|^{-(N-2)}|\nabla v_k|^2 dx \ge C_\rho \int_\Omega |x|^{-(N-2)} V v_k^2 dx + o(1). \tag{3.16}$$

Taking also into account (3.10) we obtain:

$$\int_\Omega |x|^{-(N-2)}|\nabla v_k|^2 dx \ge C_\rho \left(1 - \int_\Omega |x|^{-(N-2)} V w_0^2 dx\right) + o(1). \tag{3.17}$$

It then follows from (3.11) and (3.17) that

$$(B - C_\rho)\left(1 - \int_\Omega |x|^{-(N-2)} V w_0^2 dx\right) \ge 0,$$

whence, because of our assumption $B < C_\rho$:

$$\int_\Omega |x|^{-(N-2)} V w_0^2 dx \ge 1.$$

¿From this and (3.12) we finally arrive at:

$$0 < \frac{\int_\Omega |x|^{-(N-2)}|\nabla w_0|^2 dx}{\int_\Omega |x|^{-(N-2)} V w_0^2 dx} \le B,$$

from which it follows that B is attained by w_0. We note in particular that

$$\int_\Omega |x|^{-(N-2)} V w_0^2 dx = 1,$$

and it follows from (3.11) that w_k converges strongly in $W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$ to w_0. •

By slightly adjusting the arguments of Proposition 3.4 we can prove a more general result. Let $h \in \mathcal{A}$ be a nonnegative function. We set:

$$B_h = \inf_{\substack{w \in C_0^\infty(\Omega) \\ \int_\Omega |x|^{-(N-2)} V w^2 dx > 0}} \frac{\int_\Omega |x|^{-(N-2)}|\nabla w|^2 dx + \int_\Omega |x|^{-(N-2)} h w^2 dx}{\int_\Omega |x|^{-(N-2)} V w^2 dx}. \tag{3.18}$$

We then have:

Proposition 3.5 *Suppose that $h \geq 0$ and V are both in $\mathcal{A}$. Let B_h and $\mathcal{C}^0$ be as defined in (3.18) and (3.4) respectively. If*

$$B_h < \mathcal{C}^0,$$

then, B_h is achieved by some $w_0 \in W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$.

We will use Proposition 3.5 in Section 5.

We next look for an improvement of inequality (1.5). That is, for an inequality of the form:

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx + b \int_\Omega V u^2 dx + b_1 \int_\Omega W u^2 dx, \quad \forall\, u \in H_0^1(\Omega), \tag{3.19}$$

where V and W are both in $\mathcal{A}$.

Assuming that (3.19) holds true, the best constant b_1, is clearly given by:

$$b_1 = \inf_{\substack{u \in H_0^1(\Omega) \\ \int_\Omega W u^2 dx > 0}} \frac{\int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx - b \int_\Omega V u^2 dx}{\int_\Omega W u^2 dx}. \tag{3.20}$$

By the same argument as in Proposition 3.3, the constant b_1 is also equal to:

$$b_1 = \inf_{\substack{w \in W_0^{1,2}(\Omega;\ |x|^{-(N-2)}) \\ \int_\Omega |x|^{-(N-2)} W w^2 dx > 0}} \frac{\int_\Omega |x|^{-(N-2)} |\nabla w|^2 dx - b \int_\Omega |x|^{-(N-2)} V w^2 dx}{\int_\Omega |x|^{-(N-2)} W w^2 dx}. \tag{3.21}$$

Notice that by the properties of $b = B$ we always have that $b_1 \geq 0$.

Conversely, if one defines $b_1 \geq 0$ by (3.21) it is immediate that inequality (3.19) holds true with b_1 being the best constant. But of course, for (3.19) to be an improvement of the original inequality, we need b_1 to be strictly positive.

Our next result is a direct consequence of Proposition 3.4 and provides conditions under which the original inequality cannot be improved.

Proposition 3.6 *Suppose that $b < \mathcal{C}^0$. Let V and W be both in $\mathcal{A}$. If ϕ is the minimizer of the quotient (3.3) and*

$$\int_\Omega |x|^{-(N-2)} W \phi^2 dx > 0,$$

then $b_1 = 0$, that is, there is no further improvement of (1.5).

Proof: By our assumptions, $w = \phi$ is an admissible function in (3.21). Moreover, for $w = \phi$ the numerator of (3.21) becomes zero. In view of the fact that $b_1 \geq 0$ we conclude that $b_1 = 0$. •

It follows in particular that if $W \geq 0$, we cannot improve (1.5). Thus, Theorem B has been proved. As a consequence of Theorems A and B we have:

Corollary 3.7 *Let $D > \sup_{x \in \Omega} |x|$. Suppose V is not everywhere nonpositive, and such that $\int_\Omega |V|^{\frac{N}{2}} X^{1-N}(|x|/D) dx < \infty$. Then, $V \in \mathcal{A}$ but there is no further improvement of (1.5) with a nonnegative $W \in \mathcal{A}$.*

Proof: Applying Holder's inequality we get:

$$\int_\Omega |x|^{-(N-2)}|V|w^2dx \le \left(\int_\Omega |V|^{\frac{N}{2}}X^{1-N}dx\right)^{\frac{2}{N}}\left(\int_\Omega |x|^{-N}X^{1+\frac{N}{N-2}}|w|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}.$$

The first integral is bounded by our assumption, whereas the second integral is bounded from above by $C\int_\Omega |x|^{-(N-2)}|\nabla w|^2dx$ (cf Lemma 3.2). Thus we proved that $V\in\mathcal{A}$. Using once more Holder's inequality in B_r and the definition of C_r (cf (3.4)) we easily see that:

$$C_r \ge \frac{C}{\left(\int_{B_r}|V|^{\frac{N}{2}}X^{1-N}dx\right)^{\frac{2}{N}}} \to \infty, \qquad \text{as} \quad r\to 0,$$

whence $\mathcal{C}^0=+\infty$. Thus, all conditions of Proposition 3.6 are satisfied and the result follows. •

We next provide conditions under which the original inequality can be improved.

Proposition 3.8 *Suppose that $b<\mathcal{C}^0$. Let that V and W be both in $\mathcal{A}\cap L^p_{loc}(\Omega\setminus\{0\})$, for some $p>\frac{N}{2}$. If ϕ is the minimizer of the quotient (3.3) and*

$$\int_\Omega |x|^{-(N-2)}W\phi^2dx<0,$$

then there exists $b_1>0$ for which (3.19) holds.

Proof: Under our current assumptions on V it is standard to show that the minimizer ϕ of (3.3) is unique up to multiplication of constants. Indeed, notice that when ϕ is a minimizer, $|\phi|$ is also a minimizer. Hence, $|\phi|$ is a solution to the corresponding Euler-Lagrange equation. Using the change of variables (1.18), we see that $u_0(x)=|\phi(x)|\,|x|^{-\frac{N-2}{2}}\ge 0$ solves:

$$\Delta u+\tilde V(x)u=0, \qquad \text{in} \quad \Omega,$$

with $\tilde V(x)=\frac{\left(\frac{N-2}{2}\right)^2}{|x|^2}+b\,V(x)\in L^p_{loc}(\Omega\setminus\{0\})$, with $p>\frac{N}{2}$. It follows by the strong maximum principle (see e.g., [S], Theorem C.1.3, p. 493) that $u_0>0$ in $\Omega\setminus\{0\}$, unless $u_0=0$.

If ϕ and $\bar\phi$ are two minimizers, then $w=\phi-c\bar\phi$ is also a minimizer for any $c\in R$. Taking $c=\phi(x^*)/\bar\phi(x^*)$, for some $x^*\ne 0$ we see that $w(x^*)=0$, contradicting the fact that $|w|$ does not vanish in $\Omega\setminus\{0\}$. Hence $w=0$. This shows the simplicity of the minimizer ϕ.

We know that $b_1\ge 0$. Assuming that $b_1=0$ we will reach a contradiction.

Let $w_k\in W^{1,2}_0(\Omega;\ |x|^{-(N-2)})$ be a minimizing sequence for the quotient in (3.21). That is, for all $k=1,2,\dots$ $\int_\Omega |x|^{-(N-2)}Ww_k^2dx>0$, and:

$$\frac{\int_\Omega |x|^{-(N-2)}|\nabla w_k|^2dx-b\int_\Omega |x|^{-(N-2)}Vw_k^2dx}{\int_\Omega |x|^{-(N-2)}Ww_k^2dx}\to 0, \qquad \text{as} \quad k\to\infty. \tag{3.22}$$

We may normalize this sequence by $\int_\Omega |x|^{-(N-2)}|\nabla w_k|^2dx=1$. Since $W\in\mathcal{A}$, by Lemma 3.1 the denominator in (3.22) stays bounded away from infinity. Consequently we have that:

$$\int_\Omega |x|^{-(N-2)}Vw_k^2dx\to 1/b, \qquad \text{as} \quad k\to\infty. \tag{3.23}$$

Hence, $\{w_k\}$ is a bounded minimizing sequence for (3.3). It follows from Proposition 3.4 that (through a subsequence) w_k converges to a minimizer $w_0 \in W_0^{1,2}(\Omega;\ |x|^{-(N-2)})$ of $Q[w]$. By the simplicity of the minimizer we have that $w_0 = a\phi$ for some $\alpha \in I\!R$. Since $W \in \mathcal{A}$, in particular W satisfies (3.1). We then compute:

$$0 \leq \lim_{k\to+\infty} \int_\Omega |x|^{-(N-2)} W w_k^2 dx = \int_\Omega |x|^{-(N-2)} W w_0^2 dx = \alpha^2 \int_\Omega |x|^{-(N-2)} W \phi^2 dx < 0,$$

which is a contradiction. Hence $b_1 > 0$, and (1.5) can be further improved. •

4 Nonexistence of minimizers in $H_0^1(\Omega)$

In this Section we will give the proof of Theorem C, and we will discuss its consequences.

If we assume that the best constant b in (1.5) is achieved by some $u \in H_0^1(\Omega)$, then u would satisfy the corresponding Euler-Lagrange equation, that is, it would be an $H_0^1(\Omega)$ solution of the following problem:

$$\begin{aligned} \Delta u + \left(\frac{N-2}{2}\right)^2 \frac{u}{|x|^2} + bVu &= 0, \qquad \text{in } \Omega, \\ u > 0, \quad \text{in} \quad \Omega\setminus\{0\}, \qquad u &= 0, \qquad \text{in } \partial\Omega. \end{aligned} \tag{4.1}$$

However, by Theorem C, Problem (4.1) has no $H_0^1(\Omega)$ solution, if we assume some smoothness on V. This last condition seems to be of technical nature.

