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Critical heat kernel estimates for Schrödinger operators via Hardy–Sobolev inequalities

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Abstract

We obtain Sobolev inequalities for the Shcrödinger operator $-\Delta - V$, where V has critical behaviour $V(x) = ((N-2)/2)^2|x|^{-2}$ near the origin. We apply these inequalities to obtain point-wise estimates on the associated heat kernel, improving upon earlier results.

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1. Introduction

The purpose of this paper is to obtain some new Hardy–Sobolev inequalities and then use them in order to obtain new heat kernel estimates for the Schrödinger operator $-\Delta - V$ for positive potentials V with critical singularities, improving upon analogous estimates of this type.

As a typical example, let us consider the case of a bounded domain $\Omega \subset \mathbf{R}^N$, $N \geqslant 3$, containing the origin. We obtain upper estimates on the heat kernel of the operator

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H given formally by

$$Hu = -\Delta u - \frac{\lambda}{|x|^2} u, \quad u|_{\partial\Omega} = 0, \tag{1.1}$$

for various values of the real parameter λ (see Section 4 for the precise definition of H). It is well known that the power $|x|^{-2}$ is critical for the corresponding linear parabolic equation, as was shown in the fundamental work of Baras and Goldstein [BG]; see also the recent works of Goldstein and Zhang [GZ] as well as Vázquez and Zuazua [VZ].

Indeed, the associated heat kernel exhibits behaviour which is different from that of the case $|x|^{-\gamma}$, $\gamma < 2$, which is in the Kato class. When $\gamma = 2$ the semigroup is not ultracontractive: indeed, for $0 < \lambda < ((N-2)/2)^2$, the heat kernel of (1.1) satisfies

$$K(t,x,y) < ct^{-\frac{N}{2}} |x|^{-\alpha} |y|^{-\alpha},$$

where α denotes the smallest solution of $\alpha(N-2-\alpha) = \lambda$; see the works of Liskevich and Sobol [LS] and Milman and Semenov [MS] and references therein.

In Theorem 4.2 we extend this estimate to the critical case $\lambda = ((N-2)/2)^2$. Namely, we prove that the corresponding heat kernel satisfies

$$K(t,x,y) < ct^{-N/2} |x|^{-\frac{N-2}{2}} |y|^{-\frac{N-2}{2}}.$$

This estimate is sharp as can be seen by comparing with the results in [VZ].

We also consider operators that act on the whole of $\mathbf{R}^N$ with potentials having the critical Hardy singularity near zero, of the form

$$V_\varepsilon(x) = \begin{cases} \left(\frac{N-2}{2}\right)^2 |x|^{-2}, & |x| < 1, \\ \varepsilon f(x), & |x| > 1, \end{cases} \tag{1.2}$$

under appropriate subcritical assumptions on the positive function f. Thus, in Theorem 4.3 it is shown that if $\varepsilon > 0$ is small enough then the heat kernel of $-\Delta - V_\varepsilon$ satisfies

$$K(t,x,y) < ct^{-N/2} \max\left\{ |x|^{-\frac{N-2}{2}}, 1 \right\} \max\left\{ |y|^{-\frac{N-2}{2}}, 1 \right\}. \tag{1.3}$$

We also consider potentials that exhibit the critical behaviour $((N-2)/2)^2 |x|^{-2}$ near infinity, that is

$$\hat{V}_\varepsilon(x) = \begin{cases} \varepsilon g(x), & |x| < 1, \\ \left(\frac{N-2}{2}\right)^2 |x|^{-2}, & |x| > 1. \end{cases} \tag{1.4}$$

Under appropriate subcritical assumptions on g we obtain Sobolev estimates for $-\Delta - \hat{V}_\varepsilon$ for a sharp range of $\varepsilon > 0$. We note here that while the question of Sobolev inequalities for $-\Delta - \hat{V}_\varepsilon$ is rather similar to that for $-\Delta - V_\varepsilon$, when it comes to heat kernel estimates essential differences arise. As mentioned earlier, the Sobolev inequality for $-\Delta - V_\varepsilon$ yields estimate (1.3) for the corresponding heat kernel. On the other hand, while the short-time behaviour of the heat kernel of $-\Delta - \hat{V}_\varepsilon$ is similar to that of the Laplacian, the long-time behaviour is very different. Working on a Riemannian manifold setting, Zhang [Z] used a parabolic Harnack inequality to obtain estimates for the heat kernel of $-\Delta - V$, when V is equal near infinity to $\lambda d(x)^{-2}$, $\lambda < ((N-2)/2)^2$, $d(x) = \mathrm{dist}(x_0, x)$; however no explicit power of t was given. This complements earlier estimates given by Davies and Simon [DS] which involved the correct power of t in the Euclidean case. The corresponding problem for the critical case $\lambda = ((N-2)/2)^2$ remains open.

Going back to bounded $\Omega \subset \mathbf{R}^N$ and to the operator H given formally by (1.1), for the critical case $\lambda = ((N-2)/2)^2$ we finally consider additional singularities, that is, we consider potentials of the form $((N-2)/2)^2|x|^{-2} + V_1$, where $V_1 > 0$ is also critical; V_1 is defined as a series involving iterated logarithms (see definition (5.17)) and is critical in the sense that the following improved Hardy inequality holds

$$\int_\Omega |\nabla u|^2\,dx - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2}\,dx \geqslant \int_\Omega V_1 u^2\,dx, \quad u \in C_c^\infty(\Omega), \tag{1.5}$$

whereas this inequality is no longer true if we replace V_1 by $(1+\varepsilon)V_1$ for any $\varepsilon > 0$. It is remarkable that the extra potential V_1 does not affect the time dependence of the heat kernel estimates, but only affects the spatial singularity at the origin (cf. Theorems 4.2 and 5.3). This is in contrast with Proposition 4.1(ii) where, for $\lambda < 0$, the potential affects the time singularity of the heat kernel as well.

Throughout the paper we study a number of concrete potentials. These are chosen precisely because they are critical. By simple monotonicity one can then obtain heat kernel estimates for a whole range of other potentials, including potentials that are not radially symmetric.

To prove the above heat kernel estimates we first use an appropriate change of variables, $u = \phi w$, by means of which, the problem is reduced to obtaining uniform estimates on the heat kernel $K_\phi(t, x, y)$ of an auxiliary operator H_ϕ which acts on the function w; see, e.g., [MS]. Those estimates are in turn proved by means of some new Hardy–Sobolev inequalities.

As a typical example of such an inequality we mention the following inequality proved by Brezis and Vázquez [BV]:

$$\int_\Omega |\nabla u|^2\,dx - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2}\,dx \geqslant K\left(\int_\Omega |u|^p\,dx\right)^{2/p}, \tag{1.6}$$

valid for $u \in H_0^1(\Omega)$ and $1 < p < \frac{2N}{N-2}$; this inequality fails for the critical Sobolev exponent $p = \frac{2N}{N-2}$. To obtain sharp heat kernel estimates one needs to go up to the critical exponent. In connection with this we mention the following sharp

Hardy-Sobolev inequality established in [FT]:

$$\int_{\Omega} |\nabla u|^2 \, dx - \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{u^2}{|x|^2} \, dx \geqslant c \left(\int_{\Omega} |u|^{\frac{2N}{N-2}} X_1^{1+\frac{N}{N-2}}\left(\frac{|x|}{D}\right) dx \right)^{\frac{N-2}{N}}, \tag{1.7}$$

valid for $u \in H_0^1(\Omega)$; here $D = \sup_{\Omega} |x|$ and $X_1(t) = (1 - \log t)^{-1}$, $t \in (0, 1)$. In the present work we derive new Hardy-Sobolev inequalities that involve potentials such as the ones given in (1.2) or (1.4); see Theorems 3.4, 3.5, 5.1 and 5.2. We should mention that the validity of improved Hardy inequalities is strongly connected to the existence and large time behaviour of solutions of the heat equation with singular potential; see, e.g., [BV,CM,DD,GZ] as well as Vázquez and Zuazua [VZ].

As a byproduct of our approach we establish various results concerning improved Hardy inequalities with boundary terms. Such inequalities have recently attracted attention, see the articles by Adimurthi [Ad], Adimurthi and Esteban [AE], Wang and Zhu [WZ] and references therein.

The structure of the paper is as follows: in Section 2 we present some auxiliary results concerning improved Hardy inequalities with boundary terms. In Section 3 we prove the Hardy–Sobolev inequalities; in Section 4 we apply them to obtain heat kernel estimates. Finally, in Section 5 we prove refined Sobolev inequalities and heat kernel estimates when additional singularities are present.

2. Two minimization problems

Throughout this section, $\Omega \subset \mathbf{R}^N$, $N \geqslant 3$, is a bounded domain containing the origin with C^1 boundary. Also, we always denote by v the outward-pointing (with respect to Ω) unit vector on the surface $\partial\Omega$. In Section 2.1 we will work on Ω, while in Section 2.2 we will work on Ω^c. The results of this section will be applied in Section 3.

2.1. Bounded domains

For $\alpha > 0$ we define

$$\lambda_{\Omega}(\alpha) = \inf_{H^1(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 \, dx + \alpha \int_{\partial\Omega} \frac{x \cdot v}{|x|^2} u^2 \, dS}{\int_{\Omega} \frac{u^2}{|x|^2} \, dx}. \tag{2.1}$$

Lemma 2.1. *We have*:

(i) *If* $0 < \alpha \leqslant \frac{N-2}{2}$, *then* $\lambda_{\Omega}(\alpha) = \alpha(N-2-\alpha)$. *Moreover*, $|x|^{-\alpha} \in H^1(\Omega)$ *is a minimizer for* $0 < \alpha < \frac{N-2}{2}$, *whereas for* $\alpha = \frac{N-2}{2}$ *there is no* $H^1(\Omega)$ *minimizer*.

(ii) *If* $\alpha > \frac{N-2}{2}$ *and* Ω *is starshaped with respect to zero*, *then* $\lambda_{\Omega}(\alpha) = (\frac{N-2}{2})^2$ *and there is no* $H^1(\Omega)$ *minimizer*.

