JOURNAL OF DIFFERENTIAL EQUATIONS 116, 119–148 (1995)

Modulation Theory for the Blowup of Vector-Valued Nonlinear Heat Equations

STATHIS FILIPPAS

Université Paris VI, 75230 Paris Cedex, France

AND

FRANK MERLE

Université Cergy Pontoise and Université Paris VI, 75230 Paris Cedex, France

Received July 20, 1992

This paper is concerned with the blowup of solutions of the nonlinear vector-valued heat equation

$$U_t - \Delta U = |U|^{p-1} U, \qquad U(0) = U_0,$$

where $U(x, t) = (u_1(x, t), ..., u_m(x, t))$ is a vector-valued function from $R^n \times (0, T)$ to R^m and $1 < p < (3n+8)/(3n-4)$. Working with the equation in similarity variables, and using modulation theory and ideas from center manifold theory, we obtain the asymptotic behavior of U in a backward space–time parabola near any blowup point. © 1995 Academic Press, Inc.

1. INTRODUCTION

This paper is concerned with the blowup of solutions of the nonlinear vector-valued heat equation

$$U_t - \Delta U = |U|^{p-1} U, \qquad U(0) = U_0, \tag{1.1}$$

where $U(x, t) = (u_1(x, t), ..., u_m(x, t))$ is a vector-valued function from $R^n \times (0, T)$ to R^m, and $p > 1$. This equation has some physical interest in ferromagnetism. Working with the equation in similarity variables, and using modulation theory and ideas from center manifold theory, we obtain the asymptotic behavior of U in a backward space–time parabola near any blowup point.

0022-0396/95 $6.00

Classical theory yields the existence of a regular solution in $L^\infty(R^n)$ up to a time T, which is finite in the case where the initial data satisfy

$$E(U_0)=\frac{1}{2}\int_{R^n}|\nabla U_0|^2\,dx-\frac{1}{p+1}\int_{R^n}|U_0|^{p+1}\,dx<0;$$

see, e.g., [15].

In this case, where $T<+\infty$, we say that the solution blow up at time $t=T$. We say that b is a blowup point if there exist sequences $\{x_n\}$, $\{t_n\}$ such that $\lim_{n\to\infty} x_n=b$, $\lim_{n\to\infty} t_n=T$, and $\lim_{n\to\infty}|U(x_n,t_n)|=\infty$. If x_0 is not a blowup point then local regularity theory yields the existence of a strong limit of $U(x,t)$ as $t\to T$ for x near x_0. If x_0 is a blowup point we show that $U(x,t)$ develops a singularity in an asymptotically self-similar manner.

Much attention has recently been focussed on the real-valued version of (1.1), that is the case where $m=1$,

$$u_t-\Delta u=|u|^{p-1}u, \qquad u(0)=u_0, \tag{1.2}$$

and related equations. We refer to [1, 6, 7, 9–11] for discussions and bibliographies. We also refer to [2, 4, 5, 8, 16, 12–14, 19, 20] for the most recent results concerned with Eq. (1.2).

One particularly efficient method for studying the local properties of the blowing-up solutions of (1.2) is the method developed by Giga and Kohn [9–11] based on similarity variables. This change of both dependent and independent variables is defined by

$$\begin{aligned} w(y,s)&=(T-t)^{1/p-1}u(x,t),\\ y=(x-b)/\sqrt{T-t}, &\qquad s=-\ln(T-t), \end{aligned} \tag{1.3}$$

where b is a blowup point and T is the blowup time. If u solves (1.1) then w exists for all positive times s and solves

$$w_s=\frac{1}{\rho}\nabla(\rho\nabla w)-\frac{1}{p-1}w+|w|^{p-1}w, \tag{1.4}$$

where $\rho=\rho(y)=e^{-|y|^2/4}$. Studying the behavior of u near blowup is equivalent to studying the large time behavior of w. It follows from [9–11] that in the case where $n=1$, or $1<p<(3n+8)/(3n-4)$

$$w(y,s)\to\pm\kappa, \qquad \text{as}\quad s\to\infty, \tag{1.5}$$

uniformly on bounded sets $|y|<C$, where κ is the constant nonzero stationary solution of (1.4), i.e., $|\kappa|=(p-1)^{-1/(p-1)}$. We note that (1.5) has also been established in the case where $u\geqslant 0$ and p is subcritical, that is, $n\geqslant 2$ or $1<p<(n+2)/(n-2)$.

The way w approaches its limit in (1.5) is by now well understood (cf. [4, 5, 12, 19]). We briefly recall a few facts from [4, 5]. Assuming that w (or equivalently u) is nonnegative, we linearize equation (1.4) about $w=\kappa$. Thus, by setting

$$v(y, s)=w(y, s)-\kappa, \tag{1.6}$$

we obtain that v solves the equation

$$v_s=\frac{1}{\rho}\nabla(\rho\nabla v)+v+f(v), \tag{1.7}$$

with

$$f(v)=\frac{p}{2\kappa}v^2+g(v), \qquad |g(v)|\leqslant c|v|^3,$$

and $v\to 0$ uniformly on compact sets in y.

Let us denote by L^2_ρ the space of functions $v(y)$ for which $\int v^2\rho<+\infty$ where $\rho=e^{-|y|^2/4}$ as usual. This is easily seen to be a Hilbert space with inner product $\langle u, v\rangle=\int uv\rho$. We also denote by $\|v\|$ the L^2_ρ-norm of v, i.e., $\|v\|=(\int v^2\rho)^{1/2}$. In the case where $U=(u_1, ..., u_m)$, $V=(v_1, ..., v_m)$ are vector-valued functions, we set $\langle U, V\rangle=\sum_{i=1}^m\int u_iv_i\rho$ and $\|V\|=(\sum_{i=1}^m\int u_i^2\rho)^{1/2}$.

Then, it follows from [4, 5] (or [12, 19]) that $\|v\|\leqslant c/s$, for some positive constant c, and that $s\,|v(y, s)|$ is uniformly bounded on compact sets $|y|<C$.

Regarding the vector-valued case, however, less is known. After introducing similarity variables

$$\begin{aligned} W(y, s)&=(T-t)^{1/p-1}\,U(x, t),\\ y=(x-b)/\sqrt{T-t}, &\qquad s=-\ln(T-t), \end{aligned} \tag{1.8}$$

we have that $W(y, s)$ is a vector-valued function from $(-\ln T, \infty)\times R^n$ to R^m. Moreover it has been shown in [10] that if $n=1$ or $1<p<(3n+8)/(3n-4)$, then $W(y, s)$ stays uniformly bounded in space–time. The bounded nonzero stationary solutions of (1.4) are now the points $W(x)\equiv W_0$ where

$$|W_0|=\kappa=(p-1)^{-1/(p-1)}. \tag{1.9}$$

Thus, in contrast with the scalar case, where the bounded nonzero stationary solution of (1.4) is two distinct points ($\pm\kappa$), we now have a whole continuum of stationary points, namely an $(m-1)$-dimensional sphere of radius κ; we denote it by S^{m-1}_κ.

Due to the fact that the vector-valued case equation (1.4) still retains its scaling properties as well as its gradient structure, many of the arguments employed in [9–11] are still valid. In particular, it follows from the work of Giga and Kohn that

$$\inf_{W_0 \in S_\kappa^{m-1}} |W(y, s) - W_0| \to 0 \qquad \text{as} \quad s \to \infty, \tag{1.10}$$

uniformly for $|y| < C$, and

$$\operatorname{dist}(W(\cdot, s), S_\kappa^{m-1}) \equiv \inf_{W_0 \in S_\kappa^{m-1}} \int_{R^n} |(W(\cdot, s) - W_0|^2 \rho \to 0 \qquad \text{as} \quad s \to \infty. \tag{1.11}$$

It is to be remarked, however, that the above information does not furnish the complete analogue of (1.5). A natural question left open is whether the trajectory $W(y, s)$ approaches a specific point W_∞ on the manifold S_κ^{m-1}; if not, although it approaches S_κ^{m-1} with time, it has no limiting value.

For systems of the type

$$iU_t - \Delta U = |U|^{p-1} U, \qquad U(0) = U_0, \tag{1.12}$$

with $U(x, t) = u_1(x, t) + iu_2(x, t)$ (that is $m = 2$), after suitable rescaling, it has been observed in some cases that the asymptotic shape of the rescaled function is a periodic nonconstant solution of the type $e^{i\omega s}Q(y)$ (cf. [16]). It is of interest to know whether the coupling in a system (1.1) is weak enough so as to expect a scalar behavior as $s \to \infty$.