By the same token, neither the constant b_1 in (1.9) is achieved in $H_0^1(\Omega)$ since, by Theorem C, it would yield an $H_0^1(\Omega)$ solution of (4.1) with $b' = 1$ and $V' = bV + b_1 W$.

We next give the proof of Theorem C.

Proof of Theorem C: We will prove it by contradiction. Suppose that u is a $H_0^1(\Omega)$ positive solution of (4.1). By standard elliptic regularity we know that $u \in C^{2,\alpha}_{loc}(\Omega\setminus\{0\})$.

Let us take the surface average of u:

$$v(r) = \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r} u(x) dS = \frac{1}{N\omega_N} \int_{|\omega|=1} u(r\omega) d\omega > 0, \tag{4.2}$$

where ω_N denotes the volume of the unit ball in $I\!R^N$. Without loss of generality, we may assume that the unit ball B_1 is contained in Ω (if not, we just use a smaller ball). A standard calculation shows that:

$$v''(r) + \frac{N-1}{r} v'(r) = \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r} \Delta u(x) dS.$$

Hence, taking into account (4.1), we see that v satisfies the equation:

$$v''(r) + \frac{N-1}{r} v'(r) + \frac{\left(\frac{N-2}{2}\right)^2}{r^2} v(r) = f(r) - g(r), \qquad \text{in} \quad 0 < r \leq 1. \tag{4.3}$$

where:

$$f(r) = \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r} V_-(x) u(x) dS \geq 0, \tag{4.4}$$

$$g(r) = \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r} V_+(x) u(x) dS \geq 0. \tag{4.5}$$

We next change variables by:

$$w(r) = r^{\frac{N-2}{2}} v(r) > 0, \qquad r > 0. \tag{4.6}$$

Using equation (4.3), a straightforward calculation shows that w satisfies:

$$(rw')' = r^{\frac{N}{2}}(f(r) - g(r)) \le r^{\frac{N}{2}} f(r) \qquad \text{in} \quad 0 < r \le 1.$$

It then follows by Lemma 4.1, see below, that (under our current assumptions) there exists an r_0 small enough, and a C independent of r such that:

$$w(r) \le C r^{2-\frac{N}{q}}, \qquad 0 < r < r_0. \tag{4.7}$$

To reach a contradiction we will obtain a lower bound for $w(r)$ that is incompatible with (4.7). Working in this direction we set:

$$Q(r) = r\frac{w'(r)}{w(r)}.$$

A straightforward calculation shows that Q satisfies the ODE:

$$rQ'(r) + Q^2(r) = F(r) - G(r), \qquad \text{in} \quad 0 < r \le 1,$$

with:

$$F(r) = \frac{r^{\frac{N}{2}+1} f(r)}{w(r)} \ge 0, \qquad G(r) = \frac{r^{\frac{N}{2}+1} g(r)}{w(r)} \ge 0. \tag{4.8}$$

By Lemmas 4.2 and 4.3 (see below) we obtain that $\lim_{r\downarrow 0} Q(r) = 0$. Hence, given any $\varepsilon > 0$ there exists an $r_1 > 0$ such that:

$$Q(r) = r\frac{w'(r)}{w(r)} < \varepsilon, \qquad \text{for} \quad 0 < r < r_1.$$

Integrating this from r to r_1 we easily conclude that:

$$Cr^{\varepsilon} < w(r), \qquad \text{for} \quad 0 < r < r_1, \tag{4.9}$$

for some positive constant C depending on r_1 but independent of r. Notice however that $\varepsilon > 0$ is arbitrary and $2 - \frac{N}{q}$ is a positive quantity, hence (4.9) is contradictory to (4.7), since we can always choose an $\varepsilon < 2 - \frac{N}{q}$. •

It remains to prove the three auxiliary Lemmas we used in the proof of the Theorem. At first we have:

Lemma 4.1 *Let v, w, f be as defined in (4.2), (4.4), (4.6) respectively, with V as in Theorem C and $u \in H_0^1(\Omega)$. We also assume that $B_1 \subset \Omega$ and that w satisfies in $(0,1]$ the equation :*

$$(rw')' = r^{\frac{N}{2}}(f(r) - g(r)).$$

Then, for $r \in (0,1]$, the following representation formula holds:

$$w(r) = \int_0^r \frac{1}{t}\int_0^t s^{N/2}(f(s) - g(s))\,ds\,dt.$$

In addition, for r sufficiently small, say $r < r_0$, the following estimate holds:

$$w(r) \le Cr^{2-\frac{N}{q}},$$

for some positive constant C independent of r.

Proof: The w-equation can be easily integrated to yield:

$$w(r) = C_1 + \int_r^1 \frac{1}{t}\left(C_2 + \int_t^1 s^{N/2}(f(s)-g(s))ds\right)dt, \tag{4.10}$$

where C_1, C_2 are the constants of integration. Using the fact that V and u are elements of specific function spaces we will calculate the values of these constants.

Working in this direction we will first show that the following limit exists:

$$\lim_{t\to 0}\int_t^1 s^{N/2}(f(s)-g(s))ds = l_2 \in I\!R. \tag{4.11}$$

At first we note that $l_2 \neq -\infty$, since otherwise (4.10) would contradict the positivity of w. Hence, in view of (4.4), it is enough to show that:

$$J := \int_0^1 s^{N/2} f(s)ds = \int_0^1 r^{-\frac{N-2}{2}} \int_{\partial B_r} V_-(x)u(x)dSdr < \infty.$$

Since $u \in H_0^1(\Omega)$, by the Sobolev embedding, we also have that $u \in L^{\frac{2N}{N-2}}(\Omega)$. We then apply Holder's inequality as follows:

$$\int_{\partial B_r} V_-(x)u(x)dS \le \left(\int_{\partial B_r} V_-^q dS\right)^{1/q}\left(\int_{\partial B_r} u^{\frac{2N}{N-2}}dS\right)^{\frac{N-2}{2N}}\left(\int_{\partial B_r} 1dS\right)^{1/\theta}, \tag{4.12}$$

with

$$\frac{1}{q}+\frac{N-2}{2N}+\frac{1}{\theta}=1 \qquad \Longrightarrow \qquad \theta = \frac{2Nq}{Nq-2N+2q} > 1.$$

For $q > \frac{N}{2}$, such a θ is always well defined. Also, the last integral in (4.12) is equal to $N\omega_N r^{(N-1)}$. We next apply Holder's inequality in J to get:

$$J \le \|V_-\|_{L^q(B_1)}\|u\|_{L^{\frac{2N}{N-2}}(B_1)}\left(\int_0^1 r^{N-1-\frac{N-2}{2}\theta}dr\right)^{1/\theta} \le C\|V_-\|_{L^q(B_1)}\|u\|_{L^{\frac{2N}{N-2}}(B_1)},$$

since, for $q > \frac{N}{2}$ the last integral above is easily checked to be finite. Thus, (4.11) is proved. We note, for later use, that by the same argument, we have that:

$$\int_0^t s^{N/2} f(s)ds \le C\|V_-\|_{L^q(B_t)}\|u\|_{L^{\frac{2N}{N-2}}(B_t)} t^{\frac{N}{\theta}-\frac{N-2}{2}} \le Ct^{\frac{N}{\theta}-\frac{N-2}{2}}. \tag{4.13}$$

We next prove the following statement:

$$\textit{if there exist positive constants } C,\ r_0 \textit{ such that}$$
$$w(r) > C_0 \textit{ for } 0 < r \le r_0, \ \textit{ then } \ u \notin H_0^1(B_{r_0}). \tag{4.14}$$

We will prove it by contradiction. Since $u \in H_0^1(B_{r_0})$, we also have that $u \in L^{\frac{2N}{N-2}}(B_{r_0})$ Assuming that $w(t) > C_0$ for $t \in (0, r_0]$, it follows from the definitions of w and v (using Holder's inequalty) that:

$$C \le t^{-\frac{N}{2}}\int_{\partial B_t} udS \le (N\omega_N)^{\frac{N+2}{2N}}\left(\int_{\partial B_t} u^{\frac{2N}{N-2}}dS\right)^{\frac{N-2}{2N}} t^{\frac{N-2}{2N}},$$

Integrating this from 0 to $r \le r_0$ and using once more Holder's inequality we easily end up with $C \le \|u\|_{L^{\frac{2N}{N-2}}(B_r)}$, for some positive constant C independent of r. This is clearly a contradiction, hence (4.14) is proved.

We are now ready to compute the constants. In view of (4.11) and (4.14), it follows easily from (4.10) that we should take $C_2 = -l_2$, that is:

$$C_2 = -\int_0^1 s^{N/2}(f(s) - g(s))ds,$$

since otherwise $w(r)$ would go to infinity as r approaches zero. Hence, (4.10) can be written as:

$$w(r) = C_1 - \int_r^1 \frac{1}{t}\int_0^t s^{N/2}(f(s) - g(s))dsdt,$$

To compute t C_1, we next show that the integral above has a limit, say $l_1 \in I\!R$, as r goes to zero. Because of (4.14), $l_1 \neq -\infty$. Using (4.13) we have that:

$$\int_r^1 \frac{1}{t}\int_0^t s^{N/2} f(s)dsdt \le C\int_r^1 t^{\frac{N}{\theta}-\frac{N-2}{2}-1}dt \le C,$$

since, for $q > \frac{N}{2}$, the function $t^{\frac{N}{\theta}-\frac{N-2}{2}-1}$ is easily checked to be integrable at zero. Hence, $l_1 \in I\!R$, as claimed. In view of (4.14), we then choose $C_1 = l_1$, that is:

$$C_1 = \int_0^1 \frac{1}{t}\int_0^t s^{N/2}(f(s) - g(s))dsdt.$$

With this choice of C_1 the representation formula follows.

