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The Hardy–Morrey & Hardy–John–Nirenberg inequalities involving distance to the boundary

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Abstract

We strengthen the classical inequality of C.B. Morrey concerning the optimal Hölder continuity of functions in $W^{1,p}$ when $p > n$, by replacing the L^p-modulus of the gradient with the sharp Hardy difference involving distance to the boundary. When $p = n$ we do the same strengthening in the integral form of a well known inequality due to F. John and L. Nirenberg.

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1. Introduction and main results

Let $\Omega \subsetneq \mathbb{R}^n$, $n \geq 1$, be a domain and denote the distance function to its boundary $\partial\Omega$ by

$$d(x) := \inf_{y \in \partial\Omega} |x - y|, \quad \text{whenever } x \in \bar{\Omega}.$$

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It is proved in [3] that if Ω satisfies the following condition:

$$-\Delta d \geq 0 \text{ in the sense of distributions in } \Omega, \qquad (\mathscr{C})$$

then Hardy's inequality holds true with the best possible constant, that is

$$\int_\Omega |\nabla u|^p \mathrm{d}x \geq \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|u|^p}{d^p}\mathrm{d}x \quad \text{for all } u \in C_c^\infty(\Omega), \tag{1.1}$$

where $p > 1$ is arbitrary. Examples of domains satisfying condition $(\mathscr{C})$ are convex domains since then d is superharmonic in Ω (see [2]). Moreover, if the boundary $\partial\Omega$ is smooth enough, say uniformly of class C^2 (see Definition 2.3), then $(\mathscr{C})$ is known to be *equivalent* to the domain being mean convex, i.e. having nonnegative mean curvature everywhere on its boundary (see [25] and also [14], [19] and [12]). In view of this, we call *weakly mean convex* domain any domain satisfying condition $(\mathscr{C})$.

For $p = 2$ and Ω being the half-space, i.e. $\Omega = \mathbb{R}^n_+$ where

$$\mathbb{R}^n_+ := \{(x', x_n) \mid x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1},\ x_n > 0\},\ n \geq 2,$$

the critical Sobolev norm can be added on the right hand side of (1.1). More precisely, Maz'ya in his treatise [22] proved that for $n \geq 3$ there exists a positive constant C such that

$$\Bigg(\int_{\mathbb{R}^n_+} |\nabla u|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2}\mathrm{d}x\Bigg)^{1/2} \geq C\Bigg(\int_{\mathbb{R}^n_+} |u|^{2^*}\mathrm{d}x\Bigg)^{1/2^*} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n_+), \tag{1.2}$$

where $2^* := 2n/(n-2)$. This inequality has been extended to domains in [9]. It is proved there that if Ω is a uniformly C^2 mean convex domain with finite inner radius, that is

$$D_\Omega := \sup_{x\in\Omega} d(x) < \infty,$$

then there exists a positive constant C such that

$$\Bigg(\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{4}\int_\Omega \frac{|u|^2}{d^2}\mathrm{d}x\Bigg)^{1/2} \geq C\Bigg(\int_\Omega |u|^{2^*}\mathrm{d}x\Bigg)^{1/2^*} \quad \text{for all } u \in C_c^\infty(\Omega). \tag{1.3}$$

It is also known (see [11]) that if one strengthens assumption $(\mathscr{C})$ to convexity, then (1.3) holds true with a constant C independent of the domain Ω and without any regularity assumption on Ω.

At this point we want to compare the above result with the corresponding result for Hardy's inequality with the distance taken from a point in Ω. It is known (see [10, Theorem A] and also [1]) that if Ω is a bounded domain containing the origin, then there exists a positive constant C such that for any $u \in C_c^\infty(\Omega)$ the following estimate holds true

$$\Bigg(\int_\Omega |\nabla u|^2 \mathrm{d}x - \Big(\frac{n-2}{2}\Big)^2\int_\Omega \frac{|u|^2}{|x|^2}\mathrm{d}x\Bigg)^2 \geq C\Bigg(\int_\Omega |u|^{2^*} X^{1+2^*/2}\Big(\frac{|x|}{R_\Omega}\Big)\mathrm{d}x\Bigg)^{1/2^*}. \tag{1.4}$$

Here $R_\Omega := \sup_{x\in\Omega}|x|$ and $X(t) := (1-\log t)^{-1},\ t \in (0,1]$. The nonnegativity of the left hand side is the Hardy inequality involving distance to the origin with the best possible constant $\big((n-2)/2\big)^2$ (see for instance [22]). We stress that the exponent on the logarithmic correction in (1.4) is the optimal one, i.e. it cannot be decreased.

Coming back to the case where the distance is taken from the boundary, inequalities (1.2) and (1.3) have p-versions for any $2 < p < n$, obtained in [9] (see also [11] for convex domains, the case of the half-space being common in these two essentially different approaches).

Our main goal in this paper is to obtain the corresponding to [22] and [9] results for the case $p > n \geq 1$.

Our first result is the L^∞-Hardy–Sobolev inequality. Let us first recall *Sobolev's inequality for* $p > n$ (see for example [13, Theorem 7.10]): *If Ω has finite volume $\mathscr{H}^n(\Omega) < \infty$ and $p > n \geq 1$, then there exists a positive constant $C = C(n,p)$ depending only on n, p, such that:*

$$\sup_{x\in\Omega}|u(x)| \leq C[\mathscr{H}^n(\Omega)]^{1/n-1/p}\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)^{1/p} \quad \text{for all } u \in C_c^\infty(\Omega). \tag{1.5}$$

It turns out that in a weakly mean convex domain one can replace the L^p-norm of the right hand side by the sharp Hardy difference. More precisely we have

Theorem A. *Let $\Omega \subset \mathbb{R}^n$ be a weakly mean convex domain of finite volume $\mathscr{H}^n(\Omega) < \infty$. For $p > n \geq 1$, there exists a positive constant $C = C(n,p)$, depending only on n and p, such that*

$$\sup_{x\in\Omega}|u(x)| \leq C[\mathscr{H}^n(\Omega)]^{1/n-1/p}\left(\int_\Omega |\nabla u|^p \mathrm{d}x - \left(\frac{p-1}{p}\right)^p\int_\Omega \frac{|u|^p}{d^p}\mathrm{d}x\right)^{1/p} \quad \textit{for all } u \in C_c^\infty(\Omega). \tag{1.6}$$

Remark. The corresponding inequality in the case where the distance is taken from the origin is true as well (see [26, Theorem A]).

Next we present the following extension of the Hardy–Sobolev inequality obtained in [9].

Theorem B. *Let $\Omega \subset \mathbb{R}^n,\ n \geq 2$, be a uniformly C^2 mean convex domain of finite inner radius. For*

$$-1 < b \leq 0, \qquad 2 \leq p < \frac{n}{b+1}, \qquad \textit{and} \quad q := \frac{np}{n-p(b+1)},$$

there exists a positive constant K such that

$$\left(\int_\Omega |\nabla u|^p \mathrm{d}x - \left(\frac{p-1}{p}\right)^p\int_\Omega \frac{|u|^p}{d^p}\mathrm{d}x\right)^{1/p} \geq K\left(\int_\Omega \big(d^b|u|\big)^q \mathrm{d}x\right)^{1/q} \quad \textit{for all } u \in C_c^\infty(\Omega). \tag{1.7}$$

The inequality (1.7) remains true when Ω is the half space, that is $\Omega = \mathbb{R}^n_+$.

Remark. For $b < 0$ the exponent p is allowed to exceed the dimension n. This fact is not captured in Theorem 5.3 of [9] and will play a crucial role in the present work.

Our central result is presented next. Recall first *Morrey's inequality in* $\mathbb{R}^n$ (see for example [8, §4.5.3, Theorem 3(ii)]): *If* $p > n \geq 1$ *then*

$$[u]_{C^{0,1-n/p}} := \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \leq C\Bigg(\int_{\mathbb{R}^n} |\nabla u|^p \mathrm{d}x\Bigg)^{1/p} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n). \tag{1.8}$$

Note that from (1.8) we can derive (1.5). Also that the seminorm $[\cdot]_{C^{0,1-n/p}}$ and the norm $\||\nabla \cdot|\|_{L^p}$ involved in (1.8) are dimensionally balanced. We prove:

Theorem C *(Hardy–Morrey inequality). Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a uniformly* C^2 *mean convex domain of finite inner radius. For* $p > n$ *there exists a positive constant* C *such that*

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \leq C\Bigg(\int_\Omega |\nabla u|^p \mathrm{d}x - \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|u|^p}{d^p}\mathrm{d}x\Bigg)^{1/p} \quad \textit{for all } u \in C_c^\infty(\Omega). \tag{1.9}$$

The inequality (1.9) remains true when Ω *is the half space, that is* $\Omega = \mathbb{R}^n_+$. *In the one dimensional case, for* $p > 1$ *there exist constants* $C = C(p) > 0$, $\lambda = \lambda(p) \geq 1$, *such that for any* $\alpha < \beta$ *and all* $u \in C_c^\infty(\alpha, \beta)$

$$\sup_{\substack{x,y\in(\alpha,\beta) \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{1-1/p}} X^{1/p}\Big(\frac{|x-y|}{\lambda D}\Big) \leq C\Bigg(\int_\alpha^\beta |u'|^p \mathrm{d}x - \Big(\frac{p-1}{p}\Big)^p \int_\alpha^\beta \frac{|u|^p}{d^p}\mathrm{d}x\Bigg)^{1/p}, \tag{1.10}$$

where $X(t) := (1-\log t)^{-1}$, $t \in (0, 1]$, *and* $D = (\beta - \alpha)/2$. *Moreover, the exponent* $1/p$ *on* X *cannot be decreased.*

Remark 1. The corresponding to (1.9) inequality in the case of Hardy difference with the distance taken from the origin is not true unless a logarithmic correction in the Holder seminorm is introduced. In particular we have for any $u \in C_c^\infty(\Omega \setminus \{0\})$ that

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{1-1/p}} X^{1/p}\Big(\frac{|x-y|}{\lambda R_\Omega}\Big) \leq C\Bigg(\int_\Omega |\nabla u|^p \mathrm{d}x - \Big(\frac{p-n}{p}\Big)^p \int_\Omega \frac{|u|^p}{|x|^p}\mathrm{d}x\Bigg)^{1/p},$$

for some constants $C = C(n, p) > 0$, $\lambda = \lambda(n, p) \geq 1$ and $R_\Omega = \sup_{x\in\Omega} |x| < \infty$ (see [26, Theorem B]). Here, the exponent $1/p$ in the logarithmic correction $X^{1/p}$ cannot be decreased. The fact that in the one dimensional case of Theorem C a logarithmic correction of the Hölder seminorm is needed is not surprising, since in this case the problem behaves the same way as when the distance is taken from a point.