For parabolic systems similar to (1.1) under the assumption that the stationary solutions of the corresponding W-equation are isolated points, it has been proved that the trajectory $W(y, s)$ "chooses" one of these points as its limiting value (see [9, Theorem 5]). It is the purpose of the present work to show that this behavior stays the same, even if the stationary points form a continuum. More precisely, working with system (1.1) we will show

THEOREM A. *Let* $W(y, s) \in (s_0, \infty) \times R^n \to R^m$ *be a bounded solution of* (1.4) *for which we know that* $\operatorname{dist}(W(\cdot, s), S_\kappa^{m-1}) \to 0$ *and* $\inf_{W_0 \in S_\kappa^{m-1}} |W(y, s) - W_0| \to 0$ *as* $s \to \infty$, *uniformly for* $|y| < C$. *Then there exists a* $W_\infty \in S_\kappa^{m-1}$ *such that*

$$W(y, s) \to W_\infty, \qquad \text{as} \quad s \to \infty, \tag{1.13}$$

uniformly for $|y| < C$, *and*

$$\| W(\cdot, s) - W_\infty \| \to 0 \qquad \text{as} \quad s \to \infty. \tag{1.14}$$

As a consequence of the above theorem we also have

THEOREM B. *We assume that $n = 1$ or $1 < p < (3n+8)/(3n-4)$. Let b be a blowup point of* (1.1). *Then there exists a constant function $W_\infty \in R^m$ such that $|W_\infty| = \kappa$ and*

$$\lim_{t \uparrow T} (T-t)^{1/(p-1)} U(b + y(T-t)^{1/2}, t) = W_\infty, \tag{1.15}$$

uniformly for $|y| < C$, for any $C > 0$.

The assumptions of Theorem B on n and p guarantee that the assumptions of Theorem A are valid (cf. [10]). Once the assumptions of Theorem A hold true, (1.15) is nothing but a restatement of (1.13) in the original variables.

Remark 1. Since equation (1.1) is rotation invariant we also have that if (b, T) is a blowup point for (1.1) then given any point $W_\infty \in R^m$ such that $|W_\infty| = \kappa$ there are initial data for which (1.15) holds.

Remark 2. In proving Theorem A we will show in fact that $\| W(\cdot, s) - W_\infty \| \leqslant c/s$ and that $s\,|W(y, s) - W_\infty|$ is uniformly bounded on compact sets in y. Thus, the situation is quite similar to that of the scalar case.

Finally we remark that the same results can be shown in the case of the equation

$$U_t = \Delta U + F(|U|^2)\, U, \tag{1.16}$$

where $F(|U|^2) \simeq |U|^{p-1}$ in a suitable topology as $|U| \to +\infty$.

We close this section by discussing the way the paper is organized. In order to simplify the presentation in most of the work we concentrate in the case $m = 2$. More precisely Sections 2–4 are devoted to the study of the case $m = 2$, whereas the appropriate modifications required for the study of the general case are presented in the last section. In Section 2 we describe the way we formulate the problem by introducing a suitable parameter. In Section 3 we present a formal argument by which the desired results are obtained, while avoiding all technical difficulties. In Section 4 we give the rigorous proofs for the case $m = 2$. The general case $m \geqslant 2$ is then discussed in Section 5.

2. FORMULATION OF THE PROBLEM

In this section we will explain the approach we will follow in proving Theorem A. In order to simplify the presentation—while keeping all the essential points—we will assume throughout this section as well as Sections

3 and 4 that $m=2$. We note that in this case $U=(u_1, u_2)$ can also be thought of as a complex-valued function $U(x, t)=u_1(x, t)+iu_2(x, t)$. The appropriate modifications required for the general case $m \geqslant 2$ will then be presented in Section 5.

It is convenient (but not necessary) to use complex notation. Thus, for $m=2$ we have that $S^1_\kappa = \{\kappa_0 e^{i\theta}\}_{\theta \in S^1}$, where $\kappa_0=(p-1)^{-1/(p-1)}$, and S^1 is the 1-dimensional unit sphere. We recall that after introducing similarity variables, the equation satisfied by $W(y, s)=w_1(y, s)+iw_2(y, s)$ is

$$W_s = \frac{1}{\rho}\nabla(\rho \nabla W) - \frac{1}{p-1} W + |W|^{p-1} W. \tag{2.1}$$

A key point in our analysis is the following parametrization of the problem

$$W(y, s) = e^{i\theta(s)}(V(y, s)+\kappa_0), \qquad \kappa_0 = (p-1)^{-1/(p-1)}, \tag{2.2}$$

where $\theta=\theta(s)$ is defined for each time s by

$$\int |W(y, s) - e^{i\theta(s)}\kappa_0|^2 \rho = \min_{\theta \in S^1} \int |W(y, s) - e^{i\theta}\kappa_0|^2 \rho, \tag{2.3}$$

where $|\cdot|$ represents the complex modulus. Since S^1 is a compact set such a θ always exists. We still have to discuss the uniqueness and regularity of the parameter $\theta(s)$ just introduced. It follows from (1.11), (1.12), and (2.2) that

$$|V(y, s)| \to 0, \qquad \text{as} \quad s \to \infty, \tag{2.4}$$

uniformly for $|y| < C$, and

$$\|V\| = \|W - e^{i\theta(s)}\kappa_0\| \to 0, \qquad \text{as} \quad s \to \infty. \tag{2.5}$$

Moreover, an easy calculation gives

$$\|V\|^2 = \|W - e^{i\theta(s)}\kappa_0\|^2$$
$$= \int (|W|^2 + \kappa_0^2)\rho - 2\kappa_0\left(\cos\theta \int w_1 \rho + \sin\theta \int w_2 \rho\right). \tag{2.6}$$

We are now ready to prove

PROPOSITION 2.1. *The parameter $\theta(s)$ introduced in* (2.3) *is unique and regular for $s \geqslant s_0$, where s_0 is such that $\|V(\cdot, s)\|^2 \leqslant (\kappa_0^2/2)\int \rho$, for $s \geqslant s_0$.*

Proof. By computing the first variation of the functional in (2.3) we have that

$$\langle ie^{i\theta}, W\rangle = 0, \tag{2.7}$$

or equivalently,

$$\sin\theta \int w_1\rho - \cos\theta \int w_2\rho = 0. \tag{2.8}$$

We claim that for s large enough (2.8) always defines two values of θ (mod 2π) which differ from each other by π. Indeed, if $\int w_i\rho \neq 0$ for $i = 1, 2$ then one can solve (2.8) up to a π. The possibility that $\int w_1\rho = \int w_2\rho = 0$ is ruled out, for $s \geqslant s_0$ because in such a case the last term in (2.6) would be zero. But s_0 is chosen in such a way that the last ter in (2.6) is never zero for $s \geqslant s_0$.

The existence of two different values of $\theta(s)$ is of course explained by the fact that the functional in (2.2) attains not only a minimum but also a maximum. In order to pick up the right value of $\theta(s)$ we demand the last term in (2.6) to be negative since otherwise (2.5) would not hold. Thus, $\theta(s)$ is uniquely defined for $s \geqslant s_0$. Finally, by the implicit function theorem we conclude that it is also regular. ∎

We now turn our attention to the study of the function $V(y, s)$ defined by (2.2). It follows from (2.2) and (2.7) that for $s \geqslant s_0$

$$\langle i, V\rangle = 0, \tag{2.9}$$

that is

$$\int v_2(\cdot, s)\,\rho = 0. \tag{2.10}$$

Using (2.1) we see that the equation satisfied by V is

$$i\theta_s(s)(V+\kappa_0) + V_s = \frac{1}{\rho}\nabla(\rho\nabla v) - \frac{1}{p-1}(V+\kappa_0) + |V+\kappa_0|^{p-1}(V+\kappa_0). \tag{2.11}$$

We next compute the Taylor expansion of the last term in (2.11). An easy calculation shows that for all y and s we have

$$\begin{aligned} |V+\kappa_0|^{p-1} &= ((v_1+\kappa_0)^2 + v_2^2)^{(p-1)/2} \\ &= \kappa_0^{p-1} + \frac{1}{\kappa_0}v_1 + \frac{p-2}{2\kappa_0^2}v_1^2 + \frac{1}{2\kappa_0^2}v_2^2 + g(V), \end{aligned} \tag{2.12}$$

with $|g(V)| < C|V|^3$, for some constant C depending only on p. Thus, we can rewrite (2.11) as

$$v_{1s} = \frac{1}{\rho}\nabla(\rho\nabla v_1) + v_1 + \frac{p}{2\kappa_0} v_1^2 + \frac{1}{2\kappa_0} v_2^2 + \theta_s v_2 + g_1(V), \tag{2.13}$$

$$v_{2s} = \frac{1}{\rho}\nabla(\rho\nabla v_2) + \frac{1}{\kappa_0} v_1 v_2 - \theta_s(v_1 + \kappa_0) + g_2(V), \tag{2.14}$$

where $|g_i(V)| < C|V|^3$, for $i = 1, 2$. We next show

PROPOSITION 2.2. *For $s \geqslant s_0$, $\theta(s)$ satisfies the following ODE*:

$$\theta_s(s)\left(\int (v_1 + \kappa_0)\,\rho\right) = \frac{1}{\kappa_0}\int v_1 v_2 \rho + \int g_2(V)\,\rho. \tag{2.15}$$

Moreover, for s large enough there is a positive constant C such that

$$|\theta_s(s)| \leqslant C \int (v_1^2 + v_2^2)\,\rho = C\,\|V(\cdot, s)\|^2. \tag{2.16}$$

Proof. We multiply (2.14) by ρ and integrate over all R^n to get

$$0 = \int v_{2s}\rho = \int \frac{1}{\rho}\nabla(\rho\nabla v_2)\,\rho + \int \left(\frac{1}{\kappa_0} v_1 v_2 - \theta_s(v_1 + \kappa_0)\right)\rho + \int g_2(V)\,\rho.$$

The term in the left hand side is zero because of (2.10). The first term in the right hand side is easily seen to be zero and (2.15) follows.