Finally, the estimate on $w(r)$ follows easily from the representation formula and (4.13). •

We next prove the ODE Lemma:

Lemma 4.2 *Let $Q(r)$ be a $C^1(0,1]$ solution of:*

$$rQ'(r) + Q^2(r) = F(r) - G(r), \qquad \text{in } \ 0 < r \le 1, \tag{4.15}$$

where F, G are nonnegative continuous function and:

$$\int_0^1 \frac{F(s)}{s} < \infty.$$

Then:

$$\lim_{r\downarrow 0} Q(r) = 0.$$

Proof: After dividing equation (4.15) by r, and integrating once, we obtain:

$$Q(r) = \int_r^1 \frac{Q^2(s)}{s}ds + Q(1) + \int_r^1 \frac{G(s)}{s}ds - \int_r^1 \frac{F(s)}{s}ds. \tag{4.16}$$

We claim that:

$$\int_0^1 \frac{Q^2(s)}{s}ds < \infty. \tag{4.17}$$

Indeed, if this is not true then:

$$H(r) := \int_r^1 \frac{Q^2(s)}{s}ds \to \infty, \qquad \text{as } \ r \to 0.$$

We may then rewrite (4.16) as:

$$(-rH'(r))^{1/2} = H(r) + Q(1) + \int_r^1 \frac{G(s)}{s} ds - \int_r^1 \frac{F(s)}{s} ds.$$

By our assumptions, the last term of the right hand side is bounded, whereas $G \geq 0$, and H grows unbounded as r goes to zero. Hence, for r small we have that:

$$-rH' \geq \frac{1}{2}H^2 \quad \Leftrightarrow \quad \left(\frac{1}{H(r)} - \frac{1}{2}\ln r\right)' \geq 0,$$

that contradicts the fact that H grows to infinity as r tends to zero. Thus, (4.17) is proved. It then follows from (4.16) that $\lim_{r\downarrow 0} Q(r)$ exists. In view of (4.17) this limit should be equal to zero. •

We finally have:

Lemma 4.3 *Let $F(r)$ be as defined in (4.8) with V, u and w as before. Then:*

$$I = \int_0^1 \frac{F(s)}{s} < \infty.$$

Proof: We assume that $B_{3/2}$ is contained in Ω, and consider the domains $D = \{1/2 < |x| < 3/2\}$ and $K = \{|x| = 1\} \subset D$. Note that V is Holder continuous in D and therefore $V \in L^p(D)$, for some (in fact, for any) $p > \frac{N}{2}$. Since u satisfies (4.1) in D we may use Harnack's inequality ([S], Th. C.1.3, p. 493) to obtain:

$$u(x) \leq Cu(y), \qquad \forall\, x, y \in K,$$

where the constant C depends only on $\|V\|_{L^p(D)}$.

Using the scaling properties of the potential $1/|x|^2$ we see that $u_\lambda(x) = u(\lambda x)$, $\lambda \in (0,1]$ satisfies in D the same equation as u, with V replaced by $V_\lambda(x) = \lambda^2 V(\lambda x)$. Hence, by the same argument, we have that $u(x) \leq Cu(y)$ for all x, y for which $|x| = |y| = \lambda$; the constant C now depends only on $\|V_\lambda\|_{L^p(D)}$. But,

$$\begin{aligned} \|V_\lambda\|_{L^p(D)} &= \lambda^{2-\frac{N}{p}} \left(\int_{\lambda D} |V(y)|^p dy\right)^{1/p} = C\left(|\lambda D|^{-1+\frac{2p}{N}} \int_{\lambda D} |V(y)|^p dy\right)^{1/p} \\ &\leq C\left(\||V|^p\|_{L^{\frac{N}{2p},\infty}(\Omega)}\right)^{1/p} = C\|V\|_{L^{\frac{N}{2},\infty}(\Omega)}. \end{aligned}$$

We therefore conclude that:

$$\frac{1}{C}\sup_{\partial B_r} u(r) \leq u(x) \leq C \inf_{\partial B_r} u(r), \qquad |x| = r$$

with C independent of $r \in (0,1]$. We then have that:

$$\frac{F(r)}{r} \leq \frac{C}{r^{N-2}} \int_{\partial B_r} V_-(x) dS \leq C\left(\int_{\partial B_r} V_-^q(x) dS\right)^{1/q} r^{\frac{(N-1)(q-1)}{q}+2-N},$$

where we also used Holder's inequality. Applying Holder's inequality once more we obtain:

$$I \leq C\left(\int_0^1 \int_{\partial B_r} V_-^q(x) dS dr\right)^{1/q} \left(\int_0^1 r^{\frac{(2-N)q}{q-1}+N-1} dr\right)^{\frac{q-1}{q}} \leq C\|V_-\|_{L^q(B_1)},$$

by noticing that, since $q > \frac{N}{2}$, the second integral above is finite. •

5 The special case $V = 1$

In this Section we consider the special case $V = 1$, that is the inequality:

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx + \lambda_\Omega \int_\Omega u^2 dx, \tag{5.1}$$

with λ_Ω being the best constant. It is a consequence of Theorem C that λ_Ω is not achieved in $H_0^1(\Omega)$. On the other hand by Corollary 1.2 we cannot further improve (5.1) by adding a nonnegative potential in the right hand side.

As an application of the previous results we will obtain some information about λ_Ω. More specifically, if Ω^* is the ball centered at the origin and having the same volume as Ω, we will show the following:

Proposition 5.1 *There holds $\lambda_\Omega > \lambda_{\Omega^*}$, unless Ω is a ball centered at the origin.*

As noted in [BV] the constant λ_{Ω^*} is explicitly known, namely:

$$\lambda_{\Omega^*} = (z_0/R)^2\,,$$

where R is the radius of the ball Ω^*, and $z_0 \approx 2.4048$ is the first zero of the Bessel function $J_0(z)$.

Let us first present some Lemmas. At first we have:

Lemma 5.2 *Let $\Omega \subset I\!R^N$ be a bounded domain, containing the origin, and $f : (0,\infty) \to I\!R^+$ be a Lipschitz continuous and strictly decreasing function. We denote by $g : (0,\infty) \to I\!R^+$ the radially decreasing rearrangement of $f(|x|)$ in Ω^* with respect to the origin. If B_ρ is the largest ball centered at the origin contained in Ω, then:*

$$g(|x|) = f(|x|), \quad \forall x \in \bar{B}_\rho, \quad \text{and} \quad g(|x|) < f(|x|), \qquad \forall x \in \Omega^* - \bar{B}_\rho.$$

If in addition $g(|x|) = f(|x|)$ in Ω^*, then necessarily $\Omega^* = B_\rho = \Omega$.

Proof: By standard results, g is strictly decreasing in $(0,\infty)$ and Lipschitz continuous in every compact subinterval of $(0,\infty)$; see e.g. [K].

It follows from the definition of g that:

$$\text{meas}\{x \in \Omega : \quad f(|x|) > t\} = \text{meas}\{x \in \Omega^* : \quad g(|x|) > t\}, \qquad \forall t \geq 0.$$

If $t \geq f(\rho)$, or equivalently $f^{-1}(t) \leq \rho$, the set $\{x \in \Omega : \ f(|x|) > t\}$ is contained in B_ρ, hence: $\text{meas}\{x \in \Omega : \ f(|x|) > t\} = \omega_N (f^{-1}(t))^N$, where ω_N is the volume of the unit ball in $I\!R^N$. Similarly, we have that $\text{meas}\{x \in \Omega^* : \ g(|x|) > t\} = \omega_N (f^{-1}(t))^N$. It then follows that $g(\xi) = f(\xi), \text{for } |\xi| \leq \rho$, as claimed.

Suppose now that $0 < t < f(\rho)$, or equivalently, $f^{-1}(t) > \rho$. Then, the set $\{x \in \Omega : \ f(|x|) > t\}$ is *strictly* contained in $B_{f^{-1}(t)}(0)$ and therefore $\text{meas}\{x \in \Omega : \ f(|x|) > t\} < \omega_N (f^{-1}(t))^N$. We then obtain that: $\omega_N (g^{-1}(t))^n < \omega_N (f^{-1}(t))^N, \ \text{ for } t < f(\rho)$. Whence: $g(y) < f(y) \text{for } y > \rho$, and the second claim follows.

The last statement follows easily, since, if $g(|x|) = f(|x|)$ in Ω^* then $\Omega^* \subseteq B_\rho \subseteq \Omega$. Taking into account that Ω^* and B_ρ are concentric balls as well as the fact that $|\Omega| = |\Omega^*|$ we easily obtain that $\Omega^* = B_\rho = \Omega$. •

¿From here on we will denote by $g(x)$ the decreasing rearrangement of $\frac{1}{|x|^2}$ in Ω, with respect to zero. We also define:

$$\lambda^*_\Omega = \inf_{u \in H^1_0(\Omega^*)} \frac{\int_{\Omega^*} |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_{\Omega^*} g(x) u^2 dx}{\int_{\Omega^*} u^2 dx}. \tag{5.2}$$

Let $u \in H^1_0(\Omega)$ and u^* be its decreasing rearrangement. It is a standard fact that the decreasing rearrangement preserves the L^2 norm, decreases the H^1_0 norm and that $\int_\Omega \frac{u^2}{|x|^2} \leq \int_{\Omega^*} g(x) u^{*2} dx$. Whence

$$\lambda_\Omega \geq \lambda^*_\Omega. \tag{5.3}$$

As in the previous Sections, we would like to have an alternative characterization of the constant λ^*_Ω. To this end we define:

$$\Lambda^*_\Omega = \inf_{w \in C^\infty_0(\Omega^*)} \frac{\int_{\Omega^*} |x|^{-(N-2)} |\nabla w|^2 dx + \frac{(N-2)^2}{4} \int_{\Omega^*} |x|^{-(N-2)} (|x|^{-2} - g(x)) w^2 dx}{\int_{\Omega^*} |x|^{-(N-2)} w^2 dx}. \tag{5.4}$$

The reason for introducing Λ^*_Ω becomes clear in the following:

Lemma 5.3 $\lambda^*_\Omega = \Lambda^*_\Omega$. *Moreover* Λ^*_Ω *is achieved by some* w *in* $W^{1,2}_0(\Omega^*, |x|^{-(N-2)})$.