In case Ω is not starshaped with respect to zero, concerning the analogue of part (ii) of the above lemma, we have

Lemma 2.2. *Suppose Ω is not starshaped with respect to zero. Then, there exist finite constants $\alpha^* \geqslant N-2$ and $\alpha_* \in [\frac{N-2}{2}, \alpha^*)$ depending on Ω such that:*

(i) $\lambda_\Omega(\alpha^*) = 0$, *whereas $\lambda_\Omega(\alpha) > 0$ for all $\frac{N-2}{2} < \alpha < \alpha^*$.*

(ii) *If $\frac{N-2}{2} \leqslant \alpha \leqslant \alpha_*$, then $\lambda_\Omega(\alpha) = (\frac{N-2}{2})^2$ and there is no $H^1(\Omega)$ minimizer.*

(iii) *If $\alpha_* < \alpha < \alpha^*$, then* $\max(0, \alpha(N-2-\alpha)) \leqslant \lambda_\Omega(\alpha) < (\frac{N-2}{2})^2$, *and there exists an $H^1(\Omega)$ minimizer.*

Remark. (1) We note in particular that for any Ω and any $\alpha > 0$ there holds

$$\alpha(N-2-\alpha) \leqslant \lambda_\Omega(\alpha) \leqslant \left(\frac{N-2}{2}\right)^2. \tag{2.2}$$

(2) We do not know whether there exists a nonstarshaped domain Ω with smooth boundary so that $\alpha^* = N-2$. Similarly, we do not know whether there exists such an Ω for which $\alpha_* = (N-2)/2$.

Proof of Lemmas 2.1 and 2.2. Let $u \in C^\infty(\bar{\Omega})$ be supported outside a neighbourhood of zero. For any $\alpha > 0$ we set $u(x) = |x|^{-\alpha} v(x)$. A straightforward calculation shows that

$$\int_\Omega |\nabla u|^2\, dx = \int_\Omega |x|^{-2\alpha} |\nabla v|^2\, dx + \alpha(N-2-\alpha) \int_\Omega \frac{u^2}{|x|^2}\, dx - \alpha \int_{\partial\Omega} \frac{x \cdot v}{|x|^2} u^2\, dS, \tag{2.3}$$

therefore,

$$\int_\Omega |\nabla u|^2\, dx + \alpha \int_{\partial\Omega} \frac{x \cdot v}{|x|^2} u^2\, dS \geqslant \alpha(N-2-\alpha) \int_\Omega \frac{u^2}{|x|^2}\, dx. \tag{2.4}$$

By a simple density argument this inequality is valid for all $u \in H^1(\Omega)$. This implies in particular the lower bound on $\lambda_\Omega(\alpha)$ in (2.2).

If $0 < \alpha < \frac{N-2}{2}$, then $|x|^{-\alpha}$ is in $H^1(\Omega)$ and an easy calculation shows that it satisfies (2.4) as equality, hence it is a minimizer. If $\alpha = \frac{N-2}{2}$ then the fact that $\lambda_\Omega(\alpha) = (\frac{N-2}{2})^2$ follows by considering the functions $u_\varepsilon(x) = |x|^{-\frac{N-2}{2}+\varepsilon}$ in the limit $\varepsilon \to 0^+$.

Using the same functions, $u_\varepsilon(x) = |x|^{-\frac{N-2}{2}+\varepsilon}$, $\varepsilon > 0$, one can show that $(\frac{N-2}{2})^2 \geqslant \lambda_\Omega(\alpha)$, for any $\alpha > 0$, thus proving the upper bound in (2.2).

Suppose now that Ω is starshaped and $\alpha > \frac{N-2}{2}$. Then, using first the fact that $x \cdot \nu \geqslant 0$ on the boundary of Ω, and then (2.4) (with $\alpha = \frac{N-2}{2}$)

$$\int_\Omega |\nabla u|^2\, dx + \alpha \int_{\partial\Omega} \frac{x \cdot \nu}{|x|^2} u^2\, dS \geqslant \int_\Omega |\nabla u|^2\, dx + \frac{N-2}{2} \int_{\partial\Omega} \frac{x \cdot \nu}{|x|^2} u^2\, dS$$
$$\geqslant \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2}\, dx.$$

Hence, in this case $\lambda_\Omega(\alpha) = \left(\frac{N-2}{2}\right)^2$.

We next show that when $\lambda_\Omega(\alpha) = \left(\frac{N-2}{2}\right)^2$, there is no $H^1(\Omega)$ minimizer. Indeed assuming that there is one, then it would be a positive $H^1(\Omega)$ solution of the Euler–Lagrange equation

$$\Delta u + \frac{\left(\frac{N-2}{2}\right)^2}{|x|^2} u = 0, \quad x \in \Omega.$$

However, by Lemma 2.3 (see below), this equation has no $H^1(\Omega)$ positive solutions. Thus, Lemma 2.1 has been proved.

Suppose now that Ω is not starshaped with respect to zero. The existence of α^* follows from the continuity of $\lambda_\Omega(\alpha)$ with respect to α combined with the fact that if Ω is not starshaped with respect to zero, then one can easily find test functions making the surface integral in (2.1) negative. The fact that $\alpha^* \geqslant N-2$ follows from the lower bound in (2.2).

From Lemma 2.1(i), we have that $\lambda_\Omega(\frac{N-2}{2}) = (\frac{N-2}{2})^2$. We then define α_* as the supremum of all α for which $\lambda_\Omega(\alpha) = (\frac{N-2}{2})^2$. Assuming that $\alpha_* > \frac{N-2}{2}$, we will show that for any $\frac{N-2}{2} < \alpha < \alpha_*$ there holds $\lambda_\Omega(\alpha) = (\frac{N-2}{2})^2$. Indeed, if this is not the case then there would exist an α in the above interval and $\phi \in H^1(\Omega)$ such that

$$\frac{\int_\Omega |\nabla\phi|^2\, dx + \alpha \int_{\partial\Omega} \frac{x\cdot\nu}{|x|^2}\phi^2\, dS}{\int_\Omega \frac{\phi^2}{|x|^2}\, dx} < \left(\frac{N-2}{2}\right)^2 \tag{2.5}$$

On the other hand from Lemma 2.1(i), we have that

$$\frac{\int_\Omega |\nabla\phi|^2\, dx + (\frac{N-2}{2}) \int_{\partial\Omega} \frac{x\cdot\nu}{|x|^2}\, \phi^2\, dS}{\int_\Omega \frac{\phi^2}{|x|^2}\, dx} \geqslant \left(\frac{N-2}{2}\right)^2. \tag{2.6}$$

From the above two inequalities it follows that $\int_{\partial\Omega} \frac{x\cdot\nu}{|x|^2}\, \phi^2\, dS < 0$. Using ϕ as a test function and the fact that $\alpha_* > \alpha$ we conclude that $\lambda_\Omega(\alpha_*) < (\frac{N-2}{2})^2$, which is a contradiction. Thus, the estimates of parts (ii) and (iii) of Lemma 2.2 have been proved.

The nonexistence of $H^1(\Omega)$ minimizer of part (ii) follows exactly as in Lemma 2.1. The existence of $H^1(\Omega)$ minimizer of part (iii) will follow later from a more general result; see Proposition 2.6. $\square$

Lemma 2.3. *If Ω contains the origin then there is no $H^1(\Omega)$ positive solution of the equation*

$$\Delta u + \frac{(\frac{N-2}{2})^2}{|x|^2} u = 0, \quad x \in \Omega.$$

Proof. We note that this is a very special case of [FT, Theorem C] (although in this Theorem Dirichlet condition were imposed, the proof is independent of the boundary conditions). Since in the present case the argument is simple we sketch the proof.

Assuming that we have a positive $H^1(\Omega)$ solution we will reach a contradiction. Taking the surface average of u (over $\partial B(0,r)$)

$$v(r) = \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r} u(x)\, dS > 0,$$

an easy calculation shows that for r near zero,

$$v''(r) + \frac{N-1}{r} v'(r) + \frac{(\frac{N-2}{2})^2}{r^2} v(r) = 0.$$

Hence, $v(r) = c_1 r^{-\frac{N-2}{2}} + c_2 r^{-\frac{N-2}{2}} \ln r$ and the positivity of v implies that $v(r) \geqslant c r^{-\frac{N-2}{2}}$, for $c > 0$. From this and using Hölder's inequality we obtain that for small r,

$$\int_{\partial B_r} u^{\frac{2N}{N-2}}\, dS \geqslant \frac{c}{r},$$

from which it follows that $\int_\Omega u^{\frac{2N}{N-2}}\, dx = \infty$ contradicting the fact that $u \in H^1(\Omega)$. $\square$

2.2. Complement of bounded domains

Here we consider the complement of a bounded domain and we study the corresponding infimum, that is

$$\mu_\Omega(\alpha) = \inf_{u \in C_c^\infty(\mathbf{R}^N)|_{\Omega^c}} \frac{\int_{\Omega^c} |\nabla u|^2\, dx - \alpha \int_{\partial\Omega} \frac{x\cdot v}{|x|^2} u^2\, dS}{\int_{\Omega^c} \frac{u^2}{|x|^2}\, dx}. \tag{2.7}$$

where Ω, as before, is a bounded domain containing the origin and ν is the outward-pointing (with respect to Ω) unit vector on the surface $\partial\Omega$. Also, $C_c^\infty(\mathbf{R}^N)|_{\Omega^c}$ is the set of restrictions on Ω^c of all functions $u \in C_c^\infty(\mathbf{R}^N)$. We also introduce the following norms:

$$||u||_{\mathscr{D}^{1,2}(\Omega^c)} = \left(\int_{\Omega^c} |\nabla u|^2 \, dx\right)^{1/2} + \left(\int_{\Omega^c} |u|^{\frac{2N}{N-2}} \, dx\right)^{\frac{N-2}{2N}}, \tag{2.8}$$

$$||u||_{\mathscr{H}^1(\Omega^c)} = \left(\int_{\Omega^c} |\nabla u|^2 \, dx\right)^{1/2} + \left(\int_{\Omega^c} \frac{|u|^2}{|x|^2} \, dx\right)^{1/2}, \tag{2.9}$$

$$||u||_{\mathscr{W}(\Omega^c)} = \left(\int_{\Omega^c} |\nabla u|^2 \, dx\right)^{1/2} + \left(\int_{\partial\Omega} |u|^{\frac{2(N-1)}{N-2}} \, dS\right)^{\frac{N-2}{2(N-1)}}, \tag{2.10}$$

and we denote by $\mathscr{D}^{1,2}(\Omega^c)$, $\mathscr{H}^1(\Omega^c)$ and $\mathscr{W}(\Omega^c)$ the completion of $C_c^\infty(\mathbf{R}^N)|_{\Omega^c}$ under the corresponding norms. The space $\mathscr{W}(\Omega^c)$ is well studied by Maz'ya [M, Section 3.6, Chapter 4]. For our purposes however, the natural spaces to use are $\mathscr{D}^{1,2}(\Omega^c)$ and $\mathscr{H}^1(\Omega^c)$. In the next lemma we show that these three spaces coincide (a trivial fact if Ω^c were replaced by Ω).

