Estimates on the velocity of a rigid body moving in a fluid

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Abstract

We obtain estimates of all components of the velocity of a 3D rigid body moving in a viscous incompressible fluid without any symmetry restriction on the shape of the rigid body or the container. The estimates are in terms of suitable norms of the velocity field in a small domain of the fluid only, provided the distance h between the rigid body and the container is small. As a consequence we obtain suitable differential inequalities that control the distance h. The results are obtained using the fact that the vector field under investigation belongs to suitable function spaces, without any use of hydrodynamic equations.

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1 Introduction

Throughout the paper $\Omega \subset I\!R^3$ is a smooth domain, $S \subset \Omega$ is a bounded smooth connected domain and $F = \Omega \setminus \bar{S}$. We denote by $W_0^{1,p}(S,\,\Omega)$, $p \geq 1$, the vector function space consisting of functions

$$\boldsymbol{u} : \Omega \to I\!R^3, \qquad \boldsymbol{u} \in W_0^{1,p}(\Omega)\ ,$$

such that for a constant vector $\boldsymbol{u}_* \in I\!R^3$ and a vector $\boldsymbol{\omega} \in I\!R^3$ there holds

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_* + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_*), \quad \text{for } \boldsymbol{x} \in S\ , \tag{1.1}$$

where $\boldsymbol{x}_* \in \overline{S}$ is a fixed point.

Such function spaces arise naturally when studying the motion of a rigid body S inside a fluid region Ω, see e.g., [8, 6, 9, 11]. In this context $\boldsymbol{u}$ is the velocity field in Ω where in particular inside the rigid body the velocity is given by (1.1).

We denote by h the distance between S and $\partial\Omega$, that is,

$$h = \operatorname{dist}(S, \partial\Omega) = |PQ|,$$

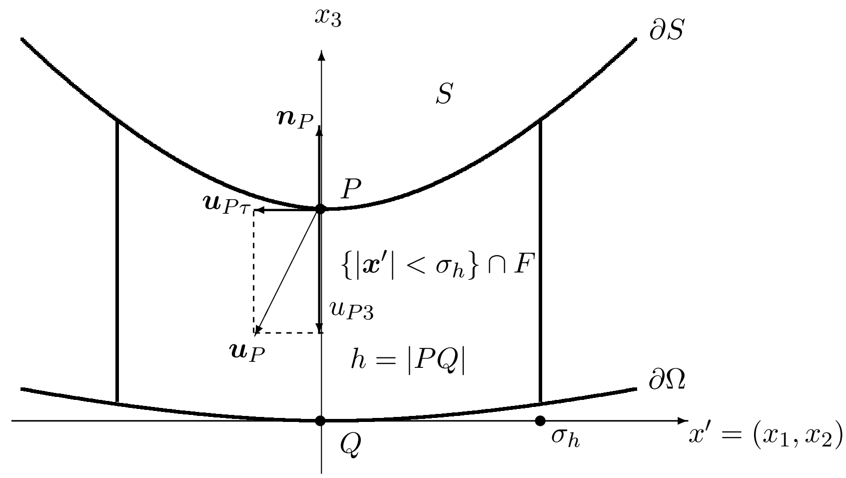

Figure 1: The coordinate system

where $P \in \partial S$ and $Q \in \partial\Omega$ is a pair of points that realize the distance. If there are more than one such pairs, we just take one of them. A similar analysis can be made for the other pairs. We are interested in the case where h is small, hence we may assume that

$$h < H := \mathrm{diam} S.$$

We introduce an orthogonal coordinate system (x_1, x_2, x_3) with the origin at the point Q. We note that the plane $\boldsymbol{x}' = (x_1, x_2)$ is tangent to $\partial\Omega$ at the point Q and parallel to the tangent plane of ∂S at the point $P = (0, 0, h)$. Let $\boldsymbol{u}_P = (\boldsymbol{u}_{P\tau}, u_{P3}) = (u_{P1}, u_{P2}, u_{P3})$ be the velocity of the rigid body at the point P and $\boldsymbol{\omega} = (\boldsymbol{\omega}_\tau, \omega_3) = (\omega_1, \omega_2, \omega_3)$ its angular velocity.

Our first result concerns vector fields that enjoy the regularity of weak solutions of the solid–fluid interaction problem in hydrodynamics cf [6, 8, 12].

Theorem 1.1. *We assume that both Ω and S are uniformly $C^{1,\alpha}$ domains, $\alpha \in (0, 1]$. Let $\boldsymbol{u} \in W_0^{1,p}(S, \Omega)$, $p \in [1, \infty)$ be such that* $\mathrm{div}\boldsymbol{u} = 0$*. Then, there exist positive constants H_*, C_w, c_0 depending only on S, Ω, p, such that, whenever $h < H_*$ and $\sigma_h := c_0 h^{\frac{1}{1+\alpha}}$,*

$$\begin{aligned}
|u_{P3}| &\leq C_w h^{\frac{1+2\alpha}{p(1+\alpha)}\left(p - \frac{3+\alpha}{1+2\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)}, \\
|\boldsymbol{u}_{P\tau}| &\leq C_w\, h^{\frac{1}{p}\left(p - \frac{3+\alpha}{1+\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)}, \\
|\boldsymbol{\omega}_\tau| &\leq C_w\, h^{\frac{2\alpha}{p(1+\alpha)}\left(p - \frac{3+\alpha}{2\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)}, \\
|\omega_3| &\leq C_w\, h^{\frac{\alpha}{p(1+\alpha)}\left(p - \frac{3+\alpha}{\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)}.
\end{aligned}$$

Estimates for $|u_{P3}|$ where first proved in [8, 11] where in the right hand side the norm of $\nabla \boldsymbol{u}$ over a larger region is required. What is new in our Theorem is that the norm of $\nabla \boldsymbol{u}$ appears in a small region of the *fluid only*. This is in agreement with the observation that, in simple geometries, knowledge of the fluid vector field in a small region of the fluid between S and $\partial\Omega$ is enough to produce significant information such as the noncollision of the rigid body and the boundary of the container, see e.g. [3, 5, 6, 10]. The rest of the estimates are altogether new. Since the velocity at any point of S is given by (1.1), one obtains a complete control of the motion of the rigid body as it approaches the boundary of Ω. The above result extends Theorem 4.6 of [2] to 3 dimensions.

Our next result concerns vector fields that enjoy the regularity of strong solutions of the solid–fluid interaction problem, cf [13, 7].

Theorem 1.2. *We assume that both Ω and S are uniformly C^3 domains. Let $\boldsymbol{u} \in W_0^{1,2}(S, \Omega)$ be such that* $\operatorname{div}\boldsymbol{u} = 0$ *and in addition $|\nabla^2 \boldsymbol{u}| \in L^p(F)$ for some $p \in [1, \infty)$. Then, there exist positive constants H_*, C_s, c_0 depending only on S, Ω, p, such that, whenever $h < H_*$ and $\sigma_h := c_0 h^{\frac{1}{2}}$,*

$$|u_{P3}| \leq C_s \left(h^{\frac{5}{2}-\frac{2}{p}} \|\nabla^2 \boldsymbol{u}\|_{L^p(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)} + h \|\nabla \boldsymbol{u}\|_{L^2(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)} \right).$$

For instance, for $p = 2$ which corresponds to the regularity of strong solutions, we have that

$$|u_{P3}| \leq C_s \left(h^{\frac{3}{2}} \|\nabla^2 \boldsymbol{u}\|_{L^2(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)} + h \|\nabla \boldsymbol{u}\|_{L^2(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)} \right).$$

This can be thought as an improvement of the corresponding estimate of Theorem 1.1, which for $\alpha = 1$, $p = 2$ takes the form

$$|u_{P3}| \leq C_w h^{\frac{1}{2}} \|\nabla \boldsymbol{u}\|_{L^2(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)}.$$

Such estimates for strong solutions were first established in [4, 7] in the case where Ω is half–space and S is a disc or a ball. In such a case only the first term in the right hand side appears due to the apparent symmetries. In the absence of symmetries however, the second term above seems to be necessary. The proof of Theorem 1.2 uses in an essential way all the estimates of Theorem 1.1. It is not clear whether the other quantities appearing in Theorem 1.1 can also be improved in a similar way and we leave it as an open question.

We next consider the dynamic case, where S is allowed to move inside Ω. Denote by $S(t) \subset \Omega$ the position of S at time t and $F = F(t) = \Omega \setminus S(t)$. We thus assume that there are functions $\boldsymbol{u}_*(t)$, $\boldsymbol{\omega}(t)$ both in $L^\infty(0, T)$ and function $\boldsymbol{x}_*(t)$ in $W^{1,\infty}(0, T)$ such that $\boldsymbol{x}_*'(t) = \boldsymbol{u}_*(t)$, so that $S(t)$ consists of points $\boldsymbol{x}(t)$ satisfying

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{u}_*(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x}(t) - \boldsymbol{x}_*(t)), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \in S_0.$$

For $p \geq 1$, $q \geq 1$ we define the space of functions $\boldsymbol{u}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \Omega$, $t \in (0, T)$,

$$\begin{aligned} &L^q(0, T; W_0^{1,p}(S(t)\,, \Omega)) := \\ &\{\boldsymbol{u} \in L^q\left(0, T; W_0^{1,p}(\Omega)\right) : \ \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}_*(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{x}_*(t)) \ \text{ for } \boldsymbol{x} \in S(t)\}. \end{aligned}$$

The distance between $S(t)$ and $\partial\Omega$ is now a function of time $h(t)$, that is,

$$h(t) = \operatorname{dist}(S(t), \partial\Omega).$$

The following is a consequence of the previous Theorems.

Theorem 1.3. (i) *We assume that both Ω and S_0 are uniformly $C^{1,\alpha}$ domains with $\alpha \in (0, 1]$. In addition Ω satisfies an inner sphere condition. Let*

$$\boldsymbol{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^1(0, T; W_0^{1,p}(S(t), \Omega)), \ \ p \in [1, \infty), \ \ \operatorname{div}\boldsymbol{u} = 0. \tag{1.2}$$

Then, there exist positive constants H_0, C_w, c_0 depending only on S, Ω, p, such that, whenever $h < H_0$, function $h = h(t)$ is Lipschitz continuous and for $\sigma_h := c_0 h^{\frac{1}{1+\alpha}}$ there holds

$$\left| \frac{dh}{dt} \right| \leq C_w \, h^{\frac{1+2\alpha}{p(1+\alpha)}\left(p - \frac{3+\alpha}{1+2\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\{|\boldsymbol{x}'|<\sigma_h\}\cap F(t))}, \quad \textit{for a.a. } t \in (0, T). \tag{1.3}$$

(ii) *We assume that both* Ω *and* S_0 *are uniformly* C^3 *domains. The vector field* $\boldsymbol{u}$ *satisfies*

$$\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;W_0^{1,2}(S(t),\Omega)), \ \ \mathrm{div}\boldsymbol{u} = 0.$$

and in addition

$$|\nabla^2 \boldsymbol{u}| \in L^p(F(t)), \quad p \in [1,\infty) \ \ \textit{for a.a. } t \in (0,T).$$

Then, there exist positive constants H_0*,* C_s*,* c_0 *depending only on* S*,* Ω*,* p*, such that, whenever* $h < H_0$*, function* $h = h(t)$ *is Lipschitz continuous and for* $\sigma_h := c_0 h^{\frac{1}{2}}$*, and for a.a.* $t \in (0,T)$*,*

$$\left| \frac{dh}{dt} \right| \le C_s \left(h^{\frac{5}{2}-\frac{2}{p}} \left\| \nabla^2 \boldsymbol{u} \right\|_{L^p(\{|\boldsymbol{x}'|<\sigma_h\}\cap F(t))} + h \left\| \nabla \boldsymbol{u} \right\|_{L^2(\{|\boldsymbol{x}'|<\sigma_h\}\cap F(t))} \right). \tag{1.4}$$

It follows from Theorem 1.3(ii) with $p = 2$, that if $\lim_{t\to T} h(t) \to 0$, one has that

$$\lim_{t\to T} \int_0^t h^{\frac{1}{2}}(\tau) \|\nabla^2 \boldsymbol{u}\|_{L^2(\{|\boldsymbol{x}'|<\sigma_h(\tau)\}\cap F(\tau))}\, d\tau = \infty,$$

see Corollary 3.2 for a more general statement. We see that the noncollision for strong solutions of the solid–fluid interaction problem cf [13, 7], is a consequence of the fact that $\boldsymbol{u}$ belongs to a specific function space.