Proof: To prove that $\lambda^*_\Omega = \Lambda^*_\Omega$ we argue as in Proposition 3.3. The last statement follows from Proposition 3.5 with $h(x) = |x|^{-2} - g(x)$. Notice that h thus defined, is equal to zero in a neighborhood of zero, by Lemma 5.2, and therefore $h \in L^q(\Omega^*)$ for any $q > \frac{N}{2}$. •

We are now ready to give the proof of Proposition 5.1

Proof of Proposition 5.1: By Lemma 5.3 and (5.3) we have that $\lambda_\Omega \geq \lambda^*_\Omega = \Lambda^*_\Omega$. We therefore need to compare Λ^*_Ω and $\lambda_{\Omega^*} = \Lambda_{\Omega^*}$. The main observation here is that Λ^*_Ω is achieved in $W^{1,2}_0(\Omega^*, |x|^{-(N-2)})$ by a positive function, say, $\bar{w}$. Recalling (5.2) and the definition of Λ_{Ω^*},

$$\Lambda_{\Omega^*} = \inf_{w \in C^\infty_0(\Omega^*)} \frac{\int_{\Omega^*} |x|^{-(N-2)} |\nabla w|^2 dx}{\int_{\Omega^*} |x|^{-(N-2)} w^2 dx},$$

we easily obtain that:

$$\Lambda^*_\Omega \geq \Lambda_{\Omega^*} + \frac{(N-2)^2}{4} \frac{\int_{\Omega^*} |x|^{-(N-2)} (|x|^{-2} - g(x)) \bar{w}^2 dx}{\int_{\Omega^*} |x|^{-(N-2)} \bar{w}^2 dx}.$$

By Lemma 5.2 the second term of the right hand side is strictly positive, unless $|x|^{-2} = g(x)$ in $\Omega^* = \Omega^*$, which happens only if Ω is a ball centered at the origin. Therefore, $\Lambda^*_\Omega > \Lambda_{\Omega^*}$, unless $\Omega = \Omega^*$, and the result follows. •

We finally point out a consequence of Proposition 5.1 reminiscent of the Faber-Krahn inequality. Since

$$\lambda_\Omega = \Lambda_\Omega = \inf_{w \in C^\infty_0(\Omega)} \frac{\int_\Omega |x|^{-(N-2)} |\nabla w|^2 dx}{\int_\Omega |x|^{-(N-2)} w^2 dx},$$

we see that λ_Ω is the first eigenvalue of the problem:

$$\begin{aligned} \mathrm{div}(|x|^{-(N-2)} \nabla w) + \lambda_\Omega |x|^{-(N-2)} w &= 0 \qquad \text{in } \Omega, \\ w &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{5.5}$$

with $w \in W^{1,2}_0(\Omega;\ |x|^{-(N-2)})$. According to Proposition 5.1 the first eigenvalue of (5.5) takes on its maximum value when Ω is a ball.

6 Infinite improvement

In this Section we will give the proof of Theorem D. Before that we will introduce some auxiliary functions, which are basically the iterated log functions. Let $X_1(t) = (1-\log t)^{-1}$ for $t \in (0,1]$. We define recursively:

$$X_k(t) = X_1(X_{k-1}(t)), \qquad k = 2,3,\ldots.$$

It is easy to see that the X_k are well defined and that for $k = 1,2,\ldots$

$$X_k(0) = 0, \qquad X_k(1) = 1, \qquad 0 < X_k(t) < 1, \qquad t \in (0,1).$$

For the reader's convenience we restate Theorem D.

Theorem 6.1 *Let $D \geq \sup_{x\in\Omega}|x|$. Then, for any $u \in H_0^1(\Omega)$ there holds:*

$$\begin{aligned}\int_\Omega |\nabla u(x)|^2 dx \; - \; & \frac{(N-2)^2}{4}\int_\Omega \frac{u^2(x)}{|x|^2}dx \geq \\ \geq \; & \frac{1}{4}\sum_{i=1}^{\infty}\int_\Omega \frac{1}{|x|^2}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})\ldots X_i^2(\frac{|x|}{D})u^2(x)dx. \qquad (6.1)\end{aligned}$$

Moreover, for each $k = 1,2,\ldots$ the constant $1/4$ is the best constant for the corresponding k- Improved Hardy inequality, that is

$$\frac{1}{4} = \inf_{u\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2dx - \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2}dx - \frac{1}{4}\sum_{i=1}^{k-1}\int_\Omega \frac{1}{|x|^2}X_1^2X_2^2\ldots X_i^2u^2dx}{\int_\Omega \frac{1}{|x|^2}X_1^2X_2^2\ldots X_k^2u^2dx}$$

Proof: We may assume that $D = 1$, since all subsequent calculations are invariant with respect to D. We also consider first the case $u \in C_0^\infty(\Omega\setminus\{0\})$. We will use a change of variables, namely, $u(x) = \phi(|x|)v(x)$. A simple calculation shows that

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega \phi^2|\nabla v|^2dx + \int_\Omega \phi'^2v^2dx + \int_\Omega \phi\phi'\frac{x}{|x|}\cdot\nabla v^2dx.$$

After integrating by parts the last term, we arrive at:

$$\begin{aligned}\int_\Omega |\nabla u|^2dx &= -\int_\Omega \phi\Delta\phi v^2dx + \int_\Omega \phi^2|\nabla v|^2dx \\ &= -\int_\Omega \frac{\Delta\phi}{\phi}u^2dx + \int_\Omega \phi^2|\nabla v|^2dx \qquad (6.2)\\ &\geq -\int_\Omega \frac{\Delta\phi}{\phi}u^2dx.\end{aligned}$$

¿From now on we set $H = \frac{N-2}{2}$. We will now make a specific choice of ϕ, so that

$$-\frac{\Delta\phi}{\phi} = \frac{1}{|x|^2}\left(H^2 + \frac{1}{4}X_1^2 + \frac{1}{4}X_1^2X_2^2 + \ldots + \frac{1}{4}X_1^2\ldots X_k^2\right). \qquad (6.3)$$

We take for $k = 1,2\ldots$:

$$\phi_k(r) = r^{-H}X_1^{-1/2}(r)X_2^{-1/2}(r)\ldots X_k^{-1/2}(r), \qquad r = |x|. \qquad (6.4)$$

We also set $\phi_0(r) = r^{-H}$, and this corresponds to the change of variables used in the previous Sections. When differentiating ϕ_k, the following (easily checked) relation is helpful:

$$X_j' = \frac{1}{r} X_1 X_2 \dots X_{j-1} X_j^2, \qquad j = 1, 2, \dots. \tag{6.5}$$

Differentiating once we obtain

$$\phi_k' = -\frac{\phi_k}{r} \left(H + \frac{1}{2} \sum_{i=1}^{k} X_1 X_2 \dots X_i \right).$$

Differentiating for a second time we have that

$$\begin{aligned}
\phi_k'' &= \frac{\phi_k}{r^2} \left(H + \frac{1}{2} \sum_{i=1}^{k} X_1 X_2 \dots X_i \right)^2 + \frac{\phi_k}{r^2} \left(H + \frac{1}{2} \sum_{i=1}^{k} X_1 X_2 \dots X_i \right) \\
&\quad - \frac{\phi_k}{2r} (\sum_{i=1}^{k} X_1 X_2 \dots X_i)' \\
&= \frac{\phi_k}{r^2} \left(H^2 + H \sum_{i=1}^{k} X_1 X_2 \dots X_i + \frac{1}{4} (\sum_{i=1}^{k} X_1 X_2 \dots X_i)^2 \right) \\
&\quad + \frac{\phi_k}{r^2} \left(H + \frac{1}{2} \sum_{i=1}^{k} X_1 X_2 \dots X_i \right) \\
&\quad - \frac{\phi_k}{2r^2} (\sum_{i=1}^{k} \sum_{j=1}^{i} X_1^2 X_2^2 \dots X_j^2 X_{j+1} \dots X_i) \\
&= \frac{\phi_k}{r^2}(H^2 + H) + \frac{\phi_k}{r^2}(H + \frac{1}{2}) \sum_{i=1}^{k} X_1 X_2 \dots X_i - \frac{\phi_k}{4r^2} \sum_{i=1}^{k} X_1^2 X_2^2 \dots X_i^2,
\end{aligned}$$

We then compute

$$\frac{\phi_k''}{\phi_k} + \frac{N-1}{r} \frac{\phi_k'}{\phi_k} = -\frac{H^2}{r^2} - \frac{1}{4r^2} \sum_{i=1}^{k} X_1^2 X_2^2 \dots X_i^2,$$

and (6.3) is proved.

In view of (6.2) we see that (6.1) has been proved for $u \in C_0^\infty(\Omega \setminus \{0\})$ if in the right hand side we have a finite series. Taking the limit as $k \to \infty$, and then using a standard density argument we see that (6.1) is valid for any $u \in H_0^1(\Omega)$.

We next prove the second part of the theorem.

We set for $k = 1, 2, \dots$:

$$I_k[u] = \int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{u^2}{|x|^2} \left(H^2 + \frac{1}{4} X_1^2 + \frac{1}{4} X_1^2 X_2^2 + \dots + \frac{1}{4} X_1^2 \dots X_k^2 \right) dx.$$

We also identify $I_0[u]$ with $I[u]$ (cf (2.2)). Clearly, there holds:

$$I_{k-1}[u] = I_k[u] + \frac{1}{4} \int_\Omega \frac{u^2}{|x|^2} X_1^2 X_2^2 \dots X_k^2 dx. \tag{6.6}$$

Using identity (6.2) and (6.6) we see that

$$I_k[u] = \int_\Omega \phi_k^2 |\nabla v|^2 dx, \tag{6.7}$$

with $u = \phi_k v$, and ϕ_k as before (cf (6.4)). Taking into account (6.6) and (6.7) we form the quotient that appears in the second part of the Theorem,

$$\frac{I_{k-1}[u]}{\int_\Omega \frac{u^2}{|x|^2} X_1^2 X_2^2 \dots X_k^2} = \frac{\int_\Omega \phi_k^2 |\nabla v|^2 dx}{\int_\Omega \phi_k^2 v^2 |x|^{-2} X_1^2 X_2^2 \dots X_k^2} + \frac{1}{4}, \tag{6.8}$$

We will now make a particular choice of v. Namely,

$$U_{\varepsilon,a}(r) = v_{\varepsilon,a}(r)\psi(r) = r^\varepsilon X_1^{a_1} X_2^{a_2} \dots X_k^{a_k} \psi(r), \qquad r = |x|. \tag{6.9}$$

The parameters ϵ, a_i will be positive and small and eventually will be sent to zero. The function $\psi(r)$ is a smooth cut-off function such that $\psi(r) = 1$ in B_δ and $\psi(r) = 0$ outside $B_{2\delta}$ for some δ small. It is easy to check that

$$u_{\varepsilon,a}^{(k)}(x) := \phi_k(r) U_{\varepsilon,a}(r) \in H_0^1(\Omega), \tag{6.10}$$

and therefore $U_{\varepsilon,a}$ is a legitimate test function for the quotient in the right hand side of (6.8).