Lemma 2.4. *Let Ω be a bounded domain in $\mathbf{R}^N$, $N \geqslant 3$, with C^1 boundary, containing the origin. Then $\mathscr{D}^{1,2}(\Omega^c) = \mathscr{H}^1(\Omega^c) = \mathscr{W}(\Omega^c)$.*

Proof. We will show that all norms are equivalent. Let $u \in C_c^\infty(\mathbf{R}^N)|_{\Omega^c}$. Under our assumptions, it follows easily as in Lemma 2.1 (cf. (2.4) with $\alpha = \frac{N-2}{2}$) that

$$\int_{\Omega^c} |\nabla u|^2 \, dx - \frac{N-2}{2} \int_{\partial\Omega} \frac{x \cdot \nu}{|x|^2} u^2 \, dS \geqslant \left(\frac{N-2}{2}\right)^2 \int_{\Omega^c} \frac{u^2}{|x|^2} \, dx, \tag{2.11}$$

whence,

$$\begin{aligned} \int_{\Omega^c} \frac{|u|^2}{|x|^2} \, dx &\leqslant C\left(\int_{\Omega^c} |\nabla u|^2 \, dx + \int_{\partial\Omega} |u|^2 \, dS\right) \\ &\leqslant C\left(\int_{\Omega^c} |\nabla u|^2 \, dx + \left(\int_{\partial\Omega} |u|^{\frac{2(N-1)}{N-2}} \, dS\right)^{\frac{N-2}{(N-1)}}\right). \end{aligned}$$

Hence, $||u||_{\mathscr{H}^1(\Omega^c)} \leqslant C||u||_{\mathscr{W}(\Omega^c)}$. To obtain the reverse inequality we note that it follows from the standard trace Theorem (e.g. [A, Theorem 5.22], Chapter V)—applied to $B\backslash\Omega$ for some ball $B\supset\Omega$—that

$$\left(\int_{\partial\Omega}|u|^{\frac{2(N-1)}{N-2}}\,dS\right)^{\frac{N-2}{2(N-1)}}\leqslant C||u||_{\mathscr{H}^1(\Omega^c)},$$

$$\left(\int_{\partial\Omega}|u|^{\frac{2(N-1)}{N-2}}\,dS\right)^{\frac{N-2}{2(N-1)}}\leqslant C||u||_{\mathscr{D}^{1,2}(\Omega^c)}.$$

From the first one it follows that $||u||_{\mathscr{W}(\Omega^c)}\leqslant C||u||_{\mathscr{H}^1(\Omega^c)}$, whence $\mathscr{H}^1(\Omega^c)=\mathscr{W}(\Omega^c)$. From the second one it follows that $||u||_{\mathscr{W}(\Omega^c)}\leqslant C||u||_{\mathscr{D}^{1,2}(\Omega^c)}$. Thus, it remains to prove that $||u||_{\mathscr{D}^{1,2}(\Omega^c)}\leqslant C||u||_{\mathscr{W}(\Omega^c)}$. This inequality follows from Corollary 1 of Section 4.11.1 [M, p. 258]. Notice that in the notation of Maz'ya $\mathscr{W}(\Omega^c)=W_{2,\frac{2(N-1)}{N-2}}(\Omega^c,\partial\Omega)$. □

An immediate consequence of the above lemma is that the infimum in (2.7) can be taken over $\mathscr{D}^{1,2}(\Omega^c)$, that is

$$\mu_\Omega(\alpha)=\inf_{u\in\mathscr{D}^{1,2}(\Omega^c)}\frac{\int_{\Omega^c}|\nabla u|^2\,dx-\alpha\int_{\partial\Omega}\frac{x\cdot v}{|x|^2}u^2\,dS}{\int_{\Omega^c}\frac{u^2}{|x|^2}\,dx}. \tag{2.12}$$

We now state the analogue of Lemma 2.1 for exterior domains. We recall that Ω is a bounded domain in $\mathbf{R}^N$, $N\geqslant 3$, containing the origin.

Lemma 2.5. *We have*:

(i) *If* $\alpha\geqslant\frac{N-2}{2}$, *then* $\mu_\Omega(\alpha)=\alpha(N-2-\alpha)$. *Moreover*, $|x|^{-\alpha}\in\mathscr{D}^{1,2}(\Omega^c)$ *is a minimizer for* $\alpha>\frac{N-2}{2}$, *whereas for* $\alpha=\frac{N-2}{2}$ *there is no* $\mathscr{D}^{1,2}(\Omega^c)$ *minimizer.*

(ii) *If* $0<\alpha<\frac{N-2}{2}$, *and* Ω *starshaped with respect to zero, then* $\mu_\Omega(\alpha)=(\frac{N-2}{2})^2$ *and there is no* $\mathscr{D}^{1,2}(\Omega^c)$ *minimizer.*

Proof. The proof is quite similar to the proof of the previous Lemmas 2.1 and 2.2. An alternative proof can be given using the Kelvin transform; see the Remark that follows. □

Remark. There is a duality between the minimization problems (2.1) and (2.7). Indeed, by means of the Kelvin transform, $u(x)=|y|^{N-2}v(y)$, $y=x/|x|^2$, $x\in\Omega^c$, the domain Ω^c is transformed to a bounded domain containing the origin that we denote by $(\Omega^c)^*$. Denoting by v^* the outward pointing normal to $\partial(\Omega^c)^*$ a straightforward calculation shows that

$$\int_{\Omega^c}|\nabla_x u|^2\,dx=\int_{(\Omega^c)^*}|\nabla_y v|^2\,dy+(N-2)\int_{\partial(\Omega^c)^*}\frac{y\cdot v^*}{|y|^2}v^2\,dS_y.$$

Also,

$$\int_{\Omega^c} \frac{|u|^2}{|x|^2}\,dx = \int_{(\Omega^c)^*} \frac{|v|^2}{|y|^2}\,dy,$$
$$\int_{\Omega^c} |u|^{\frac{2N}{N-2}}\,dx = \int_{(\Omega^c)^*} |v|^{\frac{2N}{N-2}}\,dy.$$

It can be seen from these relations that $u\in\mathscr{D}^{1,2}(\Omega^c)$ if and only if $v\in H^1((\Omega^c)^*)$. It then follows easily that $\mu_\Omega(\alpha)=\lambda_{(\Omega^c)^*}(N-2-\alpha)$, and that the existence of a minimizer for $\mu_\Omega(\alpha)$ in $\mathscr{D}^{1,2}(\Omega^c)$ is equivalent to the existence of a minimizer in $H^1(\Omega)$ for $\lambda_{(\Omega^c)^*}(N-2-\alpha)$.

2.3. *Existence of minimizers*

In this section we establish a sufficient condition for the existence of minimizers. We recall from Lemma 2.2 that when Ω is not starshaped with respect to the origin, α^* denotes the first zero of $\lambda_\Omega(\alpha)$. We also set $\alpha^*=\infty$ in case Ω is starshaped with respect to zero. Thus, in both cases we have $\lambda_\Omega(\alpha)>0$ for $0<\alpha<\alpha^*$. Given $0<\alpha<\alpha^*$, and a nonnegative measurable potential V we define

$$\lambda_\Omega(\alpha,V) := \inf_{\substack{u\in H^1(\Omega)\\ \int_\Omega Vu^2\,dx>0}} \frac{\int_\Omega |\nabla u|^2\,dx + \alpha\int_{\partial\Omega}\frac{x\cdot\nu}{|x|^2}u^2\,dS}{\int_\Omega Vu^2\,dx}. \tag{2.13}$$

Note that with this notation $\lambda_\Omega(\alpha)=\lambda_\Omega(\alpha,|x|^{-2})$. Since the numerator in (2.13) is always positive and finite when $0<\alpha<\alpha^*$, we interpret $\lambda_\Omega(\alpha,V)=0$ in case there exists $u\in H^1(\Omega)$ such that $\int_\Omega Vu^2\,dx=+\infty$. It is worth mentioning that $\lambda_\Omega(\alpha,V)$ is not monotone with respect to Ω, unlike the case of Dirichlet boundary conditions.

We denote by $B_r\subset\Omega$ the ball centered at zero with radius r. We have the following

Proposition 2.6. *Let Ω be a bounded domain in $\mathbf{R}^N$, $N\geqslant 3$, containing the origin, and let $0\leqslant V\in L^{\frac{N}{2}}_{\mathrm{loc}}(\bar\Omega\backslash\{0\})$. If for some $r>0$*

$$0<\lambda_\Omega(\alpha,V)<\lambda_{B_r}(\alpha,V) \tag{2.14}$$

then (2.13) *has an $H^1(\Omega)$ minimizer.*

Note. It is a consequence of (2.14) that $\int_\Omega Vu^2\,dx<+\infty$ for $u\in H^1(\Omega)$.

Proof. Let $\{u_j\}\in H^1(\Omega)$ be a minimizing sequence of the Rayleigh quotient in (2.13). We may normalize it so that $\int_\Omega Vu_j^2\,dx=1$. We claim that $\|u_j\|_{H^1(\Omega)}<C$. This will

follow from two inequalities. The first inequality follows from the fact that $0<\alpha<\alpha^*$ and $\lambda_\Omega(\alpha^*) = 0$ and reads

$$\int_\Omega |\nabla u|^2\,dx + \alpha \int_{\partial\Omega} \frac{x\cdot\nu}{|x|^2} u^2\,dS \geqslant \left(1 - \frac{\alpha}{\alpha^*}\right) \int_\Omega |\nabla u|^2\,dx. \tag{2.15}$$

The second one is a consequence of Lemma 2.1 and reads

$$\begin{aligned}\int_\Omega |\nabla u|^2\,dx + \alpha \int_{\partial\Omega} \frac{x\cdot\nu}{|x|^2} u^2\,dS &\geqslant \lambda_\Omega(\alpha) \int_\Omega \frac{u^2}{|x|^2}\,dx \\ &\geqslant K\lambda_\Omega(\alpha) \int_\Omega u^2\,dx.\end{aligned} \tag{2.16}$$

Thus, we may extract a subsequence such that $u_j \rightharpoonup u_0$ weakly in $H^1(\Omega)$, and $u_j \to u_0$ strongly in $L^p(\Omega)$, $1<p<\frac{2N}{N-2}$. Moreover, since $V \in L^{N/2}(\Omega \backslash B_r)$, standard results (see for instance [T]) give

$$\int_{\Omega\backslash B_r} V u_j^2\,dx \to \int_{\Omega\backslash B_r} V u_0^2\,dx. \tag{2.17}$$

Also by the trace theorem

$$\int_{\partial\Omega} \frac{x\cdot\nu}{|x|^2} u_j^2\,dS \to \int_{\partial\Omega} \frac{x\cdot\nu}{|x|^2} u_0^2\,dS. \tag{2.18}$$