Using (2.4) and the fact that $|V(y, s)| < M < +\infty$ for all y, s (this follows from the fact that W is uniformly bounded in space time) we get that

$$\int (v_1 + \kappa_0)\,\rho \to \int \kappa_0 \rho, \qquad \text{as} \quad s \to \infty,$$

and

$$\int g_2(V)\,\rho \leqslant \int |V|^3\,\rho \leqslant MC \int |V|^2\,\rho.$$

Finally using Holder's inequality we get

$$\int v_1 v_2 \rho \leqslant \int (v_1^2 + v_2^2)\,\rho,$$

and (2.16) follows from (2.15). ∎

The usefulness of the parameter $\theta(s)$ now becomes clear in the following

LEMMA 2.1. *If*

$$\int_{s_0}^{+\infty} \|V(\cdot, s)\|^2 \, ds < +\infty, \tag{2.17}$$

then the conclusions of Theorem A *hold.*

From (2.17) and (2.16) it follows that

$$\int_{s_0}^{+\infty} |\theta_s(s)| \, ds < +\infty.$$

Therefore $\lim_{s \to \infty} \theta(s) = \theta_\infty$ exists. But then it follows from (2.2), (2.3), and (2.5) that

$$|W(y, s) - W_\infty| \to 0, \qquad \text{as} \quad s \to \infty,$$

uniformly for $|y| < C$, and

$$\|W(\cdot, s) - W_\infty\| \to 0, \qquad \text{as} \quad s \to \infty,$$

where

$$W_\infty = e^{i\theta_\infty} \kappa_0.$$

Hence, Theorem A will be proved once we have shown (2.17).

3. THE LINEAR OPERATOR: A FORMAL ANALYSIS

In this section we discuss the properties of the linear operator. We then present a formal derivation of our main theorem. Throughout the formal analysis, we assume that $n = 1$ in order to simplify the argument.

As we have explained we would like to know that $\int_{s_0}^{+\infty} \|V(\cdot, s)\|^2 \, ds < +\infty$ where $V = (v_1, v_2)$. Working in this direction we will show

PROPOSITION 3.1. *Either* $\|V(\cdot, s)\|$ *tends to zero exponentially fast, or else for* s *large enough*

$$\frac{C_1}{s} \leqslant \|V(\cdot, s)\| \leqslant \frac{C_2}{s} \tag{3.1}$$

for suitable positive constants C_1, C_2.

Clearly, Theorem A is a direct consequence of the above proposition and Lemma 2.1.

We postpone the proof of Proposition 3.1 until the Section 4. In the remainder of this section we will present a formal derivation of (3.1), based on ideas from center manifold theory.

To begin with, we recall from Section 2 (cf. (2.13), (2.14)) that the equation satisfied by $V=(v_1, v_2)$ is

$$v_{1s}=\mathscr{L}v_1+\frac{p}{2\kappa_0}v_1^2+\frac{1}{2\kappa_0}v_2^2+\theta_s v_2+g_1(V), \tag{3.2}$$

$$v_{2s}=(\mathscr{L}-\mathscr{I})v_2+\frac{1}{\kappa_0}v_1v_2-\theta_s(v_1+\kappa_0)+g_2(V), \tag{3.3}$$

with $|g_i(V)|<C|V|^3$ for $i=1,2$. $\mathscr{I}$ in (3.3) represents the identity operator, and the operator $\mathscr{L}$ is defined by

$$\mathscr{L}f=\frac{1}{\rho}\nabla(\rho\nabla f)+f, \qquad f\in L^2_\rho(R^n). \tag{3.4}$$

We also know that

$$V(y,s)\to 0, \qquad \text{as} \quad s\to\infty, \tag{3.5}$$

uniformly for $|y|<C$,

$$|V(y,s)|<M \qquad \text{for all} \quad y, s, \tag{3.6}$$

and

$$\|V(\cdot,s)\|\to 0, \qquad \text{as} \quad s\to\infty. \tag{3.7}$$

Equations (3.2) and (3.3) can be written in vector form as

$$V_s=\mathbf{L}V+N(V), \tag{3.8}$$

where

$$\mathbf{L}=\begin{pmatrix}\mathscr{L} & 0\\ 0 & \mathscr{L}-\mathscr{I}\end{pmatrix}, \tag{3.9}$$

and

$$N(V)=\left(\frac{p}{2\kappa_0}v_1^2+\frac{1}{2\kappa_0}v_2^2+\theta_s v_2+g_1(V), \frac{1}{\kappa_0}v_1v_2-\theta_s(v_1+\kappa_0)+g_2(V)\right)^T. \tag{3.10}$$

A good understanding of the linear operator defined in (3.4) will be essential in our analysis. It is easy to see that $\mathscr{L}$ is a self-adjoint operator on L^2_ρ.

Let us at this point introduce the Hermite polynomials defined by

$$\tilde{H}_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k}(e^{-x^2}), \qquad k = 0, 1, ...;$$

see, e.g., [18]. It is well known that the $\tilde{H}_k$'s form an orthogonal basis for $L^2_{\gamma(x)}(R)$ with weight $\gamma(x) = e^{-x^2}$. We next define

$$h_k(y) = d_k \tilde{H}_k(y/2), \qquad \text{for} \quad k = 0, 1, ...,$$

with $d_k = (\pi^{1/2} 2^{k+1} k!)^{-1/2}$. One can easily verify that the h_k's form an orthonormal basis for $L^2_\rho(R)$. In particular

$$h_0 = c_0, \qquad h_1 = c_1 y, \qquad h_2 = c_2(\tfrac{1}{2} y^2 - 1), \tag{3.11}$$

with $c_1 = c_2 = (1/2)\pi^{-1/4}$ and $c_0 = (1/\sqrt{2})\pi^{-1/4}$.

Concerning the spectral properties of $\mathscr{L}$ we have the following

LEMMA 3.1. *In R^n, $n \geqslant 1$, the eigenvalues of $\mathscr{L}$ are given by*

$$\lambda_k = 1 - \frac{k}{2}, \qquad k = 0, 1, 2, ...$$

The corresponding normalized eigenfunctions are,

for $\lambda_0 = 1$,	h_0^n,		
for $\lambda_1 = \frac{1}{2}$,	$h_0^{n-1} h_1(y_i)$,	$i = 1, ..., n$,
for $\lambda_2 = 0$,	$h_0^{n-1} h_2(y_i)$,	$i = 1, ..., n$,
	$h_0^{n-2} h_1(y_i) h_1(y_j)$,	$i \neq j,\ i, j = 1, ..., n$,

and so forth.

This can be found in [4].

For future reference we note that the null space of $\mathscr{L}$ has dimension $n(n+1)/2 \equiv N$ and the (normalized) neutral eigenfunctions are

$$h_{ii}(y) = \frac{1}{\sqrt{2}} (4\pi)^{-n/4} \left(\frac{1}{2} y_i^2 - 1\right), \qquad i = 1, ..., n,$$

$$h_{ij}(y) = \frac{1}{2} (4\pi)^{-n/4} y_i y_j, \qquad i \neq j,\ i, j = 1, ..., n.$$

We next introduce some notation. We denote by π_+ the projection operator onto the span of the eigenfunctions of $\mathscr{L}$ which correspond to the positive eigenfunctions, and similarly for π_0 and π_-. We also set $v_{1+}=\pi_+ v_1$, $v_{10}=\pi_0 v_1$, $v_{1-}=\pi_- v_1$, so that

$$v_1 = v_{1+} + v_{10} + v_{1-},$$

and (for $n=1$)

$$v_{1+} = \beta_1(s)\, h_0(y) + \beta_2 h_1(y),$$
$$v_{10} = \alpha(s)\, h_2(y),$$
$$v_{1-} = \gamma_1(s)\, h_3(y) + \gamma_2(s)\, h_4(y) + \cdots.$$

We call the v_{1+}, v_{10}, and v_{1-} the unstable, neutral, and stable components of v_1, respectively.