Remark 2. The requirement on Ω to be uniformly of class C^2 in Theorem C, is inherited from the corresponding regularity assumption in Theorem B.

Remark 3. In our proof of Theorem C, the constant C of inequality (1.9) depends in general on the domain Ω. To prove inequality (1.9) under the assumption of mean convexity with a constant independent of the domain, remains an open question. The same remark applies for the constant K of inequality (1.7) in Theorem B and also for the constant C in Theorem D below, as well as the constant C_1 in Corollary E.

Remark 4. Since in Theorem C the domain has finite inner radius D_Ω, it follows from (1.9) that

$$\sup_{x\in\Omega}|u(x)| \le C D_\Omega^{1-n/p}\left(\int_\Omega |\nabla u|^p \mathrm{d}x - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p}\mathrm{d}x\right)^{1/p} \quad \text{for all } u \in C_c^\infty(\Omega).$$

If the domain has finite volume, then by the fact that $D_\Omega \le \left(\mathscr{H}^n(\Omega)/\omega_n\right)^{1/n}$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, we readily deduce (1.6) for uniformly C^2 domains with some positive constant C. However, in Theorem A the constant C depends only on n, p and moreover no regularity assumption on Ω is needed.

To introduce our final result we first recall that a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ has *bounded mean oscillation* and we write $u \in BMO$ if

$$\|u\|_{BMO} := \sup_B \frac{1}{\mathscr{H}^n(B)} \int_B |u - u_B|\mathrm{d}x < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. Here u_B is the average of u in the ball B, that is

$$u_B = \frac{1}{\mathscr{H}^n(B)}\int_B u\,\mathrm{d}x.$$

For example if $u \in W_0^{1,n}(\Omega)$ is extended to be zero outside Ω, then we know that $u \in BMO$ (see [7, §5.8.1]). The John–Nirenberg inequality (see [18]) in its integral form states that there exist positive constants $C_1(n)$ and $C_2(n)$ such that

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \exp\left\{C_1(n)\frac{|u-u_B|}{\|u\|_{BMO}}\right\}\mathrm{d}x \le C_2(n) \quad \text{for all } u \in BMO. \tag{1.11}$$

In the following we have extended functions in $C_c^\infty(\Omega)$ to be 0 in $\mathbb{R}^n \setminus \Omega$.

Theorem D. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a uniformly C^2 mean convex domain of finite inner radius. Then there exists a positive constant C such that*

$$\left(\int_\Omega |\nabla u|^n \mathrm{d}x - \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|u|^n}{d^n}\mathrm{d}x\right)^{1/n} \geq C\|u\|_{BMO} \quad \textit{for all } u \in C_c^\infty(\Omega).$$

The above inequality remains true when Ω *is the half space, that is* $\Omega = \mathbb{R}^n_+$.

A direct consequence of Theorem D and the John–Nirenberg inequality (1.11), is

Corollary E *(Hardy–John–Nirenberg inequality). Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a uniformly* C^2 *mean convex domain of finite inner radius. There exist a positive constant* C_1 *and a positive constant* $C_2 = C_2(n)$ *such that*

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\mathscr{H}^n(B)} \int_B \exp\left\{C_1 \frac{|u-u_B|}{\left(\int_\Omega |\nabla u|^n \mathrm{d}x - (\frac{n-1}{n})^n \int_\Omega \frac{|u|^n}{d^n}\mathrm{d}x\right)^{1/n}}\right\}\mathrm{d}x \leq C_2 \quad \textit{for all } u \in C_c^\infty(\Omega).$$

The above inequality remains true when Ω *is the half space, that is* $\Omega = \mathbb{R}^n_+$.

The paper is organized as follows: In §2 we gather all definitions that appear throughout, and also several known results on the Hardy inequality that we are going to use. In §3 and §4 we prove Theorem A and Theorem B, respectively. §5 is interesting on its own and comprises of several calculus results focused on estimating the local integral of the distance function to the boundary raised on small negative powers. We gradually build the proof of Theorem C in §6.1 and §6.2. The proof of Theorem D is given in §6.3. Finally, in §7 we prove the one dimensional version of Theorem C.

2. Preliminaries

2.1. Notation, regular boundaries and some properties of the distance function

Throughout the paper, the boundary and inner radius of a domain (open and connected set) $\Omega \subsetneq \mathbb{R}^n$, $n \geq 1$, are denoted by $\partial\Omega$ and D_Ω respectively. We write $\mathscr{H}^n$ for the Lebesgue measure in $\mathbb{R}^n$, and $\mathscr{H}^{n-1}$ for the $n-1$ Hausdorff measure in $\mathbb{R}^n$. $B_r(y)$ stands for an open ball in $\mathbb{R}^n$, $n \geq 2$, having center at $y \in \mathbb{R}^n$ and radius $r > 0$. When the center, or both the center and radius are of no importance, we simply write B_r, or B respectively. For convenience we will write ω_n in place of $\mathscr{H}^n(B_1)$, and so $\mathscr{H}^{n-1}(\partial B_1) = n\omega_n$. Also, $B_r^{n-1}(y)$ is the $n-1$-dimensional ball having center at $y \in \mathbb{R}^{n-1}$ and radius $r > 0$, and we write ω_{n-1} for $\mathscr{H}^{n-1}(B_1^{n-1})$.

By $C(\alpha, \beta, ...)$ or $c(\alpha, \beta, ...)$ we mean a positive constant that is allowed to change value from line to line but depends only on the arguments $\alpha, \beta, ...$. Sometimes we use the notation q' for the dual index of $q \in (1, \infty)$, i.e. $q' := q/(q-1)$.

Definition 2.1. Let $n \geq 2$. By a *locally* Lipschitz domain $\Omega \subsetneq \mathbb{R}^n$ (respectively *locally* C^2 domain $\Omega \subsetneq \mathbb{R}^n$), we mean that for any $x \in \partial\Omega$ there exist a neighborhood U_x of x, a system of coordinates $y_1, ..., y_n$, such that the point x is characterized by $y_1 = ... = y_n = 0$ in this system, a Lipschitz (resp. C^2) mapping $\phi_x : \mathbb{R}^{n-1} \to \mathbb{R}$, and $r_x > 0$ such that

$$U_x \cap \Omega = U_x \cap \{(y', y_n) \in B^{n-1}_{r_x}(0) \times \mathbb{R} \,:\, y_n > \phi_x(y')\}.$$

Remark 2.2. If for some $r > 0$, the set $B_r \cap \Omega$ can be written as

$$B_r \cap \Omega = B_r \cap \big\{(y', y_n) \in A \times \mathbb{R} \mid y_n > f(y')\big\},$$

and f is Lipschitz in the set $A \subseteq \mathbb{R}^{n-1}$, then

$$\mathscr{H}^{n-1}(B_r \cap \partial\Omega) = \int_A \sqrt{1 + |\nabla f(y')|^2}\mathrm{d}y'.$$

Definition 2.3. Let $n \geq 2$. We say that $\Omega \subsetneq \mathbb{R}^n$ is a *uniformly* Lipschitz domain (respectively *uniformly* C^2 domain) if there exist $\varepsilon > 0$, $L > 0$, and $M \in \mathbb{N}$ and a locally finite countable cover $\{U_i\}$ of $\partial\Omega$ with the following properties:

(i) If $x \in \partial\Omega$ then $B_\varepsilon(x) \subset U_i$ for some i.
(ii) Every point of $\mathbb{R}^n$ is contained in at most M U_is.
(iii) For each i there exist local coordinates $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and a Lipschitz (resp. C^2) function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, with $\mathrm{Lip}\, f \leq L$ (resp. $\|f\|_{C^2} < L$) such that

$$U_i \cap \Omega = U_i \cap \big\{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y_n > f(y')\big\}.$$

Stein [29, §IV, 3.3] calls uniformly Lipschitz domains *minimally smooth*.

Remark 2.4. If $\partial\Omega$ is bounded then every locally Lipschitz (resp. locally C^2) domain is uniformly Lipschitz (resp. uniformly C^2).

We refer to the *generalized* Gauss–Green theorem whenever we use Theorem 5.2 (or its consequence, Theorem 5.3) from [6] (see also [28]), and to the Gauss–Green theorem whenever we use Theorem 1 from [8, §5.8].

Recall next that the gradient of d is a bounded vector field:

$$|\nabla d| = 1 \text{ a.e. in } \Omega. \tag{2.1}$$

Condition ($\mathscr{C}$) (or *weak mean convexity* of Ω) implies in particular that $-\Delta d$ is a nonnegative Radon measure μ in Ω (see [20, Theorem 6.22]). By abuse of notation we write $(-\Delta d)\mathrm{d}x$ instead of $\mathrm{d}\mu$. From [6, Definition 2.18] we have that ∇d is a *bounded divergence-measure field* (that is $\nabla d \in \mathscr{DM}^\infty(\Omega)$) and so the generalized Gauss–Green theorem holds true.

Recall also that since any uniformly C^2 domain satisfies a uniform interior sphere condition, a uniformly C^2 domain is weakly mean convex if and only if it is mean convex (see [25, Corollary 3.6]).

2.2. *On the Hardy inequality*

Let $p > 1$ and assume that Ω is a domain in $\mathbb{R}^n$, $n \geq 1$, such that $\Omega \neq \mathbb{R}^n$. In [3] the authors obtained various auxiliary lower bounds for the *Hardy difference*:

$$I_p[u; \Omega] := \int_\Omega |\nabla u|^p \mathrm{d}x - \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|u|^p}{d^p}\mathrm{d}x; \quad u \in C_c^\infty(\Omega).$$

In particular, the substitution

$$u = d^{1-1/p} v, \tag{2.2}$$

together with standard vectorial inequalities, gives the following lower estimates on $I_p[u]$ (see [3, Lemma 3.3])

$$I_p[u;\Omega] \geq c(p) \int_\Omega d^{p-1} |\nabla v|^p \mathrm{d}x + \Big(\frac{p-1}{p}\Big)^{p-1} \int_\Omega |v|^p (-\Delta d) \mathrm{d}x, \quad \text{if } p \geq 2, \tag{2.3}$$

$$I_p[u;\Omega] \geq c(p) \int_\Omega d |v|^{p-2} |\nabla v|^2 \mathrm{d}x + \Big(\frac{p-1}{p}\Big)^{p-1} \int_\Omega |v|^p (-\Delta d) \mathrm{d}x, \quad \text{if } p \geq 2, \tag{2.4}$$

and

$$I_p[u;\Omega] \geq c(p) \int_\Omega \frac{d^{p-1}|\nabla v|^2}{\Big(|\nabla v| + \frac{p-1}{p}\frac{|v|}{d}\Big)^{2-p}} \mathrm{d}x + \Big(\frac{p-1}{p}\Big)^{p-1} \int_\Omega |v|^p (-\Delta d) \mathrm{d}x, \quad \text{if } 1 < p < 2. \tag{2.5}$$

The above estimates imply that if Ω is weakly mean convex, then *Hardy's inequality* (1.1) (that is $I_p[\cdot;\Omega] \geq 0$ in the notation introduced above) holds true. The constant $\big((p-1)/p\big)^p$ in (1.1) was known to be the best one for the case $n = 1$ (see for example [16]). It was proved first in [21] for convex domains and then in [3] for weakly mean convex domains, that this is also the case when $n > 1$.

