# Propagation of acoustic waves in fractal networks

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## 1. INTRODUCTION

We are interested by solving the wave equation in fractal domains such as human lungs, that can be modeled modulo some approximation as infinite dyadic trees (as in [4, 6]). As it is not possible to do numerical computations on the whole geometry, the idea is to truncate resolution of wave generation to a finite number of generations, and to replace remaining generations by adapted DtN operators, which is possible if one assumes that the cut subtrees are self-similar.

## 2. THEORETICAL ASPECTS

For this part, we consider that we work in $\mathbb{R}^d$ (with $d = 2$ or $d = 3$).

### 2.1. Self-similar $p$-adic tree.

We can define a self-similar $p$-adyc tree by

- a finite closed segment $\Sigma$ given by $\Sigma = \{(t, 0), 0 \leq t \leq 1\}$ (in $\mathbb{R}^2$) or $\Sigma = \{(t, 0, 0), 0 \leq t \leq 1\}$ (in $\mathbb{R}^3$),
- $p$ strictly contractant direct similitudes $(s_i)_{0 \leq i < p}$ of ratio $\alpha_i < 1$ such that $s_i(\mathbf{0}) = (\mathbf{1})$ for any $i$.

where $\mathbf{0}$ is the origin and $\mathbf{1} = (1, 0)$ (in $\mathbb{R}^2$) or $\mathbf{1} = (1, 0, 0)$ (in $\mathbb{R}^3$).

With these datas, we build the tree $\mathcal{T}$ by induction: we define $\mathcal{T}^0 = \Sigma$; given $n \in \mathbb{N}$, we define $\mathcal{T}^{n+1} = \mathcal{T}^n \cup s_0(\mathcal{T}^n) \cup \cdots \cup s_{p-1}(\mathcal{T}^n)$; finally, we define $\mathcal{T} = \bigcup \mathcal{T}^n$.

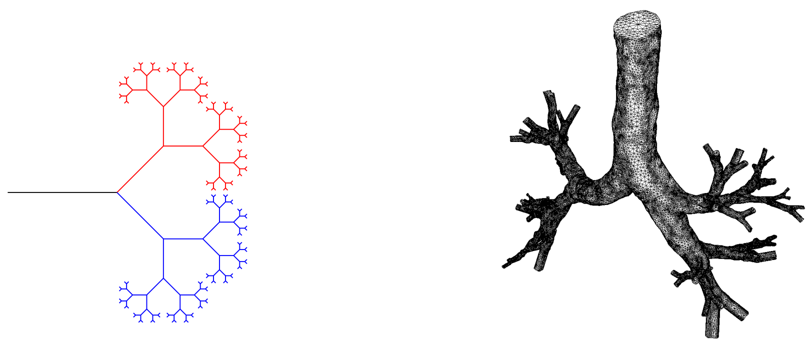

FIGURE 1. On the left: the tree $\mathcal{T}$ (whole figure), and subtrees $\mathcal{T}_0$ and $\mathcal{T}_1$. On the right: mesh associated to the partial 3D lung.

We shall denote by $\mathbb{E}(\mathcal{T})$ the set of edges of $\mathcal{T}$ and by $\mathbb{V}(\mathcal{T})$ the set of interior vertexes of $\mathcal{T}$. We also define subtrees $\mathcal{T}_i$ of $\mathcal{T}$ as $\mathcal{T}_i = s_i(\mathcal{T})$ (see figure 1 for an example of configuration with $p = 2$).

2.2. **Variational spaces and Helmholtz equations.** On $\mathcal{T}$, given $\{\mu_i > 0,\ 0 \leq i < p\}$, we define the unique piecewise constant weight function $\widetilde{\mu} : \mathcal{T} \to \mathbb{R}_+^*$ such that:

$$\begin{cases} \widetilde{\mu} = 1 \quad \text{on } \Sigma, \\ \widetilde{\mu} \circ s_i = \mu_i\, \widetilde{\mu} \end{cases}$$

and we denote by $\widetilde{\mu}_e$ the value of $\widetilde{\mu}$ on the edge $e \in \mathbb{E}(\mathcal{T})$.

We define then the weighted "broken" norms, depending of $\mu$ as, where $u'$ on $e \in \mathbb{E}(\mathcal{T})$ is the derivative of $u$ with respect to the curvilinear abscissa along $e$:

$$\|u\|^2_{\mathrm{L}^2_\mu(\mathcal{T})} = \sum_{e\in\mathbb{E}(\mathcal{T})} \widetilde{\mu}_e\, \|u\|^2_{\mathrm{L}^2(e)} \qquad |u|^2_{\mathrm{H}^1_\mu(\mathcal{T})} = \sum_{e\in\mathbb{E}(\mathcal{T})} \widetilde{\mu}_e\, \|u'\|^2_{\mathrm{L}^2(e)}$$

and the associated Sobolev spaces

$$\begin{aligned} \mathrm{H}^1_\mu(\mathcal{T}) &= \Big\{ v \text{ continuous such that } \|u\|^2_{\mathrm{L}^2_\mu(\mathcal{T})} + |u|^2_{\mathrm{H}^1_\mu(\mathcal{T})} < \infty \Big\} \\ \mathrm{H}^1_{\mu,0}(\mathcal{T}) &= \text{closure of } \big\{ v \in \mathrm{H}^1_\mu(\mathcal{T}) \text{ such that } \exists\, n \in \mathbb{N},\ v = 0 \text{ on } \mathcal{T} \setminus \mathcal{T}_n \big\} \end{aligned}$$