One can define $\tilde{\pi}_0$ to be the projection operator onto the null subspace of $(\mathscr{L} - \mathscr{I})$ and similarly for $\tilde{\pi}_-$. Note, however, that condition (2.14) on V is nothing but

$$v_{20} \equiv \tilde{\pi}_0 v_2 = 0, \qquad \text{or} \quad v_{2-} \equiv \tilde{\pi}_- v_2 = v_2.$$

In other words v_2 is identical to its stable component. We may thus write

$$v_2 = \tilde{\gamma}_1(s)\, h_1(y) + \tilde{\gamma}_2(s)\, h_2(y) + \cdots.$$

We are now ready to give a formal derivation of Proposition 3.1. We may divide our argumentation into two steps.

Step 1. *Reduction to the study of the neutral mode.*

The operator L defined in (3.9) is easily seen to be a self-adjoint operator on $(L^2_\rho(R^1))^2$ with eigenvalues $\lambda_0 = 1$, $\lambda_1 = 1/2$, $\lambda_2 = 0$, $\lambda_3 = -1/2$, etc. and eigenfunctions $H_0 = (h_0, 0)^T$, $H_1 = (h_1, 0)^T$, $H_2 = (h_2, 0)^T$, $H_j = (h_j, 0)^T$, or $H_j = (0, h_{j-2})^T$, for $j = 3, 4, \ldots$

By the same reasoning as before, we can decompose V as

$$V = V_+ + V_0 + V_-.$$

with

$$V_+ = (v_{1+}, 0), \qquad V_0 = (v_{10}, 0), \qquad V_- = (v_{1-}, v_2).$$

We next assert that for s large enough the behavior of V will be described by the behavior of its neutral part, that is

$$V(y, s) \sim \alpha(s)\, H_2(y) = (\alpha(s)\, h_2(y), 0),$$

or equivalently

$$v_1(y, s) \sim \alpha(s)\, h_2(y), \qquad v_2(y, s) \sim 0. \tag{3.12}$$

Indeed, the unstable modes (β_1, β_2) should be absent since otherwise v_{1+} (and consequently V) would grow exponentially fast contradicting the fact that V goes to zero with s. On the other hand the stable modes $(\gamma_1, \gamma_2, ..., \tilde{\gamma}_1, \tilde{\gamma}_2, ...)$ will decay exponentially fast and they will eventually become negligible compared with the neutral one (α), which as we shall see decays algebraically.

Step 2. *Study of the neutral mode.*

Substituting (3.12) into Eq. (3.1) and omitting all terms of order $O(|V|^3)$ we obtain

$$\dot{\alpha}(s)\, h_2(y) \approx \frac{p}{2\kappa_0}\, \alpha^2 h_2^2(y).$$

Projecting the above equation onto $h_2(y)$ we get

$$\dot{\alpha} \approx \frac{p}{2\kappa_0}\, \alpha^2 \int h_2^3(y)\, \rho. \tag{3.13}$$

After some integration by parts we compute that $\int h_2^3 \rho = 4c_2$ with the same c_2 as in (3.11), and (3.13) can be written as

$$\dot{\alpha} \approx \frac{2pc_2}{\kappa_0}\, \alpha^2. \tag{3.14}$$

Solving (3.14) we find that

$$\alpha(s) \approx \left[\alpha^{-1}(s_0) - \frac{2pc_2}{\kappa_0}(s - s_0) \right]^{-1}, \tag{3.15}$$

in terms of the value of α at some previous time s_0. Since $\alpha(s)$ exists for all times we conclude that $\alpha(s_0) < 0$. Putting together (3.12) and (3.15) we get that as $s \to \infty$

$$V(y, s) \sim (\alpha(s)\, h_2(y), 0) \sim \left(\frac{\kappa_0}{2ps} \left(1 - \frac{1}{2} y^2 \right), 0 \right). \tag{3.16}$$

The behavior described in (3.16) should be generically correct but there should be exceptional solutions which behave differently. In the context of center manifold theory (see, e.g., [3]) these are solutions on the stable manifold. For these solutions, which approach zero exponentially fast, the neutral mode is negligible. We thus have a formal justification of Proposition 3.1.

Although a center manifold approach gives the right picture at the formal level, a straightforward application of the theory is not possible in our problem. The reason for this is that the nonlinear term in (3.10) does not have required properties in L^2_ρ. We refer to [4] for an extensive discussion on that.

4. Proof of the Main Theorem for $m=2$

In this section we will give the proof of Theorem A in the case where $m=2$. We remind the reader that we are studying a blowing-up solution of the nonlinear vector-valued heat equation (1.1). We assume that the solution of (1.1) when written in similarity variables (cf. (1.8)) satisfies:

(i) W is uniformly bounded in space–time,

(ii) $\operatorname{dist}(W(\cdot, s), S^1_\kappa) \to 0$, where S^1_κ is the one-dimensional sphere of radius $\kappa = (p-1)^{-1/(p-1)}$, and

(iii) $\inf_{W_0 \in S^1_\kappa} |W(y, s) - W_0| \to 0$ as $s \to \infty$ uniformly for $|y| < C$.

All the above conditions are known to be valid for any solution of the Cauchy problem (1.1) provided that (a) U is uniformly bounded at infinity (e.g., $U \to 0$ at infinity) (b) $n=1$ and $p>1$, or if $n \geqslant 2$, $1 < p < (3n+8)/(3n-4)$, and (c) the center of scaling is a blowup point (see [9–11]).

As we explained in the Section 3, Theorem A will be proved once we have shown Proposition 3.1. Our aim in this section is to give the proof of Proposition 3.1. For convenience we recall the statement of that proposition.

PROPOSITION 4.1. *Either $\|V(\cdot, s)\|$ goes to zero exponentially fast or else, for s large enough*

$$\frac{C_1}{s} \leqslant \|V(\cdot, s)\| \leqslant \frac{C_2}{s}, \tag{4.1}$$

for suitable positive constants C_1, C_2.

If $\|V(\cdot, s)\|$ decays exponentially fast, we are done. Assuming throughout the rest of this section that $\|\mathbf{V}(\cdot, \mathbf{s})\|$ does not decay exponentially fast we will prove that (4.1) holds.

We will divide the proof into 3 steps.

Step 1. *Reduction to the study of the neutral modes.*

Using the notation of the Section 3 we first show

PROPOSITION 4.2. *There holds*

$$\|v_{1+}\| + \|v_{1-}\| + \|v_2\| = o(\|v_{10}\|), \qquad \text{as} \quad s \to \infty. \tag{4.2}$$

In other words (if the solution does not decay exponentially fast) the neutral component of V dominates its large time behavior.

In proving Proposition 4.2 the following ODE lemma will play an important role

LEMMA 4.1. *Let $x(t)$, $y(t)$, $z(t)$ be absolutely continuous, real-valued functions which are nonnegative and satisfy*

$$\dot{z} \geqslant c_0 z - \varepsilon(x+y)$$

$$|\dot{x}| \leqslant \varepsilon(x+y+z)$$

$$\dot{y} \leqslant -c_0 y + \varepsilon(x+z),$$

$$x, y, z \to 0, \qquad \text{as} \quad t \to \infty,$$

where c_0 is any positive constant and ε is a sufficiently small positive constant. Then either

(i) *$x, y, z \to 0$ exponentially fast or else*

(ii) *there exists a time t_0 after which $z + y < b\varepsilon x$, where b is a positive constant depending only on c_0.*

This is Lemma 3.1 in [4]. We are now ready to give the proof of Proposition 4.1.

Proof of Proposition 4.1. We note that the proof we will present here is quite similar to the proof of [4, Theorem A] to which we refer for more details. At the technical level there are two differences between [4] and here. First, there is one more component here (v_2) which is absent in [4] and second in the equation for V there are terms involving the factor θ_s. The presence of v_2 causes no problem since it can be handled in exactly the same way as v_{1-}. Finally, terms involving θ_s can be easily estimated using Lemma 2.1. With this in mind all the ideas in [4] carry over in the present situation.