The focus of the paper is in the 3D case, but similar results hold for the 2D case as well. We present some of them in the form of Remarks. We also note that although our results are obtained for a solenoidal vector field, a similar approach can be applied in the nonsolenoidal case.

The paper is organized as follows. In section 2 the proof of Theorem 1.1 is given and in section 3 the proof of Theorems 1.2 and 1.3 are given. In section 4 we present a 3D example showing the optimality of Theorem 1.1. In addition, in this example a smooth ($\alpha = 1$) rigid body hits the boundary of the container with nonzero speed.

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2 Estimates for weak solutions

Here we will give the proof of Theorem 1.1. We say that a domain $D \subset I\!R^3$ is *uniformly* $C^{1,\alpha}$, $\alpha \in (0,1]$, if there exist constants $k > 0$ and $\sigma_0 > 0$ s.t. for each $x_0 \in \partial D$ there is an isometry $\mathcal{R} : I\!R^3 \to I\!R^3$ s.t.

$$\mathcal{R}(0) = \boldsymbol{x}_0, \quad \mathcal{R}(\{\boldsymbol{x} \in I\!R^3 :\ k|\boldsymbol{x}'|^{1+\alpha} < |x_3| < k\sigma_0^{1+\alpha}, \quad |\boldsymbol{x}'| < \sigma_0\}) \cap \partial D = \emptyset.$$

This definition garantees that the above parabolas with *fixed constants* σ_0 and k fit in both sides of ∂D *no matter what the point* $\boldsymbol{x}_0$ on the boundary is.

If D is a bounded $C^{1,\alpha}$ domain then it is a uniformly $C^{1,\alpha}$ domain. For instance, this follows from Corollary 3.14 of [1]. We note that in the terminology of [1], our definition of a uniformly $C^{1,\alpha}$ domain is the same as the uniform hour–glass condition with shape function $\omega(\tau) = k\tau^\alpha$.

Throughout the rest of the Section we assume that Ω and S are uniformly $C^{1,\alpha}$ domains, with the same constants $k > 0$ and $\sigma_0 > 0$. Recall that if $P \in \partial S$ and $Q \in \partial\Omega$ is a pair of points that realize the distance, our orthogonal coordinate system (x_1, x_2, x_3) has its origin at the point $Q = (0,0,0)$. We also recall that the plane $\boldsymbol{x}' = (x_1, x_2)$ is tangent to $\partial\Omega$ at the point Q and parallel to the tangent plane of ∂S at the point $P = (0,0,h)$, see fig.1. We set $r = |\boldsymbol{x}'| = |(x_1, x_2)|$.

We will first prove a weaker version of Theorem 1.1. To this end we fix a $\sigma \in [0, \sigma_0/2]$ and let $0 < \rho \leq \sigma$. We denote by $\mathcal{C}_\rho$ the cylinder of radius ρ and height $h + 3k\sigma^{1+\alpha}$,

$$\mathcal{C}_\rho \quad = \quad \{(r\cos\theta, r\sin\theta, x_3):\ 0 < r < \rho,\ \ 0 \leq \theta < 2\pi,\ \ -k\sigma^{1+\alpha} < x_3 < h + 2k\sigma^{1+\alpha}\}.$$

The choice of the cylinder $\mathcal{C}_\rho$ is such that its top surface lies inside the rigid body S whereas the bottom surface lies outside Ω. Such a construction is possible due to the previous assumptions. A depiction of the cylinder $\mathcal{C}_\rho$, along with other domains, is shown in fig. 3. We also mention the obvious relation

$$\mathcal{C}_\sigma \cap F = \{|\boldsymbol{x}'| < \sigma\} \cap F. \tag{2.1}$$

We have

Theorem 2.1. *Assume that both Ω and S are uniformly $C^{1,\alpha}$ domains, $\alpha \in (0,1]$. Let $\boldsymbol{u} \in W_0^{1,p}(S,\,\Omega)$ be such that $\mathrm{div}\boldsymbol{u} = 0$. Then, there exist positive constant C_w depending only on S and Ω, such that if $h \in [0, H]$ and $\sigma \in (0, \sigma_0/2]$ there holds*

$$\begin{aligned}
|u_{P3}| \quad &\leq \quad C_w \sigma^{-1-\frac{2}{p}} (h + 3k\sigma^{1+\alpha})^{2-\frac{1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)},\\
|\boldsymbol{u}_{P\tau}| \quad &\leq \quad C_w \sigma^{-1-\alpha-\frac{2}{p}} (h + 3k\sigma^{1+\alpha})^{2-\frac{1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)},\\
|\boldsymbol{\omega}_\tau| \quad &\leq \quad C_w \sigma^{-2-\frac{2}{p}} (h + 3k\sigma^{1+\alpha})^{2-\frac{1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)},\\
|\omega_3| \quad &\leq \quad C_w \sigma^{-2-\alpha-\frac{2}{p}} (h + 3k\sigma^{1+\alpha})^{2-\frac{1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}.
\end{aligned}$$

Proof: Let $0 < \rho \leq \sigma \leq \sigma_0/2$. We note that σ will be fixed throughout the proof. In addition we assume that $0 < \gamma < \gamma_0 \leq \pi$. We first fix some notation.

<u>Notation.</u> We denote by $\mathcal{C}_{\rho,\gamma}$ the half cylinder given by

$$\mathcal{C}_{\rho,\gamma} := \{(r\cos\theta, r\sin\theta, x_3):\ 0 < r < \rho,\ \gamma \leq \theta < \pi + \gamma,\ \ -k\sigma^{1+\alpha} < x_3 < h + 2k\sigma^{1+\alpha}\}.$$

Clearly we have $\mathcal{C}_\rho = \mathcal{C}_{\rho,\gamma} \cup \mathcal{C}_{\rho,\pi+\gamma}$. We also set

$$\delta\mathcal{C}_{\rho,\gamma} := \mathcal{C}_{\rho,\gamma} \setminus \mathcal{C}_{\rho/2,\gamma}.$$

To describe the boundary surfaces of $\mathcal{C}_{\rho,\gamma}$ we use the notation (upper, lateral curved, lateral flat, lower)

$$\begin{aligned}
\Gamma^+_{\rho,\gamma} \quad &= \quad \{(r\cos\theta, r\sin\theta, x_3):\ 0 < r < \rho,\ \gamma \leq \theta < \pi + \gamma,\ x_3 = h + 2k\sigma^{1+\alpha}\},\\
\Gamma^0_{\rho,\gamma} \quad &= \quad \{(r\cos\theta, r\sin\theta, x_3):\ r = \rho,\ \gamma < \theta < \pi + \gamma,\ \ -k\sigma^{1+\alpha} < x_3 < h + 2k\sigma^{1+\alpha}\},\\
\Gamma^1_{\rho,\gamma} \quad &= \quad \{(r\cos\theta, r\sin\theta, x_3):\ 0 < r < \rho,\ \ \theta = \gamma,\ \ \ -k\sigma^{1+\alpha} < x_3 < h + 2k\sigma^{1+\alpha}\},\\
\Gamma^-_{\rho,\gamma} \quad &= \quad \{(r\cos\theta, r\sin\theta, x_3):\ 0 < r < \rho,\ \gamma \leq \theta < \pi + \gamma,\ x_3 = -k\sigma^{1+\alpha}\}.
\end{aligned}$$

The (total) lateral surface of $\mathcal{C}_{\rho,\gamma}$ is given by

$$\Gamma_{\rho,\gamma} := \Gamma^0_{\rho,\gamma} \cup \Gamma^1_{\rho,\gamma} \cup \Gamma^1_{\rho,\pi+\gamma}.$$

Concerning $\delta\mathcal{C}_{\rho,\gamma}$ we have that the upper surface is given by

$$\delta\Gamma^+_{\rho,\gamma} := \Gamma^+_{\rho,\gamma} \setminus \Gamma^+_{\rho/2,\gamma}.$$

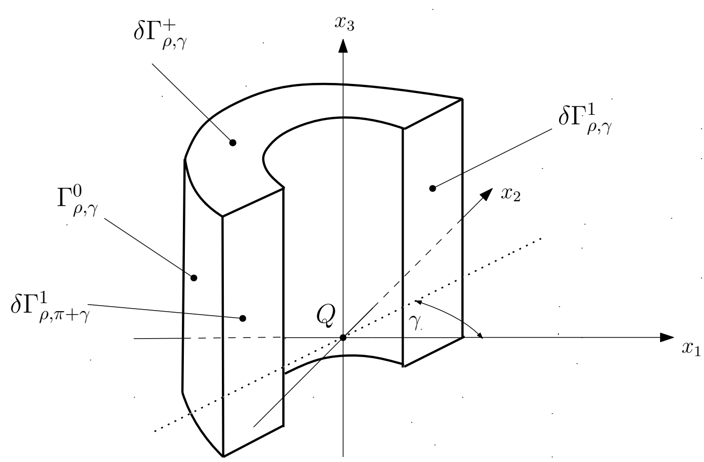

Figure 2: The domain $\delta\mathcal{C}_{\rho,\gamma}$ and its boundary surfaces.

The (total) lateral surface of $\delta\mathcal{C}_{\rho,\gamma} := \mathcal{C}_{\rho,\gamma} \setminus \mathcal{C}_{\rho/2,\gamma}$ is given by

$$\delta\Gamma_{\rho,\gamma} := \Gamma^0_{\rho,\gamma} \cup \Gamma^0_{\rho/2,\gamma} \cup \delta\Gamma^1_{\rho,\gamma} \cup \delta\Gamma^1_{\rho,\pi+\gamma},$$

where

$$\delta\Gamma^1_{\rho,\gamma} := \Gamma^1_{\rho,\gamma} \setminus \Gamma^1_{\rho/2,\gamma}.$$

Plan of the proof. The estimate of $|u_{P3}| = |\boldsymbol{u}_P \cdot \boldsymbol{n}_P|$ is essentially given in [11] Theorem 3.1, see pp 321-322. In the sequel we will first estimate the tangential part of $\boldsymbol{\omega}$, that is, $\boldsymbol{\omega_\tau} = (\omega_1, \omega_2, 0)$, then the tangential part of $\boldsymbol{u}_P$, that is $\boldsymbol{u}_{P\tau} = (u_{P1}, u_{P2}, 0)$ and finally the third component of $\boldsymbol{\omega}$, that is $|\omega_3|$. We follow this order in the proof, because at each step we use the estimates of the previous steps.

Estimate of $\boldsymbol{\omega_\tau}$. We will apply divergence Theorem in the piecewise smooth domain $\delta\mathcal{C}_{\rho,\gamma} = \mathcal{C}_{\rho,\gamma} \setminus \mathcal{C}_{\rho/2,\gamma}$, see e.g. [14], p. 278. By choosing this non symmetric domain the tangential component of $\boldsymbol{\omega}$ appears in the calculations. The removal of $\mathcal{C}_{\rho/2,\gamma}$ is done for technical reasons and will be explained later in the proof.