Setting $u_j = v_j + u_0$ we easily see that as $j \to \infty$,

$$\int_\Omega |\nabla u_j|^2\,dx = \int_\Omega |\nabla u_0|^2\,dx + \int_\Omega |\nabla v_j|^2\,dx + o(1) \tag{2.19}$$

and

$$1 = \int_\Omega V u_j^2\,dx = \int_\Omega V u_0^2\,dx + \int_\Omega V v_j^2\,dx + o(1). \tag{2.20}$$

It then follows from (2.13) that

$$\begin{aligned}\lambda_\Omega(\alpha, V) &= \int_\Omega |\nabla v_j|^2\,dx + \int_\Omega |\nabla u_0|^2\,dx + \alpha \int_{\partial\Omega} \frac{x\cdot\nu}{|x|^2} u_0^2\,dS + o(1) \\ &\geqslant \int_\Omega |\nabla v_j|^2\,dx + \lambda_\Omega(\alpha, V) \int_\Omega V u_0^2\,dx + o(1).\end{aligned} \tag{2.21}$$

We then have

$$\begin{aligned}\int_\Omega |\nabla v_j|^2\,dx &\geqslant \int_{B_r} |\nabla v_j|^2\,dx\\ &\geqslant \lambda_{B_r}(\alpha,V)\int_{B_r} Vv_j^2\,dx-\alpha\int_{\partial B_r}\frac{x\cdot\nu}{|x|^2}v_j^2\,dS\\ &=\lambda_{B_r}(\alpha,V)\int_{B_r} Vv_j^2\,dx+o(1)\\ &=\lambda_{B_r}(\alpha,V)\int_{\Omega} Vv_j^2\,dx+o(1)\\ &=\lambda_{B_r}(\alpha,V)\left(1-\int_\Omega Vu_0^2\,dx\right)+o(1)\quad (j\to\infty).\end{aligned}\tag{2.22}$$

Using this and (2.21) we end up with

$$(\lambda_\Omega(\alpha,V)-\lambda_{B_r}(\alpha,V))\left(1-\int_\Omega Vu_0^2\,dx\right)\geqslant 0,\tag{2.23}$$

whence, since $\lambda_\Omega(\alpha,V)<\lambda_{B_r}(\alpha,V)$, it follows that $\int_\Omega Vu_0^2\,dx\geqslant 1$. By lower semi continuity we conclude that $\int_\Omega Vu_0^2\,dx=1$. It then follows that u_0 is a minimizer for (2.13). □

As a consequence we have:

Completion of Proof of Lemma 2.12(iii) (*Existence of a minimizer*): Since $\frac{N-2}{2}\leqslant\alpha_*<\alpha<\alpha^*$ it follows from Lemma 2.2 that $0<\lambda_\Omega(\alpha)<(\frac{N-2}{2})^2$. If $B_r\subset\Omega$ is a ball centred at zero it follows by Lemma 2.1 that $\lambda_{B_r}(\alpha)=(\frac{N-2}{2})^2$. By Proposition 2.6, $\lambda_\Omega(\alpha)$ is attained by an $H^1(\Omega)$ function. □

We next state the corresponding result for the exterior of a bounded domain Ω. For $0<a<N-2$ we define

$$\mu_\Omega(\alpha,V):=\inf_{\substack{u\in\mathscr{D}^{1,2}(\Omega^c)\\ \int_{\Omega^c} Vu^2\,dx>0}}\frac{\int_{\Omega^c}|\nabla u|^2\,dx-\alpha\int_{\partial\Omega}\frac{x\cdot\nu}{|x|^2}u^2\,dS}{\int_{\Omega^c} Vu^2\,dx}.\tag{2.24}$$

We then have

Proposition 2.7. *Let Ω be a bounded domain in $\mathbf{R}^N$, $N\geqslant 3$, containing the origin, and let $0\leqslant V\in L^{\frac{N}{2}}_{\mathrm{loc}}(\overline{\Omega^c})$. Also let $0<\alpha<N-2$. If for some ball $B_R\supset\Omega$ centered at zero*

$$0<\mu_\Omega(\alpha,V)<\mu_{B_R}(\alpha,V),\tag{2.25}$$

then (2.24) *has a $\mathscr{D}^{1,2}(\Omega^c)$ minimizer.*

The proof is similar to that of the previous proposition.

3. Hardy–Sobolev inequalities

3.1. Auxiliary inequalities

We begin this section with two known Sobolev-type inequalities that will be used in the sequel. In Theorems 3.4 and 3.5 we then prove two new Sobolev inequalities.

By the classical inequality of Caffarelli–Kohn–Nirenberg [CKN] we have that

$$\int_{\mathbf{R}^N} |\nabla w|^2 |x|^{-2\alpha}\, dx \geqslant c\left(\int_{\mathbf{R}^N} |w|^p |x|^{-p\beta}\, dx\right)^{2/p}, \quad w \in C_c^\infty(\mathbf{R}^N), \tag{3.1}$$

with $p = 2N/(N-2+2(\beta-\alpha))$, provided $\alpha < (N-2)/2$ and $0 \leqslant \beta - \alpha \leqslant 1$.

At the critical case $\alpha = \beta = (N-2)/2$ inequality (3.1) fails. A sharp substitute for bounded Ω was obtained in [FT], where it was shown that, with

$$X_1(t) = (1 - \log t)^{-1}, \quad t \in (0,1), \tag{3.2}$$

and $D = \sup_\Omega |x|$ there holds

$$\int_\Omega |\nabla w|^2 |x|^{2-N}\, dx \geqslant c\left(\int_\Omega |w|^{\frac{2N}{N-2}} |x|^{-N} X_1^{\frac{2N-2}{N-2}}\left(\frac{|x|}{D}\right) dx\right)^{(N-2)/N}, \tag{3.3}$$

for all $w \in C_c^\infty(\Omega)$, where the exponent $\frac{2N-2}{N-2}$ of $X_1(|x|/D)$ is optimal.

In the sequel we will make essential use of the following one-dimensional result, which is a special case of a more general statement by Maz'ya, cf. [M, Theorem 3, Section 1.3.1, p. 44]:

Proposition 3.1. *Let $A(r)$, $B(r)$ nonnegative functions such that $1/A(r)$ and $B(r)$ are integrable in (r, ∞) and $(0, r)$, respectively, for all positive $r < \infty$. Then, for $q \geqslant 2$ the Sobolev inequality*

$$\int_0^\infty (v'(r))^2 A(r)\, dr \geqslant c\left(\int_0^\infty |v(r)|^q B(r)\, dr\right)^{2/q}$$

is valid for all $v \in C^1(0, \infty)$ that vanish near infinity, if and only if

$$\sup_{r>0}\left(\int_0^r B(t)\, dt\right)\left(\int_r^\infty \frac{dt}{A(t)}\right)^{q/2} < +\infty.$$

The above proposition will be applied to higher dimensions by means of the following

Lemma 3.2. *Let* $N \geqslant 2$. *Suppose that* $V \in L^{\infty}_{\mathrm{loc}}(\mathbf{R}^N \backslash \{0\}) \cap L^1_{\mathrm{loc}}(\mathbf{R}^N)$ *is a radially symmetric function. We further assume that inequality*

$$\int_{\mathbf{R}^N} |\nabla u|^2 \, dx - \int_{\mathbf{R}^N} V u^2 \, dx \geqslant 0, \tag{3.4}$$

is valid for all radially symmetric functions $u \in C_c^{\infty}(\mathbf{R}^N)$.

(i) *Then*, (3.4) *is also valid for nonradial functions*, *that is*, *for all* $u \in C_c^{\infty}(\mathbf{R}^N)$.

(ii) *If*, *in addition*,

$$0 < \operatorname*{ess\,sup}_{x \in \mathbf{R}^N} |x|^2 V(x) = \theta < \infty, \tag{3.5}$$

then the following improved inequality holds:

$$\begin{aligned} &\int_{\mathbf{R}^N} |\nabla u|^2 \, dx - \int_{\mathbf{R}^N} V u^2 \, dx \\ &\quad \geqslant \int_{\mathbf{R}^N} |\nabla u_0|^2 \, dx - \int_{\mathbf{R}^N} V u_0^2 \, dx + \frac{N-1}{N-1+\theta} \int_{\mathbf{R}^N} |\nabla (u - u_0)|^2 \, dx, \end{aligned} \tag{3.6}$$

where $u_0(r)$ *denotes the spherical average of* $u \in C_c^{\infty}(\mathbf{R}^N)$, *that is*

$$u_0(r) = \frac{1}{N \omega_N r^{N-1}} \int_{\partial B_r(0)} u(x) \, dS_x, \quad r > 0. \tag{3.7}$$

Proof. Let $u \in C_c^{\infty}(\mathbf{R}^N)$ and let

$$u(x) = \sum_{m=0}^{\infty} f_m(\sigma) u_m(r)$$

be its decomposition into spherical harmonics; here f_m are orthogonal in $L^2(S^{N-1})$, normalized by $\frac{1}{N\omega_N} \int_{S^{N-1}} f_i(\sigma) f_j(\sigma) \, dS = \delta_{ij}$. In particular $f_0(\sigma) = 1$ and the first term in the above decomposition is given by (3.7). The f_m's are eigenfunctions of the Laplace–Beltrami operator, with corresponding eigenvalues $c_m = m(N - 2 + m)$, $m \geqslant 0$. An easy calculation shows that

$$\begin{aligned} \int_{\mathbf{R}^N} (|\nabla u|^2 - V u^2) \, dx &= \sum_{m=0}^{\infty} \int_{\mathbf{R}^N} \left\{ |\nabla u_m|^2 + \left(\frac{c_m}{|x|^2} - V \right) u_m^2 \right\} dx \\ &= \int_{\mathbf{R}^N} (|\nabla u_0|^2 - V u_0^2) \, dx \\ &\quad + \sum_{m=1}^{\infty} \int_{\mathbf{R}^N} \left\{ |\nabla u_m|^2 + \left(\frac{c_m}{|x|^2} - V \right) u_m^2 \right\} dx. \end{aligned} \tag{3.8}$$

Part (i) follows immediately since $c_m > 0$ and $u_m = u_m(r)$, $r = |x|$. To prove part (ii) we first observe that

$$\int_{\mathbf{R}^N} |\nabla(u-u_0)|^2\,dx = \sum_{m=1}^{\infty} \int_{\mathbf{R}^N} \left\{ |\nabla u_m|^2 + \frac{c_m}{|x|^2} u_m^2 \right\} dx.$$

In view of this and (3.8) it is enough to establish that for any $m \geqslant 1$, there holds

$$\int_{\mathbf{R}^N} \left\{ |\nabla u_m|^2 + \left(\frac{c_m}{|x|^2} - V \right) u_m^2 \right\} dx \geqslant \frac{N-1}{N-1+\theta} \int_{\mathbf{R}^N} \left\{ |\nabla u_m|^2 + \frac{c_m}{|x|^2} u_m^2 \right\} dx \tag{3.9}$$

or, equivalently,

$$\int_{\mathbf{R}^N} |\nabla u_m|^2\,dx \geqslant \int_{\mathbf{R}^N} u_m^2 \left\{ \frac{N-1+\theta}{\theta} V - \frac{c_m}{|x|^2} \right\} dx.$$

Since $c_m \geqslant c_1 = N-1$ it is enough to establish this for $c_m = N-1$. By the definition of θ, cf. (3.5), it follows easily that

$$\frac{N-1+\theta}{\theta} V - \frac{N-1}{|x|^2} \geqslant V,$$

and the result follows from (3.4). □

As a consequence of this we next establish the following result.