Let us first introduce some notation. We set

$$z \equiv \|v_{1+}\|, \quad x \equiv \|v_{10}\|, \quad y_1 \equiv \|v_{1-}\|, \quad y_2 \equiv \|v_2\|,$$

$$N_1 \equiv \left\| \frac{p}{2\kappa_0} v_1^2 + \frac{1}{2\kappa_0} v_2^2 + \theta_s v_2 + g_1(V) \right\|,$$

$$N_2 \equiv \left\| \frac{1}{\kappa_0} v_1 v_2 - \theta_s (v_1 + \kappa_0) + g_2(V) \right\|.$$

Projecting (3.2) onto the unstable subspace of $\mathscr{L}$ forming the L^2_ρ-inner product with v_{1+} and using standard inequalities we get

$$\dot{z} \geqslant \tfrac{1}{2} z - N_1.$$

Working similarly with v_{10}, v_{1-}, v_2 we arrive at the system

$$\begin{aligned} &\dot{z} \geqslant \tfrac{1}{2} z - N_1 \\ &|\dot{x}| \leqslant N_1 \\ &\dot{y}_1 \leqslant -\tfrac{1}{2} y_1 + N_1 \\ &\dot{y}_2 \leqslant -\tfrac{1}{2} y_2 + N_2, \end{aligned} \tag{4.3}$$

From (3.5), (3.7), and the definition of the N_i's it follows after some straightforward calculations that

$$N_1^2 + N_2^2 \leqslant C \int |V|^4 \rho, \tag{4.4}$$

for some positive constant C. Thus, it follows from (4.3) that

$$\begin{aligned} &\dot{z} \geqslant \tfrac{1}{2} z - CN \\ &|\dot{x}| \leqslant CN \\ &\dot{y} \leqslant -\tfrac{1}{2} y + CN, \end{aligned} \tag{4.5}$$

where

$$y \equiv y_1 + y_2, \qquad N^2 \equiv \int |V|^4 \rho. \tag{4.6}$$

If we knew that for s large enough

$$N \leqslant \varepsilon(x + y + z), \tag{4.7}$$

which is equivalent to $\int |V|^4 \rho \leqslant \varepsilon^2 \int |V|^2 \rho$, we could use the ODE Lemma 4.1 to conclude (4.2). The meaning of estimate (4.7) is essentially that the

L^2_ρ-norm of the quadratic term $(|V|^2)$ is small compared to the norm of the linear term $(|V|)$. However, we do not have this information at this stage. As a matter of fact (4.7) will follows from the proof of Proposition 4.2 (cf. Corollary 4.1).

We thus estimate N as follows. Given any $\varepsilon > 0$ and any $\delta > 0$ (both will be chosen small in the sequel) there is a time s_* such that for $i = 1, 2$

$$\int |V|^4 \rho = \int_{|y| > \delta^{-1}} |V|^4 \rho + \int_{|y| < \delta^{-1}} |V|^4 \rho \leqslant \delta^k \int |V|^4 |y|^k \rho + \varepsilon^2 \int |V|^2 \rho, \quad \text{for} \quad s \geqslant s_*. \tag{4.8}$$

Here we use the fact that $V(y, s)$ goes to zero uniformly on the compact set $|y| < \delta^{-1}$. The exponent k which appears in (4.8) is an arbitrary positive integer (later we will choose it to be big). We set

$$J^2 \equiv \int |V|^4 |y|^k \rho,$$

so that (4.8) can be rewritten as

$$\int |V|^4 \rho \leqslant \delta^k J^2 + \varepsilon^2 \int |V|^2 \rho, \qquad \text{for} \quad s \geqslant s_*, \quad i = 1, 2. \tag{4.9}$$

From (4.6) and (4.9) we get that

$$N \leqslant \delta^{k/2} J + \varepsilon(x + y + z), \qquad \text{for} \quad s \geqslant s_*. \tag{4.10}$$

We next estimate J. Let c denote a positive constant not necessarily the same in each occurence. Multiplying (3.2) by $v_1 |V|^2 |y|^k \rho$, and (3.3) by $v_2 |V|^2 |y|^k \rho$, integrating over all R^n we get after some calculations (see [4] for details)

$$\dot{J} \leqslant -\theta J + \varepsilon'(x + y + z) + c(x + y + z)^2, \tag{4.11}$$

where

$$\theta = \frac{k}{4} - c - \frac{k\delta^2}{2}(k + n - 2), \tag{4.12}$$

$$\varepsilon' = \frac{1}{2} \varepsilon \delta^{2 - k/2} k(k + n - 2). \tag{4.13}$$

Using the fact that $x, y, z \to 0$ as $s \to \infty$ we end up with

$$\dot{J} \leqslant -\theta J + 4\varepsilon'(x + y + z), \tag{4.14}$$

where θ is still given by (4.12) with a different value of the constant c. We can now finish the proof in exactly the same way as it is done in [4]. If we choose k large enough (certainly $k > 4$) then there exists a $\delta^*(k) > 0$ such that for $0 < \delta < \delta^*$ we have that $\theta \geqslant \frac{1}{2}$. From (4.5), (4.10), and (4.14) we obtain

$$\dot{z} \geqslant (\tfrac{1}{2} - \hat{\varepsilon})\, z - \hat{\varepsilon}(x + \tilde{y})$$
$$|\dot{x}| \leqslant \hat{\varepsilon}(x + \tilde{y} + z)$$
$$\dot{\tilde{y}} \leqslant -(\tfrac{1}{2} - \hat{\varepsilon})\, \tilde{y} + \hat{\varepsilon}(x + z),$$

where

$$\tilde{y} \equiv y + J, \qquad \hat{\varepsilon} \equiv C \max(\varepsilon + \varepsilon\delta^{2-k/2}, \delta^{k/2}).$$

Note that $\hat{\varepsilon}$ can be made arbitrarilly small by choosing first δ and then ε sufficiently small. We are now able to use Lemma 4.1 to conclude that either x, $\tilde{y}$, z to go to zero exponentially fast or else $z + \tilde{y} \leqslant b\hat{\varepsilon}x$, that is

$$z + y_1 + y_2 + J \leqslant b\hat{\varepsilon}x, \tag{4.15}$$

or equivalently

$$\|v_{1+}\| + \|v_{1-}\| + \|v_2\| \leqslant b\hat{\varepsilon}\, \|v_{10}\|, \tag{4.16}$$

and Proposition 4.2 has been proved. ∎

As a consequence of (4.15) we also have the following

COROLLARY 4.1. *Given any $\varepsilon_0 > 0$ there exists an s_0 such that for $s \geqslant s_0$*

$$\int |V|^4 \mid \rho \leqslant \varepsilon_0^2 \int v_{10}^2 \rho. \tag{4.17}$$

Proof. Using our previous notation (4.17) is equivalent to

$$N \leqslant \varepsilon_0 x. \tag{4.18}$$

From (4.10), (4.15) we have that

$$N \leqslant \delta^{k/2} J + \varepsilon(y + z) + \varepsilon x \leqslant \delta^{k/2} b\hat{\varepsilon}x + \varepsilon b\hat{\varepsilon}x + \varepsilon x,$$

and (4.18) follows. ∎

Step 2. *Derivation of the* ODE *satisfied by the neutral modes*

Here we will obtain an ODE which is satisfied by the neutral modes of v_1. Our main result is

PROPOSITION 4.3. *The neutral modes* $\{\alpha_l\}_{l=1}^{N}$ *of* v_{10} *satisfy for s large enough*

$$\dot{\alpha}_l = \frac{p}{2\kappa_0}\,\pi_l^0(v_{10}^2) + o\left(\sum_{l=1}^{N} \alpha_l^2\right), \tag{4.19}$$

where π_l^0 *denotes the orthogonal projection onto the neutral eigenfunction* e_l^0 *of* $\mathscr{L}$ *and* v_{10} *is the neutral component of* v_{10}, *that is*

$$v_{10} = \sum_{l=1}^{N} \alpha_l(s)\, e_l^0(y), \qquad N = \frac{n(n+1)}{2}. \tag{4.20}$$

The proof of the above proposition is quite similar to the proof of [4, Theorem B]. We first quote an auxiliary result. By slightly adapting the arguments in [4, Lemma 5.1] we obtain

LEMMA 4.2. *There exists* $\delta_0 > 0$ *and an integer* $k > 4$ *with the following property: given any* $0 < \delta < \delta_0$, *there exists a time* s^* *such that*

$$\int |V|^2\, |y|^k \rho \leqslant c_0(k)\, \delta^{4-k} \int v_{10}^2 \rho, \qquad \text{for} \quad s \geqslant s^*, \tag{4.21}$$

where $c_0(k)$ *is a positive constant depending only on k.*

We are now ready to give the proof of Proposition 4.2.