Remark 2.5. When $n = 1$ and $\Omega = (\alpha, \beta)$ for some $-\infty < \alpha < \beta < \infty$ then $-\Delta d = 2\delta\big((\alpha+\beta)/2\big)$, where $\delta(x_0)$ denotes Dirac's delta measure concentrated at $x_0 \in \mathbb{R}$. In particular, ignoring the first term on the right hand side of (2.3) and (2.5), we get

$$\Big|v\Big(\frac{\alpha+\beta}{2}\Big)\Big| \leq c(p)\big(I_p[u;(\alpha,\beta)]\big)^{1/p}, \quad \text{if } p > 1. \tag{2.6}$$

We will use this estimate in §3 and §7 when arguing for the one dimensional case of Theorem A and Theorem C, respectively.

Also, in proving Theorem A for the case $n = 1$, we make use of the following result taken from [3].

Proposition 2.6. [3, Proposition 3.4] *Let $1 < p < 2$. For any $u \in C_c^\infty(\alpha, \beta)$ the following inequality is valid*

$$I_p[u;(\alpha,\beta)] \geq c(p) \int_\alpha^\beta \big(d(t)\big)^{p-1} |v'(t)|^p X^{2-p}(d(t)/D) \mathrm{d}t + 2\Big(\frac{p-1}{p}\Big)^{p-1} \Big|v\Big(\frac{\alpha+\beta}{2}\Big)\Big|^p,$$

where $X(t) := (1 - \log t)^{-1}$, $t \in (0,1]$, $D = (\beta - \alpha)/2$ and v given by (2.2).

For domains with finite inner radius, one can add remainder terms of the form $\int_\Omega |u|^p W \mathrm{d}x$, in Hardy's inequality (1.1). Clearly, W has to be of lower order than d^{-p} (see [4] for $p=2$ and [3] for the general case). In particular we will need the following case of the central theorem of [3].

Theorem 2.7. [3, Theorem A] *If $p>1$ and $\Omega\subset\mathbb{R}^n$, $n\geq 1$, is a weakly mean convex domain of finite inner radius, then*

$$I_p[u;\Omega]\geq c(p)\int_\Omega \frac{|v|^p}{d}X^2(d/D_\Omega)\mathrm{d}x \quad \textit{for all } u\in C_c^\infty(\Omega),$$

where $X(t):=(1-\log t)^{-1}$, $t\in(0,1]$, and v given by (2.2). Moreover, the exponent 2 on X cannot be decreased.

A direct consequence of the above theorem and estimate (2.4) is

Proposition 2.8. *If $p\geq 2$ and $\Omega\subset\mathbb{R}^n$, $n\geq 1$, is a weakly mean convex domain of finite inner radius, then*

$$I_p[u;\Omega]\geq c(p)\int_\Omega |v|^{p-1}|\nabla v|X(d/D_\Omega)\mathrm{d}x \quad \textit{for all } u\in C_c^\infty(\Omega),$$

where $X(t):=(1-\log t)^{-1}$, $t\in(0,1]$, and v given by (2.2).

Proof. We write

$$\int_\Omega |v|^{p-1}|\nabla v|X(d/D_\Omega)\mathrm{d}x=\int_\Omega \left\{d^{-1/2}|v|^{p/2}X(d/D_\Omega)\right\}\left\{d^{1/2}|v|^{p/2-1}|\nabla v|\right\}\mathrm{d}x,$$

and use the Cauchy–Schwartz inequality. □

We will also need the following lemma.

Lemma 2.9. *Let $\Omega\subsetneq\mathbb{R}^n$, $n\geq 1$, be a domain and V be a locally Lipschitz domain in $\mathbb{R}^n$, such that $\Omega\cap V\neq\emptyset$. Denote by $\nu(x)$ the exterior unit normal vector defined at almost every $x\in\partial V$. For all $q\geq 1$, all $s\neq 1$ and any $v\in C_c^\infty(\Omega)$, there holds*

$$\int_V \frac{|\nabla v|^q}{d^{s-q}}\mathrm{d}x-\frac{s-1}{q}\left|\frac{s-1}{q}\right|^{q-2}\Big(\int_V \frac{|v|^q}{d^{s-1}}(-\Delta d)\mathrm{d}x+\int_{\partial V}\frac{|v|^q}{d^{s-1}}\nabla d\cdot\nu\,\mathrm{d}\mathscr{H}^{n-1}(x)\Big)$$
$$\geq\left|\frac{s-1}{q}\right|^q\int_V\frac{|v|^q}{d^s}\mathrm{d}x. \tag{2.7}$$

Proof. The generalized Gauss–Green theorem gives

$$\int_V \nabla|v| \cdot \frac{\nabla d}{d^{s-1}} \mathrm{d}x = -\int_V |v| \operatorname{div}\Big(\frac{\nabla d}{d^{s-1}}\Big) \mathrm{d}x + \int_{\partial V} |v| \frac{\nabla d}{d^{s-1}} \cdot \nu \mathrm{d}\mathscr{H}^{n-1}(x),$$

and since $\operatorname{div}(\nabla d/d^{s-1}) = (1-s)/d^s - (-\Delta d)/d^{s-1}$ for a.e. $x \in \Omega$, we get

$$\int_V \frac{|\nabla v|}{d^{s-1}} \mathrm{d}x - \int_V \frac{|v|}{d^{s-1}}(-\Delta d)\mathrm{d}x - \int_{\partial V} \frac{|v|}{d^{s-1}} \nabla d \cdot \nu \mathrm{d}\mathscr{H}^{n-1}(x) \geq (s-1)\int_V \frac{|v|}{d^s}\mathrm{d}x, \quad \text{if } s > 1,$$

$$\int_V \frac{|\nabla v|}{d^{s-1}} \mathrm{d}x + \int_V \frac{|v|}{d^{s-1}}(-\Delta d)\mathrm{d}x + \int_{\partial V} \frac{|v|}{d^{s-1}} \nabla d \cdot \nu \mathrm{d}\mathscr{H}^{n-1}(x) \geq (1-s)\int_V \frac{|v|}{d^s}\mathrm{d}x, \quad \text{if } s < 1,$$

where we have also used the fact that $|\nabla|v(x)|| = |\nabla v(x)|$ for a.e. $x \in V$. We may write both inequalities in one as follows

$$\int_V \frac{|\nabla v|}{d^{s-1}} \mathrm{d}x - \frac{s-1}{|s-1|}\Big(\int_V \frac{|v|}{d^{s-1}}(-\Delta d)\mathrm{d}x + \int_{\partial V} \frac{|v|}{d^{s-1}} \nabla d \cdot \nu \mathrm{d}\mathscr{H}^{n-1}(x)\Big) \geq |s-1|\int_V \frac{|v|}{d^s}\mathrm{d}x.$$

This is inequality (2.7) for $q = 1$. Substituting v by $|v|^q$ with $q > 1$, we arrive at

$$\frac{q}{|s-1|}\int_V \frac{|\nabla v||v|^{q-1}}{d^{s-1}} \mathrm{d}x - \frac{s-1}{|s-1|^2}\Big(\int_V \frac{|v|^q}{d^{s-1}}(-\Delta d)\mathrm{d}x + \int_{\partial V} \frac{|v|^q}{d^{s-1}} \nabla d \cdot \nu \mathrm{d}\mathscr{H}^{n-1}(x)\Big)$$
$$\geq \int_V \frac{|v|^q}{d^s}\mathrm{d}x. \tag{2.8}$$

The first term on the left of (2.8) can be written as follows

$$\frac{q}{|s-1|}\int_V \frac{|\nabla v||v|^{q-1}}{d^{s-1}} \mathrm{d}x = \int_V \Big\{\frac{q}{|s-1|}\frac{|\nabla v|}{d^{s/q-1}}\Big\}\Big\{\frac{|v|^{q-1}}{d^{s-s/q}}\Big\}\mathrm{d}x$$
$$\leq \frac{1}{q}\Big|\frac{q}{s-1}\Big|^q \int_V \frac{|\nabla v|^q}{d^{s-q}}\mathrm{d}x + \frac{q-1}{q}\int_V \frac{|v|^q}{d^s}\mathrm{d}x,$$

by Young's inequality with conjugate exponents q and $q/(q-1)$. Thus (2.8) becomes

$$\frac{1}{q}\Big|\frac{q}{s-1}\Big|^q \int_V \frac{|\nabla v|^q}{d^{s-q}}\mathrm{d}x - \frac{s-1}{|s-1|^2}\Big(\int_V \frac{|v|^q}{d^{s-1}}(-\Delta d)\mathrm{d}x + \int_{\partial V} \frac{|v|^q}{d^{s-1}} \nabla d \cdot \nu \mathrm{d}\mathscr{H}^{n-1}(x)\Big)$$
$$\geq \frac{1}{q}\int_V \frac{|v|^q}{d^s}\mathrm{d}x.$$

Rearranging the constants we arrive at the inequality we sought for. □

Remark 2.10. The choice $V = \Omega$ is acceptable since $v \in C_c^\infty(\Omega)$, and taking also $q = s = p > 1$ in the above lemma, leads to inequalities (2.3)–(2.5) without the first terms on their right hand side (terms involving $|\nabla v|$). In particular we have obtained another proof of the Hardy inequality (1.1).

3. Proof of Theorem A

We will first reformulate Theorem A in the notation introduced above.

Theorem 3.1. *Let Ω be a weakly mean convex domain of finite volume $\mathscr{H}^n(\Omega) < \infty$. Then for $p > n \geq 1$ there exists a positive constant $C(n, p)$ such that*

$$\sup_{x\in\Omega} |u(x)| \leq C(n, p)[\mathscr{H}^n(\Omega)]^{1/n-1/p}\big(I_p[u; \Omega]\big)^{1/p} \quad \textit{for all } u \in C_c^\infty(\Omega).$$

For the proof we will need the following lemma.