Since $\boldsymbol{u}$ is divergence free and $\boldsymbol{u} = 0$ on $\Gamma^-_{\rho,\gamma}$, applying the divergence Theorem in $\delta\mathcal{C}_{\rho,\gamma}$ we have that

$$\int_{\delta\Gamma^+_{\rho,\gamma}} \boldsymbol{u} \cdot \boldsymbol{n}\, dS + \int_{\delta\Gamma_{\rho,\gamma}} \boldsymbol{u} \cdot \boldsymbol{n}\, dS = 0. \tag{2.2}$$

We next compute the integral over $\delta\Gamma^+_{\rho,\gamma} = \Gamma^+_{\rho,\gamma} \setminus \Gamma^+_{\rho/2,\gamma}$. If $\boldsymbol{x}_P$ is the radius vactor of the point P, we have that $\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_P)$ and $\boldsymbol{n} = \boldsymbol{n}_p = (0, 0, 1)$. Then,

$$\begin{aligned}
\int_{\delta\Gamma^+_{\rho,\gamma}} \boldsymbol{u} \cdot \boldsymbol{n}\, dS &= \int_{\delta\Gamma^+_{\rho,\gamma}} \boldsymbol{u}_P \cdot \boldsymbol{n}_P\, dS + \int_{\delta\Gamma^+_{\rho,\gamma}} \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_P) \cdot \boldsymbol{n}\, dS \\
&= \frac{3}{8}\pi\rho^2 \boldsymbol{u}_P \cdot \boldsymbol{n}_P + \boldsymbol{\omega} \cdot \int_{\delta\Gamma^+_{\rho,\gamma}} (\boldsymbol{x} - \boldsymbol{x}_P) \times \boldsymbol{n}\, dS,
\end{aligned}$$

here we used the identity $\boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_P) \cdot \boldsymbol{n} = (\boldsymbol{x} - \boldsymbol{x}_P) \times \boldsymbol{n} \cdot \boldsymbol{\omega}$. To continue we note that

$$\begin{aligned} \boldsymbol{x} - \boldsymbol{x}_P &= (r\cos\theta, r\sin\theta, 2k\sigma^{1+\alpha}) \\ (\boldsymbol{x} - \boldsymbol{x}_P) \times \boldsymbol{n} &= (r\sin\theta, -r\cos\theta, 0), \end{aligned}$$

so that finally we have

$$\int_{\delta\Gamma^+_{\rho,\gamma}} \boldsymbol{u} \cdot \boldsymbol{n}\, dS = \frac{3}{8}\pi\rho^2 \boldsymbol{u}_P \cdot \boldsymbol{n}_P + \frac{7\rho^3}{12}(\omega_1 \cos\gamma + \omega_2 \sin\gamma).$$

From this and (2.2) we get

$$\frac{7\rho^3}{12}|\omega_1 \cos\gamma + \omega_2 \sin\gamma| \leq \frac{3\pi\rho^2}{8}|\boldsymbol{u}_P \cdot \boldsymbol{n}_P| + \int_{\Gamma^0_{\rho,\gamma} \cup \Gamma^0_{\rho/2,\gamma}} |\boldsymbol{u}| dS + \int_{\delta\Gamma^1_{\rho,\gamma} \cup \delta\Gamma^1_{\rho,\pi+\gamma}} |\boldsymbol{u}|\, dS. \quad (2.3)$$

To estimate the terms in the right hand side we will integrate the above estimate from $\gamma = 0$ to $\gamma_0 \leq \pi$ and then from $\rho = \sigma/2$ to σ. In the calculations that follow we use cylindrical coordinates. By abuse of notation we write $\boldsymbol{u}(r, \theta, x_3)$. We first note that

$$\begin{aligned} \int_0^{\gamma_0} \int_{\Gamma^0_{\rho,\gamma}} |\boldsymbol{u}|\, dS d\gamma &= \int_0^{\gamma_0} \int_\gamma^{\pi+\gamma} \int_{-k\sigma^{1+\alpha}}^{h+2k\sigma^{1+\alpha}} |\boldsymbol{u}(\rho, \theta, x_3)|\, \rho\, dx_3 d\theta d\gamma \\ &\leq \int_0^{\gamma_0} \int_0^{2\pi} \int_{-k\sigma^{1+\alpha}}^{h+2k\sigma^{1+\alpha}} |\boldsymbol{u}(\rho, \theta, x_3)|\, \rho\, dx_3 d\theta d\gamma \\ &\leq \pi \int_0^{2\pi} \int_{-k\sigma^{1+\alpha}}^{h+2k\sigma^{1+\alpha}} |\boldsymbol{u}(\rho, \theta, x_3)|\, \rho\, dx_3 d\theta. \end{aligned}$$

We next inegrate from $\rho = \sigma/2$ to σ to get

$$\begin{aligned} \int_{\sigma/2}^{\sigma} \int_0^{\gamma_0} \int_{\Gamma^0_{\rho,\gamma}} |\boldsymbol{u}|\, dS d\gamma d\rho &\leq \pi \int_{\mathcal{C}_\sigma \setminus \mathcal{C}_{\sigma/2}} |\boldsymbol{u}|\, d\boldsymbol{x} \leq \pi \int_{\mathcal{C}_\sigma} |\boldsymbol{u}|\, d\boldsymbol{x} \\ &\leq \pi\, \|\boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}\, |\mathcal{C}_\sigma|^{\frac{p-1}{p}} \\ &\leq \pi\, C_1\, (h + 3k\sigma^{1+\alpha})\, \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}\, (\sigma^2(h + 3k\sigma^{1+\alpha}))^{\frac{p-1}{p}} \\ &= \pi\, C_1\, \sigma^{2-\frac{2}{p}}\, (h + 3k\sigma^{1+\alpha})^{2-\frac{1}{p}}\, \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}, \end{aligned}$$

for some universal positive constant C_1. Working similarly for the integral over $\Gamma^0_{\rho/2,\gamma}$ we end up with

$$\begin{aligned} \int_{\sigma/2}^{\sigma} \int_0^{\gamma_0} \int_{\Gamma^0_{\rho,\gamma} \cup \Gamma^0_{\rho/2,\gamma}} |\boldsymbol{u}|\, dS d\gamma d\rho &\leq 3\pi \int_{\mathcal{C}_\sigma} |\boldsymbol{u}|\, d\boldsymbol{x} \\ &\leq 3\pi\, C_1\, \sigma^{2-\frac{2}{p}}\, (h + 3k\sigma^{1+\alpha})^{2-\frac{1}{p}}\, \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}. \end{aligned}$$

Concerning the flat parts we have

$$\begin{aligned}
\int_0^{\gamma_0}\int_{\sigma/2}^{\sigma}\int_{\delta\Gamma^1_{\rho,\gamma}} |\boldsymbol{u}|\, dS d\rho d\gamma &= \int_0^{\gamma_0}\int_{-k\sigma^{1+\alpha}}^{h+2k\sigma^{1+\alpha}}\int_{\sigma/2}^{\sigma}\int_{\rho/2}^{\rho} |\boldsymbol{u}(r,\gamma,x_3)|\, dr d\rho dx_3 d\gamma \\
&\leq \int_0^{\gamma_0}\int_{-k\sigma^{1+\alpha}}^{h+2k\sigma^{1+\alpha}}\int_{\sigma/2}^{\sigma}\int_{\sigma/4}^{\sigma} |\boldsymbol{u}|\, dr d\rho dx_3 d\gamma \\
&= \frac{\sigma}{2}\int_0^{\gamma_0}\int_{-k\sigma^{1+\alpha}}^{h+2k\sigma^{1+\alpha}}\int_{\sigma/4}^{\sigma} \frac{|\boldsymbol{u}|}{r}\, r\, dr dx_3 d\gamma \\
&\leq \frac{\sigma}{2}\int_{\mathcal{C}_\sigma\setminus\mathcal{C}_{\sigma/4}} \frac{|\boldsymbol{u}|}{r} d\boldsymbol{x} \\
&\leq 2\int_{\mathcal{C}_\sigma} |\boldsymbol{u}|\, d\boldsymbol{x} \\
&\leq 2C_1\, \sigma^{2-\frac{2}{p}}\, (h+3k\sigma^{1+\alpha})^{2-\frac{1}{p}}\, \|\nabla\boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}.
\end{aligned}$$

It is in the above calculation that the removal of $\mathcal{C}_{\rho/2,\gamma}$ is required. The same estimate holds true for the integral over $\delta\Gamma^1_{\rho,\pi+\gamma}$.

On the other hand examination of the proof of Theorem 3.1 in [11] (see p. 321-322) shows that the following estimate is true

$$|u_{P3}| = |\boldsymbol{u}_P\cdot\boldsymbol{n}_P| \leq C_2\sigma^{-1-\frac{2}{p}}(h+3k\sigma^{1+\alpha})^{2-\frac{1}{p}}\|\nabla\boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}, \tag{2.4}$$

for a universal positive constant C_2.

Integrating twice (2.3) and employing the above estimates we get

$$|(\omega_1\,\sin\gamma_0 + \omega_2\,(1-\cos\gamma_0)| \leq C_0\sigma^{-2-\frac{2}{p}}(h+3k\sigma^{1+\alpha})^{2-\frac{1}{p}}\|\nabla\boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}.$$

Since $\gamma_0\in(0,\pi)$ is arbitrary, we conclude

$$|\boldsymbol{\omega}_\tau| \leq |\omega_1| + |\omega_2| \leq C\sigma^{-2-\frac{2}{p}}(h+3k\sigma^{1+\alpha})^{2-\frac{1}{p}}\|\nabla\boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}, \tag{2.5}$$

for a universal positive constant C.

Estimate of $\boldsymbol{u}_{P\tau}$. We will use a similar argument, but this time instead of working with the half cylinder $\mathcal{C}_{\rho,\gamma}$ we will work with a half cylindrical domain with $\cup$–like top surface given by

$$\mathcal{C}^{\cup}_{\rho,\gamma} := \{(r\cos\theta, r\sin\theta, x_3):\ 0<r<\rho,\ \gamma\leq\theta<\pi+\gamma,\ -k\sigma^{1+\alpha}<x_3<h+kr^{1+\alpha}\}.$$

Such a choice allows for the tangential part of $\boldsymbol{u}_P$ to enter the calculations.

To describe the boundary surfaces of $\mathcal{C}^{\cup}_{\rho,\gamma}$ we use, in complete analogy with the previous case, the notation

$$\Gamma^{\cup,+}_{\rho,\gamma} = \{(r\cos\theta, r\sin\theta, x_3):\ 0<r<\rho,\ \gamma\leq\theta<\pi+\gamma,\ x_3=h+kr^{1+\alpha}\},$$

for the upper surface and quite similarly $\Gamma^{\cup,0}_{\rho,\gamma}$, $\Gamma^{\cup,1}_{\rho,\gamma}$, $\Gamma^{\cup,-}_{\rho,\gamma}=\Gamma^{-}_{\rho,\gamma}$ for the lateral curved, lateral flat, and lower surfaces respectively. Again, we apply divergence Theorem in $\delta\mathcal{C}^{\cup}_{\rho,\gamma} := \mathcal{C}^{\cup}_{\rho,\gamma}\setminus\mathcal{C}^{\cup}_{\rho/2,\gamma}$ to get the analogue of (2.2)

$$\int_{\Gamma^{\cup,+}_{\rho,\gamma}\setminus\Gamma^{\cup,+}_{\rho/2,\gamma}} \boldsymbol{u}\cdot\boldsymbol{n}\, dS + \int_{\Gamma^{\cup,0}_{\rho,\gamma}\cup\Gamma^{\cup,0}_{\rho/2,\gamma}} \boldsymbol{u}\cdot\boldsymbol{n}\, dS + \int_{\delta\Gamma^{\cup,1}_{\rho,\gamma}\cup\delta\Gamma^{\cup,1}_{\rho,\pi+\gamma}} \boldsymbol{u}\cdot\boldsymbol{n}\, dS = 0. \tag{2.6}$$