Proof of Proposition 4.2. Projecting Eq. (3.2) onto e_l^0 we get

$$\dot{\alpha}_l = \frac{p}{2\kappa_0}\,\pi_l^0(v_{10}^2) + \frac{p}{2\kappa_0}\,\pi_l^0(v_1^2 - v_{10}^2) + \frac{1}{2\kappa_0}\,\pi_l^0(v_2^2) + \frac{1}{2\kappa_0}\,\theta_s \pi_l^0(v_2) + \pi_l^0(g_1(V))$$

$$= \frac{p}{2\kappa_0}\,\pi_l^0(v_{10}^2) + \frac{p}{2\kappa_0}\,\mathscr{E}_1 + \frac{1}{2\kappa_0}\,\mathscr{E}_3 + \mathscr{E}_4,$$

with

$$\mathscr{E}_1 \equiv \pi_l^0(v_1^2 - v_{10}^2), \quad \mathscr{E}_2 \equiv \pi_l^0(v_2^2), \quad \mathscr{E}_3 \equiv \theta_s \pi_l^0(v_2), \quad \mathscr{E}_4 \equiv \pi_l^0(g_1(V)).$$

Thus, (4.19) will be proved once we have shown

$$\mathscr{E}_i = o(\|v_{10}\|^2), \qquad \text{as} \quad s \to \infty, \quad i = 1, 2, 3, 4,$$

where, of course, $\|v_{10}\|^2 = \sum_{l=1}^{N} \alpha_l^2$. These estimates are quite similar with each other. Let us give in detail the estimate for $\mathscr{E}_1$ which is the most technical. Using the identity

$$v_1^2 - v_{10}^2 = (v_{1+} + v_{1-})^2 + 2v_{10}(v_{1+} + v_{1-})$$

and the fact that $|e_l^0| \leqslant c_1 + c_2|y|^2$ for suitable constants c_1, c_2 we write

$$|\mathscr{E}_1| \equiv \left|\int (v_1^2 - v_{10}^2)\, e_l^0 \rho\right| \leqslant c_1 \int (v_{1+} + v_{1-})^2 \rho + c_2 \int (v_{1+} + v_{1-})^2 |y|^2 \rho$$

$$+ c_1 \int |v_{10}|\, |v_{1+} + v_{1-}|\, \rho + c_2 \int |v_{10}|\, |v_{1+} + v_{1-}|\, |y|^2 \rho$$

$$\equiv c_1 I_1 + c_2 I_2 + c_1 I_3 + c_2 I_4.$$

From (4.2) it follows that given any $\varepsilon > 0$

$$I_1 \leqslant \varepsilon \left(\int v_{10}^2 \rho \right),$$

for s sufficiently large. To estimate I_2 we write for $0 < \delta < \delta_0$ (The same δ_0 as in Lemma 4.2)

$$I_2 = \int_{|y| \leqslant \delta^{-1}} |v_{1+} + v_{1-}|^2 |y|^2 \rho + \int_{|y| \geqslant \delta^{-1}} |v_1 - v_{10}|^2 |y|^2 \rho$$

$$\leqslant \delta^{-2} \int |v_{1+} + v_{1-}|^2 \rho + 2\delta^{k-2} \int v_1^2 |y|^k \rho + 2\delta^{k-2} \int v_{10}^2 |y|^k \rho$$

$$\leqslant \varepsilon\delta^{-2} \int v_{10}^2 \rho + 2\delta^{4-k} \delta^{k-2} c_0(k) \int v_{10}^2 \rho + 2\delta^{k-2} c \int v_{10}^2 \rho$$

$$= (\varepsilon\delta^{-2} + 2\delta^2 c_0(k) + 2c\delta^{k-2}) \int v_{10}^2 \rho,$$

where we used (4.2), Lemma 4.2, and the fact that all norms of v_{10} are equivalent. The exponent k which appears in the above calculations is the same as in Lemma 4.2.

Using once more (4.2) and the fact that all norms of v_{10} are equivalent we get

$$I_3 \leqslant \left(\int v_{10}^2 \rho \right)^{1/2} \left(\int (v_{1+} + v_{1-})^2 \rho \right)^{1/2} \leqslant \varepsilon \int v_{10}^2 \rho,$$

$$I_4 \leqslant \left(\int v_{10}^2 |y|^2 \rho \right)^{1/2} \left(\int (v_{1+} + v_{1-})^2 \rho \right)^{1/2} \leqslant c\varepsilon \int v_{10}^2 \rho.$$

Thus, by first choosing δ and then ε sufficiently small we end up with

$$|\mathscr{E}_1| = o(\|v_{10}\|^2), \qquad \text{as} \quad s \to \infty.$$

By arguments quite similar to the ones used above one can show that

$$|\mathscr{E}_i| = o(\|v_{10}\|^2), \qquad \text{as} \quad s \to \infty.$$

for $i = 2, 3, 4$. We simply note that in estimating $\mathscr{E}_3$ one has to use Lemma 3.2 whereas in estimating $\mathscr{E}_4$ one has to use (4.17). We omit further details. ∎

Step 3. Dynamics of the neutral modes.

Teh final step in studying the ODE system (4.19) satisfied by the neutral modes. Our goal here is to show

PROPOSITION 4.4. *Assume that the neutral modes of v_1 satisfy the system (4.19). Then for s large enought there exist positive constants C_1, C_2 such that*

$$\frac{C_1}{s} \leqslant \|v_{10}(\cdot, s)\| \leqslant \frac{C_2}{s}. \tag{4.22}$$

System (4.19) has been studied in detail in [5] from which we now recall a few results.

We first note that as we explained in Section 3 the neutral eigenfunctions of $\mathscr{L}$ are either of the form

$$h_{ii}(y) = \frac{1}{\sqrt{2}}(4\pi)^{-n/4}\left(\frac{1}{2}y_i^2 - 1\right), \qquad i = 1, ..., n,$$

or of the form

$$h_{ij}(y) = \tfrac{1}{2}(4\pi)^{-4/n}\, y_i y_j, \qquad i \neq j, \quad i, j = 1, ..., n.$$

We next expand $v_{10}(y, s)$,

$$v_{10}(y, s) = \frac{1}{\sqrt{2}}\sum_{i=1}^{n} a_{ii}(s)\, h_{ii}(y) + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}(s)\, h_{ij}(y), \tag{4.23}$$

with $a_{ij}(s) = a_{ji}(s)$, and define the coefficient matrix-function $A(s)$ to be

$$A(s) = (a_{ij}(s))_{i,j=1}^{n}.$$

In view of (4.20) and (4.23) the α_l are equal to either a_{ii} or $(1/\sqrt{2})\, a_{ij}$ $(i \neq j)$. The reason for introducing $A(s)$ becomes apparent in the following lemmas. After some rather lenghty (but straightforward) calculations (cf. [5, Lemma 3.1]) we obtain

LEMMA 4.3. *Let v_{10} denote the neutral component of v_1; then*

$$v_{10}(y, s) = \tfrac{1}{2}(4\pi)^{-n/4}\left(\tfrac{1}{2}y^T A(s)\, y - \operatorname{tr} A(s)\right),$$

where $A(s)$ satisfies

$$\dot{A}(s) = c_n A^2(s) + o(\|A(s)\|^2), \tag{4.24}$$

with $c_n = p/k(4\pi)^{-n/4}$ *and* $\|A\| = (\sum_{i,j=1}^{n} a_{ij}^2)^{1/2}$.

We note that if the error term in (4.24) were absent one could solve (4.24) in an elementary way. The presence of the error term, however, introduces some technical difficulties.

To complete the proof of Proposition 4.4. we finally quote from [5] (see Lemma 3.1 there)

LEMMA 4.4. *For s large enough there exist positive constants C_1, C_2 such that*

$$\frac{C_1}{s} \leqslant \|A(s)\| \leqslant \frac{C_2}{s}. \tag{4.25}$$

Proposition 4.4 now follows at once from (4.25) and the fact that $\|A(s)\|$ is an equivalent norm for $\|v_{10}(\cdot, s)\|$.

5. THE GENERAL CASE

In this section we will give the proof of Theorem A in the general case, that is, $m \geqslant 2$. Although at the technical level things here are slightly more complicated than in the case $m = 2$, the logic of the previous sections stays the same. We recall that Eq. (1.1) when written in similarity variables is

$$W_s = \frac{1}{\rho} \nabla(\rho \nabla W) - \frac{1}{p-1} W + |W|^{p-1} W, \tag{5.1}$$

where $W = (w_1, ..., w_m)$ satisfies the assumptions of Theorem A.

We remark that all the results presented here have their analogue in Section 2.

Let us first introduce some notation. We denote by N_i the unit vector along the direction of the i-axis ($i = 1, 2, ..., m$) and by $\langle \cdot, \cdot \rangle_{R^m}$ the usual Euclidean inner product. We nex set

$$R_i \equiv \begin{pmatrix} \cos\theta_i & 0 & \cdots & -\sin\theta_i & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \sin\theta_i & 0 & \cdots & \cos\theta_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}. \tag{5.2}$$

R_i is an $m \times m$ orthonormal matrix which rotates the (N_1, N_i)-plane by an angle θ_i and leaves all other directions invariant. It is constructed if in the $m \times m$ identity matrix the first and ith row are replaced as shown in (5.2). We also set

$$R_\theta \equiv R_2 R_3 \cdots R_m, \tag{5.3}$$

where $\theta = (\theta_2, \theta_3, ..., \theta_m)$. Clearly, R_θ is an $m \times m$ orthonormal matrix.

We next introduce $m-1$ parameters $\theta_2(s)$, $\theta_3(s)$, ..., $\theta_m(s)$ defined at each time s by

$$\int |W(\cdot, s) - \kappa_0 R_{\theta(s)} N_1|^2 \rho = \min_{\theta \in (S^1)^{m-1}} \int |W(\cdot, s) - \kappa_0 R_\theta N_1|^2 \rho, \tag{5.4}$$

where $\kappa_0 = (p-1)^{-1/(p-1)}$ and $N_1 = (1, 0, ..., 0)^T$. Since $(S^1)^{m-1}$ is a compact set the existence of the θ_i's follows at once.