Lemma 3.2. *Let Ω be a weakly mean convex domain in $\mathbb{R}^n$ with $\mathscr{H}^n(\Omega) < \infty$. Denote by μ the nonnegative Radon measure in Ω defined by the nonnegative distribution $-\Delta d$. Then*

$$\int_U d \, \mathrm{d}\mu \leq \mathscr{H}^n(\Omega) \quad \textit{for any } U \Subset \Omega.$$

Proof. It is easy to see that $d \in W_0^{1,1}(\Omega)$ (see for example [17, Lemma 1.26]). There exists thus a sequence of nonnegative functions $\{\phi_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that

$$\int_\Omega |\phi_k - d|\mathrm{d}x + \int_\Omega |\nabla\phi_k - \nabla d|\mathrm{d}x \to 0, \quad \text{as } k \to \infty.$$

Since the limit function d is continuous in Ω, we may assume that $\phi_k \to d$ uniformly on compact subsets of Ω. Thus

$$\left|\int_U (\phi_k - d)\mathrm{d}\mu\right| \leq \sup_U |\phi_k - d|\mu(U) \to 0, \quad \text{as } k \to \infty,$$

therefore

$$\int_U d\mathrm{d}\mu = \lim_{k\to\infty} \int_U \phi_k \mathrm{d}\mu. \tag{3.1}$$

Since both μ and ϕ_k are nonnegative we have

$$\int_U \phi_k \mathrm{d}\mu \leq \int_\Omega \phi_k \mathrm{d}\mu. \tag{3.2}$$

Using the fact that $|\nabla d| = 1$ a.e. in Ω we also have

$$\Big|\int_\Omega (\nabla\phi_k - \nabla d)\cdot\nabla d\mathrm{d}x\Big| \le \int_\Omega |\nabla\phi_k - \nabla d|\mathrm{d}x \to 0, \quad \text{as } k\to\infty,$$

and consequently

$$\int_\Omega \nabla\phi_k\cdot\nabla d\ \mathrm{d}x \ \to \mathscr{H}^n(\Omega), \quad \text{as } k\to\infty.$$

We now note that

$$\int_\Omega \phi_k\mathrm{d}\mu = \int_\Omega \phi_k(-\Delta d)\mathrm{d}x = \int_\Omega \nabla\phi_k\cdot\nabla d,$$

from which we get

$$\lim_{k\to\infty}\int_\Omega \phi_k\mathrm{d}\mu = \mathscr{H}^n(\Omega). \tag{3.3}$$

The result now follows from (3.1), (3.2) and (3.3). □

Proof of Theorem 3.1. Assume first that $n\ge 2$. From Lemma 7.14 in [13], we have for all $x\in\Omega$

$$|u(x)| \le \frac{1}{n\omega_n}\int_\Omega \frac{|\nabla u(z)|}{|x-z|^{n-1}}\mathrm{d}z.$$

Setting $u = d^{1-1/p}v$ we arrive at

$$n\omega_n|u(x)| \le \underbrace{\int_\Omega \frac{\big(d(z)\big)^{1-1/p}|\nabla v(z)|}{|x-z|^{n-1}}\mathrm{d}z}_{=:K(x)} + \frac{p-1}{p}\underbrace{\int_\Omega \frac{|v(z)|}{\big(d(z)\big)^{1/p}|x-z|^{n-1}}\mathrm{d}z}_{=:L(x)}. \tag{3.4}$$

Using Hölder's inequality we get

$$K(x) \le \Bigg(\int_\Omega |x-z|^{-(n-1)p'}\mathrm{d}z\Bigg)^{1/p'}\Bigg(\int_\Omega d^{p-1}|\nabla v|^p\mathrm{d}z\Bigg)^{1/p}. \tag{3.5}$$

By an elementary symmetrization argument (see [13, §7.8, eq. (7.31)]) we have

$$\sup_{x\in E}\int_E |x-z|^{-(n-1)s}\mathrm{d}z \le \frac{\omega_n^{s/n'}}{1-s/n'}[\mathscr{H}^n(E)]^{1-s/n'} \quad \text{for all } 0\le s<n',\ E\subset\mathbb{R}^n. \tag{3.6}$$

Applying this for $E = \Omega$ and $s = p' < n'$ (since $p > n$), we get

$$K(x) \le C(n,p)[\mathscr{H}^n(\Omega)]^{1/n-1/p}\left(\int_\Omega d^{p-1}|\nabla v|^p \mathrm{d}z\right)^{1/p}. \tag{3.7}$$

From (3.7) and (2.3) we conclude that

$$K(x) \le C(n,p)[\mathscr{H}^n(\Omega)]^{1/n-1/p}\big(I_p[u;\Omega]\big)^{1/p}. \tag{3.8}$$

We next estimate $L(x)$. Using Hölder's inequality with conjugate exponents $p/(p-1-\varepsilon)$ and $p/(1+\varepsilon)$, where $0 < \varepsilon < (p-n)/p$ is fixed and depending only on n, p, we get

$$L(x) \le \left(\int_\Omega |x-z|^{-(n-1)p/(p-1-\varepsilon)}\mathrm{d}z\right)^{1-(1+\varepsilon)/p}\left(\int_\Omega \frac{|v|^{p/(1+\varepsilon)}}{d^{1/(1+\varepsilon)}}\mathrm{d}z\right)^{(1+\varepsilon)/p}.$$

Using (3.6) with $s = p/(p-1-\varepsilon)$, $E = \Omega$ for the first factor and Lemma 2.9 with $V = \Omega$, $s = 1/(1+\varepsilon)$, $q = p/(1+\varepsilon)$ for the second, we obtain

$$L(x) \le C(n,p)[\mathscr{H}^n(\Omega)]^{1/n-1/p-\varepsilon/p}\left[\left(\frac{p}{\varepsilon}\right)^{p/(1+\varepsilon)}\int_\Omega d^{(p-1)/(1+\varepsilon)}|\nabla v|^{p/(1+\varepsilon)}\mathrm{d}z + \frac{p}{\varepsilon}\int_\Omega d^{\varepsilon/(1+\varepsilon)}|v|^{p/(1+\varepsilon)}(-\Delta d)\mathrm{d}z\right]^{(1+\varepsilon)/p}.$$

Using once more Hölder's inequality with conjugate exponents $(1+\varepsilon)/\varepsilon$ and $1+\varepsilon$ in both terms inside brackets we get

$$\begin{aligned}
L(x) &\le C(n,p)[\mathscr{H}^n(\Omega)]^{1/n-1/p-\varepsilon/p}\Bigg[[\mathscr{H}^n(\Omega)]^{\varepsilon/(1+\varepsilon)}\left(\int_\Omega d^{p-1}|\nabla v|^p\mathrm{d}z\right)^{1/(1+\varepsilon)} \\
&\qquad + \left(\int_{\mathrm{sprt}\{v\}} d\,\mathrm{d}\mu\right)^{\varepsilon/(1+\varepsilon)}\left(\int_\Omega |v|^p(-\Delta d)\mathrm{d}z\right)^{1/(1+\varepsilon)}\Bigg]^{(1+\varepsilon)/p} \\
&\le C(n,p)[\mathscr{H}^n(\Omega)]^{1/n-1/p-\varepsilon/p}\big(I_p[u;\Omega]\big)^{1/p} \\
&\qquad \times \left[[\mathscr{H}^n(\Omega)]^{\varepsilon/(1+\varepsilon)} + \left(\int_{\mathrm{sprt}\{v\}} d\,\mathrm{d}\mu\right)^{\varepsilon/(1+\varepsilon)}\right]^{(1+\varepsilon)/p},
\end{aligned}$$

by (2.3). Using Lemma 3.2 we easily conclude

$$L(x) \le C(n,p)[\mathscr{H}^n(\Omega)]^{1/n-1/p}\big(I_p[u;\Omega]\big)^{1/p}. \tag{3.9}$$

The proof follows inserting (3.9) and (3.8) in (3.4).

For $n = 1$ it suffices to assume $\Omega = (-1, 1)$. The result for an arbitrary finite interval follows then by a translation and a dilation. For any $x \in (-1, 1)$ we have

$$|u(x)| \leq \frac{1}{2} \int_{-1}^{1} |u'| \mathrm{d}t,$$

and setting $u = d^{1-1/p} v$ we arrive at

$$|u(x)| \leq \frac{1}{2} \int_{-1}^{1} d^{1-1/p} |v'| \mathrm{d}t + \frac{p-1}{2p} \int_{-1}^{1} d^{-1/p} |v| \mathrm{d}t.$$

Applying Lemma 2.9 for $s = 1/p$ and $q = 1$ on the second term of the right hand side (recalling Remark 2.5) we obtain

$$\begin{aligned} |u(x)| &\leq \frac{1}{2} \int_{-1}^{1} d^{1-1/p} |v'| \mathrm{d}t + \frac{1}{2} \Big(\int_{-1}^{1} d^{1-1/p} |v'| \mathrm{d}t + 2d(0)^{1-1/p} |v(0)| \Big), \\ &= \int_{-1}^{1} d^{1-1/p} |v'| \mathrm{d}t + |v(0)|. \end{aligned}$$

Because of (2.6), we only need to estimate the first term. For this we use Hölder's inequality and then (2.3) if $p \geq 2$, or Proposition 2.6 if $1 < p < 2$. We omit the details. □

4. Proof of Theorem B

The proof of Theorem B follows by coupling estimates (2.3) and the one provided in the following proposition which is an extension of [9, Theorem 2.5] and [9, Theorem 4.5 with $k = 1$].

Proposition 4.1. *Let either $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniformly C^2 mean convex domain of finite inner radius, or $\Omega = \mathbb{R}^n_+$; $n \geq 2$. For*

$$-1 < b \leq 0, \qquad 1 \leq p < \frac{n}{b+1}, \qquad \text{and} \quad q := \frac{np}{n - p(b+1)}, \tag{4.1}$$

there exists a positive constant C such that

$$C\bigg(\int_{\Omega} \Big(d^{b+1/p'} |v| \Big)^q \mathrm{d}x \bigg)^{p/q} \leq \int_{\Omega} d^{p-1} |\nabla v|^p \mathrm{d}x + \int_{\Omega} |v|^p (-\Delta d) \mathrm{d}x \quad \text{for all } u \in C_c^\infty(\Omega). \tag{4.2}$$

Proof. The proof is a variation of the proof of Theorem 4.5 in [9] (with $k = 1$) to which we refer for more details. The only difference between [9] and here is in the range of the parameter p.