Lemma 3.3. *Let* $N \geqslant 3$. *Suppose that* $V \in L^\infty_{\mathrm{loc}}(\mathbf{R}^N \backslash \{0\}) \cap L^1_{\mathrm{loc}}(\mathbf{R}^N)$ *is a radially symmetric function, such that*

$$0 < \operatorname*{ess\,sup}_{x \in \mathbf{R}^N} |x|^2 V(x) = \theta < \infty,$$

and $W \in L^\infty(\mathbf{R}^N)$ *is a positive radially symmetric function. We further assume that the inequality*

$$\int_{\mathbf{R}^N} |\nabla u|^2\,dx - \int_{\mathbf{R}^N} V u^2\,dx \geqslant c \left(\int_{\mathbf{R}^N} |u|^{2N/(N-2)} W\,dx \right)^{(N-2)/N} \tag{3.10}$$

is valid for all radially symmetric functions $u \in C_c^\infty(\mathbf{R}^N)$. *Then inequality* (3.10) *is true for all* $u \in C_c^\infty(\mathbf{R}^N)$ (*without radial symmetry*), *provided the constant c is replaced by a new constant C depending on c, N,* θ *and* $\|W\|_{L^\infty}$.

Proof. Starting from (3.6) we compute

$$\begin{aligned}
&\int_{\mathbf{R}^N} \{|\nabla u|^2 - Vu^2\}\,dx \\
&\geqslant \int_{\mathbf{R}^N} (|\nabla u_0|^2 - Vu_0^2)\,dx + \frac{N-1}{N-1+\theta}\int_{\mathbf{R}^N} |\nabla(u-u_0)|^2\,dx \\
&\geqslant c\left(\int_{\mathbf{R}^N} |u_0|^{2N/(N-2)} W\,dx\right)^{(N-2)/N} + c'\left(\int_{\mathbf{R}^N} |u-u_0|^{2N/(N-2)}\,dx\right)^{(N-2)/N} \\
&\geqslant C\left(\int_{\mathbf{R}^N} |u|^{2N/(N-2)} W\,dx\right)^{(N-2)/N},
\end{aligned}$$

where, for the last inequalities we used the standard Sobolev inequality, the boundedness of W and the triangle inequality. □

3.2. *Hardy–Sobolev inequalities*

In this section we prove improved Hardy–Sobolev inequalities for potentials that are critical either near zero or near infinity. We first consider a potential which is critical near zero. For $\varepsilon > 0$ we define

$$V_\varepsilon(x) = \begin{cases} \left(\frac{N-2}{2}\right)^2 |x|^{-2}, & |x| < 1, \\ \varepsilon f(x), & |x| \geqslant 1, \end{cases} \tag{3.11}$$

where f is a nonnegative, continuous and radially symmetric function on $\{|x| \geqslant 1\}$. Moreover we assume f to be subcritical, satisfying

$$f(x) \leqslant K|x|^{-2-\sigma}, \quad |x| \geqslant 1, \tag{3.12}$$

for some $\sigma, K > 0$.

Also, for X_1 as in (3.2) we define the auxiliary function

$$\tilde{X}_1(|x|) = \begin{cases} X_1(|x|), & |x| < 1, \\ 1, & |x| > 1. \end{cases} \tag{3.13}$$

We shall henceforth denote by B the unit ball in $\mathbf{R}^N$ centered at zero, by B^c its complement, and, as before, we denote by $C_c^\infty(\mathbf{R}^N)|_{B^c}$ the set of restrictions on B^c of all functions $u \in C_c^\infty(\mathbf{R}^N)$. We also denote by ν the outward-pointing unit vector on the surface ∂B. We have the following:

Theorem 3.4. *Let*

$$\varepsilon_0 = \inf_{u \in \mathscr{H}^1(B^c)} \frac{\int_{B^c} |\nabla u|^2\,dx - \frac{N-2}{2}\int_{\partial B} u^2\,dS}{\int_{B^c} f u^2\,dx}. \tag{3.14}$$

Then $\varepsilon_0 > 0$ *and for any* $\varepsilon \in (0, \varepsilon_0)$ *there holds*

$$\int_{\mathbf{R}^N} |\nabla u|^2\, dx - \int_{\mathbf{R}^N} V_\varepsilon u^2\, dx \geqslant c\left(\int_{\mathbf{R}^N} |u|^{2N/(N-2)} \tilde{X}_1^{(2N-2)/(N-2)}(|x|)\, dx\right)^{(N-2)/N}, \tag{3.15}$$

for all $u \in C_c^\infty(\mathbf{R}^N)$. *Moreover*, (3.15) *fails for* $\varepsilon = \varepsilon_0$.

Proof. Since $f(x) \leqslant K|x|^{-2}$ the positivity of ε_0 follows from Lemma 2.5 with $a = \frac{N-2}{2}$, yielding in fact $\varepsilon_0 \geqslant K^{-1}((N-2)/2)^2$.

Let us now fix $\varepsilon \in (0, \varepsilon_0)$. By Lemma 3.3, it is enough to prove (3.15) in the case where u is radially symmetric, $u = u(r)$. Now, there exists a radially symmetric and positive function $\tilde{\psi}$ on B^c which solves the Robin problem

$$\Delta\tilde{\psi} + \varepsilon f \tilde{\psi} = 0, \quad |x| > 1,$$

$$\frac{\partial \tilde{\psi}}{\partial v} = -\frac{N-2}{2}\tilde{\psi}, \quad |x| = 1.$$

The existence of such a $\tilde{\psi}$ can be easily derived, for example, by a shooting argument from $\{|x| = 1\}$. We assume that $\tilde{\psi}$ is normalized so that $\tilde{\psi} = 1$ on $\{|x| = 1\}$. The function

$$\psi(x) = \begin{cases} |x|^{-(N-2)/2}, & |x| < 1, \\ \tilde{\psi}(x), & |x| > 1, \end{cases} \tag{3.16}$$

then lies in $C^1(\mathbf{R}^N \backslash \{0\})$, is positive, radially symmetric and satisfies $\Delta\psi + V_\varepsilon \psi = 0$ in $\mathbf{R}^N$. Following [FT] we change variables, $u = \psi v$, and (3.15) for radially symmetric functions is then written as

$$\int_0^\infty (v')^2 \psi^2 r^{N-1}\, dr \geqslant c\left(\int_0^\infty |v|^{2N/(N-2)} \psi^{2N/(N-2)} \tilde{X}_1^{2(N-1)/(N-2)}\, dr\right)^{(N-2)/N}. \tag{3.17}$$

We claim that $\psi(r)$ has a positive limit as $r \to +\infty$. Indeed, since $(r^{N-1}\psi')' = -r^{N-1} V_\varepsilon \psi < 0$ and $\psi'(1) < 0$, $\psi(r)$ is decreasing on $(1, +\infty)$. If the limit $\lim_{r\to+\infty} \psi(r)$ were zero it would then follow from [LN, Theorem 2.9] that $\psi(r) < cr^{2-N}$ near infinity, which then easily implies $\psi \in \mathscr{H}^1(B^c)$. Hence ψ can be taken as a test function for the infimum in the right-hand side of (3.14), in which case the value of the Rayleigh quotient is $\varepsilon < \varepsilon_0$, contradicting the definition of ε_0. Hence $\lim_{r\to+\infty} \psi(r) = l > 0$. Using this we deduce (3.17) from Proposition 3.1. Hence (3.15) has been proved.

We finally show that (3.15) fails for $\varepsilon = \varepsilon_0$. For this we will use Proposition 2.7. Let $B_R \supset B_1$. Then $V_{\varepsilon_0}(x) \leqslant \varepsilon_0 K R^{-\sigma} |x|^{-2}$ on $(B_R)^c$, hence

$$\mu_{B_R}(\alpha, V_{\varepsilon_0}) \geqslant \frac{R^\sigma}{K} \mu_{B_R}(\alpha).$$

By Lemma 2.5 (i), for $\alpha\in(0,\ N-2)\ \mu_{B_R}(\alpha)$, is positive and independent of R; taking R large enough we have $\mu_B(\alpha, V_{\varepsilon_0})<\mu_{B_R}(\alpha, V_{\varepsilon_0})$. Hence, by Proposition 2.7—with $\alpha=(N-2)/2$—there exists an $\mathscr{H}^1(B^c)$-minimizer ϕ to (3.14). It is standard to show that ϕ is simple, radial and of one sign; we normalize it by $\phi|_{|x|=1}=1$ and for $\theta>0$ we define the function $u_\theta\in H^1(\mathbf{R}^N)$ by

$$u_\theta(x)=\begin{cases}|x|^{-\frac{N-2}{2}+\theta}, & |x|<1,\\ \phi(x), & |x|>1.\end{cases}$$

We then compute the left-hand side of (3.15): in B there holds $\Delta u_\theta+V_{\varepsilon_0}u_\theta=\theta^2 u_\theta$, hence

$$\begin{aligned}&\int_{\mathbf{R}^N}(|\nabla u_\theta|^2-V_{\varepsilon_0}u_\theta^2)\,dx\\ &\quad=-\int_B(u_\theta\Delta u_\theta+V_{\varepsilon_0}u_\theta^2)\,dx+\int_{\partial B}u_\theta\frac{\partial u_\theta}{\partial v}\,dS+\int_{B^c}(|\nabla u_\theta|^2-V_{\varepsilon_0}u_\theta^2)\,dx\\ &\quad=-\theta^2\int_B\frac{u_\theta^2}{r^2}\,dx+N\omega_N\left(-\frac{N-2}{2}+\theta\right)+\frac{N-2}{2}\int_{\partial B}u_\theta^2\,dS\\ &\quad=\frac{N\omega_N\theta}{2}.\end{aligned}$$