The first integral above is computed explicitly. Since the surface $\Gamma^{\cup,+}_{\rho,\gamma}$ lies in the rigid body, we have $\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_P)$. Straightforward calculations show that,

$$\begin{aligned} \boldsymbol{n} &= \frac{(-k(1+\alpha)r^\alpha \cos\theta,\ -k(1+\alpha)r^\alpha \sin\theta,\ 1)}{(k^2(1+\alpha)^2 r^{2\alpha}+1)^{\frac{1}{2}}} \\ \boldsymbol{x} - \boldsymbol{x}_P &= (r\cos\theta, r\sin\theta, kr^{1+\alpha}) \\ dS &= (k^2(1+\alpha)^2 r^{2\alpha}+1)^{\frac{1}{2}}\, r\, dr\, d\theta \\ \boldsymbol{n}_P &= (0,\ 0,\ 1). \end{aligned}$$

Then,

$$\begin{aligned} \int_{\Gamma^{\cup,+}_{\rho,\gamma}} \boldsymbol{u}\cdot\boldsymbol{n}\, dS &= \int_{\Gamma^{\cup,+}_{\rho,\gamma}} \boldsymbol{u}_P\cdot\boldsymbol{n}\, dS + \int_{\Gamma^{\cup,+}_{\rho,\gamma}} \boldsymbol{\omega}\times(\boldsymbol{x}-\boldsymbol{x}_P)\cdot\boldsymbol{n}\, dS \\ &= \boldsymbol{u}_P \cdot \int_0^\rho \int_\gamma^{\pi+\gamma} (-k(1+\alpha)r^\alpha\cos\theta,\ -k(1+\alpha)r^\alpha \sin\theta,\ 1)\, r\, dr\, d\theta \\ &+ \boldsymbol{\omega}\cdot \int_0^\rho \int_\gamma^{\pi+\gamma} (\sin\theta + k^2(1+\alpha)r^{2\alpha}\sin\theta,\ -\cos\theta - k^2(1+\alpha)r^{2\alpha}\cos\theta,\ 0)\, r^2\, dr\, d\theta, \end{aligned}$$

so that

$$\begin{aligned} \int_{\Gamma^{\cup,+}_{\rho,\gamma}} \boldsymbol{u}\cdot\boldsymbol{n}\, dS &= \frac{1}{2}\pi\rho^2 \boldsymbol{u}_P\cdot\boldsymbol{n}_P + \frac{2k(1+\alpha)}{2+\alpha}\rho^{2+\alpha}\boldsymbol{u}_P\cdot(\sin\gamma, -\cos\gamma, 0) \\ &+ \left(\frac{2}{3} + \frac{2k^2(\alpha+1)}{2\alpha+3}\rho^{2\alpha}\right)\rho^3\, \boldsymbol{\omega}\cdot(\cos\gamma, \sin\gamma, 0), \end{aligned}$$

and similarly for the integral over $\Gamma^{\cup,+}_{\rho/2,\gamma}$. For the remaining two integrals in (2.6) we note that

$$\Gamma^{\cup,0}_{\rho,\gamma} \subset \Gamma^0_{\rho,\gamma} \quad \text{and similarly} \quad \delta\Gamma^{\cup,1}_{\rho,\gamma} \subset \delta\Gamma^1_{\rho,\gamma}.$$

From (2.6) then, we obtain the analogue of (2.3) which reads

$$\begin{aligned} A_1\rho^{2+\alpha}|u_{P1}\sin\gamma - u_{P2}\cos\gamma| &\le A_2\rho^2|\boldsymbol{u}_P\cdot\boldsymbol{n}_P| + A_3\rho^3|\boldsymbol{\omega}_\tau| \\ &+ \int_{\Gamma^0_{\rho,\gamma}\cup\Gamma^0_{\rho/2,\gamma}} |\boldsymbol{u}|dS + \int_{\delta\Gamma^1_{\rho,\gamma}\cup\delta\Gamma^1_{\rho,\pi+\gamma}} |\boldsymbol{u}|\, dS, \end{aligned} \tag{2.7}$$

with positive constants A_i, $i = 1,2,3$, depending only on k, α. Again, we will integrate the above estimate from $\gamma = 0$ to $\gamma_0 \le \pi$ and then from $\rho = \sigma/2$ to σ to reach the analogue of (2.5),

$$|\boldsymbol{u}_{P\tau}| \le |u_{P1}| + |u_{P2}| \le C\sigma^{-1-\alpha-\frac{2}{p}}(h+3k\sigma^{1+\alpha})^{2-\frac{1}{p}}\|\nabla\boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}, \tag{2.8}$$

with a positive constant $C = C(k,\alpha)$.

Estimate of ω_3. We will use the same argument as before, but in a modified domain that allows for the appearance of ω_3. To simplify the calculations we choose our coordinate system $x_1x_2x_3$ so that $\boldsymbol{\omega} = (0, \omega_2, \omega_3)$. This can be done by rotating our coordinate system around the x_3 axis so that $\boldsymbol{\omega}_\tau$ is along the x_2 axis. We next cut the cylinder $\mathcal{C}_{\rho,\gamma}$, $\rho \le \sigma$ with the plane

$$x_3 = (x_2 - \sigma)\tan\phi + h + 2k\sigma^{1+\alpha} =: \hat{x}_3(x_2),$$

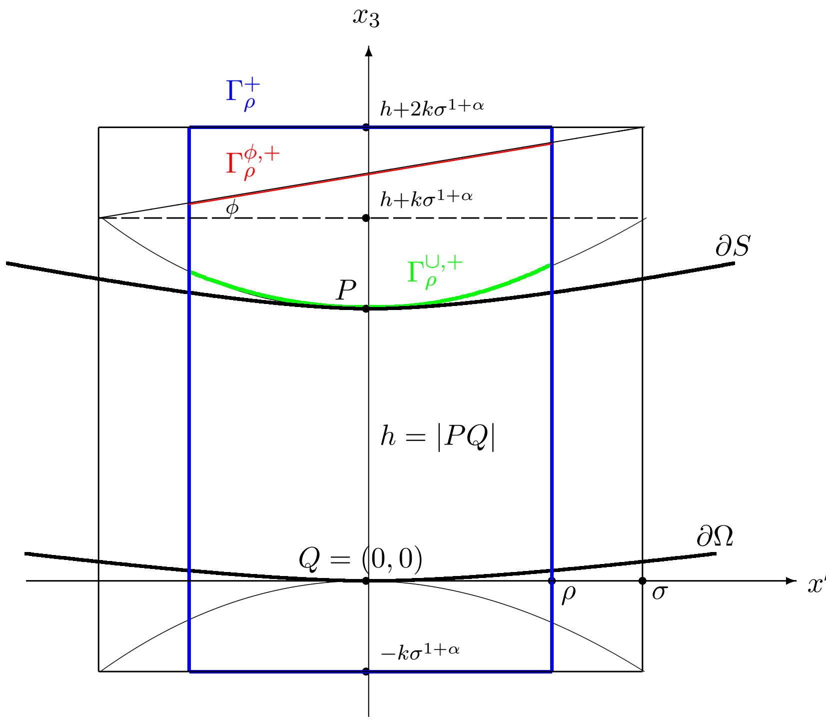

Figure 3: *A 2d depiction of the cylinder C_ρ (blue) as well as the top surface of $\mathcal{C}_\rho^\cup$ (green) and $\mathcal{C}_\rho^\phi$(red). The bottom surface of all three domains is the same.*

and we denote the resulting domain by

$$\mathcal{C}_{\rho,\gamma}^\phi := \{(r\cos\theta, r\sin\theta, x_3) : \ 0 < r < \rho, \ \gamma \le \theta < \pi + \gamma, \ -k\sigma^{1+\alpha} < x_3 < \hat{x}_3(r\sin\theta)\}.$$

We also denote the various parts of the boundary of $\mathcal{C}_{\rho,\gamma}^\phi$ by $\Gamma_{\rho,\gamma}^{\phi,+}$, $\Gamma_{\rho,\gamma}^{\phi,0}$, $\Gamma_{\rho,\gamma}^{\phi,1}$, $\Gamma_{\rho,\gamma}^{\phi,-} = \Gamma_{\rho,\gamma}^-$ and $\delta\Gamma_{\rho,\gamma}^{\phi,1}$ as usual.

The angle ϕ is so chosen that $\tan\phi = \frac{k\sigma^{1+\alpha}}{2\sigma} = \frac{1}{2}k\sigma^\alpha$. In particular we have

$$\mathcal{C}_{\rho,\gamma}^\cup \subset \mathcal{C}_{\rho,\gamma}^\phi \subset \mathcal{C}_{\rho,\gamma} \quad \text{with} \quad \Gamma_{\rho,\gamma}^{\phi,+} \subset S.$$

We will apply the divergence Theorem in $\mathcal{C}_{\rho,\gamma}^\phi \setminus \mathcal{C}_{\rho/2,\gamma}^\phi$. Again, the integral over the top surface of $\mathcal{C}_{\rho,\gamma}^\phi$ is computed explicitly,

$$\int_{\Gamma_{\rho,\gamma}^{\phi,+}} \boldsymbol{u}\cdot\boldsymbol{n}\, dS = \frac{\pi\rho^2}{2\cos\phi}(u_{P2}\sin\phi + u_{P3}\cos\phi) + \frac{2}{3}\rho^3\omega_2\sin\gamma - \frac{2}{3}\rho^3\omega_3\tan\phi\sin\gamma.$$

For the remaining integrals we note that

$$\Gamma_{\rho,\gamma}^{\phi,0} \subset \Gamma_{\rho,\gamma}^0 \quad \text{and similarly} \quad \delta\Gamma_{\rho,\gamma}^{\phi,1} \subset \delta\Gamma_{\rho,\gamma}^1.$$

We then obtain the analogue of (2.3) or (2.7) which now reads

$$\begin{aligned} A_1\rho^3\,\tan\phi\,\sin\gamma\,|\omega_3| \ &\le\ A_2\rho^2|u_{P3}|\cos\phi + A_2\rho^2|u_{P2}|\sin\phi + A_3\rho^3\sin\gamma\,|\omega_2| \\ &\quad + \int_{\Gamma_{\rho,\gamma}^0\cup\Gamma_{\rho/2,\gamma}^0} |\boldsymbol{u}| dS + \int_{\delta\Gamma_{\rho,\gamma}^1\cup\delta\Gamma_{\rho,\pi+\gamma}^1} |\boldsymbol{u}|\, dS, \end{aligned}$$

with positive constants A_i, $i = 1, 2, 3$, depending only on k, α. Integrating the above estimate from $\gamma = 0$ to $\gamma_0 \leq \pi$ and then from $\rho = \sigma/2$ to σ, and employing the previous estimates, we get

$$|\omega_3| \leq C\sigma^{-2-\alpha-\frac{2}{p}}(h + 3k\sigma^{1+\alpha})^{2-\frac{1}{p}}\|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}, \tag{2.9}$$

with $C = C(k, \alpha)$.

Completion of the proof. Putting together (2.5), (2.8) and (2.9) we have shown that there exists a positive constant $C_w = C_w(k, \alpha)$ depending at most on k, α, s.t. for any $h \in [0, H]$ and any $\sigma \in (0, \sigma_0/2]$ the inequalities of the statement hold true.

□

Remark 1. It is an immediate consequence of Theorem 2.1 and (2.4) that for $h \leq H$ and $\sigma \in (0, \sigma_0/2]$ there holds

$$\begin{aligned} |\boldsymbol{u}_P| &\leq C_w\sigma^{-1-\alpha-\frac{2}{p}}(h + 3k\sigma^{1+\alpha})^{2-\frac{1}{p}}\|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}, \\ |\boldsymbol{\omega}| &\leq C_w\sigma^{-2-\alpha-\frac{2}{p}}(h + 3k\sigma^{1+\alpha})^{2-\frac{1}{p}}\|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma)}, \end{aligned} \tag{2.10}$$

for a positive constant $C_w = C_w(k, \alpha)$. This provides an upper estimate on the kinetic condition of the rigid body as it approaches the boundary.