We start with the case where Ω is a uniformly C^2 mean convex domain of finite inner radius. We recall that we first work near the boundary

$$\Omega_\delta := \{x \in \Omega \quad \text{such that} \quad d(x) < \delta\},$$

for $\delta > 0$ sufficiently small but fixed, to obtain an L^1 interpolation estimate. More precisely, under the assumptions

$$\bar{a} \neq 0, \qquad \bar{a} - 1 < \bar{b} \leq \bar{a}, \qquad \bar{q} := \frac{n}{n - (\bar{b} - \bar{a} + 1)}, \tag{4.3}$$

we get (see [9, Lemma 4.3])

$$C\|d^{\bar{b}} v\|_{L^{\bar{q}}(\Omega_\delta)} \leq \int_{\Omega_\delta} d^{\bar{a}} |\nabla v| \mathrm{d}x \quad \text{for all } v \in C_c^\infty(\Omega_\delta). \tag{4.4}$$

Working similarly in $\Omega \setminus \Omega_{\delta/2}$ and noting that $\delta/2 < d < D_\Omega$ there, we get

$$C\|d^{\bar{b}} v\|_{L^{\bar{q}}(\Omega \setminus \Omega_{\delta/2})} \leq \int_{\Omega \setminus \Omega_{\delta/2}} d^{\bar{a}} |\nabla v| \mathrm{d}x \quad \text{for all } v \in C_c^\infty(\Omega \setminus \Omega_{\delta/2}). \tag{4.5}$$

Putting estimates (4.4) and (4.5) together we obtain the existence of a constant $C = C(\bar{a}, \bar{b}, n, \delta/D_\Omega) > 0$ such that (cf. (2.21) of [9]),

$$C\|d^{\bar{b}} v\|_{L^{\bar{q}}(\Omega)} \leq \int_{\Omega} d^{\bar{a}} |\nabla v| \mathrm{d}x + \int_{\Omega_\delta \setminus \Omega_{\delta/2}} d^{\bar{a}-1} |v| \mathrm{d}x \quad \text{for all } v \in C_c^\infty(\Omega). \tag{4.6}$$

We next derive the corresponding $L^p - L^q$ estimates, with b, p and q as in (4.1). To this end we replace v by $|v|^s$ in (4.6) with

$$s = q\frac{p-1}{p} + 1 = \frac{p\big(n - (b+1)\big)}{n - p(b+1)} > 1,$$

to obtain:

$$C\left(\int_{\Omega} d^{\bar{b}\bar{q}} |v|^{\bar{q}s} \mathrm{d}x\right)^{1/\bar{q}} \leq s\int_{\Omega} d^{\bar{a}} |v|^{s-1} |\nabla v| \mathrm{d}x + \int_{\Omega_\delta \setminus \Omega_{\delta/2}} d^{\bar{a}-1} |v|^s \mathrm{d}x. \tag{4.7}$$

We choose $\bar{a}$ such that

$$\bar{a} = \frac{p-1}{p}\Big[\Big(b + \frac{p-1}{p}\Big)q + 1\Big] = \frac{(p-1)(b+1)(n-1)}{n - p(b+1)} > 0,$$

and $\bar{b}$ such that

$$\bar{b}\bar{q} = \Big(b + \frac{p-1}{p}\Big)q.$$

Straightforward calculations show that

$$\bar{b} = \frac{\big(p(b+1)-1\big)\big(n-(b+1)\big)}{n-p(b+1)}.$$

It is easy to check that $\bar{a} - \bar{b} = -b$ and therefore the conditions on $\bar{b}$ imposed by (4.3) are satisfied. Moreover one easily verifies that $\bar{q}s = q$.

In view of the above choices we rewrite (4.7) as

$$C\|d^{b+1/p'}v\|_{L^q(\Omega)}^{1+q/p'} \le s\int_\Omega d^{\bar{a}}|v|^{s-1}|\nabla v|\mathrm{d}x + \int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{\bar{a}-1}|v|^s\mathrm{d}x. \tag{4.8}$$

We next apply Holder inequality in both terms of the right hand side to get

$$\begin{aligned}\int_\Omega d^{\bar{a}}|v|^{s-1}|\nabla v|\mathrm{d}x &= \int_\Omega \Big\{d^{1/p'}|\nabla v|\Big\}\Big\{d^{(b+1/p')q/p'}|v|^{q/p'}\Big\}\mathrm{d}x \\ &\le \|d^{1-1/p}|\nabla v|\|_{L^p(\Omega)}\|d^{b+1/p'}v\|_{L^q(\Omega)}^{q/p'},\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{\bar{a}-1}|v|^s\mathrm{d}x &= \int_{\Omega_\delta\setminus\Omega_{\delta/2}} \Big\{d^{-1/p}|v|\Big\}\Big\{d^{(b+1/p')q/p'}|v|^{q/p'}\Big\}\mathrm{d}x \\ &\le \|d^{-1/p}v\|_{L^p(\Omega_\delta\setminus\Omega_{\delta/2})}\|d^{b+1/p'}v\|_{L^q(\Omega)}^{q/p'}.\end{aligned}$$

Substituting into (4.8) we get after simplifying,

$$C\|d^{b+1/p'}v\|_{L^q(\Omega)}^{p} \le \int_\Omega d^{p-1}|\nabla v|^p\mathrm{d}x + \int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{-1}|v|^p(-\Delta d)\mathrm{d}x. \tag{4.9}$$

To conclude the proof we need to estimate the last term in (4.9). This is done exactly as in [9] (cf. (4.39) of [9]) to finally obtain:

$$C\int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{-1}|v|^p\mathrm{d}x \le \int_\Omega d^{p-1}|\nabla v|^p\mathrm{d}x + \int_\Omega |v|^p(-\Delta d)\mathrm{d}x. \tag{4.10}$$

Combining (4.9) and (4.10) the result follows.

Next we discuss the case where $\Omega = \mathbb{R}^n_+$. In this case the proof is easier due to the fact that it is enough to work in Ω_δ only, since the easy geometry (note that $-\Delta d(x) = 0$ for all $x \in \mathbb{R}^n_+$) allows us to take δ arbitrarily large. Thus one first obtains the L^1 estimate

$$C(\bar{a}, \bar{b}, n)\|d^{\bar{b}} v\|_{L^{\bar{q}}(\mathbb{R}^n_+)} \le \int_{\mathbb{R}^n_+} d^{\bar{a}} |\nabla v| \mathrm{d}x \quad \text{for all } v \in C_c^\infty(\mathbb{R}^n_+). \tag{4.11}$$

We then conclude as before. Estimate (4.10) in not needed in this case. □

Remark 4.2. When $\Omega = \mathbb{R}^n_+$ we have $-\Delta d(x) = 0$ for all $x \in \mathbb{R}^n_+$ and Proposition 4.1 reads:
Let b, p, q be as in (4.1). There exists a positive constant C such that

$$C\left(\int_{\mathbb{R}^n_+} \left(x_n^{b+1/p'} |v|\right)^q \mathrm{d}x\right)^{p/q} \le \int_{\mathbb{R}^n_+} x_n^{p-1} |\nabla v|^p \mathrm{d}x \quad \text{for all } v \in C_c^\infty(\mathbb{R}^n_+). \tag{4.12}$$

This is to be compared with the case where the monomial weight in [5, Theorem 1.3] (see also [24]), degenerates to the distance from the boundary of the half-space. In particular, by the choice $A_i = 0$ for all $i = 1, ..., n-1$ and $A_n = p-1$ in [5], one deduces the following weighted Sobolev inequality

$$C\left(\int_{\mathbb{R}^n_+} x_n^{p-1} |v|^{p(p+n-1)/(n-1)} \mathrm{d}x\right)^{(n-1)/(p+n-1)} \le \int_{\mathbb{R}^n_+} x_n^{p-1} |\nabla v|^p \mathrm{d}x \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n),$$

which for $u \in C_c^\infty(\mathbb{R}^n_+)$ is a special case of (4.12), as one can easily check by taking $b = -(p-1)/(p+n-1)$. Let us mention that the best constant C in the above inequality is obtained in [5].

5. Some calculus lemmas

Here we prove several calculus estimates which under our assumptions on the domain Ω show that we have the correct growth of the local integral of the distance function to the boundary on small negative powers. We start with:

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a uniformly Lipschitz domain and set*

$$Q_r := \frac{\mathscr{H}^{n-1}(B_r \cap \partial\Omega)}{n\omega_n r^{n-1}}.$$

(i) There exist positive constants ρ_0, A_0 such that

$$Q_r \le A_0, \quad \text{for all } r \le \rho_0. \tag{5.1}$$

(ii) If Ω is bounded, then (5.1) holds true for any $r > 0$.

Proof. Let ε be as in Definition 2.3 and suppose $r \le \varepsilon/3$. Let $B_r(z)$ be a ball such that $B_r(z) \cap \partial\Omega \neq \emptyset$. Then taking any point $x \in B_r(z) \cap \partial\Omega$ we have

$$B_r(z) \subset B_{2r}(x) \subset B_\varepsilon(x) \subset U_i,$$

for some U_i as in the definition of the uniformly Lipschitz domain. Then by the monotonicity of $\mathscr{H}^{n-1}$ and Remark 2.2 we obtain

$$\begin{aligned}\mathscr{H}^{n-1}\big(B_r(z) \cap \partial\Omega\big) &\le \mathscr{H}^{n-1}\big(B_{2r}(x) \cap \partial\Omega\big)\\ &\le \int\limits_{|y'|<2r} \sqrt{1+|\nabla f(y')|^2}\mathrm{d}y'\\ &= \sqrt{1+L^2}\omega_{n-1}(2r)^{n-1}.\end{aligned}$$

This shows (i) with $\rho_0 = \varepsilon/3$ and $A_0 = \sqrt{1+L^2}\omega_{n-1}2^{n-1}$.

Suppose now that $r > \varepsilon/3$ and Ω is bounded. We may consider that $\{B_{\varepsilon/3}(x_i)\}_{i=1}^{N_0}$ covers Ω, for suitable $\{x_i\}_{i=1}^{N_0} \in \mathbb{R}^n$, and $N_0 \in \mathbb{N}$. For any $z \in \mathbb{R}^n$ such that $B_r(z) \cap \partial\Omega \neq \emptyset$, we have

$$\begin{aligned}\mathscr{H}^{n-1}\big(B_r(z) \cap \partial\Omega\big) &\le \sum_{i=1}^{N_0} \mathscr{H}^{n-1}\big(B_{\varepsilon/3}(x_i) \cap \partial\Omega\big)\\ &\le N_0\sqrt{1+L^2}\omega_{n-1}2^{n-1}(\varepsilon/3)^{n-1}\\ &\le N_0\sqrt{1+L^2}\omega_{n-1}2^{n-1}r^{n-1},\end{aligned}$$

and the proof of (ii) is complete with $A_0 = N_0\sqrt{1+L^2}\omega_{n-1}2^{n-1}$. □

The next lemma shows that there are also unbounded domains for which (5.1) is true for any $r > 0$.