On the other hand for the right-hand side of (3.15) we have

$$\int_{\mathbf{R}^N}u_\theta^{2N/(N-2)}\tilde{X}_1^{\frac{2N-2}{N-2}}\,dx\geqslant\int_{B^c}u_\theta^{2N/(N-2)}\,dx=\int_{B^c}\phi^{2N/(N-2)},$$

the last term being independent of θ. Letting $\theta\to 0$ we conclude that (3.15) fails for $\varepsilon=\varepsilon_0$. $\square$

We close this section proving a Sobolev inequality which involves radial potentials with critical behaviour $((N-2)/2)^2|x|^{-2}$ near infinity. Let g be a nonnegative, radially symmetric, and continuous in $B\backslash\{0\}$ function that is subcritical near zero, satisfying

$$g(x)\leqslant K|x|^{-2+\sigma},\quad |x|\leqslant 1,$$

for some σ, $K>0$. For $\varepsilon>0$ we define

$$\hat{V}_\varepsilon(x)=\begin{cases}\varepsilon g(x), & |x|<1,\\ \left(\frac{N-2}{2}\right)^2|x|^{-2}, & |x|>1.\end{cases}\tag{3.18}$$

We also set

$$\tilde{Y}_1(|x|) = \begin{cases} 1, & |x| < 1, \\ (1 + \ln|x|)^{-1}, & |x| > 1. \end{cases}$$

We then have

Theorem 3.5. *Let*

$$\bar{\varepsilon}_0 = \inf_{u \in H^1(B)} \frac{\int_B |\nabla u|^2 \, dx + \frac{N-2}{2} \int_{\partial B} u^2 \, dS}{\int_B g u^2 \, dx}.$$

Then $\bar{\varepsilon}_0 > 0$ *and for any* $\varepsilon \in (0, \bar{\varepsilon}_0)$ *there holds*

$$\int_{\mathbf{R}^N} |\nabla u|^2 \, dx - \int_{\mathbf{R}^N} \hat{V}_\varepsilon u^2 \, dx \geqslant c \left(\int_{\mathbf{R}^N} |u|^{2N/(N-2)} \tilde{Y}_1^{2(N-1)/(N-2)}(|x|) \, dx \right)^{(N-2)/N}, \tag{3.19}$$

for all $u \in C_c^\infty(\mathbf{R}^N)$. *Moreover*, (3.19) *fails for* $\varepsilon = \bar{\varepsilon}_0$.

Proof. The proof of (3.19) follows closely that of Theorem 3.4, reversing essentially the role of B and B^c while making the necessary adjustments; in particular, we now use Lemma 2.1 instead of Lemma 2.5. In fact, an alternative and simpler proof consists in simply taking the Kelvin transform of (3.15). The optimality of $\bar{\varepsilon}_0$ is also proven analogously; we omit the details. □

4. Heat kernel estimates

In this section we shall apply the Sobolev inequalities of Section 3 to obtain heat kernel estimates for the Schrödinger operator

$$Hu = -\Delta u - Vu, \quad u|_{\partial\Omega} = 0,$$

for various critical potentials V. We still assume that $\Omega \subset \mathbf{R}^N$, $N \geqslant 3$, is a domain containing the origin and we will consider the case of bounded Ω as well as the case $\Omega = \mathbf{R}^N$. The operator H is defined via quadratic forms, with initial domain $C_c^1(\Omega \backslash \{0\})$; it will always be the case that $H \geqslant 0$. Note that, equivalently, we could have set $C_c^1(\Omega)$ as the initial domain.

We shall use the standard technique of transference to a weighted L^2 space, which we now describe briefly. Let $\phi \in C^1(\Omega \backslash \{0\})$ be positive and such that $\Delta\phi \in L^1_{\mathrm{loc}}(\Omega \backslash \{0\})$. The unitary map

$$L^2(\Omega) \ni u \mapsto w = \frac{u}{\phi} \in L^2_\phi := L^2(\Omega, \phi^2 \, dx) \tag{4.1}$$

satisfies

$$\int_\Omega (|\nabla u|^2 - Vu^2)\,dx = \int_\Omega \left(|\nabla w|^2 - Vw^2 - \frac{\Delta\phi}{\phi} w^2\right)\phi^2\,dx \tag{4.2}$$

for all $u\in C_c^1(\Omega\backslash\{0\})$. Hence, if in addition ϕ satisfies $\Delta\phi + V\phi = 0$ (weakly) on $\Omega\backslash\{0\}$, then

$$\int_\Omega (|\nabla u|^2 - Vu^2)\,dx = \int_\Omega |\nabla w|^2\phi^2\,dx \tag{4.3}$$

for all $u\in C_c^1(\Omega\backslash\{0\})$. Hence H is unitarily equivalent via (4.1) to the self-adjoint operator H_ϕ on L^2_ϕ, defined initially on $C_c^1(\Omega\backslash\{0\})$ and given formally by

$$H_\phi w = -\frac{1}{\phi^2}\mathrm{div}(\phi^2\nabla w), \quad w|_{\partial\Omega} = 0.$$

The space $C_c^1(\Omega\backslash\{0\})$ is invariant under multiplication by either ϕ or $1/\phi$ and hence it is a form core also for H_ϕ. Moreover, a Sobolev inequality of the form

$$\int_\Omega (|\nabla u|^2 - Vu^2)\,dx \geqslant c\left(\int_\Omega |u|^q W\,dx\right)^{2/q}$$

is valid for all $u\in C_c^1(\Omega\backslash\{0\})$ if and only if

$$\int_\Omega |\nabla w|^2\phi^2\,dx \geqslant c\left(\int_\Omega |w|^q\phi^q W\,dx\right)^{2/q}$$

for all $w\in C_c^1(\Omega\backslash\{0\})$. Finally, the heat kernels of H and H_ϕ are related by

$$K(t,x,y) = \phi(x)\phi(y)K_\phi(t,x,y), \quad t>0,\ x,y\in\Omega, \tag{4.4}$$

and hence one can obtain estimates on $K(t,x,y)$ via estimates on $K_\phi(t,x,y)$.

Example. As a typical example let us consider the case of a bounded domain Ω in $\mathbf{R}^N$, $N\geqslant 3$, and let $V(x) = \lambda|x|^{-2}$, $\lambda\leqslant((N-2)/2)^2$. Let $\phi(x) = |x|^{-\alpha}$, α being the smallest solution of $\alpha(N-2-\alpha) = \lambda$. Then $\Delta\phi + V\phi = 0$ on $\Omega\backslash\{0\}$ and therefore

$$\int_\Omega (|\nabla u|^2 - \lambda\frac{u^2}{|x|^2})\,dx = \int_\Omega |\nabla w|^2|x|^{-2\alpha}\,dx \tag{4.5}$$

for all $u \in C_c^1(\Omega \setminus \{0\})$ or, equivalently, for all $w \in C_c^1(\Omega \setminus \{0\})$. Moreover, the heat kernel of $H = -\Delta - V$ is related to the heat kernel of H_ϕ by

$$K(t,x,y) = |x|^{-\alpha}|y|^{-\alpha}K_\phi(t,x,y).$$

We note here that a simple approximation argument shows that for $\lambda < ((N-2)/2)^2$ the form domain of H is $H_0^1(\Omega)$, but at the critical case $\lambda = ((N-2)/2)^2$ the form domain is strictly larger than $H_0^1(\Omega)$; see also [FT].

Sobolev inequalities are related to heat kernel estimates by the following standard result [D, Theorem 2.4.2]: for any $q > 2$,

$$\begin{cases} \text{the upper bound} \\ \qquad K_\phi(t,x,y) < ct^{-q/2}, \quad t>0,\ x,y \in \Omega \\ \text{is equivalent to the Sobolev inequality} \\ \qquad \int_\Omega |\nabla w|^2 \phi^2\, dx \geqslant c\left(\int_\Omega |w|^{\frac{2q}{q-2}}\phi^2\, dx\right)^{(q-2)/q}, \quad w \in C_c^1(\Omega\setminus\{0\}). \end{cases} \tag{4.6}$$

In the rest of this section we shall apply the Hardy–Sobolev inequalities of Section 3 in order to obtain upper estimates on the heat kernel $K(t,x,y)$ of the operator $-\Delta - V$ for critical and subcritical potentials V. For this we shall use (4.4) for appropriate functions ϕ, together with uniform estimates on $K_\phi(t,x,y)$, obtained by means of (4.6). We initially present on-diagonal estimates, and add the Gaussian factor in Proposition 4.4.

We assume that Ω is a domain in $\mathbf{R}^N$, $N \geqslant 3$. We retain the notation introduced in the last example, and, in particular, we have $H = -\Delta - \lambda|x|^{-2}$, subject to Dirichlet boundary conditions on $\partial\Omega$. We first consider the subcritical case. Although the result is known, see [LS,MS], we include the proof for the sake of completeness.

Proposition 4.1 (Subcritical case). *Let $K(t,x,y)$ be the heat kernel of $H = -\Delta - \lambda\frac{1}{|x|^2}$, subject to Dirichlet boundary conditions on $\partial\Omega$. For $\lambda < ((N-2)/2)^2$, let α be the smallest solution of $\alpha(N-2-\alpha) = \lambda$.*

(i) *If Ω is bounded and $0 \leqslant \lambda < ((N-2)/2)^2$ then*

$$K(t,x,y) < ct^{-N/2}|x|^{-\alpha}|y|^{-\alpha}, \quad t>0,\ x,y \in \Omega. \tag{4.7}$$

(ii) *For any $\Omega \subset \mathbf{R}^N$ (bounded or unbounded) and $\lambda \leqslant 0$ there holds*

$$K(t,x,y) < ct^{-N/2}\min\left\{1, \left(\frac{|x|}{t^{1/2}}\right)^{-\alpha}\right\}\min\left\{1, \left(\frac{|y|}{t^{1/2}}\right)^{-\alpha}\right\}, \quad t>0,\ \ x,y \in \Omega. \tag{4.8}$$

Proof. (i) The boundedness of Ω together with (3.1) imply

$$\int_\Omega |\nabla w|^2 |x|^{-2\alpha}\,dx \geqslant c\left(\int_\Omega |w|^{\frac{2N}{N-2}} |x|^{-2\alpha}\,dx\right)^{(N-2)/N}, \quad w\in C_c^\infty(\Omega). \tag{4.9}$$

By (4.6) this implies $K_\phi(t,x,y) < ct^{-N/2}$, from which (4.7) follows using (4.4).