Remark 2. In the special case where $\alpha = 1$ and $p = 2$ and under the assumptions of Theorem 2.1 we have that for a positive constant $C_w = C_w(k)$,

$$\begin{aligned} |u_{P3}| &\leq C_w\sigma^{-2}(h + 3k\sigma^3)^{\frac{3}{2}}\|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_\sigma)}, \\ |\boldsymbol{u}_{P\tau}|, |\boldsymbol{\omega}_\tau| &\leq C_w\sigma^{-3}(h + 3k\sigma^2)^{\frac{3}{2}}\|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_\sigma)}, \\ |\omega_3| &\leq C_w\sigma^{-4}(h + 3k\sigma^2)^{\frac{3}{2}}\|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_\sigma)}. \end{aligned}$$

To prove Theorem 1.1 we first present the following Lemma

Lemma 2.2. *We assume that both Ω and S are uniformly $C^{1,\alpha}$ domains. Let $\boldsymbol{u} \in W_0^{1,p}(S, \Omega)$ be such that* $\mathrm{div}\boldsymbol{u} = 0$. *There exists positive constants $H_* \leq H$ and $\sigma_* \leq \frac{\sigma_0}{2}$ depending only on $\alpha, k, \sigma_0, H, p$ such that for*

$$h \leq H_* \quad \textit{and} \quad \frac{\sigma_0}{2}\left(\frac{h}{H}\right)^{\frac{1}{1+\alpha}} \leq \sigma \leq \sigma_*,$$

there holds

$$\int_{\mathcal{C}_\sigma} |\nabla \boldsymbol{u}|^p dx \leq 2\int_{\mathcal{C}_\sigma \cap F} |\nabla \boldsymbol{u}|^p dx.$$

Proof: Let $h < H$ and $\sigma \leq \frac{\sigma_0}{2}$. For later use we also require require that

$$h\sigma^{-1-\alpha} \leq H\left(\frac{2}{\sigma_0}\right)^{1+\alpha}. \tag{2.11}$$

All conditions are satisfied provided that

$$h \leq H \quad \text{and} \quad \frac{\sigma_0}{2}\left(\frac{h}{H}\right)^{\frac{1}{1+\alpha}} \leq \sigma \leq \frac{\sigma_0}{2}. \tag{2.12}$$

We next compute

$$\begin{aligned}\int_{\mathcal{C}_\sigma} |\nabla \boldsymbol{u}|^p dx &= \int_{\mathcal{C}_\sigma \cap F} |\nabla \boldsymbol{u}|^p dx + \int_{\mathcal{C}_\sigma \cap S} |\nabla \boldsymbol{u}|^p dx \\ &= \int_{\mathcal{C}_\sigma \cap F} |\nabla \boldsymbol{u}|^p dx + 2^{\frac{p}{2}} |\boldsymbol{\omega}|^p |\mathcal{C}_\sigma \cap S| \\ &\leq \int_{\mathcal{C}_\sigma \cap F} |\nabla \boldsymbol{u}|^p dx + 2^{\frac{p}{2}}\, 3\, k\, |\boldsymbol{\omega}|^p \pi \sigma^{3+\alpha}.\end{aligned}$$

The last inequality follows since the domain $\mathcal{C}_\sigma \cap S$ is contained in a cylinder of radius σ and height $3k\sigma^{1+\alpha}$. To continue, we first use (2.10) and then (2.11) to get

$$\begin{aligned}\int_{\mathcal{C}_\sigma} |\nabla \boldsymbol{u}|^p dx &\leq \int_{\mathcal{C}_\sigma \cap F} |\nabla \boldsymbol{u}|^p dx \\ &\quad + 2^{\frac{p}{2}}\, 3\, k\, \pi\, C_w^p\, \sigma^{\alpha p} (h\sigma^{-1-\alpha} + 3k)^{2p-1} \int_{\mathcal{C}_\sigma} |\nabla \boldsymbol{u}|^p dx \\ &\leq \int_{\mathcal{C}_\sigma \cap F} |\nabla \boldsymbol{u}|^p dx \\ &\quad + 2^{\frac{p}{2}}\, 3\, k\, \pi\, C_w^p\, \sigma^{\alpha p} \left[H \left(\frac{2}{\sigma_0} \right)^{1+\alpha} + 3k \right]^{2p-1} \int_{\mathcal{C}_\sigma} |\nabla \boldsymbol{u}|^p dx \\ &= \int_{\mathcal{C}_\sigma \cap F} |\nabla \boldsymbol{u}|^p dx + \sigma^{\alpha p}\, B \int_{\mathcal{C}_\sigma} |\nabla \boldsymbol{u}|^p dx;\end{aligned}$$

here $B = B(p, H, \sigma_0, k, \alpha)$ and is given by

$$B := 2^{\frac{p}{2}}\, 3\, k\, \pi\, C_w^p \left[H \left(\frac{2}{\sigma_0} \right)^{1+\alpha} + 3k \right]^{2p-1}.$$

We will choose σ small enough so that

$$\sigma^{\alpha p}\, B \leq \frac{1}{2} \quad \Leftrightarrow \quad \sigma \leq (2\, B)^{-\frac{1}{\alpha p}}.$$

For such a choice to be compatible with (2.12) we need to have

$$\frac{\sigma_0}{2} \left(\frac{h}{H} \right)^{\frac{1}{1+\alpha}} \leq (2\, B)^{-\frac{1}{\alpha p}}.$$

This imposes a new smalness condition on h, namely that

$$h \leq \left(\frac{2}{\sigma_0} \right)^{1+\alpha} (2\, B)^{-\frac{1+\alpha}{\alpha p}}\ H.$$

Hence for

$$H_* := \min\{H,\ \left(\frac{2}{\sigma_0} \right)^{1+\alpha} (2\, B)^{-\frac{1+\alpha}{\alpha p}}\ H\}, \quad \sigma_* := \min\{\frac{\sigma_0}{2}, (2B)^{-\frac{1}{\alpha p}}\},$$

and under the assumptions of the Lemma, we have that

$$\int_{\mathcal{C}_\sigma} |\nabla \boldsymbol{u}|^p dx \leq \int_{\mathcal{C}_\sigma \cap F} |\nabla \boldsymbol{u}|^p dx + \frac{1}{2} \int_{\mathcal{C}_\sigma} |\nabla \boldsymbol{u}|^p dx,$$

and the result follows.

□

Remark 3. We note that the same result holds true in the 2D case where Ω, $S \subset I\!R^2$. The proof is essentially the same.

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1 : We first recall (2.1). Our starting point are the estimates of Theorem 2.1. Let H_* be as in Lemma 2.2. For $h \leq H_*$ and $\sigma = \sigma_h := \frac{\sigma_0}{2}\left(\frac{h}{H}\right)^{\frac{1}{1+\alpha}}$ conditions of Lemma 2.2 are fulfilled, therefore

$$\|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_{\sigma_h})} \leq 2^{\frac{1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_{\sigma_h} \cap F)}.$$

On the other hand, to show the estimate of $|u_{P3}|$, elementary calculations show that

$$\sigma^{-1-\frac{2}{p}}(h + 3k\sigma^{1+\alpha})^{2-\frac{1}{p}} = \left(\frac{\sigma_0}{2H^{\frac{1}{1+\alpha}}}\right)^{1+2\alpha-\frac{3+\alpha}{p}} \left[H\left(\frac{2}{\sigma_0}\right)^{1+\alpha} + 3k\right]^{2-\frac{1}{p}} h^{\frac{1+2\alpha}{p(1+\alpha)}\left(p-\frac{3+\alpha}{1+2\alpha}\right)},$$

from which the estimate for $|u_{P3}|$ follows from the corresponding estimate of Theorem 2.1. We similarly treat the other estimates of Theorem 1.1.

□

Remark 4. In the special case where $\alpha = 1$ and $p = 2$ and under the assumptions of Theorem 1.1 we have that

$$\begin{aligned} |u_{P3}| &\leq C_w h^{\frac{1}{2}} \|\nabla \boldsymbol{u}\|_{L^2(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)}, \\ |\boldsymbol{u}_{P\tau}|, |\boldsymbol{\omega}_\tau| &\leq C_w \|\nabla \boldsymbol{u}\|_{L^2(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)}, \\ |\omega_3| &\leq C_w h^{-\frac{1}{2}} \|\nabla \boldsymbol{u}\|_{L^2(\{|\boldsymbol{x}'|<\sigma_h\}\cap F)}. \end{aligned}$$

Remark 5. In the 2D case (cf Theorem 4.6 in [2]) the analogous estimates of Theorem 1.1 are

$$\begin{aligned} |u_{P2}| &\leq C_w h^{\frac{1+2\alpha}{p(1+\alpha)}\left(p-\frac{2+\alpha}{1+2\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_{\sigma_h}\cap F)}, \\ |u_{P1}| &\leq C_w\, h^{\frac{1}{p}\left(p-\frac{2+\alpha}{1+\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_{\sigma_h}\cap F)}, \\ |\omega| &\leq C_w\, h^{\frac{2\alpha}{p(1+\alpha)}\left(p-\frac{2+\alpha}{2\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_{\sigma_h}\cap F)}, \end{aligned}$$

where u_{P2} and u_{P1} are the vertical and tangential components of $\boldsymbol{u}_P$ respecively and ω is the scalar angular velocity. In particular for $\alpha = 1$, $p = 2$,

$$|u_{P2}| \leq C_w h^{\frac{3}{4}} \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma_h}\cap F)}, \quad \text{and} \quad |u_{P1}|, |\omega| \leq C_w h^{\frac{1}{4}} \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma_h}\cap F)}.$$

3 Estimates for strong solutions

Here we will give the proof of Theorems 1.2 and 1.3. Assume that both S and Ω are uniformy C^3. More precisely, for $Q \in \partial\Omega$ we recall our orthogonal coordinate system with the origin at $Q = (0, 0, 0)$ and such that $\boldsymbol{x}' = (x_1, x_2)$ lies on the tangent plane to $\partial\Omega$ at the point Q. Then the function that describes $\partial\Omega$ locally near $\boldsymbol{x}' = 0$ is given by

$$x_3 = g_\Omega(\boldsymbol{x}'), \qquad |\boldsymbol{x}'| < \sigma_0,$$

where g_Ω is C^3 and such that

$$|\nabla^i g_\Omega(\boldsymbol{x}')| \le K|\boldsymbol{x}'|^{2-i}, \quad i = 0, 1, 2 \ \text{ for } |\boldsymbol{x}'| < \sigma_0,$$

as well as

$$|\nabla^3 g_\Omega(\boldsymbol{x}')| \le K, \ \text{ for } |\boldsymbol{x}'| < \sigma_0,$$

with positive constants σ_0, K independent of the choice of the point $Q \in \partial\Omega$. In connection with this, we use the notation

$$f(\boldsymbol{x}') = O(K|\boldsymbol{x}'|^m), \quad |\boldsymbol{x}'| < \sigma_0, \quad m = 1, 2, \dots,$$

to mean that $|f(\boldsymbol{x}')| \le C(K)|\boldsymbol{x}'|^m$ with a constant $C(K)$ that depends only on K (and not on the specific point on the boundary).

Without loss of generality we assume that

$$K \ge \max\{1, k\};$$

we also take σ_0 to be the same as the one in Section 2. The same estimates are assumed to be true for ∂S. In particular, if $P \in \partial S$ is a point nearest to Q so that $h = |PQ|$, then

$$x_3 = h + g(\boldsymbol{x}'), \qquad |\boldsymbol{x}'| < \sigma_0,$$

and g satisfies the same estimates as g_Ω.