Lemma 5.2. *If $\Omega \subsetneq \mathbb{R}^n$, $n \ge 2$, is convex then (5.1) holds true for any $r > 0$.*

Proof. Denote first the distance function to $\bar{\Omega}$ by

$$d_c(x) := \inf_{y \in \bar{\Omega}} |x-y|, \quad \text{whenever } x \in \Omega^c,$$

where the exponent c means complement in $\mathbb{R}^n$. It is well known (see [2]) that Ω being convex is equivalent to d_c being convex. Thus $-\Delta d_c(x) \ge 0$ is a nonnegative Radon measure in $\overline{\Omega}^c$. In particular $\nabla d_c \in \mathscr{DM}(\overline{\Omega}^c)$ and the generalized Gauss–Green theorem gives

$$\begin{aligned}0 &\ge -\int\limits_{B_r \cap \bar{\Omega}^c} \Delta d_c \,\mathrm{d}x\\ &= -\int\limits_{B_r \cap \partial\Omega} \nabla d_c \cdot \nu \,\mathrm{d}\mathscr{H}^{n-1}(x) - \int\limits_{\partial B_R \cap \bar{\Omega}^c} \nabla d_c \cdot \nu \,\mathrm{d}\mathscr{H}^{n-1}(x).\end{aligned}$$

Since $\nabla d_c \cdot \nu = -1$ on $B_r \cap \partial\Omega$ we deduce

$$\begin{aligned}\mathscr{H}^{n-1}(B_r \cap \partial\Omega) &\le \int_{\partial B_r \cap \bar{\Omega}^c} \nabla d_c \cdot \nu \, \mathrm{d}\mathscr{H}^{n-1}(x) \\ &\le \mathscr{H}^{n-1}(\partial B_r \cap \bar{\Omega}^c) \\ &\le n\omega_n r^{n-1},\end{aligned}$$

where we have used the fact that $|\nabla d_c \cdot \nu| \le 1$ on $\partial B_r \cap \bar{\Omega}^c$ and also the monotonicity of $\mathscr{H}^{n-1}$. □

Proposition 5.3. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a locally Lipschitz, weakly mean convex domain. For any $\theta < 1$ and any $r > 0$ set*

$$M_r(\theta) := \int_{B_r \cap \Omega} d^{-\theta} \mathrm{d}x.$$

Then if $\theta \in (0, 1)$ we have the relation

$$M_r(\theta) \le \frac{\omega_n}{1-\theta}\big(n + 1 - \theta + nQ_r\big)r^{n-\theta}.$$

Proof. Writing $\{d < r\}$ for the set $\{x \in \Omega : d(x) < r\}$ and $\{d \ge r\}$ for its complement in Ω, we have

$$M_r(\theta) = \int_{B_r \cap \{d \ge r\}} d^{-\theta} \mathrm{d}x + \int_{B_r \cap \{d < r\}} d^{-\theta} \mathrm{d}x. \tag{5.2}$$

For the first integral we have by the monotonicity of $\mathscr{H}^n$

$$\begin{aligned}\int_{B_r \cap \{d \ge r\}} d^{-\theta} \mathrm{d}x &\le r^{-\theta} \mathscr{H}^n\big(B_r \cap \{d \ge r\}\big) \\ &\le \omega_n r^{n-\theta}.\end{aligned} \tag{5.3}$$

For the second integral we note first that since $1 - \theta > 0$, the generalized Gauss–Green theorem gives[1]

$$(1-\theta) \int_{B_r \cap \{d < r\}} d^{-\theta} \mathrm{d}x = \int_{B_r \cap \{d < r\}} d^{1-\theta}(-\Delta d)\mathrm{d}x + \int_{\partial(B_r \cap \{d < r\})} d^{1-\theta} \nabla d \cdot \nu \, \mathrm{d}\mathscr{H}^{n-1}(x). \tag{5.4}$$

Because of condition $(\mathscr{C})$, the first term on the right is estimated as follows

[1] Since $d \in \mathrm{BV}(\Omega)$, its level sets $\{d < r\}$ are of finite perimeter for a.e. $r \in (0, \infty)$ (see [8, Theorem 1, §5.5]).

$$\int_{B_r\cap\{d<r\}} d^{1-\theta}(-\Delta d)\mathrm{d}x \le r^{1-\theta}\int_{B_r\cap\{d<r\}}(-\Delta d)\mathrm{d}x$$
$$= -r^{1-\theta}\int_{\partial(B_r\cap\{d<r\})}\nabla d\cdot\nu\,\mathrm{d}\mathscr{H}^{n-1}(x),$$

where we have used the generalized Gauss–Green theorem once more. Inserting this in (5.4) we arrive at

$$(1-\theta)\int_{B_r\cap\{d<r\}} d^{-\theta}\mathrm{d}x \le \int_{\partial(B_r\cap\{d<r\})}\big(d^{1-\theta}-r^{1-\theta}\big)\nabla d\cdot\nu\,\mathrm{d}\mathscr{H}^{n-1}(x).$$

Since $\nabla d\cdot\nu = -1$ a.e. on $\partial\Omega$ we obtain

$$\begin{aligned}
&(1-\theta)\int_{B_r\cap\{d<r\}} d^{-\theta}\mathrm{d}x \\
&\le \int_{\partial B_r\cap\{d<r\}}\big(r^{1-\theta}-d^{1-\theta}\big)|\nabla d\cdot\nu|\mathrm{d}\mathscr{H}^{n-1}(x) + r^{1-\theta}\mathscr{H}^{n-1}(B_r\cap\partial\Omega)\\
&\le r^{1-\theta}\mathscr{H}^{n-1}\big(\partial B_r\cap\{d<r\}\big) + r^{1-\theta}\mathscr{H}^{n-1}(B_r\cap\partial\Omega)\\
&\le n\omega_n r^{n-\theta} + r^{1-\theta}\mathscr{H}^{n-1}(B_r\cap\partial\Omega), \qquad (5.5)
\end{aligned}$$

where in the last estimate we have used the monotonicity of $\mathscr{H}^{n-1}$. We conclude by coupling (5.3) and (5.5) with (5.2). □

A direct consequence of Lemma 5.1, Lemma 5.2 and Proposition 5.3 is

Corollary 5.4. *Let $\Omega\subset\mathbb{R}^n$, $n\ge 2$, be a uniformly Lipschitz, weakly mean convex domain.*
(i) There exist positive constants ρ_0, A_0 such that

$$M_r(\theta)\le A_0 r^{n-\theta} \quad \textit{for all } \theta\in(0,1) \textit{ and all } r\le\rho_0. \qquad (5.6)$$

(ii) If in addition Ω is bounded or convex, then (5.6) holds true for all $\theta\in(0,1)$ and all $r>0$ (for bounded domains this is to be compared with [15, Lemma 6]).

6. The Hardy–Morrey inequality for $n\ge 2$ & proof of Theorem D

In the first subsection, by imposing an extra assumption (see (6.1) below), we give a considerably shorter proof of the Hardy–Morrey inequality of Theorem C. From the results of the previous section, this extra assumption is satisfied when the domain is weakly mean convex and bounded, or just convex. In the second subsection we give the proof of Theorem C as stated in the introduction. As a byproduct, in the third subsection we obtain the Hardy–John–Nirenberg estimate (Corollary E).

6.1. A weaker version of the Hardy–Morrey inequality

We prove the following version of Theorem C:

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniformly C^2 mean convex domain of finite inner radius. Suppose in addition that Ω is such that for some $\theta_0 \in (0,1)$ there exists a positive constant A_0 so that*

$$M_r(\theta_0) = \int_{B_r \cap \Omega} d^{-\theta_0} \mathrm{d}x \leq A_0 r^{n-\theta_0} \quad \textit{for all } r > 0. \tag{6.1}$$

Then for $p > n$, there exists a positive constant C, such that

$$\sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^{1-n/p}} \leq C\big(I[u;\Omega]\big)^{1/p} \quad \textit{for all } u \in C_c^\infty(\Omega). \tag{6.2}$$

Remark 6.2. Because of Corollary 5.4-(*ii*), the above theorem directly implies Theorem C for convex or, bounded mean convex domains. To obtain it for domains with finite inner radius the proof is more delicate and we present it in the next subsection. The reason is that only Corollary 5.4-(*i*) is available when the domain has finite inner radius, and so in the next subsection we will present a different argument to handle balls with arbitrary large radius.

Remark 6.3. Assumption (6.1) in Theorem 6.1 easily implies that

$$M_r(\theta) \leq 2A_0 r^{n-\theta} \quad \text{for all } \theta \in (0, \theta_0) \text{ and all } r > 0.$$

To prove Theorem 6.1 we need to recall the well known Morrey's "Dirichlet growth" theorem (see [23, Theorem 3.5.2] or [13, Theorem 7.19]).

Theorem 6.4. *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 1$. Let $u \in C_c^\infty(\Omega)$ and suppose that for some $M > 0$ and $\alpha \in (0,1]$ the following estimate is true for all $B_r \subset \mathbb{R}^n$*

$$\int_{B_r} |\nabla u| \mathrm{d}x \leq M r^{n-1+\alpha}. \tag{6.3}$$

Then there exists $c(n,\alpha) > 0$ such that for all $B_r \subset \mathbb{R}^n$

$$\sup_{x,y \in B_r} |u(x) - u(y)| \leq cMr^\alpha,$$

or, equivalently (since u is compactly supported)

$$\sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq cM.$$

Proof of Theorem 6.1. In view of the above theorem it is enough to establish the following estimate

$$\int_{B_r} |\nabla u| \mathrm{d}x \le c\big(I[u;\Omega]\big)^{1/p} r^{n(1-1/p)},$$

for all $r > 0$ and for some positive constant c not depending on r. To this end, let $B_r \subset \mathbb{R}^n$ such that $B_r \cap \Omega \neq \emptyset$. Setting $u = d^{1-1/p} v$ we have

$$\int_{B_r} |\nabla u| \mathrm{d}x \le \underbrace{\int_{B_r} d^{1-1/p} |\nabla v| \mathrm{d}x}_{=:K_r} + \frac{p-1}{p} \underbrace{\int_{B_r} d^{-1/p} |v| \mathrm{d}x}_{=:L_r}.$$

Using first Hölder's inequality and then (2.3) we get

$$\begin{aligned} K_r &\le \Big(\int_{B_r} d^{p-1} |\nabla v|^p \mathrm{d}x\Big)^{1/p} (\omega_n r^n)^{1-1/p} \\ &\le C(n,p)\big(I_p[u;\Omega]\big)^{1/p} r^{n(1-1/p)}. \end{aligned} \tag{6.4}$$

We will next estimate L_r. To this end, we return first in the original function u, and for some $b \in (-1, 0)$ that we will chose later, and $q := np/\big(n - p(b+1)\big)$, we get by Holder's inequality

$$\begin{aligned} L_r &= \int_{B_r} d^b |u| d^{-b-1} \mathrm{d}x \\ &\le \Big(\int_{B_r} \big(d^b |u|\big)^q \mathrm{d}x\Big)^{1/q} \big(M_r(\theta)\big)^{1/q'}; \quad \theta := (b+1)q'. \end{aligned}$$

Taking b sufficiently close to -1, we may assume $\theta \in (0, \theta_0)$, so that $M_r(\theta)$ is bounded by $2A_0 r^{n-\theta}$ (see Remark 6.3). Using also the Hardy–Sobolev inequality of Theorem B, we arrive at

$$L_r \le C(n, p, b, A_0, K)\big(I_p[u;\Omega]\big)^{1/p} r^{(n-\theta)/q'}.$$

This is the desired estimate since

$$\frac{n-\theta}{q'} = n\Big(1 - \frac{1}{p}\Big). \quad \square$$

6.2. Proof of Theorem C

Now we prove Theorem C for domains with finite inner radius.