We will show that as the small parameters tend to zero (in a specific order) the fraction in the right hand side of (6.8) tends to zero, that is

$$\frac{\int_\Omega \phi_k^2 |\nabla U_{\varepsilon,a}|^2 dx}{\int_\Omega \phi_k^2 U_{\varepsilon,a}^2 |x|^{-2} X_1^2 X_2^2 \dots X_k^2} \to 0. \tag{6.11}$$

An immediate consequence of this is that

$$\inf_{u \in H_0^1(\Omega)} \frac{I_{k-1}[u]}{\int_\Omega \frac{u^2}{|x|^2} X_1^2 X_2^2 \dots X_k^2} \le \frac{1}{4},$$

which shows the optimality of $\frac{1}{4}$.

Consider first the denominator in (6.11). It is easy to check that as the small parameters ϵ, a_i approach zero (for δ fixed) we have

$$\int_\Omega \phi_k^2 U_{\varepsilon,a}^2 |x|^{-2} X_1^2 X_2^2 \dots X_k^2 = \int_{B_\delta} r^{-N+2\varepsilon} X_1^{1+2a_1} \dots X_k^{1+2a_k} dr + O(1); \tag{6.12}$$

that is, the integral over $B_{2\delta} \setminus B_\delta$ (not written above) stays bounded. Concerning the numerator we write, by a similar argument

$$\begin{aligned} \int_\Omega \phi_k^2 |\nabla U_{\varepsilon,a}|^2 dx &= \int_{B_\delta} \phi_k^2 v_{\varepsilon,a}^{'2}(r) dx + \int_{B_{2\delta} \setminus B_\delta} \phi_k^2 (2 v_{\varepsilon,a}' v_{\varepsilon,a} \psi' \psi + v_{\varepsilon,a} \psi'^2 + v_{\varepsilon,a}'^2 \psi^2) dx \\ &= \int_{B_\delta} \phi_k^2 v_{\varepsilon,a}^{'2}(r) dx + O(1), \end{aligned} \tag{6.13}$$

as the small parameters ϵ, a_i tend to zero.

In view of (6.5) we easily compute for $r \in B_\delta$:

$$v_{\varepsilon,a}'(r) = v_{\varepsilon,a} r^{-1} (\varepsilon + \sum_{j=1}^k a_j X_1 \dots X_j).$$

Using this and the specific value of ϕ_k we compute (we introduce spherical coordinates)

$$\begin{aligned}
\frac{1}{N\omega_N}\int_{B_\delta}\phi_k^2 v'^2_{\varepsilon,a}(r)dx = & \\
& \varepsilon^2\int_0^\delta r^{-1+2\varepsilon}X_1^{-1+2a_1}X_2^{-1+2a_2}\dots X_k^{-1+2a_k}dr \\
& +\sum_{j=1}^k a_j^2\int_0^\delta r^{-1+2\varepsilon}X_1^{1+2a_1}\dots X_j^{1+2a_j}X_{j+1}^{-1+2a_{j+1}}\dots X_k^{-1+2a_k}dr \\
& +2\varepsilon\sum_{j=1}^k a_j\int_0^\delta r^{-1+2\varepsilon}X_1^{2a_1}\dots X_j^{2a_j}X_{j+1}^{-1+2a_{j+1}}\dots X_k^{-1+2a_k}dr \\
& +2\sum_{j=1}^{k-1}\sum_{i=j+1}^{k}a_ia_j\int_0^\delta r^{-1+2\varepsilon}X_1^{1+2a_1}\dots X_j^{1+2a_j}X_{j+1}^{2a_{j+1}}\dots X_i^{2a_i}\cdot \\
& \qquad X_{i+1}^{-1+2a_{i+1}}\dots X_k^{-1+2a_k}dr.
\end{aligned} \tag{6.14}$$

We intend to take the limit $\varepsilon\to 0$ (keeping the a_i's fixed) in (6.14). It is not clear however what will happen to the first and third term in the right hand side. To this end we derive two identities. Concerning the first term, we integrate by parts and use (6.5) to get

$$\begin{aligned}
& \varepsilon\int_0^\delta r^{-1+2\varepsilon}X_1^{-1+2a_1}X_2^{-1+2a_2}\dots X_k^{-1+2a_k}dr = \\
& = \frac{1}{2}\int_0^\delta (r^{2\varepsilon})'X_1^{-1+2a_1}X_2^{-1+2a_2}\dots X_k^{-1+2a_k}dr = \\
& = O(1)-\sum_{i=1}^k(-\frac{1}{2}+a_i)\int_0^\delta r^{-1+2\varepsilon}X_1^{2a_1}\dots X_i^{2a_i}X_{i+1}^{-1+2a_{i+1}}\dots X_k^{-1+2a_k}dr
\end{aligned} \tag{6.15}$$

A similar integration by parts yields the second identity

$$\begin{aligned}
& \varepsilon\int_0^\delta r^{-1+2\varepsilon}X_1^{2a_1}\dots X_i^{2a_i}X_{i+1}^{-1+2a_{i+1}}\dots X_k^{-1+2a_k}dr = O(1)- \\
& -\sum_{j=1}^i a_j\int_0^\delta r^{-1+2\varepsilon}X_1^{1+2a_1}\dots X_j^{1+2a_j}X_{j+1}^{2a_{j+1}}\dots X_i^{2a_i}X_{i+1}^{-1+2a_{i+1}}\dots X_k^{-1+2a_k}dr \\
& -\sum_{j=i+1}^k(-\frac{1}{2}+a_j)\int_0^\delta r^{-1+2\varepsilon}X_1^{1+2a_1}\dots X_i^{1+2a_i}X_{i+1}^{2a_{i+1}}\dots X_j^{2a_j}\cdot \\
& \qquad X_{j+1}^{-1+2a_{j+1}}\dots X_k^{-1+2a_k}dr
\end{aligned} \tag{6.16}$$

It is convenient at this point to introduce the following notation

$$\begin{aligned}
A_i &= \int_0^\delta r^{-1+2\varepsilon}X_1^{1+2a_1}\dots X_i^{1+2a_i}X_{i+1}^{-1+2a_{i+1}}\dots X_k^{-1+2a_k}dr, \\
\Gamma_{ji} &= \int_0^\delta r^{-1+2\varepsilon}X_1^{1+2a_1}\dots X_j^{1+2a_j}X_{j+1}^{2a_{j+1}}\dots X_i^{2a_i}X_{i+1}^{-1+2a_{i+1}}\dots X_k^{-1+2a_k}dr,
\end{aligned}$$

with $\Gamma_{ii}=A_i$.

We now return to (6.14). We use (6.15) and then (6.16) to replace the first term of the right hand side. We also use (6.16) to replace the third term. After grouping similar terms, we rewrite (6.14) as

$$\frac{1}{N\omega_N}\int_{B_\delta}\phi_k^2 v_{\varepsilon,a}'^2(r)dx = O(1) - \frac{1}{2}\sum_{j=1}^{k-1}\sum_{i=j+1}^{k}(-\frac{1}{2}+a_i)\Gamma_{ji} - \frac{1}{2}\sum_{j=1}^{k} a_j A_j. \tag{6.17}$$

Taking into account the definition of A_j and Γ_{ji} we see that we can now take the limit $\varepsilon \to 0$ in (6.17) by simply setting $\varepsilon = 0$ in the A_j's and Γ_{ji}'s.

Our next step will be to take the limit $a_1 \to 0$ (keeping the $a_2, \dots a_k$ fixed). Again, it is not clear that all terms in the right hand side of (6.17) have a limit. More precisely in the terms Γ_{1i}, $i = 2, \dots k$ as well as $a_1 A_1$ we cannot take the limit in a straightforward way (e.g setting $a_1 = 0$). By distinguishing these terms from the rest we rewrite (6.17) as

$$\begin{aligned}\frac{1}{N\omega_N}\int_{B_\delta}\phi_k^2 v_{0,a}'^2(r)dx &= O(1) - \frac{1}{2}\sum_{j=2}^{k-1}\sum_{i=j+1}^{k}(-\frac{1}{2}+a_i)\Gamma_{ji} - \frac{1}{2}\sum_{j=2}^{k} a_j A_j \\ &- \frac{1}{2}\left(a_1 A_1 + \sum_{i=2}^{k}(-\frac{1}{2}+a_i)\Gamma_{1i}\right). \end{aligned}\tag{6.18}$$

To estimate the last parenthesis above we will derive a new identity, relating A_1 and Γ_{1i} (with $\varepsilon = 0$). A simple integration by parts yields

$$\begin{aligned} a_1 A_1 &= a_1\int_0^\delta r^{-1}X_1^{1+2a_1}X_2^{-1+2a_2}\dots X_k^{-1+2a_k}dr \qquad (6.19)\\ &= \frac{1}{2}\int_0^\delta (X_1^{2a_1})' X_2^{-1+2a_2}\dots X_k^{-1+2a_k}dr \\ &= O(1) - \sum_{i=2}^{k}(-\frac{1}{2}+a_i)\int_0^\delta r^{-1}X_1^{1+2a_1}X_2^{2a_2}\dots X_i^{2a_i}X_{i+1}^{-1+2a_{i+1}}\dots X_k^{-1+2a_k}dr \\ &= O(1) - \sum_{i=2}^{k}(-\frac{1}{2}+a_i)\Gamma_{1i}. \end{aligned}$$

Thus, we have that

$$a_1 A_1 + \sum_{i=2}^{k}(-\frac{1}{2}+a_i)\Gamma_{1i} = O(1), \tag{6.20}$$

and we can now set $a_1 = 0$ in (6.18). We can continue this process in the same way. For instance to take the limit as $a_2 \to 0$ we will use the identity

$$a_2 A_2 + \sum_{i=3}^{k}(-\frac{1}{2}+a_i)\Gamma_{2i} = O(1),$$

relating A_2 and Γ_{2i} (with $\varepsilon = a_1 = 0$), that is derived in the same way as (6.20). We can then simply set $a_2 = 0$ in the remaining terms of (6.17), and so on.