Although the point $R_\theta N_1 \in S^{m-1}$ is always unique there is more than one combination of $\theta = (\theta_2, ..., \theta_m)$ which represents the same point $R_\theta N_1$ on the sphere. For instance, for $m = 3$ one may choose either (θ_2, θ_3) or $(\theta_2 + \pi, \pi - \theta_3)$. For general m $(\geqslant 3)$ there are 2^{m-2} different combinations of θ which yield the same point $(R_\theta N_1)$ on the sphere S^{m-1}.

We also note that if $\cos \theta_j = 0$ for some $j \geqslant 3$ then the $\theta_2, ..., \theta_{j-1}$ can be chosen arbitrarily. For instance if $\cos \theta_m = 0$, that is $\theta_m = \pm \pi/2 \pmod{2\pi}$, then $R_\theta N_1 = \pm (0, ..., 1)^T$, no matter what the values of $\theta_2, ..., \theta_{m-1}$ are.

In any case we have the following

PROPOSITION 5.1. *For s large enough $R_\theta N_1(s) \equiv R_{\theta(s)} N_1$ is unique and varies smoothly with time. Moreover we can choose the $m-1$ parameters introduced in* (5.4) *so that $\theta_i(s)$ $(2 \leqslant i \leqslant m)$ are* a.e. *differentiable.*

Proof. We have that

$$\int |W(\cdot, s) - \kappa_0 R_{\theta(s)} N_1|^2 \rho = \int (|W|^2 + \kappa_0^2) \rho - 2\kappa_0 \langle W, R_\theta N_1(s) \rangle.$$

Therefore the minimum of the functional in (5.4) is achieved when

$$\langle W(\cdot, s), R_\theta N_1(s) \rangle = \left\langle \int W(\cdot, s) \rho, R_\theta N_1(s) \right\rangle_{R^m}, \tag{5.5}$$

achieves its maximum. This happens when $R_\theta N_1(s)$ is parallel to $\int W(\cdot, s) \rho$. Clearly for each time s there are two different $R_\theta N_1(s)$ with this property. As in the case $m = 2$ one of them is excluded as yielding the minimum of (5.5). Finally, since $\int W(\cdot, s)\rho$ is smooth we conclude that $R_\theta N_1(s)$ is also smooth.

An easy calculation shows that

$$R_\theta N_1 = (\cos\theta_2 \cos\theta_3 \cdots \cos\theta_m,$$
$$\sin\theta_2 \cos\theta_3 \cdots \cos\theta_m, ..., \sin\theta_{m-1} \cos\theta_m, \sin\theta_m)^T.$$

Thus, the angles $\theta_2, \theta_3, ..., \theta_m$ are smoothly defined so long as $\cos\theta_m \cdots \cos\theta_3 \neq 0$. In particular θ_m is always smoothly defined.

Suppose now that $\cos\theta_m = 0$. Then, if this is true along a time interval, the other angles can be arbitrarily defined so that they are smooth along this interval. If this is true at isolated point then these points are possibly points of discontinuity for the other angles. Arguing similarly in the case where for some other angle we have $\cos\theta_j = 0$ $(j = 3, ..., m)$ we conclude the proof of the proposition. ∎

We next introduce $V(y, s) = (v_1(y, s), ..., v_m(y, s))$ as follows:

$$W(y, s) = R_{\theta(s)}(V(y, s) + \kappa_0 N_1). \tag{5.6}$$

As in the case $m = 2$ we have that

$$|V(y, s)| \to 0, \qquad \text{as} \quad s \to \infty, \tag{5.7}$$

uniformly for $|y| < C$, and

$$\|V(\cdot, s)\| = \|W(\cdot, s) - R_{\theta(s)}\kappa_0 N_1\| \to 0, \qquad \text{as} \quad s \to \infty. \tag{5.8}$$

We next show

LEMMA 5.1. *Let $V = (v_1, ..., v_m)$ be as defined in (5.6). Then for s large enough we have that*

$$\int v_j(\cdot, s)\,\rho = 0, \qquad j = 2, ..., m. \tag{5.9}$$

Proof. Computing the first variation of the functional in (5.4) we have that for $i = 2, 3, ..., m$:

$$\left\langle W, \frac{\partial R_\theta}{\partial \theta_i} N_1 \right\rangle = 0.$$

Using (5.6) we get that

$$\left\langle R_\theta V, \frac{\partial R_\theta}{\partial \theta_i} N_1 \right\rangle = 0,$$

or, equivalently

$$\langle V, A_i N_1 \rangle = 0, \tag{5.10}$$

where

$$A_i \equiv R_\theta^{-1} \frac{\partial R_\theta}{\partial \theta_i}. \tag{5.11}$$

The matrices A_i can be explicitly calculated. It follows from (5.11) and straightforward calculations that $A_i = (a_{i,jk})_{j,k=1}^m$ is antisymmetric (that is $A_i^T = -A_i$) and that all of its entries are zero except the ith row and ith column for which we have

$$\begin{aligned} a_{i,i1} &= \cos\theta_i \cdots \cos\theta_m, \\ a_{i,i2} &= a_{i,i3} = \cdots = a_{i,ii} = 0, \\ a_{i,i(i+1)} &= -\sin\theta_{i+1}, \\ a_{i,i(i+2)} &= -\sin\theta_{i+2}\cos\theta_{i+1}, \\ \cdots & \quad \cdots \\ a_{i,in} &= -\sin\theta_n \cos\theta_{n-1} \cdots \cos\theta_{i+1}. \end{aligned} \tag{5.12}$$

An easy computation shows that (5.10) is equivalent to

$$\int v_m \rho = 0$$

$$\cos\theta_m \int v_{m-1}\rho = 0$$

$$\cdots$$

$$\cos\theta_m \cos\theta_{m-1} \cdots \cos\theta_3 \int v_2 \rho = 0.$$

Thus, if $\cos\theta_3 \cos\theta_4 \cdots \cos\theta_m \neq 0$ then (5.9) follows.

We next show that even if some of the $\cos\theta_i$'s are zero (5.9) still holds true. To see this suppose that $\cos\theta_m = 0$, that is $\theta_m = \pm\pi/2 \pmod{2\pi}$. Let us assume for definiteness that $\theta_m = \pi/2$. Then $R_\theta N_1 = N_m \equiv (0, 0, ..., 1)^T$ and the minimum of

$$\| W - \kappa_0 R_\theta N_1 \|^2 = \int (|W|^2 + \kappa_0)\rho - 2\kappa_0 \langle W, R_\theta N_1 \rangle,$$

or equivalently, the maximum of

$$\langle W, R_\theta N_1 \rangle,$$

is achieved when $R_\theta N_1 = N_m$. For $i = 1, ..., m-1$, consider the vector $P_i = (0, ..., 0, \sin\phi_i, 0, ..., 0, \cos\phi_i)^T \in S^{m-1}$ which lies on the (N_i, N_m)-plane. We then have that

$$\langle W, P_i \rangle = \sin\phi_i \int w_i\rho + \cos\phi_i \int w_m\rho$$

attains its maximum when $\phi_i = 0$ (in which case $P_i = N_m$). Computing the first derivative (w.r.t. ϕ_i) of $\langle W, P_i \rangle$ and setting $\phi_i = 0$ we get

$$\int w_i\rho = 0, \qquad i = 1, ..., m-1. \tag{5.13}$$

It then follows from (5.6) and (5.13) that

$$R_\theta \int V\rho = \int W\rho - N_m \int \rho = \text{multiple of } N_m = cN_m,$$

and

$$\int V\rho = cR_\theta^{-1} N_m = cN_1.$$

In particular we get

$$\int v_j\rho = 0, \qquad j = 2, ..., m.$$

Arguing similarly in the case where some of the other angles is zero we conclude the proof. ∎

We next find the equation satisfied by V. From (5.7) we get that

(5.14)

Computing the Taylor expansion of the nonlinear term we get

$$|V + \kappa_0 N_1|^{p-1} (V + \kappa_0 N_1)$$

$$= \begin{pmatrix} v_1 + \kappa_0 + \kappa_0^{p-1} v_1 \\ \kappa_0^{p-1} v_2 \\ \vdots \\ \kappa_0^{p-1} v_m \end{pmatrix} + \frac{1}{\kappa_0} \begin{pmatrix} \frac{1}{2} v_1^2 + \frac{p}{2} \sum_{i=2}^{m} v_i^2 \\ v_1 v_2 \\ \vdots \\ v_1 v_m \end{pmatrix} + G(V), \tag{5.15}$$

with $|G(V)| < C|V|^3$. From (5.15), (5.14), and (2.1) we get

$$V_s = (\mathscr{L} - \mathscr{I})\, V + \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \frac{1}{\kappa_0} \begin{pmatrix} \frac{1}{2} v_1^2 + \frac{p}{2} \sum_{i=2}^{m} v_i^2 \\ v_1 v_2 \\ \vdots \\ v_1 v_m \end{pmatrix}$$

$$- \sum_{i=2}^{m} \theta_{is} A_i \begin{pmatrix} v_1 + \kappa_0 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} + G(V), \qquad (5.16)$$

where the matrices A_i are defined in (5.11), (5.12).