Theorem 6.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniformly C^2 mean convex domain of finite inner radius. For $p > n$, there exists a positive constant C such that*

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \leq C\big(I_p[u;\Omega]\big)^{1/p} \quad \textit{for all } u \in C_c^\infty(\Omega).$$

Proof. We will use again Dirichlet's growth Theorem 6.4, that is, we will prove

$$\int_{B_r} |\nabla u| \mathrm{d}x \leq c\big(I_p[u;\Omega]\big)^{1/p} r^{n(1-1/p)} \quad \text{for all } r > 0. \tag{6.5}$$

By Corollary 5.4-(i) we have that the inequality

$$M_r(\theta) \leq A_0 r^{n-\theta} \quad \text{for all } \theta \in (0,1) \text{ and all } r \leq \rho_0,$$

holds true for some positive constants ρ_0, A_0. Arguing exactly as in the proof of Theorem 6.1, we see that (6.5) is true for all $r \leq \rho_0$. In the sequel we will prove (6.5) for balls of radius $r > \rho_0$.

Setting $u = d^{1-1/p} v$ we have

$$\int_{B_r} |\nabla u| \mathrm{d}x \leq \underbrace{\int_{B_r} d^{1-1/p} |\nabla v| \mathrm{d}x}_{=:K_r} + \frac{p-1}{p} \underbrace{\int_{B_r} d^{-1/p} |v| \mathrm{d}x}_{=:L_r}.$$

Using first Hölder's inequality and then (2.3) we get

$$\begin{aligned} K_r &\leq \left(\int_{B_r} d^{p-1} |\nabla v|^p \mathrm{d}x \right)^{1/p} (\omega_n r^n)^{1-1/p} \\ &\leq C(n,p)\big(I_p[u;\Omega]\big)^{1/p} r^{n(1-1/p)}. \end{aligned} \tag{6.6}$$

We will next estimate L_r. Using Lemma 2.9 for $V = B_r$, $s = 1/p$ and $q = 1$, we obtain

$$\frac{p-1}{p} L_r \leq K_r + \underbrace{\int_{B_r} d^{1-1/p} |v| (-\Delta d) \mathrm{d}x}_{=:N_r} + \underbrace{\int_{\partial B_r} d^{1-1/p} |v| \nabla d \cdot \nu \, \mathrm{d}\mathscr{H}^{n-1}(x)}_{=:P_r}. \tag{6.7}$$

K_r was estimated in (6.6) so we only need to estimate N_r and P_r. For N_r, since condition ($\mathscr{C}$) holds we may apply Hölder's inequality as follows

$$N_r \le \left(\int\limits_{B_r \cap \Omega} d(-\Delta d) \mathrm{d}x \right)^{1-1/p} \left(\int\limits_{\Omega} |v|^p (-\Delta d) \mathrm{d}x \right)^{1/p}.$$

Using the generalized Gauss–Green theorem in the first integral, and applying (2.3) in the second, we obtain

$$\begin{aligned} N_r &\le c(p) \left(\int\limits_{B_r \cap \Omega} \nabla d \cdot \nabla d \, \mathrm{d}x - \int\limits_{\partial(B_r \cap \Omega)} d \nabla d \cdot \nu \, \mathrm{d}\mathscr{H}^{n-1}(x) \right)^{1-1/p} \big(I_p[u;\Omega]\big)^{1/p} \\ &\le c(p) \left(\mathscr{H}^n(B_r \cap \Omega) + \int\limits_{\partial B_r \cap \Omega} d \, \mathrm{d}\mathscr{H}^{n-1}(x) \right)^{1-1/p} \big(I_p[u;\Omega]\big)^{1/p} \\ &\le c(p) \Big(\mathscr{H}^n(B_r) + \frac{D_\Omega}{\rho_0} r \mathscr{H}^{n-1}\big(\partial B_r \cap \Omega\big)\Big)^{1-1/p} \big(I_p[u;\Omega]\big)^{1/p} \\ &\le c\big(n, p, D_\Omega/\rho_0\big) r^{n(1-1/p)} \big(I_p[u;\Omega]\big)^{1/p}, \end{aligned} \tag{6.8}$$

where in the last two inequalities we have used first the fact that

$$d(x) \le \frac{D_\Omega}{\rho_0} r \quad \text{for all } x \in \Omega, \tag{6.9}$$

and then the monotonicity of $\mathscr{H}^{n-1}$. We finally estimate P_r. By elementary considerations and Hölder's inequality

$$\begin{aligned} P_r &\le \int\limits_{\partial B_r} d^{1-1/p} |v| \mathrm{d}\mathscr{H}^{n-1}(x) \\ &\le \left(\int\limits_{\partial B_r \cap \Omega} d X^{-1/(p-1)}(d/D_\Omega) \mathrm{d}\mathscr{H}^{n-1}(x) \right)^{1-1/p} \left(\int\limits_{\partial B_r} |v|^p X(d/D_\Omega) \mathrm{d}\mathscr{H}^{n-1}(x) \right)^{1/p}, \end{aligned} \tag{6.10}$$

where $X(t) = (1 - \log t)^{-1}$; $t \in (0, 1]$. The function $t X^{-1/(p-1)}(t/D_\Omega)$ is increasing in $[0, r]$, for any $r \in (0, D_\Omega]$. We may thus estimate the first factor by

$$r^{1-1/p} X^{-1/p}\big(r/D_\Omega\big)\big[\mathscr{H}^{n-1}(\partial B_r \cap \Omega)\big]^{1-1/p} \le C\big(n, p, D_\Omega/\rho_0\big) r^{n(1-1/p)},$$

where we have used the monotonicity of $\mathscr{H}^{n-1}$, the fact that $X^{-1/p}\big(r/D_\Omega\big) \le X^{-1/p}\big(\rho_0/D_\Omega\big)$ for $r \ge \rho_0$ (note that $X^{-1/p}(t)$ is decreasing in $(0, 1]$), and also the elementary inequality $X^{-1/p}\big(\rho_0/D_\Omega\big) \le \big(\rho_0/D_\Omega\big)^{-1/p}$. To estimate the second factor we notice first that the function

$$\frac{x - \xi_x}{r}; \; x \in \bar{B}_r,$$

when restricted to ∂B_r gives the unit outer normal $\nu_{\partial B_r}$ to ∂B_r. So we use the Gauss–Green theorem as follows

$$\begin{aligned}
&\int_{\partial B_r} |v|^p X(d/D_\Omega) \mathrm{d}\mathscr{H}^{n-1}(x) \\
&= \int_{\partial B_r} |v|^p X(d/D_\Omega) \nu_{\partial B_r} \cdot \nu_{\partial B_r} \mathrm{d}\mathscr{H}^{n-1}(x) \\
&= \int_{B_r} \operatorname{div}\left\{|v|^p X(d/D_\Omega) \frac{x-\xi_x}{r}\right\} \mathrm{d}x \\
&\leq p \int_\Omega |v|^{p-1} |\nabla v| X(d/D_\Omega) \mathrm{d}x + \int_\Omega \frac{|v|^p}{d} X^2(d/D_\Omega) \mathrm{d}x + \frac{n}{r} \int_{B_r} |v|^p X(d/D_\Omega) \mathrm{d}x,
\end{aligned}$$

where we have used the fact that $|(x-\xi_x)/r| \leq 1$ for all $x \in B_r$. By Proposition 2.8 and Theorem 2.7, the first two summands are bounded by $c(p) I_p[u; \Omega]$. To estimate the last summand, again by the elementary inequality $X(t) \geq t$ for all $t \in (0, 1]$, we get $dX^{-1}(d/D_\Omega) \leq D_\Omega$ for all $x \in \Omega$. Since $r > \rho_0$ we deduce

$$\begin{aligned}
\frac{n}{r} \int_{B_r} |v|^p X(d/D_\Omega) \mathrm{d}x &\leq n \frac{D_\Omega}{\rho_0} \int_{B_r} \frac{|v|^p}{d} X^2(d/D_\Omega) \mathrm{d}x \\
&\leq C\big(n, p, D_\Omega/\rho_0\big) I_p[u; \Omega],
\end{aligned}$$

by Theorem 2.7. The above estimates when inserted to (6.10) give

$$P_r \leq C\big(n, p, D_\Omega/\rho_0\big) r^{n(1-1/p)} \big(I_p[u; \Omega]\big)^{1/p}. \tag{6.11}$$

In turn, estimates (6.6), (6.8) and (6.11), give

$$L_r \leq C\big(n, p, D_\Omega/\rho_0\big) r^{n(1-1/p)} \big(I_p[u; \Omega]\big)^{1/p},$$

for $r > \rho_0$. This together with (6.6) implies (6.5) for balls of radius $r > \rho_0$. This completes the proof of the theorem. □

6.3. *A Hardy–John–Nirenberg inequality*

We now proceed in the proof of Theorem D. Recall the seminorm

$$\|u\|_{BMO} = \sup_B \frac{1}{\mathscr{H}^n(B)} \int_B |u - u_B| \mathrm{d}x,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and u_B is the average of u in the ball B, that is

$$u_B = \frac{1}{\mathscr{H}^n(B)} \int_B u \, \mathrm{d}x.$$

Theorem 6.6. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniformly C^2 mean convex domain of finite inner radius. There exists a positive constant C such that*

$$\|u\|_{BMO} \leq C\big(I_n[u;\Omega]\big)^{1/n} \quad \textit{for all } u \in C_c^\infty(\Omega).$$

Proof. Using first the L^1-Poincaré inequality (see [8, Theorem 2, §4.5]) and then the substitution $u = d^{1-1/n}v$, we have that

$$\begin{aligned}\int_{B_r} |u - u_{B_r}| \mathrm{d}y &\leq C(n) r \int_{B_r} |\nabla u| \mathrm{d}y \\ &\leq C(n) r \underbrace{\int_{B_r} d^{1-1/n} |\nabla v| \mathrm{d}y}_{=:\mathscr{K}_r} + C(n) r \underbrace{\int_{B_r} d^{-1/n} |v| \mathrm{d}y}_{=:\mathscr{L}_r}.\end{aligned}$$

The result will follow once we establish the following estimates

$$\mathscr{K}_r,\ \mathscr{L}_r \leq c\big(I_n[u;\Omega]\big)^{1/n} r^{n-1} \quad \text{for all } r > 0.$$

These estimates are proved in exactly the same way as in estimating K_r and L_r in the proof of Theorem 6.1 for $r \leq \rho_0$ and in the proof of Theorem 6.5 for $r > \rho_0$. We note that one can take $p = n$ in the proofs of these theorems without any change. We omit further details. □

As a consequence the above Theorem and (1.11) we obtain Corollary E of the introduction.