To prove Theorem 1.2 let us first show

Proposition 3.1. *We assume that both Ω and S are uniformly C^3 domains. Let $\boldsymbol{u} \in W_0^{1,2}(S, \Omega)$ be such that $\mathrm{div}\boldsymbol{u} = 0$ and in addition $\boldsymbol{u} \in W^{2,2}(F)$. Then there exists a positive constant $C(K)$ depending only on K, such that for any $h \in [0, H]$ and $\sigma \in (0, \sigma_0/2]$ there holds*

$$\begin{aligned} |u_{P3}| &\le 3\pi^{-\frac{1}{2}}\,\sigma^{-2}\,(h + 2K\sigma^2)^{\frac{5}{2}}\,\|\nabla^2\boldsymbol{u}\|_{L^2(\mathcal{C}_\sigma \cap F)} + C(K)\sigma^2|\boldsymbol{u}_{P\tau}| \\ &\quad + C(K)\,\sigma\,(h + 2K\sigma^2)|\boldsymbol{\omega}_\tau|. \end{aligned}$$

Proof: We apply divergence Theorem in the domain $\mathcal{C}_\rho \cap F$. Setting

$$\tilde\Gamma_\rho^+ \ = \ \{|\boldsymbol{x}'| < \rho, \ x_3 = h + g(\boldsymbol{x}')\},$$

we have

$$\int_{\tilde\Gamma_\rho^+} \boldsymbol{u}\cdot\boldsymbol{n}\,dS = -\int_{\Gamma_\rho^0 \cap F} \boldsymbol{u}\cdot\boldsymbol{n}\,dS. \tag{3.1}$$

Estimates on $\tilde\Gamma_\rho^+$. Here $\boldsymbol{n} = (1 + |\nabla g|^2)^{-\frac{1}{2}}(-g_{x_1}, -g_{x_2}, 1)$. As usual we will write $\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{\omega}\times(\boldsymbol{x} - \boldsymbol{x}_P) = \boldsymbol{u}_P + \boldsymbol{\omega}\times(x_1, x_2, g(x_1, x_2))$.

$$\begin{aligned} \int_{\tilde\Gamma_\rho^+} \boldsymbol{u}\cdot\boldsymbol{n}\,dS \ &= \ \int_{|\boldsymbol{x}'|<\rho} \boldsymbol{u}\cdot(-g_{x_1}, -g_{x_2}, 1)\,d\boldsymbol{x}' \\ &= \int_{|\boldsymbol{x}'|<\rho} \boldsymbol{u}_P\cdot\boldsymbol{n}_P\,d\boldsymbol{x}' + \int_{|\boldsymbol{x}'|<\rho} \boldsymbol{u}_P\cdot(-g_{x_1}, -g_{x_2}, 0)\,d\boldsymbol{x}' \\ &\quad + \boldsymbol{\omega}\cdot\int_{|\boldsymbol{x}'|<\rho} (x_1, x_2, g)\times(-g_{x_1}, -g_{x_2}, 0)\,d\boldsymbol{x}' + \boldsymbol{\omega}\cdot\int_{|\boldsymbol{x}'|<\rho} (x_1, x_2, g)\times(0, 0, 1)\,d\boldsymbol{x}' \\ &= \pi\rho^2\boldsymbol{u}_P\cdot\boldsymbol{n}_P - \boldsymbol{u}_{P\tau}\cdot\int_{|\boldsymbol{x}'|<\rho} (g_{x_1}, g_{x_2})dx_1dx_2 \\ &\quad + \boldsymbol{\omega}\cdot\int_{|\boldsymbol{x}'|<\rho} (gg_{x_2}, -gg_{x_1}, x_2g_{x_1} - x_1g_{x_2})\,dx_1dx_2 + \boldsymbol{\omega}\cdot\int_{|\boldsymbol{x}'|<\rho} (x_2, -x_1, 0)\,dx_1dx_2 \\ &=: \pi\rho^2\boldsymbol{u}_P\cdot\boldsymbol{n}_P - \boldsymbol{u}_{P\tau}\cdot\boldsymbol{I}_1 + \boldsymbol{\omega}\cdot\boldsymbol{I}_2 + \boldsymbol{\omega}\cdot\boldsymbol{I}_3. \end{aligned}$$

It is easily seen that $\boldsymbol{I}_3 = 0$. We also note that for $\boldsymbol{x}' = (x_1, x_2) = (r\cos\theta, r\sin\theta)$ small, we have by Taylor expansion

$$\begin{aligned} g(x_1,x_2) &= \frac{1}{2}[g_{x_1x_1}(0,0)x_1^2 + 2g_{x_1x_2}(0,0)x_1x_2 + g_{x_2x_2}(0,0)x_2^2] + O(K|\boldsymbol{x}'|^3) \\ &= \frac{r^2}{2}\left[g_{x_1x_1}(0,0)\cos^2\theta + 2g_{x_1x_2}(0,0)\cos\theta\sin\theta \right. \\ &\quad \left. + g_{x_2x_2}(0,0)\sin\theta^2\right] + O(Kr^3). \end{aligned} \tag{3.2}$$

Then, using the above Taylor expansion we have

$$\begin{aligned} \int_{|\boldsymbol{x}'|<\rho} g_{x_1}\,dx_1dx_2 &= \int_{-\rho}^{\rho} g((\rho^2-x_2^2)^{1/2},x_2) - g(-(\rho^2-x_2^2)^{1/2},x_2)\,dx_2 \\ &= 2g_{x_1x_2}(0,0)\int_{-\rho}^{\rho} x_2(\rho^2-x_2^2)^{1/2}\,dx_2 + O(K\rho^4). \end{aligned}$$

The last integral is zero (the integrand is odd), we therefore end up with

$$|\boldsymbol{I}_1| \le C(K)\rho^4.$$

Concerning the integral $\boldsymbol{I}_2$, we first note that

$$\begin{aligned} \int_{|\boldsymbol{x}'|<\rho} (x_2g_{x_1} - x_1g_{x_2})\,dx_1dx_2 &= \int_{|\boldsymbol{x}'|<\rho} (x_2,-x_1)\cdot\nabla g\,dx_1dx_2 \\ &= \int_{|\boldsymbol{x}'|<\rho} \operatorname{div}(x_2,-x_1)\; g\; dx_1dx_2 + \int_{|\boldsymbol{x}'|=\rho} (x_2,-x_1)\cdot\frac{(x_1,x_2)}{(x_1^2+x_2^2)^{\frac{1}{2}}}\; g\; dS \\ &= 0. \end{aligned}$$

One the other hand from (3.2) we have that

$$g^2(x_1,x_2) = \frac{1}{4}[g_{x_1x_1}(0,0)x_1^2 + 2g_{x_1x_2}(0,0)x_1x_2 + g_{x_2x_2}(0,0)x_2^2]^2 + O(Kr^5).$$

Using this we calculate

$$\begin{aligned} \int_{|\boldsymbol{x}'|<\rho} g\,g_{x_1}\,dx_1dx_2 &= \frac{1}{2}\int_{|\boldsymbol{x}'|<\rho} (g^2)_{x_1}\,dx_1dx_2 \\ &= \frac{1}{2}\int_{-\rho}^{\rho} g^2((\rho^2-x_2^2)^{1/2},x_2) - g^2(-(\rho^2-x_2^2)^{1/2},x_2)\,dx_2 \\ &= \int_{-\rho}^{\rho} [g_{x_1x_1}(0,0)(\rho^2-x_2^2) + g_{x_2x_2}(0,0)x_2^2]\; g_{x_1x_2}(0,0)\; (\rho^2-x_2^2)^{\frac{1}{2}}\; x_2\,dx_2 + O(K\rho^5) \\ &= O(K\rho^5), \end{aligned}$$

since the last integral is zero (the integrand is odd). The same estimate holds true for the integral of $g\,g_{x_2}$ so that eventually

$$|\boldsymbol{I}_2| \le C(K)\rho^5.$$

Hence,

$$\int_{\tilde{\Gamma}_\rho^+} \boldsymbol{u}\cdot\boldsymbol{n}\,dS = \pi\rho^2\boldsymbol{u}_P\cdot\boldsymbol{n}_P + R, \quad |R| \le C(K)(|\boldsymbol{u}_{P\tau}| + |\boldsymbol{\omega}_\tau|\rho)\rho^4. \tag{3.3}$$

Estimates on $\Gamma^0_\rho \cap F$. In the subsequent calculations we will use cylindrical coordinates. By abuse of notation, we will write $\boldsymbol{u}(r,\theta,x_3)$ to mean $\boldsymbol{u}(r\cos\theta, r\sin\theta, x_3)$. In what follows we will change variables by

$$\xi = \frac{x_3 - g_\Omega}{h + g - g_\Omega}, \qquad \tilde{\boldsymbol{u}}(r,\theta,\xi) = \boldsymbol{u}(r,\theta,x_3). \tag{3.4}$$

$$\begin{aligned}
\int_{\Gamma^0_\rho \cap F} \boldsymbol{u}\cdot\boldsymbol{n}\, dS &= \rho \int_0^{2\pi}\int_{g_\Omega}^{h+g} \boldsymbol{u}\cdot\boldsymbol{n}\, dx_3\, d\theta = \rho\int_0^{2\pi}\int_0^1 \tilde{\boldsymbol{u}}\cdot\boldsymbol{n}\,(h+g-g_\Omega)\, d\xi\, d\theta \\
&= \rho\int_0^1\int_0^{2\pi} \tilde{\boldsymbol{u}}\cdot\boldsymbol{n}\,(h+g-g_\Omega)\, d\theta\, d\xi \\
&= \rho\int_0^1 \Phi(\rho,\xi)\, d\xi,
\end{aligned}$$

with

$$\Phi(\rho,\xi) := \int_0^{2\pi} \tilde{\boldsymbol{u}}\cdot\boldsymbol{n}\,(h+g-g_\Omega)\, d\theta.$$

We recall that $\tilde{\boldsymbol{u}}$ depends on ρ, θ, ξ, whereas g, g_Ω depend (in polar coordinates) on ρ, θ and $\boldsymbol{n}$ depends on θ. We have that $\Phi(\rho,0)=0$, since $\tilde{\boldsymbol{u}}(r,\theta,0) = \boldsymbol{u}(r,\theta,g_\Omega) = \boldsymbol{u}\big|_{\partial\Omega} = 0$, therefore by mean value Theorem there exists $\zeta_* \in (0,1)$ s.t. $\Phi_\xi(\rho,\zeta_*) = \Phi(\rho,1)$. We then write

$$\Phi(\rho,\xi) = \int_0^\xi \Phi_\zeta(\rho,\zeta)\, d\zeta = \int_0^\xi\int_{\zeta_*}^\zeta \Phi_{\tau\tau}(\rho,\tau)\, d\tau\, d\zeta + \xi\Phi(\rho,1).$$

As a consequence

$$\int_{\Gamma^0_\rho\cap F} \boldsymbol{u}\cdot\boldsymbol{n}\, dS = \rho\int_0^1\int_0^\xi\int_{\zeta_*}^\zeta \Phi_{\tau\tau}(\rho,\tau)\, d\tau\, d\zeta\, d\xi + \frac{1}{2}\rho\Phi(\rho,1),$$

and therefore

$$\left|\int_{\Gamma^0_\rho\cap F} \boldsymbol{u}\cdot\boldsymbol{n}\, dS\right| \le \rho\int_0^1 |\Phi_{\tau\tau}(\rho,\tau)|\, d\tau + \frac{1}{2}\rho|\Phi(\rho,1)|.$$

We estimate the first term in the right hand side,

$$\begin{aligned}
\rho\int_0^1 |\Phi_{\tau\tau}(\rho,\tau)|\, d\tau &\le \rho\int_0^{2\pi}\int_0^1 |\tilde{\boldsymbol{u}}_{\tau\tau}|(h+g-g_\Omega)\, d\tau d\theta \\
&= \rho\int_0^{2\pi}\int_{g_\Omega}^{h+g} |\boldsymbol{u}_{x_3x_3}|\,(h+g-g_\Omega)^2\, dx_3 d\theta \\
&\le (h+2K\rho^2)^2 \int_0^{2\pi}\int_{g_\Omega}^{h+g} |\boldsymbol{u}_{x_3x_3}|\rho\, dx_3 d\theta.
\end{aligned}$$