(ii) Comparison with the Laplacian implies that $K(t,x,y) < ct^{-N/2}$. Moreover, inequality (3.1) for $\beta p = 2\alpha$ reads

$$\int_\Omega |\nabla w|^2 |x|^{-2\alpha}\,dx \geqslant c\left(\int_\Omega |w|^{\frac{2(N-2\alpha)}{N-2-2\alpha}} |x|^{-2\alpha}\,dx\right)^{\frac{N-2-2\alpha}{N-2\alpha}}.$$

By means of (4.6) we deduce that $K_\phi(t,x,y) < ct^{-N/2+\alpha}$, $t>0$, $x,y\in\Omega$. Hence

$$K(t,x,y) < ct^{-N/2}\min\left\{1, \left(\frac{|x|}{t^{1/2}}\right)^{-\alpha}\left(\frac{|y|}{t^{1/2}}\right)^{-\alpha}\right\}, \quad t>0,\ x,y\in\Omega. \tag{4.10}$$

This proves (4.8) when $x=y$. The general case follows from the semigroup property since

$$\begin{aligned} K(t,x,y) &= \int_\Omega K(t/2,x,z)K(t/2,z,y)dz \\ &\leqslant \left(\int_\Omega K(t/2,x,z)^2 dz\right)^{1/2}\left(\int_\Omega K(t/2,z,y)^2 dz\right)^{1/2} \\ &= K(t,x,x)^{1/2}K(t,y,y)^{1/2}. \qquad \square \end{aligned}$$

We now consider the critical case.

Theorem 4.2 (Critical case). *Let Ω be a bounded domain and $K(t,x,y)$ be the heat kernel of $H = -\Delta - (\frac{N-2}{2})^2\frac{1}{|x|^2}$, subject to Dirichlet boundary conditions on $\partial\Omega$. Then*

$$K(t,x,y) < ct^{-\frac{N}{2}}|x|^{-\frac{N-2}{2}}|y|^{-\frac{N-2}{2}}, \quad t>0, \ x,y\in\Omega. \tag{4.11}$$

Proof. Estimate (3.3) implies the weaker inequality

$$\int_\Omega |\nabla w|^2 |x|^{2-N}\,dx \geqslant c\left(\int_\Omega |w|^{\frac{2N}{N-2}} |x|^{2-N}\,dx\right)^{(N-2)/N}. \tag{4.12}$$

Hence $K_\phi(t,x,y) < ct^{-\frac{N}{2}}$ and (4.11) follows. $\square$

We next consider the case where $\Omega = \mathbf{R}^N$ and the potential is critical at zero. More precisely, we consider the potential V_ε defined by (3.11), that is

$$V_\varepsilon(x) = \begin{cases} \left(\frac{N-2}{2}\right)^2 |x|^{-2}, & |x| < 1, \\ \varepsilon f(x), & |x| \geqslant 1, \end{cases}$$

where f is a nonnegative, continuous and radially symmetric function on $\{|x| \geqslant 1\}$. Moreover we assume f to be subcritical, that is it satisfies (3.12)

$$f(x) \leqslant K|x|^{-2-\sigma}, \quad |x| \geqslant 1,$$

for some $\sigma, K > 0$.

We retain the notation of Section 3.1, and in particular we recall definition (3.14) of ε_0. We have

Theorem 4.3 (The operator $-\Delta - V_\varepsilon$ on $\mathbf{R}^N$). *For any $\varepsilon \in (0, \varepsilon_0)$ the heat kernel of the operator $-\Delta - V_\varepsilon$ satisfies*

$$K(t,x,y) < ct^{-N/2} \max\{|x|^{-\frac{N-2}{2}}, 1\} \max\{|y|^{-\frac{N-2}{2}}, 1\}, \quad t > 0, \ \ x, y \in \mathbf{R}^N. \tag{4.13}$$

Proof. Let $\psi(x)$ be as in the proof of Theorem 3.4, cf (3.16). It follows from (3.15) that

$$\int_{\mathbf{R}^N} |\nabla v|^2 \psi^2 \, dx \geqslant c \left(\int_{\mathbf{R}^N} |v|^{2N/(N-2)} \psi^{2N/(N-2)} \tilde{X}_1^{(2N-2)/(N-2)} \, dx \right)^{(N-2)/N}, \tag{4.14}$$

for all $v \in C_c^1(\mathbf{R}^N \backslash \{0\})$. Since $\psi^{2N/(N-2)} \tilde{X}_1^{(2N-2)/(N-2)} \geqslant c\psi^2$ and $C_c^1(\mathbf{R}^N \backslash \{0\})$ is a form core for H_ψ and we conclude that $K_\psi(t,x,y) < ct^{-N/2}$ whence,

$$K(t,x,y) < ct^{-N/2} \psi(x) \psi(y).$$

The required estimate on $K(t,x,y)$ follows if we note that

$$c_1 \max\{|x|^{-(N-2)/2}, 1\} \leqslant \psi(x) \leqslant c_2 \max\{|x|^{-(N-2)/2}, 1\}, \quad x \in \mathbf{R}^N. \qquad \square$$

It is well known that the estimates of the above theorems can be improved to yield Gaussian decay of the heat kernel. We have

Proposition 4.4. *Proposition* 4.1 *as well as Theorems* 4.2 *and* 4.3 *can be improved by adding a factor* $c_\delta \exp\{-|x-y|^2/((4+\delta)t)\}$ *to the corresponding right-hand sides.*

Proof. The proof is standard. One can use Grigoryan's argument [G] or Davies's method of exponential perturbation [D] as adapted in [S-C, Section 4.2]. Note that the argument is applied to the operator H_ϕ—not to H. We omit the proof since it follows exactly the proof in [S-C]. □

5. Logarithmic refinements

Our aim in this section is to obtain refined versions of the improved Hardy–Sobolev inequalities of Section 3. As an application, we prove heat kernel estimates for $H = -\Delta - V - V_1$ where V is one of the potentials studied in Section 4 (that is, $V(x) = ((N-2)/2)^2|x|^{-2}$ near zero) but $V_1 > 0$ is also critical. The criticality of V_1 is meant in the sense that the following improved Hardy inequality holds:

$$\int_\Omega |\nabla u|^2\,dx - \int_\Omega Vu^2\,dx \geqslant \int_\Omega V_1 u^2\,dx, \quad u \in C_c^\infty(\Omega),$$

whereas this inequality is no longer true if we replace V_1 by $(1+\varepsilon)V_1$. Of course, V_1 is of lower order with respect to $|x|^{-2}$ (near $x=0$) since $((N-2)/2)^2|x|^{-2}$ is already critical for the validity of the (simple) Hardy inequality. It is remarkable that the addition of the extra potential V_1 does not affect the time dependence of the heat kernel estimates, but only affects the spatial singularity at the origin, which is increased by a logarithmic factor; see Theorems 5.3 and 5.4.

More precisely, recalling that $X_1(t) = (1-\log t)^{-1}$, let us introduce the functions

$$X_{k+1}(t) = X_1(X_k(t)), \quad k = 1, 2, \dots, \quad t \in (0,1). \tag{5.15}$$

These are iterated logarithmic functions that vanish at an increasingly slow rate at $t=0$ and are equal to one at $t=1$. In [FT] the following improved Hardy inequality was obtained for a bounded domain Ω with $D = \sup_\Omega |x|$:

$$\begin{aligned}\int_\Omega \left\{ |\nabla u|^2 - \left(\frac{N-2}{2}\right)^2 \frac{u^2}{|x|^2} - \frac{u^2}{4|x|^2} \sum_{i=1}^{k} X_1^2\left(\frac{|x|}{D}\right) \dots X_i^2\left(\frac{|x|}{D}\right) \right\} dx \\ \geqslant \frac{1}{4} \int_\Omega \frac{u^2}{|x|^2} X_1^2\left(\frac{|x|}{D}\right) \dots X_{k+1}^2\left(\frac{|x|}{D}\right) dx, \quad u \in C_c^\infty(\Omega).\end{aligned} \tag{5.16}$$

The potentials in the left-hand side of (5.16) are critical for each k, in the sense that (5.16) is sharp: the term X_{k+1}^2 cannot be replaced by $c_\varepsilon X_{k+1}^{2-\varepsilon}$ for any $\varepsilon > 0$, and the constant $1/4$ in the right-hand side is also optimal. In Theorem 5.3 and for bounded Ω we obtain upper estimates on the heat kernel of the operator

$$H = -\Delta - \left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2} - \frac{1}{4|x|^2} \sum_{i=1}^{k-1} X_1^2 \dots X_i^2 - \frac{\mu}{|x|^2} X_1^2 \dots X_k^2 \tag{5.17}$$

for $\mu > 1/4$, as well as for the critical case $\mu = 1/4$; for this we use results obtained in [FT]. For the critical case $\mu = 1/4$ we also consider operators defined on $\mathbf{R}^N$, in analogy to the operator $-\Delta - V_\varepsilon$ of Theorem 4.3; for this we use Theorem 5.1 below, and the corresponding heat kernel estimate is given in Theorem 5.4.

5.1. Refined Hardy–Sobolev inequalities

In this subsection we prove two theorems that are refined versions of Theorems 3.4 and 3.5 correspondingly. We recall definition (5.15) and set

$$\tilde{X}_k(|x|) = \begin{cases} X_k(|x|), & |x| < 1, \\ 1, & |x| > 1, \end{cases}$$

$$\begin{cases} Y_k(|x|) = X_k(1/|x|), & |x| > 1, \\ \tilde{Y}_k(|x|) = \tilde{X}_k(1/|x|), & |x| > 0. \end{cases}$$

We point out the differentiation rules for $X_k(r)$ and $Y_k(r)$:

$$\frac{d}{dr} X_k^a = \frac{a}{r} X_1 \ldots X_{k-1} X_k^{a+1}, \quad \frac{d}{dr} Y_k^a = -\frac{a}{r} Y_1 Y_2 \ldots Y_{k-1} Y_k^{a+1}, \quad r = |x|, \tag{5.18}$$

valid for $0 < r < 1$ and $r > 1$, respectively, which are easily proved by induction.