7. The Hardy–Morrey inequality for $n = 1$

As mentioned in the introduction, the one dimensional case of Theorem C is close to the case where the distance in the Hardy inequality is taken from a point. In this case a logarithmic correction is needed, see [26]. Following the ideas there, we prove the following sharp substitute of Theorem C when $n = 1$.

Theorem 7.1. *There exist constants $\lambda = \lambda(p) \geq 1$ and $c = c(p) > 0$ such that for all $u \in C_c^\infty(\alpha, \beta)$*

$$\sup_{\substack{x,y \in (\alpha,\beta) \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{1-1/p}} X^{1/p}\Big(\frac{|x-y|}{\lambda D}\Big) \right\} \leq c(p)\big(I_p[u;(\alpha,\beta)]\big)^{1/p}, \tag{7.1}$$

where $X(t) := (1 - \log t)^{-1}$, $t \in (0, 1]$ and $D = (\beta - \alpha)/2$. The exponent $1/p$ on X cannot be decreased.

For the proof it suffices to restrict ourselves to the case $\alpha = -1$ and $\beta = 1$ (note then that $D = 1$).

Lemma 7.2. *Let $q > 1$, $\beta > 1 - q$. There exists a constant $c = c(q,\beta) > 0$, such that for any absolutely continuous function v in $(-1, 1)$, and any $\lambda \geq 1$*

$$\sup_{x\in(-1,1)} \Big\{|v(x)|X^{(\beta+q-1)/q}\big(d(x)/\lambda\big)\Big\} \leq c\Bigg(\int_{-1}^{1} d^{q-1}|v'|^q X^{\beta}\big(d/\lambda\big)\mathrm{d}t + |v(0)|^q\Bigg)^{1/q}.$$

Proof. Letting $\lambda \geq 1$, we have for any $x \in (-1, 1)$

$$\begin{aligned} v(x) &= \int_0^x v'\mathrm{d}t + v(0) \\ &\leq \left|\int_0^x d^{-1}X^{-\beta/(q-1)}\big(d/\lambda\big)\mathrm{d}t\right|^{1/q'}\Bigg(\int_{-1}^{1} d^{q-1}|v'|^q X^{\beta}\big(d/\lambda\big)\mathrm{d}t\Bigg)^{1/q} + |v(0)|, \end{aligned} \tag{7.2}$$

by Hölder's inequality. For any $x \in (-1, 1)$ we compute

$$\left|\int_0^x d^{-1}X^{-\beta/(q-1)}\big(d/\lambda\big)\mathrm{d}t\right| = \frac{1}{\theta}\Big[X^{-\theta}\big(d(x)/\lambda\big) - X^{-\theta}\big(1/\lambda\big)\Big]; \quad \theta := \frac{\beta+q-1}{q-1} > 0.$$

Inserting this in (7.2) we arrive at

$$\begin{aligned} |v(x)|X^{\theta/q'}\big(d(x)/\lambda\big) &\leq c\Big[1 - \Big(\frac{X(d(x)/\lambda)}{X(1/\lambda)}\Big)^{\theta}\Big]^{1/q'}\Bigg(\int_{-1}^{1} d^{q-1}|v'|^q X^{\beta}\big(d/\lambda\big)\mathrm{d}t\Bigg)^{1/q} + |v(0)| \\ &\leq c\Bigg(\int_{-1}^{1} d^{q-1}|v'|^q X^{\beta}\big(d/\lambda\big)\mathrm{d}t\Bigg)^{1/q} + |v(0)|, \end{aligned}$$

where $c = c(q,\beta) = \theta^{-1/q'}$, and we have used twice the fact that $0 < X(t) \leq 1$ for all $t \in (0, 1]$. The desired inequality follows using $a^{1/q} + b^{1/q} \leq 2^{1-1/q}(a+b)^{1/q}$, for all $q > 1$ and $a, b \geq 0$. $\square$

Proposition 7.3. *Let $p > 1$. There exists a constant $c = c(p) > 0$, such that for any $\lambda \geq 1$*

$$\sup_{x\in(-1,1)} \Big\{\frac{|u(x)|}{(d(x))^{1-1/p}}X^{1/p}\big(d(x)/\lambda\big)\Big\} \leq c\big(I_p[u; (-1,1)]\big)^{1/p} \quad \textit{for all } u \in C_c^{\infty}(-1,1). \tag{7.3}$$

Proof. Let $u \in C_c^\infty(-1,1)$ and define v by $u(x) = \big(d(x)\big)^{1-1/p} v(x)$. If $1 < p < 2$, by Lemma 7.2 for $q = p$ and $\beta = 2-p$, we have that for any $\lambda \ge 1$

$$|v(x)| X^{1/p}\big(d(x)/\lambda\big) \le c\left(\int_{-1}^{1} d^{p-1}|v'|^p X^{2-p}\big(d/\lambda\big)dt + |v(0)|^p\right)^{1/p}.$$

The result follows by Proposition 2.6. If $p \ge 2$, by Lemma 7.2 for $q = 2$ and $\beta = 0$ we have

$$|w(x)| X^{1/2}\big(d(x)/\lambda\big) \le c\left(\int_{-1}^{1} d|w'|^2 dt + |w(0)|^2\right)^{1/2},$$

for an absolutely continuous function w in $(-1,1)$ and any $\lambda \ge 1$. For $w(x) = |v(x)|^{p/2}$ we obtain

$$|v(x)| X^{1/p}\big(d(x)/\lambda\big) \le c\left(\int_{-1}^{1} d|v|^{p-2}|v'|^2 dt + |v(0)|^p\right)^{1/p}.$$

The result follows by (2.4). □

Proof of Theorem 7.1. For $-1 < y < x < 1$ we have

$$\begin{aligned} |u(x) - u(y)| &= \left|\int_y^x u' \mathrm{d}t\right| \\ (\text{setting } u(t) = (d(t))^{1-1/p} v(t)) &\le \underbrace{\int_y^x d^{1-1/p}|v'|\mathrm{d}t}_{=:\mathrm{K}(x,y)} + \frac{p-1}{p}\underbrace{\int_y^x d^{-1/p}|v|\mathrm{d}t}_{=:\Lambda(x,y)}. \end{aligned} \tag{7.4}$$

To estimate $\mathrm{K}(x,y)$ we use Hölder's inequality to get

$$\begin{aligned} \mathrm{K}(x,y) &\le (x-y)^{1-1/p}\left(\int_{-1}^{1} d^{p-1}|v'|^p \mathrm{d}t\right)^{1/p} \\ (\text{by (2.3)}) &\le c(p)(x-y)^{1-1/p}\big(I_p[u;(-1,1)]\big)^{1/p}. \end{aligned} \tag{7.5}$$

To estimate $\Lambda(x,y)$ we return to the original variable by $v(t) = \big(d(t)\big)^{1/p-1} u(t)$. Thus

$$\Lambda(x,y) = \int_y^x \frac{|u|}{d}\mathrm{d}t.$$

Inserting (7.3) in $\Lambda(x,y)$ we obtain

$$\Lambda(x,y) \le c(p)\big(I_p[u;(-1,1)]\big)^{1/p} \int_y^x d^{-1/p} X^{-1/p}\big(d/\lambda\big)\mathrm{d}t$$
$$\le c(p)(x-y)^{1-1/p} X^{-1/p}((x-y)/\lambda)\big(I_p[u;(-1,1)]\big)^{1/p}, \tag{7.6}$$

for some $\lambda \ge 1$, by virtue of Lemma 2.5 of [26]. Coupling (7.5) and (7.6) with (7.4) we obtain the desired estimate.

To prove that the exponent $1/p$ cannot be decreased, we will follow the argument introduced in [27]: Assuming that the exponent can be decreased we will violate the optimal homogeneously improved Hardy inequality (obtained in [4, Lemma A.2] for the case $p=2$ and in [3] in the general case). To this end, let $\epsilon \in (0,1]$, $c>0$ and $\lambda \ge 1$, be such that

$$\sup_{\substack{x,y\in(-1,1)\\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^{1-1/p}} X^{(1-\epsilon)/p}\Big(\frac{|x-y|}{\lambda}\Big)\right\} \le c\big(I_p[u;(-1,1)]\big)^{1/p} \quad \text{for all } u \in C_c^\infty(-1,1).$$

Restricting on functions $u \in C_c^\infty(0,1)$ and taking $y=0$, we obtain

$$\sup_{x\in(0,1)} \frac{|u(x)|}{x^{1-1/p}} X^{(1-\epsilon)/p}\Big(\frac{x}{\lambda}\Big) \le c\big(J_p[u;(0,1)]\big)^{1/p} \quad \text{for all } u \in C_c^\infty(0,1),$$

where $J_p[u;(0,1)] := \int_0^1 |u'|^p \mathrm{d}t - (1-1/p)^p \int_0^1 |u|^p/t^p \mathrm{d}t$. This readily implies that

$$\int_0^1 \frac{|u(t)|^p}{t^p} X^{2-\epsilon/2}\Big(\frac{t}{\lambda}\Big)\mathrm{d}t \le c J_p[u;(0,1)] \int_0^1 t^{-1} X^{1+\epsilon/2}\Big(\frac{t}{\lambda}\Big)\mathrm{d}t \quad \text{for all } u \in C_c^\infty(0,1). \tag{7.7}$$

Clearly, since $\epsilon > 0$ the integral on the right is a finite constant depending only on ϵ and λ. Thus we have violated the optimality of the exponent 2 on the remainder term of the one dimensional case of the improved Hardy inequality of [3, Theorem A] (for $k=n=1$ there). □

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