Concerning $\Phi(\rho,1)$, for $\boldsymbol{n} = (\cos\theta, \sin\theta, 0)$ we have

$$\begin{aligned}
\Phi(\rho,1) &= \int_0^{2\pi} \tilde{\boldsymbol{u}}(\rho,\theta,1)\cdot\boldsymbol{n}\,(h+g-g_\Omega)\, d\theta \\
&= \int_0^{2\pi} \boldsymbol{u}(\rho,\theta,h+g)\cdot\boldsymbol{n}\,(h+g-g_\Omega)\, d\theta \\
&= \int_0^{2\pi} \boldsymbol{u}_P\cdot\boldsymbol{n}\,(h+g-g_\Omega)\, d\theta + \boldsymbol{\omega}\cdot\int_0^{2\pi} (\rho\cos\theta, \rho\sin\theta, g)\times\boldsymbol{n}\,(h+g-g_\Omega)\, d\theta \\
&:= I_1 + I_2.
\end{aligned}$$

To estimate I_1, we use (3.2) to obtain

$$|I_1| \leq C(K)\,|\boldsymbol{u}_{P\tau}|\rho^3,$$

for a positive constant C depending only on K. Concerning I_2 we have that

$$\begin{aligned}(x_1, x_2, g) \times \boldsymbol{n} &= (\rho\cos\theta, \rho\sin\theta, g) \times (\cos\theta, \sin\theta, 0)\\ &= (-\sin\theta, \cos\theta, 0)\, g(\rho\cos\theta, \rho\sin\theta),\end{aligned}$$

from which it follows

$$\begin{aligned}|I_2| &\leq 2\pi K\,|\boldsymbol{\omega}_\tau|\rho^2(h + 2K\rho^2)\\ &\leq 2\pi K\,|\boldsymbol{\omega}_\tau|\sigma^2(h + 2K\sigma^2).\end{aligned}$$

Putting everything together, and recalling that $\rho \leq \sigma$, we have

$$\begin{aligned}\Big|\int_{\Gamma_\rho^0\cap F} \boldsymbol{u}\cdot\boldsymbol{n}\,dS\Big| &\leq (h + 2K\sigma^2)^2 \int_0^{2\pi}\int_{g_\Omega}^{h+g} |\boldsymbol{u}_{x_3x_3}|\rho\,dx_3d\theta\\ &\quad + C(K)\,|\boldsymbol{u}_{P\tau}|\sigma^4 + 2\pi K\,|\boldsymbol{\omega}_\tau|\sigma^3(h + 2K\sigma^2). \qquad (3.5)\end{aligned}$$

<u>The estimate on $|\boldsymbol{u}_{P3}|$:</u> Combining (3.1), (3.3) and (3.5) we get

$$\begin{aligned}\pi\rho^2|u_{P3}| &\leq (h + 2K\sigma^2)^2 \int_0^{2\pi}\int_{g_\Omega}^{h+g} |\boldsymbol{u}_{x_3x_3}|\rho\,dx_3d\theta\\ &\quad + C(K)\,|\boldsymbol{u}_{P\tau}|\sigma^4 + C(K)\,|\boldsymbol{\omega}_\tau|\sigma^3(h + 2K\sigma^2).\end{aligned}$$

We integrate this from $\rho = 0$ to $\rho = \sigma$ to get after simplifying,

$$\begin{aligned}|u_{P3}| &\leq \frac{3\sigma^{-3}}{\pi}(h + 2K\sigma^2)^2\,\|\nabla^2\boldsymbol{u}\|_{L^1(\mathcal{C}_\sigma\cap F)} + C(K)\sigma^2|\boldsymbol{u}_{P\tau}|\\ &\quad + C(K)\,\sigma\,(h + 2K\sigma^2)|\boldsymbol{\omega}_\tau|.\end{aligned}$$

By Hölder inequality

$$\begin{aligned}\|\nabla^2\boldsymbol{u}\|_{L^1(\mathcal{C}_\sigma\cap F)} &\leq |\mathcal{C}_\sigma \cap F|^{\frac{p-1}{p}}\,\|\nabla^2\boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma\cap F)}\\ &\leq \pi^{\frac{p-1}{p}}\,\sigma^{\frac{2(p-1)}{p}}(h + 2K\sigma^2)^{\frac{p-1}{p}}\,\|\nabla^2\boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma\cap F)}.\end{aligned}$$

Hence

$$\begin{aligned}|u_{P3}| &\leq 3\pi^{-\frac{1}{p}}\,\sigma^{-1-\frac{2}{p}}\,(h + 2K\sigma^2)^{3-\frac{1}{p}}\,\|\nabla^2\boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma\cap F)} + C(K)\sigma^2|\boldsymbol{u}_{P\tau}|\\ &\quad + C(K)\,\sigma\,(h + 2K\sigma^2)|\boldsymbol{\omega}_\tau|,\end{aligned}$$

and the proof is complete. □

Remark 1 Examination of the proof shows that in case both g and g_Ω are radially symmetric, the terms involving $|\boldsymbol{u}_{P\tau}|$ and $|\boldsymbol{\omega}_\tau|$ are absent. Thus we recover the result of Proposition 8 of [7]. However, in the absence of symmetries these terms are present. To proceed then with the estimation of $|u_{P3}|$ we will use the estimates for $|\boldsymbol{u}_{P\tau}|$ and $|\boldsymbol{\omega}_\tau|$ from Theorem 2.1 (the case of weak solutions).

Remark 2 In the 2D case, under similar assumptions there exists a positive constant $C(K)$ depending only on K, such that for any $h \in [0, H]$ and $\sigma \in (0, \sigma_0/2]$ there holds

$$\begin{aligned} |u_{P2}| &\leq 2^{\frac{p-1}{p}} \sigma^{-1-\frac{1}{p}} (h + 2K\sigma^2)^{3-\frac{1}{p}} \|\nabla^2 \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma \cap F)} + C(K)\sigma^2 |\boldsymbol{u}_{P\tau}| \\ &\quad + C(K)\, \sigma^2\, (h + 2K\sigma^2)|\boldsymbol{\omega}_\tau|. \end{aligned}$$

Proof of Theorem 1.2: Let $h < H$ and $\sigma \in (0, \sigma_0/2]$. We combine the estimate of Proposition 3.1 with the estimates for the weak solutions for $|\boldsymbol{u}_{P\tau}|$ and $|\boldsymbol{\omega}_\tau|$, with $\alpha = 1$, $p = 2$, cf Remark 2 after the proof of Theorem 2.1 namely

$$|\boldsymbol{u}_{P\tau}|,\ |\boldsymbol{\omega}_\tau| \leq C_w(k)\, \sigma^{-3}\, (h + 3k\sigma^2)^{\frac{3}{2}}\, \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_\sigma)}.$$

Taking also into account that $K \geq k$ we obtain

$$\begin{aligned} |u_{P3}| &\leq C(k)\, \sigma^{-1-\frac{2}{p}} (h + 3K\sigma^2)^{3-\frac{1}{p}} \|\nabla^2 \boldsymbol{u}\|_{L^p(\mathcal{C}_\sigma \cap F)} \\ &\quad + C(k, K)\, \sigma^{-1} (h + 3K\sigma^2)^{\frac{3}{2}} \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_\sigma)} \\ &\quad + C(k, K)\, \sigma^{-2} (h + 3K\sigma^2)^{\frac{5}{2}} \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_\sigma)}. \end{aligned}$$

At this point we further restrict h by $h \leq H_* \leq H$ where H_* is defined in Lemma 2.2. In addition we choose $\sigma = \sigma_h := \left(\frac{h}{H}\right)^{\frac{1}{2}} \frac{\sigma_0}{2}$. With these choices the assumptions of Lemma 2.2 are satisfied, therefore

$$\|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma_h})} \leq \sqrt{2}\, \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma_h} \cap F)}.$$

On the other hand,

$$h + 3K\sigma_h^2 = \sigma_h^2 (h\sigma_h^{-2} + 3K) \leq \sigma_h^2 \left[H \left(\frac{2}{\sigma_0} \right)^2 + 3K \right].$$

Hence,

$$|u_{P3}| \leq C_s \left(\sigma_h^{5-\frac{4}{p}} \|\nabla^2 \boldsymbol{u}\|_{L^p(\mathcal{C}_{\sigma_h} \cap F)} + \sigma_h^2 \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma_h} \cap F)} \right),$$

with a positive constant $C_s = C_s(k, \sigma_0, H, K)$. Recalling the definition of σ_h the result follows. □

Remark 3 In the 2D case, under assumptions similar to those stated in Theorem 1.2, there exist positive constants H_*, C_s, c_0 depending only on S, Ω and p, such that, whenever $h < H_*$ and $\sigma_h := c_0 h^{\frac{1}{2}}$,

$$|u_{P2}| \leq C_s \left(h^{\frac{5}{2} - \frac{3}{2p}} \|\nabla^2 \boldsymbol{u}\|_{L^p(\mathcal{C}_{\sigma_h} \cap F)} + h^{\frac{5}{4}} \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma_h} \cap F)} \right).$$

We next give the proof of Theorem 1.3.

Proof of Theorem 1.3: It is a consequence of Theorems 1.1, 1.2 and Lemma 4.5 of [2]. The last Lemma concerns the 2D case. In the 3D case the arguments are the same with the exception of the proof that $\boldsymbol{u}_*$ and $\boldsymbol{\omega}$ are $L^\infty(0, T)$. We next present the proof, as is suggested in [11], for completeness. If $\boldsymbol{x}_*(t)$ is the center of mass of S, then $\boldsymbol{u} = \boldsymbol{u}_* + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_*)$ and

$$\int_S |\boldsymbol{u}|^2 d\boldsymbol{x} = \int_S [|\boldsymbol{u}_*|^2 + |\boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_*)|^2 + 2\boldsymbol{u}_* \cdot (\boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_*))]\, d\boldsymbol{x},$$

but

$$\int_S \boldsymbol{u}_* \cdot (\boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_*)) = (\boldsymbol{u}_* \times \boldsymbol{\omega}) \cdot \int_S (\boldsymbol{x} - \boldsymbol{x}_*) = 0,$$

whence

$$\int_S [|\boldsymbol{u}_*|^2 + |\boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_*)|^2] d\boldsymbol{x} = \int_S |\boldsymbol{u}|^2 d\boldsymbol{x} \leq \int_\Omega |\boldsymbol{u}|^2 d\boldsymbol{x}. \tag{3.6}$$

We next note that

$$\int_S |\boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_*)|^2 d\boldsymbol{x} = |\boldsymbol{\omega}|^2 \int_S |\boldsymbol{x} - \boldsymbol{x}_*|^2 \sin^2 \left(\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}, \boldsymbol{x} - \boldsymbol{x}_* \right) d\boldsymbol{x},$$

and

$$\inf_{\hat{\boldsymbol{e}} \in S^2} \int_S |\boldsymbol{x} - \boldsymbol{x}_*|^2 \sin^2 (\hat{\boldsymbol{e}}, \boldsymbol{x} - \boldsymbol{x}_*) \, d\boldsymbol{x} =: a > 0.$$

It then follows from (3.6) that

$$|S|(|\boldsymbol{u}_*|^2 + a|\boldsymbol{\omega}|^2) \leq \int_\Omega |\boldsymbol{u}|^2 d\boldsymbol{x}.$$

Since $\boldsymbol{u} \in L^\infty(0, T; L^2(\Omega))$, then both $\boldsymbol{u}_*$ and $\boldsymbol{\omega}$ are in $L^\infty(0, T)$. The rest of the arguments are the same as in [2]. In particular, if R is the radius of the inner sphere of Ω and $h(t) < R$ then $h(t)$ is Lipschitz continuous and for almost all t

$$\left| \frac{dh(t)}{dt} \right| = |\boldsymbol{u}_P \cdot \boldsymbol{n}_P| = |u_{P3}|.$$

Therefore if $h < H_0 := \min\{R, H_*\}$ the result follows.