We next show the analogue of Proposition 2.2.

PROPOSITION 5.2. *For s sufficiently large $\theta_j(s)$ $(j = 2, ..., m)$ satisfy the following* ODE's.

$$\frac{d\theta_j(s)}{ds} \langle A_j N_j, N_1 \rangle_{R^m} \left(\int (v_1 + \kappa_0)\, \rho \right) = \frac{1}{\kappa_0} \int v_1 v_j \rho + \int g_j(V)\, \rho. \qquad (5.17)$$

Moreover, there is a positive constant C such that

$$\left| \frac{d\theta_j(s)}{ds} \langle A_j N_j, N_1 \rangle_{R^m} \right| \leqslant C\, \|V(\cdot, s)\|^2, \qquad (5.18)$$

or, equivalently (*if we make explicit the left hand side of* (5.18))

$$\left| \frac{d\theta_m(s)}{ds} \right| \leqslant C\, \|V(\cdot, s)\|^2,$$

$$\left| \frac{d\theta_{m-1}(s)}{ds} \cos \theta_m \right| \leqslant C\, \|V(\cdot, s)\|^2, \qquad (5.19)$$

$$\cdots \quad \cdots$$

$$\left| \frac{d\theta_2(s)}{ds} \cos \theta_m \cos \theta_{m-1} \cdots \cos \theta_3 \right| \leqslant C\, \|V(\cdot, s)\|^2.$$

Proof. Forming the L^2_ρ-inner product of N_j and V_s (cf. (5.16)) we have

$$0=\int v_{js}\rho=\int(\mathcal{L}-\mathcal{I})\,v_j\rho+\frac{1}{\kappa_0}\int v_1 v_j\rho$$

$$-\sum_{i=2}^{m}\theta_{is}\,\langle A_i(V+\kappa_0 N_1),N_j\rangle+\int g_j(V)\,\rho. \tag{5.20}$$

The term on the right hand side is zero because of (5.9). The first term on the left hand side is easily seen to be zero. For the third term of the left hand side we note that since $\int V\rho=N_1\int v_1\rho$ (this is a consequence of (5.9)) we can write

$$\langle A_i(V+\kappa_0 N_1),N_j\rangle=\langle A_iV,N_j\rangle+\langle A_iN_1,N_j\rangle_{R^m}\int\kappa_0\rho$$

$$=\left(\int(v_1+\kappa_0)\,\rho\right)\langle A_iN_1,N_j\rangle_{R^m}.$$

Because of the special structure of A_i we also have that

$$\langle A_iN_1,N_j\rangle_{R^m}=0,\qquad\text{for}\quad i\neq j,$$

and (5.17) follows from (5.20).

The second part of the statement is proved in exactly the same way as in the case $m=2$. ∎

We next have

PROPOSITION 5.3. *Either $\|V(\cdot,s)\|$ goes to zero exponentially fast or else for s large enough there are positive constants C_1, C_2 such that*

$$\frac{C_1}{s}\leqslant\|V(\cdot,s)\|\leqslant\frac{C_1}{s}.$$

Proof. The proof is the same as the proof of Proposition 4.2 with minor changes. ∎

To complete the proof of Theorem A we finally have

LEMMA 5.2. *As $s\to\infty$ we have that*

$$R_{\theta(s)}N_1\to R_{\theta\infty}N_1\in S^{m-1}. \tag{5.21}$$

Proof. From (5.19) and Proposition 5.3 we have that

$$\left|\frac{d\theta_m(s)}{ds}\right| \leqslant c/s^2.$$

Therefore $\lim_{s\to\infty} \theta_m(s) = \theta_{m\infty}$ exists. We distinguish two cases.

Assume first that $\theta_{m\infty} = \pm\pi/2 \pmod{2\pi}$. Then, $R_{\theta(s)}N_1 \to R_{\theta_\infty}N_1 = \pm N_m$, and (5.21) has been proved.

Suppose now that $\theta_{m\infty} \neq \pm\pi/2 \pmod{2\pi}$. Then, for s large enough we will have that $|\cos\theta_m(s)|$ is bounded away from zero. In particular $\theta_{m-1}(s)$ is now smoothly defined for all s large enough. It follows from (5.19) and Proposition 5.3 that

$$\left|\frac{d\theta_{m-1}(s)}{ds}\right| \leqslant c/s^2,$$

and therefore $\lim_{s\to\infty} \theta_{m-1}(s) = \theta_{(m-1)\infty}$ exists. We now argue as before, that is, if

$$\theta_{(m-1)\infty} = \pm\frac{\pi}{2} \qquad \pmod{2\pi}, \tag{5.22}$$

then (5.21) is true with $R_{\theta(s)}N_1 \to R_{\theta_\infty}N_1 = (0, ..., 0, \pm\cos\theta_{m\infty}, \sin\theta_{m\infty})$. If (5.22) is not the case we repeat the previous argument with $\theta_{m-2}(s)$ and so on. ∎

From (5.6)–(5.8) and (5.21) we have that

$$|W(y,s) - W_\infty| \to 0, \qquad \text{as} \quad s\to\infty,$$

uniformly for $|y| < C$, and

$$\|W(\cdot,s) - W_\infty\| \to 0, \qquad \text{as} \quad s\to\infty,$$

with

$$W_\infty \equiv \kappa_0 R_{\theta_\infty} N_1,$$

and Theorem A has been proved.

References

1. J. Bebernes and D. Eberly, A description of self similar blow up for dimensions $n \geqslant 3$, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **5** (1988), 1–22.
2. X.-Y. Chen and H. Matano, Convergence, asymptotic periodicity, and finite-point blowup in one-dimensional semilinear heat equations, *J. Differential Equations* **78** (1989), 160–190.

3. J. CARR, "Applications of Centre Manifold Theory," Springer-Verlag, New York, 1981.
4. S. FILIPPAS AND R. V. KOHN, Refined asymptotics for the blowup of $u_t - \Delta u = u^p$, *Comm. Pure Appl. Math.* **45** (1992), 821–869.
5. S. FILIPPAS AND W. LIU, On the blowup of multidimensional semilinear heat equations, *Ann. Inst. H. Poincaré*, to appear.
6. A. FRIEDMAN, Blow-up of solutions of nonlinear heat and wave equations, preprint.
7. A. FRIEDMAN AND B. MCLEOD, Blowup of positive solutions of semilinear heat equations, *Indiana Univ. Math. J.* **34** (1985), 425–447.
8. V. A. GALAKTIONOV, M. A. HERRERO AND J. J. L. VELÁZQUEZ, The space structure near a blowup point for semilinear heat equations: A formal approach, *U.S.S.R. Comput. Math. and Math. Phys.* No. **31**, 3 (1991), 399–411.
9. Y. GIGA AND R. V. KOHN, Asymptotically self similar blowup of semilinear heat equations, *Comm. Pure Appl. Math.* **38** (1985), 297–319.
10. Y. GIGA AND R. V. KOHN, Characterising blow up using similarity variables, *Indiana Univ. Math. J.* **36** (1987), 1–40.
11. Y. GIGA AND R. V. KOHN, Nondegeneracy of blowup for semilinear heat equations, *Comm. Pure Appl. Math.* **42** (1989), 845–884.
12. M. A. HERRERO AND J. J. L. VELÁZQUEZ, Blow-up behaviour of one-dimensional semilinear parabolic equations, *Ann. Inst. H. Poincaré*, to appear.
13. M. A. HERRERO AND J. J. L. VELÁZQUEZ, Flat blow-up in one-dimensional semilinear parabolic equations, *Differential Integral Equations*, to appear.
14. M. A. HERRERO AND J. J. L VELÁZQUEZ, Blow-up profiles in one-dimensional semi-linear parabolic equations, *Comm. Partial Differential Equations*, to appear.
15. H. A. LEVINE, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$, *Arch. Rational Mech. Anal.* **51** (1973), 371–386.
16. F. MERLE, On uniqueness and continuation properties after blowup time of self-similar solutions of nonlinear Schrodinger equation with critical exponent and critical mass, *Comm. Pure Appl. Math.* **45** (1992), 203–254.
17. F. MERLE, Solution of a nonlinear heat equation with arbitrarily given blowup points, *Comm. Pure Appl. Math.* **40** (1992), 263–300.
18. G. SZEGO, "Othogonal Polynomials," A.M.S. Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, RI, 1967.
19. J. J. L. VELÁZQUEZ, Classification of singularities for blowing up solutions in higher dimensions, *Trans. Amer. Math. Soc.*, to appear.
20. J. J. L. VELÁZQUEZ, Estimates on the $(N-1)$-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation, preprint.