After taking the limit $a_{k-1} \to 0$ we end up with

$$\frac{1}{N\omega_N}\int_{B_\delta}\phi_k^2 v_{0,a_k}'^2(r)dx = -\frac{1}{2}a_k A_k + O(1),$$

where in the A_k we have set $\varepsilon = a_1 = \dots a_{k-1} = 0$. That is,

$$\int_{B_\delta} \phi_k^2 v_{0,a_k}'^2(r)dx = -N\omega_N \frac{1}{2} a_k \int_0^\delta r^{-1} X_1 X_2 \dots X_{k-1} X_k^{1+2a_k} dr + O(1). \tag{6.21}$$

We are now in position to give the proof of (6.11). We form the quotient and take the limit as ε, $a_1, \dots a_{k-1}$ tend to zero in this order. In view of (6.12), (6.13) and (6.21) we arrive at

$$\frac{\int_\Omega \phi_k^2 |\nabla U_{0,a_k}|^2 dx}{\int_\Omega \phi_k^2 U_{0,a_k}^2 |x|^{-2} X_1^2 X_2^2 \dots X_k^2 dx} = \frac{-\frac{1}{2} a_k \int_0^\delta r^{-1} X_1 X_2 \dots X_{k-1} X_k^{1+2a_k} dr + O(1)}{\int_0^\delta r^{-1} X_1 X_2 \dots X_{k-1} X_k^{1+2a_k} dr + O(1)}.$$

Since

$$\begin{aligned} \int_0^\delta r^{-1} X_1 X_2 \dots X_{k-1} X_k^{1+2a_k} dr &= \frac{1}{2a_k} \int_0^\delta (X_k^{2a_k})' dr \\ &= \frac{1}{2a_k} X^{2a_k}(\delta) \to +\infty, \quad \text{as } \ a_k \to 0, \end{aligned} \tag{6.22}$$

we conclude that

$$\frac{\int_\Omega \phi_k^2 |\nabla U_{0,ka}|^2 dx}{\int_\Omega \phi_k^2 U_{0,a_k}^2 |x|^{-2} X_1^2 X_2^2 \dots X_k^2 dx} = \frac{O(1)}{\frac{1}{2a_k} X^{2a_k}(\delta)} \to 0, \quad \text{as } \ a_k \to 0,$$

as required. •

If we cut the series at the k step we obtain the k-Improved Hardy inequality, that is, $I_k[u] \geq 0$. To obtain from this the $(k+1)$-improved Hardy inequality we add the potential

$$V_k = |x|^{-2} X_1^2 \dots X_{k+1}^2.$$

We will show that this potential is "marginally" contained in the class $\mathcal{A}_k$, in the sense that a potential more singular than this (at zero) is outside $\mathcal{A}_k$. More precisely, let:

$$V_k^{(\gamma)}(x) = \frac{1}{|x|^2} X_1^2 \dots X_k^2 X_{k+1}^\gamma.$$

We then have:

Lemma 6.2 *Suppose that $\gamma < 2$. Then, there exists no $b_k > 0$ such that:*

$$I_k[u] \geq b_k \int_\Omega V_k^{(\gamma)} u^2 dx, \qquad \forall\ u \in H_0^1(\Omega).$$

Proof: Assuming that $b_k > 0$ we will reach a contradiction. Taking into account (6.6) we have that for all $u \in H_0^1(\Omega)$:

$$0 < b_k \leq \frac{I_k[u]}{\int_\Omega V_k^{(\gamma)} u^2 dx} = \frac{I_{k+1}[u] + \frac{1}{4} \int_\Omega \frac{u^2}{|x|^2} X_1^2 \dots X_{k+1}^2 dx}{\int_\Omega \frac{u^2}{|x|^2} X_1^2 \dots X_k^2 X_{k+1}^\gamma dx}. \tag{6.23}$$

To obtain a contradiction we will now use the test function $u = u_{\varepsilon,a}^{(k+1)}(x)$ introduced by (6.9), (6.10). Recall that in the proof of Theorem D we have shown that as $(\varepsilon, a_1, \dots, a_{k+1}) \to (0, \dots, 0)$ there holds: (cf (6.21) and (6.22)):

$$I_{k+1}[u_{\varepsilon,a}^{(k+1)}] = O(1).$$

The integrals appearing in (6.23) can be easily estimated. Thus, for the integral in the denominator after taking the limit $\varepsilon \to 0$, $a_1 \to 0$, $\dots$ $a_{k-1} \to 0$, keeping a_k and a_{k+1} fixed, we get (we omit the superscript $(k+1)$):

$$\int_\Omega \frac{u_{\varepsilon,a}^2}{|x|^2} X_1^2 \dots X_k^2 X_{k+1}^\gamma dx = N\omega_N \int_0^\delta r^{-1} X_1 X_2 \dots X_k^{1+2a_k} X_{k+1}^{\gamma-1+2a_{k+1}} dr + O(1).$$

A similar calculation for the numerator yields that, after taking the limits of $\varepsilon, a_1, \dots a_k$, going to zero keeping a_{k+1} fixed:

$$\int_\Omega \frac{u_{\varepsilon,a}^2}{|x|^2} X_1^2 \dots X_k^2 X_{k+1}^2 dx = \frac{N\omega_N}{2a_{k+1}} X_{k+1}^{2a_{k+1}}(\delta) + O(1);$$

here we also used (6.22). To obtain a contradiction in (6.23) we will now take the limit $a_k \to 0$ for a_{k+1} small but fixed. The numerator then is easily seen to be of order $O(1)$. Concerning the denominator, since $\gamma < 2$ we choose an $a_{k+1} > 0$ such that $\gamma - 1 + 2a_{k+1} < 1$. It then follows that as $a_k \to 0$ the integral of the denominator diverges to $+\infty$. Hence,

$$0 < b_k \leq \frac{I_k[u]}{\int_\Omega V_k^{(\gamma)} u^2 dx} \to 0, \qquad \text{as } a_k \to 0,$$

which is a contradiction. •

It is evident that different choices of ϕ in (6.2) lead to different inequalities. We now derive an inequality that we will use in the next Section.

Lemma 6.3 *Let $\mu < \frac{N-2}{2}$. Then, for any $u \in H_0^1(\Omega)$, the following inequality holds for any $k = 1, 2, \dots$*

$$\int_\Omega |\nabla u(x)|^2 dx \geq \mu(N-2-\mu) \int_\Omega \frac{u^2(x)}{|x|^2} dx +$$
$$+ \quad (\frac{1}{4} + \frac{N-2}{2} - \mu) \sum_{i=1}^k \int_\Omega \frac{1}{|x|^2} X_1^2(\frac{|x|}{D}) X_2^2(\frac{|x|}{D}) \dots X_i^2(\frac{|x|}{D}) u^2(x) dx. \tag{6.24}$$

Proof: In (6.2) we take $\phi = r^{-\mu} X_1^{-1/2}(r) X_2^{-1/2}(r) \dots X_k^{-1/2}(r)$. A straight forward calculation shows that

$$-\frac{\Delta\phi}{\phi} = \frac{\mu(N-2-\mu)}{r^2} + \frac{(N-2-2\mu)}{2r^2} \sum_{i=1}^k X_1 X_2 \dots X_i + \frac{1}{4r^2} \sum_{i=1}^k X_1^2 X_2^2 \dots X_i^2.$$

Since $X_1 X_2 \dots X_i \leq 1$, the result follows from (6.2). •

7 On the optimality of the series expansion

Using the notation of the previous Section we set for $k = 1, 2, \dots$:

$$I_k[u] = \int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx - \frac{1}{4} \sum_{i=1}^k \int_\Omega \frac{1}{|x|^2} X_1^2 X_2^2 \dots X_i^2 u^2 dx. \tag{7.1}$$

We may also identify $I_0[u]$ with $I[u]$ (cf (2.2)). We then consider the k-Improved Hardy inequality with best constant, that is:

$$I_k[u] \geq 0.$$

As we have seen this can be further improved. One then may ask what kind of potentials $V_k \in \mathcal{A}_k$ (cf Definition 1.3), one may add in the right hand side (besides the ones in Theorem 6.1), so that an inequality of the form holds true:

$$I_k[u] \geq b_k \int_\Omega V_k u^2 dx, \qquad \forall\, u \in H_0^1(\Omega), \tag{7.2}$$

with b_k being the best constant, that is

$$b_k = \inf_{\substack{u \in H_0^1(\Omega) \\ \int_\Omega V_k u^2 dx > 0}} R_k[u], \qquad R_k[u] := \frac{I_k[u]}{\int_\Omega V_k u^2 dx} > 0. \tag{7.3}$$

As we shall see there is a great variety of potentials $V_k \in \mathcal{A}_k$ for which (7.2) holds.

Before that we will establish the k-improved Hardy-Sobolev inequality with critical exponent, that is, the analogue of Theorem A.

We first present a Lemma similar to Lemma 2.2.

Lemma 7.1 *For any $q \geq 2$, there exists a $c > 0$ such that*

$$\int_0^1 |v'(r)|^2 r \left(\Pi_{i=1}^k X_i(r)\right)^{-1} dr \geq c \left(\int_0^1 |v(r)|^q r^{-1} \Pi_{i=1}^k X_i(r) X_{k+1}^{1+q/2}(r) dr\right)^{2/q}, \tag{7.4}$$

for any $v \in C_0^\infty(0,1)$.