□

As a consequence of Theorem 1.3(ii) we have the following noncollision result for strong solutions.

Corollary 3.2. *We assume that both Ω and S_0 are uniformly C^3 domains. Let the vector field $\mathbf{u}$ be such that*

$$\boldsymbol{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^1(0, T; W_0^{1,2}(S(t), \Omega)), \quad \mathrm{div}\boldsymbol{u} = 0,$$

and in addition for some $p \in [1, \infty)$,

$$\int_0^t \|\nabla^2 \boldsymbol{u}\|_{L^p(F(s))} ds < \infty, \qquad \forall t \in [0, T).$$

If $h(0) > 0$ then $h(t)$ remains positive for $t \in [0, T)$. Moreover, if $\liminf_{t \to T} h(t) = 0$, then

$$\lim_{t \to T} \int_0^t h^{\frac{3}{2} - \frac{2}{p}}(\tau) \|\nabla^2 \boldsymbol{u}\|_{L^p(\mathcal{C}_{\sigma_h(\tau)} \cap F(\tau))} \, d\tau = \infty.$$

In particular, if $\liminf_{t \to T} h(t) = 0$ and $p \geq 4/3$ then

$$\lim_{t \to T} \int_0^t \|\nabla^2 \boldsymbol{u}\|_{L^p(\mathcal{C}_{\sigma_h(\tau)} \cap F(\tau))} \, d\tau = \infty.$$

Proof: By Theorem 1.3(ii) we have that for a.a. t,

$$\left|\frac{1}{h}\frac{dh}{dt}\right| \le C_s \left(h^{\frac{3}{2}-\frac{2}{p}} \|\nabla^2 \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma_h}\cap F)} + \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma_h}\cap F)}\right).$$

Integrating this we get

$$\left|\ln\frac{h(t)}{h(0)}\right| \le C_s \int_0^t h^{\frac{3}{2}-\frac{2}{p}} \|\nabla^2 \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma}\cap F)}\, d\tau + C_s \int_0^t \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma}\cap F)}\, d\tau,$$

from which the results follow easily.

□

Remark 4 Similar results to Theorem 1.3 as well as to Corollary 3.2 hold true in the 2D case. In particular, under the assumptions of Theorem 1.3, the analogue of (1.3) is

$$\left|\frac{dh}{dt}\right| \le C_w h^{\frac{1+2\alpha}{p(1+\alpha)}\left(p-\frac{2+\alpha}{1+2\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{C}_{\sigma_h}\cap F)},$$

whereas the analogue of (1.4) is

$$\left|\frac{dh}{dt}\right| \le C_s \left(h^{\frac{5}{2}-\frac{3}{2p}} \|\nabla^2 \boldsymbol{u}\|_{L^p(\mathcal{C}_{\sigma_h}\cap F)} + h^{\frac{5}{4}} \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma_h}\cap F)}\right).$$

4 Example and optimality of Theorem 1.1

Let Ω be the half space $I\!R^3_+ = \{(x_1, x_2, x_3) : \ x_3 > 0\}$. For $r = |\boldsymbol{x}'| = |(x_1, x_2)|$, the lower part of the boundary of the body S is given by $x_3 = h(t) + r^{1+\alpha}$ so that $h(t)$ equals the distance between S and $\partial\Omega$ at each time t. Let

$$\boldsymbol{F} = \frac{1}{2}(-\dot{h}x_2,\ \dot{h}x_1, -\omega_3 r^2),$$

so that the velocity of the solid is given by

$$\boldsymbol{u}_S = \nabla \times \boldsymbol{F} = (-\omega_3 x_2, \omega_3 x_1, \dot{h}) = (0, 0, \dot{h}) + (0, 0, \omega_3) \times (x_1, x_2, x_3 - h).$$

This represents rotation about the x_3 axis with angular velocity ω_3 and a vertical speed $\dot{h}$. To define the velocity field everywhere in Ω we set

$$\begin{aligned}
\boldsymbol{u} &= \nabla \times (\boldsymbol{F}\Psi) \\
&= \frac{1}{2}(-\omega_3 x_2(2\Psi + r\Psi_r) - \dot{h}x_1\Psi_{x_3},\ \omega_3 x_1(2\Psi + r\Psi_r) - \dot{h}x_2\Psi_{x_3},\ \dot{h}(2\Psi + r\Psi_r)),
\end{aligned}$$

so that $\mathrm{div}\boldsymbol{u} = 0$. Here

$$\Psi = \Psi(r, x_3) = \phi\left(\frac{x_3}{h + r^{1+\alpha}}\right)\psi_1(r)\psi_2(x_3),$$

and ϕ, ψ_1, ψ_2 are suitable smooth cut off functions such that

$$\phi(\tau) = \begin{cases} 1, & \tau \ge 1, \\ 0, & \tau \le 0 \end{cases}, \quad \psi_1(r) = \begin{cases} 1, & r < \rho, \\ 0, & r > 2\rho \end{cases}, \quad \psi_2(x_3) = \begin{cases} 1, & x_3 < H, \\ 0, & x_3 > 2H \end{cases}.$$

By standard arguments we calculate for a small but fixed γ.

$$\begin{aligned}\int_{\mathbb{R}^3_+} |\nabla \boldsymbol{u}|^2 d\boldsymbol{x} &\lesssim \int_0^\infty \int_0^{h+r^{1+\alpha}} \dot{h}^2 r^3 \Psi_{x_3x_3}^2 + \omega_3^2 r^3 (\Psi_{x_3} + r\Psi_{rx_3})^2 \, dx_3 \, dr \\ &\lesssim \int_0^\gamma \int_0^{h+r^{1+\alpha}} \dot{h}^2 r^3 \phi_{x_3x_3}^2 + \omega_3^2 r^3 (\phi_{x_3} + r\phi_{rx_3})^2 \, dx_3 dr + O_h(1) \\ &\lesssim \int_0^\gamma \frac{\dot{h}^2 r^3}{(h+r^{1+\alpha})^3} + \frac{\omega_3^2 r^3}{h + r^{1+\alpha}} \, dr + O_h(1).\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^3_+} |\nabla \boldsymbol{u}|^2 d\boldsymbol{x} \lesssim \begin{cases} \dot{h}^2 + \omega_3^2, & 0 < \alpha < 1/3, \\ \dot{h}^2 \, |\ln h| + \omega_3^2, & \alpha = 1/3, \\ \dot{h}^2 \, h^{\frac{1-3\alpha}{1+\alpha}} + \omega_3^2, & 1/3 < \alpha \leq 1 \, . \end{cases}$$

We similarly have

$$\int_{\mathcal{C}_{\sigma_h} \cap F} |\nabla \boldsymbol{u}|^2 d\boldsymbol{x} \lesssim \dot{h}^2 \, h^{\frac{1-3\alpha}{1+\alpha}} + \omega_3^2 \, h^{\frac{3-\alpha}{1+\alpha}}. \tag{4.1}$$

$$\int_{\mathbb{R}^3_+} |\boldsymbol{u}|^2 d\boldsymbol{x} \lesssim \dot{h}^2 + \omega_3^2,$$

Optimality of the estimates of Theorem 1.1 Taking $\omega_3 = 0$ we have from (4.1) that

$$|\dot{h}| \gtrsim h^{\frac{3\alpha-1}{2(1+\alpha)}} \|\nabla \boldsymbol{u}\|_{L^2(\mathcal{C}_{\sigma_h} \cap F)},$$

which shows the optimality of the u_{P3} estimate of Theorem 1.1. Similarly for $\dot{h} = 0$ we get the optimality of the ω_3 estimate. Starting with the vector field

$$\boldsymbol{F} = (0, \, -\frac{1}{2}\omega_2(x_1^2 + (x_3 - h)^2), \, x_2 v_1(t)),$$

so that

$$\boldsymbol{u}_S = \nabla \times \boldsymbol{F} = (v_1, 0, 0) + (0, \omega_2, 0) \times (x_1, x_2, x_3 - h),$$

one can show the optimality of the rest of the estimates.

Non zero collision speed If we interprete $\boldsymbol{u}$ as the velocity field related to the motion of a rigid body inside an icompressible fluid, then $\boldsymbol{u}$ satisfies $\mathrm{div}\boldsymbol{u} = 0$ and

$$\boldsymbol{u}_t - \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{g}; \tag{4.2}$$

cf e.g. [6, 11]. We first choose ω_3 to be a smooth function in $[0, T]$. We next discuss under what conditions on h, our function $\boldsymbol{u}$ is a weak solution of the above equation with $g \in L^2(0, T; H^{-1}(\Omega))$.

At first, we require $\boldsymbol{u}$ to belong to the energy space of weak solutions of the solid–fluid interaction problem, that is

$$\boldsymbol{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W_0^{1,2}(S(t), \Omega)). \tag{4.3}$$

We easily see that $\boldsymbol{u} \in L^\infty(0, T; L^2(\Omega))$ whenever

$$|\dot{h}| < C \quad \text{for} \quad t \in (0, T], \tag{4.4}$$

whereas for $\alpha \in (1/3, 1]$ one has $\boldsymbol{u} \in L^2(0,T; W_0^{1,2}(S(t),\Omega))$ provided that

$$\int_0^T \dot{h}^2\, h^{\frac{1-3\alpha}{1+\alpha}}\, dt < \infty, \qquad 1/3 < \alpha \le 1. \tag{4.5}$$

To have $\boldsymbol{g} \in L^2(0,T;H^{-1}(\Omega))$, it is enough that each term in the left hand side of (4.2) is in this space. Because of (4.3), $\boldsymbol{u} \in L^2(0,T;H_0^1(\Omega))$ hence $\Delta\boldsymbol{u} \in L^2(0,T;H^{-1}(\Omega))$. For the term $\boldsymbol{u}_t$ to be in the same space it is enough to have $(\boldsymbol{F}\Psi)_t \in L^2(0,T;L^2(\Omega))$ which after direct calculations, is true provided that

$$\int_0^T (\ddot{h}^2 + \dot{h}^2)\, dt < \infty. \tag{4.6}$$

Finally for the nonlinear term it is enough for the integral

$$\int_0^T \int_\Omega \boldsymbol{u} \otimes \boldsymbol{u} : D(\boldsymbol{w})\, d\boldsymbol{x} dt, \qquad \forall \boldsymbol{w} \in L^2(0,T;H^1(\Omega)),$$

to be well defined. This is the case provided that $|\boldsymbol{u}| \in L^4((0,T)\times\Omega)$, which follows from

$$\int_0^T \dot{h}^4\, dt < \infty, \quad \text{if } \ 0 < \alpha < 1 \ \text{ or } \ \int_0^T \dot{h}^4 |\ln h|\, dt < \infty, \quad \text{if } \ \alpha = 1. \tag{4.7}$$

Hence, if conditions (4.4)–(4.7) are satisfied then $\boldsymbol{g} \in L^2(0,T;H^{-1}(\Omega))$.

We next take $h(t) = (T-t)^\theta$, $\theta \ge 1$. One can easily check that all conditions (4.4)–(4.7) are satisfied provided that

- either $\alpha = 1$ and $\theta > \frac{3}{2}$,
- or else $\alpha \in \left(\frac{1}{3}, 1\right)$ and $\theta = 1$ or $\theta > \frac{3}{2}$.

Consequently, for $\alpha \in \left(\frac{1}{3}, 1\right)$ and $\theta = 1$ we have a nonzero speed of collision ($\dot{h} = -1$), whereas for $\alpha = 1$ and $\theta > \frac{3}{2}$ we have a zero collision speed, in agreement with Theorem 3.2 in [11]. In connection with this, we recall that the assumption that the rigid body is smooth seems to lead to various results that are in disagreement with experimental observation, such as the noncollision of the rigid body with the boundary of the container, see e.g [3, 5, 6], as well as [10] p. 324.

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