As in Theorem 3.4, we assume that f is a nonnegative, continuous and radially symmetric function on B^c satisfying (3.12), that is,

$$f(x) \leqslant K|x|^{-2-\sigma}, \quad |x| \geqslant 1,$$

for some $\sigma, K > 0$. For $\varepsilon > 0$ we also define

$$V_{k,\varepsilon}(x) = \begin{cases} \left(\dfrac{N-2}{2}\right)^2 |x|^{-2} + \dfrac{1}{4} |x|^{-2} \displaystyle\sum_{i=1}^{k} X_1^2(|x|) \ldots X_i^2(|x|), & |x| < 1, \\ \varepsilon f(x), & |x| > 1. \end{cases} \tag{5.19}$$

We then have

Theorem 5.1. *Assume that $k < N-2$ and define*

$$\varepsilon_{k,0} = \inf_{u \in \mathscr{H}^1(B^c)} \frac{\int_{B^c} |\nabla u|^2 \, dx - \frac{N-2+k}{2} \int_{\partial B} u^2 \, dS}{\int_{B^c} f u^2 \, dx}. \tag{5.20}$$

Then $\varepsilon_{k,0} > 0$ and for $\varepsilon \in (0, \varepsilon_{k,0})$ there holds

$$\int_{\mathbf{R}^N} |\nabla u|^2 \, dx - \int_{\mathbf{R}^N} V_{k,\varepsilon} u^2 \, dx \geqslant c \left(\int_{\mathbf{R}^N} |u|^{2N/(N-2)} (\tilde{X}_1 \ldots \tilde{X}_{k+1})^{\frac{2N-2}{N-2}} \, dx \right)^{(N-2)/N}, \tag{5.21}$$

for all $u \in C_c^\infty(\mathbf{R}^N)$.

Remark. The constant $\varepsilon_{k,0}$ is optimal in the sense that inequality (5.21) fails for $\varepsilon = \varepsilon_{k,0}$. Also the exponent $(2N-2)/(N-2)$ in (5.21) is sharp in the sense that it cannot be replaced by a smaller exponent. The proof of these two facts is rather involved; see [FT] for similar arguments. We do not use these facts in the sequel.

Proof. The proof follows closely that of Theorem 3.4, so we only give a sketch of it. The positivity of $\varepsilon_{k,0}$ follows from Lemma 2.5(i), yielding $\varepsilon_{k,0} \geqslant K^{-1}\mu_B((N-2+k)/2)$. Now let $\varepsilon \in (0, \varepsilon_{k,0})$ be fixed and let $\tilde{\psi}$ be the radially symmetric solution to the problem

$$\Delta\tilde{\psi} + V_{k,\varepsilon}\tilde{\psi} = 0, \quad |x| > 1,$$

$$\frac{\partial\tilde{\psi}}{\partial \nu} = -\frac{N-2+k}{2}\tilde{\psi}, \quad |x| = 1,$$

normalized so that $\tilde{\psi} = 1$ on $\{|x| = 1\}$. The function

$$\psi(x) = \begin{cases} |x|^{-(N-2)/2} X_1^{-1/2} \dots X_k^{-1/2}, & |x| < 1, \\ \tilde{\psi}(x), & |x| > 1 \end{cases} \tag{5.22}$$

is then C^1, radially symmetric and a direct computation which uses (5.18) shows that $\Delta\psi + V_{k,\varepsilon}\psi = 0$ in $\mathbf{R}^N$. Exactly as in Theorem 3.4, ψ is positive, radially symmetric and has a positive limit as $r \to +\infty$. We then prove (5.21) in the case where u is radially symmetric, using once again Proposition 3.1. The validity of (5.21) for general $u \in C_c^\infty(\mathbf{R}^N)$ follows from Lemma 3.3. □

We finally prove a refined version of Theorem 3.5. Let us fix a nonnegative, continuous and radially symmetric function g on $B = \{|x| < 1\}$, such that

$$g(x) \leqslant K|x|^{-2+\sigma}, \quad |x| < 1,$$

for some $\sigma, K > 0$. Further for $\varepsilon > 0$ we define

$$\hat{V}_{k,\varepsilon}(x) = \begin{cases} \varepsilon g(x), & |x| < 1, \\ \left(\frac{N-2}{2}\right)^2 |x|^{-2} + \frac{1}{4|x|^2}\sum_{i=1}^k Y_1^2(|x|)\dots Y_i^2(|x|), & |x| > 1. \end{cases}$$

We then have

Theorem 5.2. *Assume that $k < N-2$ and define*

$$\bar{\varepsilon}_{k,0} = \inf_{u \in H^1(B)} \frac{\int_B |\nabla u|^2\,dx + \frac{N-2-k}{2}\int_{\partial B} u^2\,dS}{\int_B g u^2\,dx}.$$

Then $\bar{\varepsilon}_{k,0}>0$ *and for* $\varepsilon\in(0,\bar{\varepsilon}_{k,0})$ *there holds*

$$\int_{\mathbf{R}^N}|\nabla u|^2\,dx-\int_{\mathbf{R}^N}\hat{V}_{k,\varepsilon}u^2\,dx\geqslant c\left(\int_{\mathbf{R}^N}|u|^{2N/(N-2)}(\tilde{Y}_1\ldots\tilde{Y}_{k+1})^{\frac{2N-2}{N-2}}\,dx\right)^{(N-2)/N}, \tag{5.23}$$

for all $u\in C_c^\infty(\mathbf{R}^N)$.

Proof. We omit the proof, which is similar to that of Theorem 3.5. □

5.2. *Refined heat kernel estimates*

In Theorems 4.2 and 4.3 we obtained heat kernel estimates for operators $-\Delta-V$ where $V(x)=((N-2)/2)^2|x|^{-2}$ near the origin. We shall now prove estimates for $-\Delta-V-V_1$, with V_1 also critical near the origin. In Theorems 5.3 and 5.4 we consider the cases Ω bounded and $\Omega=\mathbf{R}^N$, respectively.

For $k\geqslant1$ and $\mu\leqslant1/4$ we define

$$V_k^\mu(x)=\frac{\left(\frac{N-2}{2}\right)^2}{|x|^2}+\frac{1}{4|x|^2}\sum_{i=1}^{k-1}X_1^2\ldots X_i^2+\frac{\mu}{|x|^2}X_1^2\ldots X_k^2,\quad x\in\Omega \tag{5.24}$$

$(X_i=X_i(|x|/D),\ D=\sup_\Omega|x|)$ and consider the operator $H=-\Delta-V_k^\mu$ subject to Dirichlet boundary conditions on $\partial\Omega$. In [FT, Proposition 7.2] the Hardy–Sobolev inequality

$$\int_\Omega(|\nabla u|^2-V_k^{1/4}u^2)\,dx \geqslant c\left(\int_\Omega|u|^{2N/(N-2)}(X_1\ldots X_{k+1})^{\frac{2N-2}{N-2}}\,dx\right)^{(N-2)/N},\quad u\in C_c^\infty(\Omega), \tag{5.25}$$

was obtained. Let β be the largest solution of $\beta(1-\beta)=\mu$ and define

$$\phi_{k,\beta}(x)=|x|^{-\frac{N-2}{2}}X_1^{-1/2}\ldots X_{k-1}^{-1/2}X_k^{-\beta}. \tag{5.26}$$

Using (5.18) we verify that $\Delta\phi_{k,\beta}+V_k^\mu\phi_{k,\beta}=0$ and hence the change of variables $u=\phi_{k,\beta}w$ yields

$$\int_\Omega(|\nabla u|^2-V_k^\mu u^2)\,dx=\int_\Omega|\nabla w|^2\phi_{k,\beta}^2\,dx \tag{5.27}$$

for all $w\in C_c^\infty(\Omega)$. We have

Theorem 5.3. *Let Ω be bounded, $1 \leqslant k < N-2$, and $0 < \mu \leqslant 1/4$. The heat kernel of $H = -\Delta - V_k^\mu$ satisfies the estimate*

$$K(t,x,y) < ct^{-N/2}\phi_{k,\beta}(x)\phi_{k,\beta}(y), \quad t>0, \; x,y \in \Omega, \tag{5.28}$$

here V_k^μ is given by (5.24) *and $\phi_{k,\beta}(x)$ by* (5.26).

Proof. For the proof we distinguish two cases.

(1) *Case* $\mu < 1/4$. For $w \in C_c^\infty(\Omega)$ we have

$$\begin{aligned}\int_\Omega |\nabla w|^2 \phi_{k,\beta}^2\,dx &= \int_\Omega |\nabla u|^2\,dx - \int_\Omega V_k^\mu u^2\,dx \\ &\geqslant c\left(\int_\Omega (|\nabla u|^2 - V_{k-1}^{1/4}u^2)\,dx\right) \quad (\text{by } (5.16)) \\ &\geqslant c\left(\int_\Omega |u|^{2N/(N-2)}(X_1 \dots X_k)^{\frac{2N-2}{N-2}}\,dx\right)^{(N-2)/N} \quad (\text{by } (5.25)) \\ &= c\left(\int_\Omega |w|^{2N/(N-2)}|x|^{-N}(X_1 \dots X_{k-1})X_k^{\frac{2N-2-2N\beta}{N-2}}\,dx\right)^{(N-2)/N} \\ &\geqslant c\left(\int_\Omega |w|^{2N/(N-2)}\phi_{k,\beta}^2\,dx\right)^{(N-2)/N}.\end{aligned}$$

This implies that $K_{\phi_{k,\beta}}(t,x,y) < ct^{-N/2}$ and (5.28) follows.

(2) *Case* $\mu = 1/4$. By [FT, Lemma 7.5] the following Sobolev inequality holds:

$$\begin{aligned}&\int_\Omega |\nabla w|^2 |x|^{2-N} X_1^{-1} \dots X_k^{-1}\,dx \\ &\geqslant c\left(\int_\Omega |w|^{\frac{2N}{N-2}}|x|^{-N}X_1 \dots X_k X_{k+1}^{\frac{2N-2}{N-2}}\,dx\right)^{(N-2)/N}, \quad w \in C_c^\infty(\Omega).\end{aligned}$$

This implies in particular

$$\int_\Omega |\nabla w|^2 \phi_{k,\beta}^2\,dx \geqslant c\left(\int_\Omega |w|^{\frac{2N}{N-2}}\phi_{k,\beta}^2\,dx\right)^{(N-2)/N}, \quad w \in C_c^\infty(\Omega)$$

and hence we have the uniform estimate $K_{\phi_{k,\beta}}(t,x,y) < ct^{-N/2}$ as required. □

We finally have the following consequence of Theorem 5.1, where we retain the notation of that theorem:

Theorem 5.4. *Let* $1 \leqslant k < N-2$, *and* $\varepsilon \in (0, \varepsilon_{k,0})$ *with* $\varepsilon_{k,0}$ given by (5.20). *the heat kernel of the operator* $-\Delta - V_{k,\varepsilon}$ *satisfies*

$$K(t,x,y) < ct^{-N/2}\psi(x)\psi(y), \quad t>0, \ x,y \in \mathbf{R}^N,$$

here $V_{k,\varepsilon}$ *is given by* (5.19) *and* $\psi(x)$ *by* (5.22).

Proof. The result follows directly from Theorem 5.1 by means of (4.6). □

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