Proof: It follows from [M], Theorem 3, p. 44, with $d\nu = r\left(\Pi_{i=1}^k X_i(r)\right)^{-1} \chi_{[0,1]} dr$ and $d\mu = r^{-1}\Pi_{i=1}^k X_i(r) X_{k+1}^{1+q/2}(r)\chi_{[0,1]} dr$. •

We then give the proof of Theorem A':

Proposition 7.2 *Let $D \geq \sup_{x\in\Omega}|x|$. Then, for any $u \in H_0^1(\Omega)$ there holds:*

$$I_k[u] \geq c \left(\int_\Omega |u|^{\frac{2N}{N-2}} \left(\Pi_{i=1}^{k+1} X_i(\frac{|x|}{D})\right)^{1+\frac{N}{N-2}} dx\right)^{(N-2)/N}, \qquad k = 1, 2, \ldots. \tag{7.5}$$

Proof: The argument parallels that of Theorem 2.3. Suppose first that Ω is the unit ball. Separating the radial part of u (u_0) from its non radial part ($u - u_0$) we will first establish the analogue of (2.10), namely

$$I_k[u] \geq I_k[u_0] + \lambda \int_B |\nabla(u - u_0)|^2 dx, \qquad \lambda > 0. \tag{7.6}$$

Let $H = \frac{N-2}{2}$. Using the decomposition of u (cf (2.8)) we calculate that:

$$I_k[u] = I_k[u_0] + \sum_{m=1}^\infty \int_B \left(|\nabla u_m|^2 - (H^2 - c_m)\frac{u_m^2}{|x|^2} - \frac{1}{4}\sum_{i=1}^k \frac{u_m^2}{|x|^2} X_1^2 \ldots X_i^2\right) dx.$$

To estimate the infinite sum we will use the inequalities

$$\int_B \left(|\nabla u_m|^2 - (H^2 - c_m)\frac{u_m^2}{|x|^2} - \frac{1}{4}\sum_{i=1}^{k} \frac{u_m^2}{|x|^2} X_1^2 \dots X_i^2 \right) dx \geq$$

$$\geq \lambda \int_B \left(|\nabla u_m|^2 + c_m \frac{u_m^2}{|x|^2} \right) dx, \tag{7.7}$$

valid for any for every $k, m = 1, 2 \dots$ and some $\lambda \in (0,1)$. Let us accept this at the moment and continue. In view of (7.7) we can estimate the infinite sum from below by $\lambda \int_B |\nabla(u - u_0)|^2 dx$, and (7.6) follows.

We then continue as in Theorem 2.3: The radial part $I_k[u_0]$ is reduced to a one dimensional integral, via the transformation $u_0(r) = \phi_k(r) w_0(r)$, with ϕ_k as in (6.4), that is

$$I_k[u_0] = \omega_N \int_0^1 w_0'^2(r) r X_1^{-1} \dots X_k^{-1} dr,$$

and then estimated from below by Lemma 7.1, with $q = 2N/(N-2)$. For the non radial part we use the standard Sobolev embedding with critical exponent and the fact that $X_i \leq 1$. Combining both estimates we conclude the proof in the case where Ω is the unit ball. The general case follows as before. We omit the details.

It remains to justify inequality (7.7). We will do so using (6.24). More precisely, we will show that there exists a $\lambda \in (0,1)$ such that (7.7) is true for every $k, m = 1, 2 \dots$. Taking into account that $c_m \geq N-1$, for $m \geq 1$, elementary calculations show that it is enough to establish the following:

$$\int_B |\nabla u_m|^2 dx \geq \left(\frac{H^2}{1-\lambda} - (N-1) \right) \int_B \frac{u_m^2}{|x|^2} dx + \frac{1}{4(1-\lambda)} \sum_{i=1}^{k} \int_B \frac{u_m^2}{|x|^2} X_1^2 \dots X_i^2 dx.$$

In view of (6.24) it is enough to show that there exists a $\mu < \frac{N-2}{2}$ such that if λ is defined by

$$\frac{H^2}{1-\lambda} - (N-1) = \mu(N-2-\mu), \tag{7.8}$$

then $\lambda \in (0,1)$ and in addition

$$\frac{1}{4} + \frac{N-2}{2} - \mu \geq \frac{1}{4(1-\lambda)}. \tag{7.9}$$

An elementary analysis of (7.8) by quadrature reveals that in order to have $\lambda \in (0,1)$ one should choose a μ satisfying $\frac{N-2}{2} - (N-1)^{1/2} < \mu < \frac{N-2}{2}$. If we solve (7.8) for λ and plug in this value in (7.9), a similar analysis shows that in order for (7.9) to hold true, we should have $\mu < \frac{N-2}{2} + (\frac{(N-2)^2}{2})(1 - (1 + 4(N-1)(N-2)^{-4})^{1/2})$. It is easy to check that for any $N \geq 3$ there exist μ satisfying both restrictions and the result follows. •

Remark By the same argument as in Lemma 6.2 we can show that (7.5) is sharp in the sense that $X_{k+1}^{1+\frac{N}{N-2}}$ cannot be replaced by a smaller power of X_{k+1}.

It is now easy to find potentials for which (7.2) holds. For instance, we have:

Lemma 7.3 *Let $D \geq \sup_{x \in \Omega} |x|$. Suppose V_k is such that that*

$$\int_\Omega |V_k|^{\frac{N}{2}} (X_1(\frac{|x|}{D}) \dots X_{k+1}(\frac{|x|}{D}))^{1-N} dx < \infty.$$

Then, there exists $b_k > 0$ such that (7.2) holds.

Proof: Applying Holder's inequality and then Proposition 7.2 we get:

$$\begin{aligned}\int_\Omega |V_k|u^2dx &\le C\left(\int_\Omega |u|^{\frac{2N}{N-2}}\left(\Pi_{i=1}^{k+1}X_i\right)^{1+\frac{N}{N-2}}dx\right)^{(N-2)/N}\cdot\\ &\quad\left(\int_\Omega |V_k|^{\frac{N}{2}}(X_1X_2\dots X_{k+1})^{1-N}dx\right)^{2/N}\\ &\le CI_k[u],\end{aligned}$$

and the result follows. •

Suppose now that we have chosen a potential $V_k\in\mathcal{A}_k$ for which (7.2) is true with b_k as its best constant. We ask again whether this can be further improved. That is, whether there are potentials $W_k\in\mathcal{A}_k$ for which the following holds:

$$I_k[u]\ge b_k\int_\Omega V_ku^2dx+b_{k+1}\int_\Omega W_ku^2dx. \qquad (7.10)$$

The situation is now analogous to the one in Section 3. In particular, the class of potentials V_k for which (7.2) can be further improved is dramatically reduced.

We will use the same strategy as before. Our first step will be to reformulate the problem by means of a change of variables. As in the previous Section, for $D\ge\sup_{x\in\Omega}|x|$ we set:

$$u(x)=\phi_k(r)v(x)=r^{-H}X_1^{-1/2}(\frac{r}{D})X_2^{-1/2}(\frac{r}{D})\dots X_k^{-1/2}(\frac{r}{D})v(x),\qquad r=|x|. \qquad (7.11)$$

Then, there holds (cf (6.2)):

$$I_k[u]=\int_\Omega |x|^{-(N-2)}X_1^{-1}\dots X_k^{-1}|\nabla v|^2dx \qquad (7.12)$$

We set

$$\rho_k(x)=\phi_k^2(r)=|x|^{-(N-2)}X_1^{-1}\dots X_k^{-1},$$

and we define the (Hilbert) space $W_0^{1,2}(\Omega;\rho_k)$ as the completion of $\mathcal{C}_0^\infty(\Omega)$ under the norm $\int_\Omega\rho_kv^2+\int_\Omega\rho_k|\nabla v|^2dx$. Working as in Section 2 we can show that $(\int_\Omega\rho_k|\nabla v|^2dx)^{1/2}$ is an equivalent norm for $W_0^{1,2}(\Omega;\rho_k)$. Also, if $u\in H_0^1(\Omega)$ then $v=\phi_k^{-1}u\in W_0^{1,2}(\Omega;\rho_k)$.

The inequality (1.15) that characterizes the k-admissible potentials is equivalent to the following inequality:

$$\int_\Omega |x|^{-(N-2)}X_1^{-1}\dots X_k^{-1}|\nabla v|^2dx\ge C\int_\Omega |x|^{-(N-2)}X_1^{-1}\dots X_k^{-1}|V_k|v^2dx, \qquad (7.13)$$

valid for any $v\in W_0^{1,2}(\Omega;\rho_k)$. In particular we have the analogue of Lemma 3.1:

Lemma 7.4 *The best constant of inequalities (1.15) and (7.13) are equal.*

Similarly the k- Hardy Sobolev inequality reads:

Lemma 7.5 *Let $D\ge\sup_{x\in\Omega}|x|$. Then, there exists $c>0$ such that for all $v\in W_0^{1,2}(\Omega;\rho_k)$ there holds:*

$$\begin{aligned}&\int_\Omega |x|^{-(N-2)}(\Pi_{i=1}^kX_i(\frac{|x|}{D}))^{-1}|\nabla v|^2dx\\ &\qquad\ge c\left(\int_\Omega |x|^{-N}|v|^{\frac{2N}{N-2}}\Pi_{i=1}^kX_i(\frac{|x|}{D})X_{k+1}^{1+\frac{N}{N-2}}(\frac{|x|}{D})dx\right)^{(N-2)/N}.\end{aligned}$$

We then define:

$$Q_k[v] := \frac{\int_\Omega |x|^{-(N-2)} X_1^{-1} \dots X_k^{-1} |\nabla v|^2 dx}{\int_\Omega |x|^{-(N-2)} X_1^{-1} \dots X_k^{-1} V_k v^2 dx},$$

and

$$B_k = \inf_{\substack{v \in W_0^{1,2}(\Omega;\rho_k) \\ \int_\Omega |x|^{-(N-2)} X_1^{-1} \dots X_k^{-1} V_k v^2 dx > 0}} Q_k[v] = \inf_{\substack{v \in \mathcal{C}_0^\infty(\Omega) \\ \int_\Omega |x|^{-(N-2)} X_1^{-1} \dots X_k^{-1} V_k v^2 dx > 0}} Q_k[v]. \tag{7.14}$$

Finally the analogue of Proposition 3.3 is

Proposition 7.6 *There holds:* $b_k = B_k$.

The proofs of Lemmas 7.4, 7.5 and Proposition 7.6 are practically the same. The proof is similar in spirit to the proof of Lemma 3.1 but technically much more involved. We therefore sketch the proof of one of these:

Proof of Proposition 7.6 : The inequality $b_k \geq B_k$ follows easily. We now sketch the proof of the reverse inequality. Let $v_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$ such that $Q_k[v_\varepsilon] \leq B_k + \varepsilon$. We set $u_{a,\varepsilon} = |x|^{-a_0} X_1^{-a_1} \dots X_k^{-a_k} v_\varepsilon \in H_0^1(\Omega)$, with $0 < a_0 < H$, $0 < a_i < 1/2$, $i = 1, \dots k$.. We intend to take the limit as $a_0 \to H$, $a_1 \to 1/2, \dots a_k \to 1/2$, in this order, keeping ε fixed. It is easy to take this limit in the denominator of $R_k[u_{a,\varepsilon}]$, but one has to be careful with the numerator. We will work as in the proof of Theorem 6.1.

A straight forward calculation shows that (we drop the subscript ε for simplicity):

$$\begin{aligned}
I_k[u_{a,\varepsilon}] &= (a_0^2 - H^2) \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1} \dots X_k^{-2a_k} v^2 dx \\
&+ \sum_{i=1}^k (a_i^2 - \frac{1}{4}) \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+2} \dots X_i^{-2a_i+2} X_{i+1}^{-2a_{i+1}} \dots X_k^{-2a_k} v^2 dx \\
&+ \int_\Omega |x|^{-2a_0} X_1^{-2a_1} \dots X_k^{-2a_k} |\nabla v|^2 dx \\
&+ 2a_0 \sum_{i=1}^k a_i \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+1} \dots X_i^{-2a_i+1} X_{i+1}^{-2a_{i+1}} \dots X_k^{-2a_k} v^2 dx \\
&- 2a_0 \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1} \dots X_k^{-2a_k} v x \cdot \nabla v \, dx \\
&- 2 \sum_{i=1}^k a_i \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+1} \dots X_i^{-2a_i+1} X_{i+1}^{-2a_{i+1}} \dots X_k^{-2a_k} v x \cdot \nabla v \, dx \\
&+ 2 \sum_{j=1}^{k-1} \sum_{i=j+1}^k a_i a_j \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+2} \dots X_j^{-2a_j+2} X_{j+1}^{-2a_{j+1}+1} \dots X_i^{-2a_i+1} \cdot \\
&\qquad X_{i+1}^{-2a_{i+1}} \dots X_k^{-2a_k} v^2 dx.
\end{aligned} \tag{7.15}$$

In order to take the limit $a_0 \to H$ we will use two identities. Observing that $2(H - a_0)|x|^{-2a_0-2} = \mathrm{div}(x|x|^{-2a_0-2})$, an integration by parts yields the first identity:

$$(H - a_0) \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1} \dots X_k^{-2a_k} v^2 dx =$$

$$\sum_{i=1}^{k} a_i \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+1} \dots X_i^{-2a_i+1} X_{i+1}^{-2a_{i+1}} \dots X_k^{-2a_k} v^2 dx \tag{7.16}$$
$$- \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1} \dots X_k^{-2a_k} v x \cdot \nabla v \, dx.$$

A similar integration by parts yields the second identity:

$$\begin{aligned}
(H-a_0) \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+1} \dots X_i^{-2a_i+1} X_{i+1}^{-2a_{i+1}} \dots X_k^{-2a_k} v^2 dx = \\
- \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+1} \dots X_i^{-2a_i+1} X_{i+1}^{-2a_{i+1}} \dots X_k^{-2a_k} v x \cdot \nabla v \, dx \qquad (7.17) \\
+ \sum_{j=1}^{i} (a_j - \frac{1}{2}) \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+2} \dots X_j^{-2a_j+2} X_{j+1}^{-2a_{j+1}+1} \dots X_i^{-2a_i+1} \cdot \\
X_{i+1}^{-2a_{i+1}} \dots X_k^{-2a_k} v^2 dx \\
+ \sum_{j=i+1}^{k} a_j \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+2} \dots X_i^{-2a_i+2} X_{i+1}^{-2a_{i+1}+1} \dots X_j^{-2a_j+1} \cdot \\
X_{j+1}^{-2a_{j+1}} \dots X_k^{-2a_k} v^2 dx.
\end{aligned}$$

We introduce for convenience the following notation:

$$\begin{aligned}
A_i &= \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+2} \dots X_i^{-2a_i+2} X_{i+1}^{-2a_{i+1}} \dots X_k^{-2a_k} v^2 dx \\
B_i &= \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+1} \dots X_i^{-2a_i+1} X_{i+1}^{-2a_{i+1}} \dots X_k^{-2a_k} v x \cdot \nabla v \, dx \\
\Gamma_{ji} &= \int_\Omega |x|^{-2a_0-2} X_1^{-2a_1+2} \dots X_j^{-2a_j+2} X_{j+1}^{-2a_{j+1}} \dots X_i^{-2a_i+1} X_{i+1}^{-2a_{i+1}} \dots X_k^{-2a_k} v^2 dx,
\end{aligned}$$

with $\Gamma_{ii} = A_i$.

We use the two identities to replace the first and fourth terms of (7.15). We then take the limit $a_0 \to H$ to obtain:

$$\begin{aligned}
I_k[u_{a,\varepsilon}] &= -\sum_{i=1}^{k} a_i B_i + \frac{1}{2} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} a_j \Gamma_{ij} + \sum_{i=1}^{k} \frac{1}{2} (a_i - \frac{1}{2}) A_i \\
&\quad + \int_\Omega |x|^{-2H} X_1^{-2a_1} \dots X_k^{-2a_k} |\nabla v|^2 dx,
\end{aligned} \tag{7.18}$$

where we have set $a_0 = H$ in the A_i, B_i, Γ_{ij}. In order to take the limit $a_1 \to 1/2, \dots a_k \to 1/2$, we will use successively similar identities. More precisely, observing that $(-2a_i+1)|x|^{-N} X_1 \dots X_{i-1} X_i^{-2a_i+2} = \mathrm{div}(x|x|^{-N} X_i^{-2a_1+1})$, $i = 1, \dots k$, we get by an integration by parts:

$$B_i = (a_i - \frac{1}{2}) A_i + \sum_{j=i+1}^{k} a_j \Gamma_{ij}, \qquad i = 1, 2, \dots k-1; \tag{7.19}$$

here for each fixed i we have set $a_0 = H$, $a_1 = \dots a_{i-1} = 1/2$ in the A_i, B_i, Γ_{ij}. Then, using (7.19) with $i = 1$ we can take the limit $a_1 \to 1/2$ in (7.18). We then use (7.19) with $i = 2$ to take the limit $a_2 \to 1/2$ and so on. After taking the limit $a_k \to 1/2$, we see that only the last term in (7.18) survives:

$$I_k[u_{a,\varepsilon}] \to \int_\Omega |x|^{-(N-2)} X_1^{-1} \dots X_k^{-1} |\nabla v_\varepsilon|^2 dx. \tag{7.20}$$

We note that the right hand side of (7.20) is the numerator of $Q_k[v_\varepsilon]$. Hence we have shown that $R_k[u_{a,\varepsilon}] \to Q_k[v_\varepsilon]$ as $(a_0, a_1, \ldots a_k) \to (H, 1/2, \ldots 1/2)$. We then complete the proof as in Lemma 3.1 •

We next define the local best constant of inequality (7.2) near zero:

$$\mathcal{C}_k^0 := \lim_{r \downarrow 0} C_{k,r}, \tag{7.21}$$

where,

$$C_{k,r} = \inf_{\substack{v \in C_0^\infty(B_r) \\ \int_\Omega |x|^{-(N-2)} X_1^{-1} \ldots X_k^{-1} V_k v^2 dx > 0}} \frac{\int_{B_r} |x|^{-(N-2)} X_1^{-1} \ldots X_k^{-1} |\nabla v|^2 dx}{\int_{B_r} |x|^{-(N-2)} X_1^{-1} \ldots X_k^{-1} V_k v^2 dx}.$$

Working as in Proposition 3.4 we establish (we omit the proof):

Proposition 7.7 *Suppose $V_k \in \mathcal{A}_k$. Let B_k and $\mathcal{C}_k^0$ be as defined in (7.14) and (7.21) respectively. If*

$$B_k < \mathcal{C}_k^0, \tag{7.22}$$

every bounded in $W_0^{1,2}(\Omega; \rho_k)$ minimizing sequence of (7.14) has a strongly in $W_0^{1,2}(\Omega; \rho_k)$ convergent subsequence. In particular B_k is achieved by some $v_0 \in W_0^{1,2}(\Omega; \rho_k)$.

¿From this Proposition, and using the same argument as in Proposition 3.6, Theorem B' follows easily.

A consequence of Theorems A' and B' is the following:

Corollary 7.8 *Let $D \geq \sup_{x \in \Omega} |x|$. Suppose V_k is not everywhere nonpositive, and such that*

$$\int_\Omega |V_k|^{\frac{N}{2}} \left(\Pi_{i=1}^{k+1} X_i(\frac{|x|}{D}) \right)^{1-N} dx < \infty. \tag{7.23}$$

Then, $V_k \in \mathcal{A}_k$, and therefore (7.2) holds, but there is no further improvement of (7.2) by a nonnegative $W_k \in \mathcal{A}_k$.

Proof: The fact that $V_k \in \mathcal{A}_k$ has been shown in Lemma 7.3. To prove the last statement we will show that $\mathcal{C}_k^0 = \infty$. Applying Holder's inequality in B_r as in Lemma 7.3 and recalling (7.21) we easily find that:

$$C_{k,r} \geq C \left(\int_{B_r} |V_k|^{\frac{N}{2}} \left(\Pi_{i=1}^{k+1} X_i(\frac{|x|}{D}) \right)^{1-N} dx \right)^{-\frac{2}{N}} \to \infty, \qquad \text{as } r \to 0,$$

and the result follows from Theorem B'. •

We finally make some comments on the optimality of the series of Theorem D. Consider the potential

$$V_k^{(\gamma)}(x) = \frac{1}{|x|^2} X_1^2 \ldots X_k^2 X_{k+1}^\gamma.$$

An elementary calculation shows that $V_k^{(\gamma)}$ satisfies (7.23) if and only if $\gamma > 2$. According to Corollary 7.8, at the k step ($k = 0, 1, \ldots$) we could add $V_k^{(\gamma)}(x)$ with $\gamma > 2$ (or a less singular at zero potential) but that would force the series to terminate. On the other hand by Lemma 6.2 we cannot add $V_k^{(\gamma)}(x)$ with $\gamma < 2$ (or a more singular at zero potential) since we are lead outside the k-admissible class $\mathcal{A}_k$. Hence, the main singularities (at zero) that the "improving" potentials are allowed to have, are the ones appearing in Theorem D.

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