Moreover we define the following Besov spaces

$$\begin{aligned} \mathcal{H}^1_\mu(\mathcal{T}) &= \Big\{ v \text{ continuous such that } |v(\mathbf{0})|^2 + |u|^2_{\mathrm{H}^1_\mu(\mathcal{T})} < \infty \Big\} \\ \mathcal{H}^1_{\mu,0}(\mathcal{T}) &= \text{closure of } \big\{ v \in \mathcal{H}^1_\mu(\mathcal{T}) \text{ such that } \exists\, n \in \mathbb{N},\ v = 0 \text{ on } \mathcal{T} \setminus \mathcal{T}_n \big\} \end{aligned}$$

We also define Helmholtz problem with "Neumann" or "Dirichlet" condition at infinity: find $u \in \mathrm{H}^1_\mu(\mathcal{T})$ (resp. $u \in \mathrm{H}^1_{\mu,0}(\mathcal{T})$) such that $u(\mathbf{0}) = 1$ and, for any test function $v \in \mathrm{H}^1_\mu(\mathcal{T})$ (resp. $v \in \mathrm{H}^1_{\mu,0}(\mathcal{T})$):

$$\text{(1)} \qquad \int_{\mathcal{T}} \widetilde{\mu} u' v' - \omega^2 \int_{\mathcal{T}} \widetilde{\mu} u v = 0, \qquad \text{where } \omega \in \mathbb{C} \text{ is the wave pulsation}$$

This formulation automatically implies homogeneous wave equation on each edge $e \in \mathbb{E}(\mathcal{T})$, and standard Kirchhoff conditions on each interior vertex $v \in \mathbb{V}(\mathcal{T})$. Standard Kirchhoff conditions are detailed in [5, 3].

**Remark.** *The particular choice $\mu_i = \alpha_i^{d-1}$ (the associated tree is called a d-geometric tree) is obtained by considering $\mathcal{T}$ as the limit of $\mathcal{T}^\varepsilon$ when $\varepsilon$ tends to 0, where $\mathcal{T}^\varepsilon$ is built as $\mathcal{T}$ - the only difference is that $\Sigma^\varepsilon$ is a d-dimensional domain which tends to $\Sigma$ when $\varepsilon$ tends to 0. Then (1) appears as the limit model for the solution of the d-dimensional homogeneous Helmholtz equation on $\mathcal{T}^\epsilon$.*

2.3. **Results.** When $\Im(\omega) \neq 0$, problem (1) admits a unique solution $u_{\mathfrak{n}} \in \mathrm{H}^1_\mu(\mathcal{T})$ (resp. $u_{\mathfrak{d}} \in \mathrm{H}^1_{\mu,0}(\mathcal{T})$). Moreover, one has

$$u_{\mathfrak{n}} \neq u_{\mathfrak{d}} \iff \mathrm{H}^1_\mu(\mathcal{T}) \neq \mathrm{H}^1_{\mu,0}(\mathcal{T}) \iff \sum \frac{\mu_i}{\alpha_i} > 1 \tag{2}$$

In the following, we assume that (2) is satisfied (this is the interesting case). By denoting $\Lambda_{\mathfrak{n}}(\omega)$ the value of $u'_{\mathfrak{n}}(\mathbf{0})$ (resp. $\Lambda_{\mathfrak{d}}(\omega)$ the value of $u'_{\mathfrak{d}}(\mathbf{0})$), we can replace the Helmoltz equation on $\mathcal{T}$ by a transparent DtN condition

$$u'(\mathbf{0}) = \Lambda_{\mathfrak{n}}(\omega)u(\mathbf{0}) \qquad (\text{resp. } u'(\mathbf{0}) = \Lambda_{\mathfrak{d}}(\omega)u(\mathbf{0})) \tag{3}$$

**Proposition 2.1.** *$\Lambda_{\mathfrak{n}}$ and $\Lambda_{\mathfrak{d}}$, as functions of $\omega$, satisfy the following quadratic relation (obtained by looking at the problem satisfied on each subtree $\mathcal{T}_i$, this approach is similar to the approach done in* [1]*)*

$$\Lambda(\omega)\cos(\omega) - \omega\sin(\omega) = \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i}\left(\cos(\omega) + \frac{\Lambda(\omega)}{\omega}\sin(\omega)\right)\Lambda(\alpha_i\omega) \tag{4}$$

For $\omega = 0$, (4) becomes

$$\Lambda(0) = \Lambda(0)\Big(1 + \Lambda(0)\Big)\sum_{i=0}^{p-1}\frac{\mu_i}{\alpha_i} \tag{5}$$

whose solutions are $\Lambda_{\mathfrak{n}}(0) = 0$ and $\Lambda_{\mathfrak{d}}(0) = (1 - \sum \mu_i/\alpha_i)/(\sum \mu_i/\alpha_i)$.

**Conjecture 2.1.** *There exists at most two homeomorphic functions $\Lambda$ satisfying (4), and the unicity is given knowning $\Lambda(0)$.*

If the length of $\Sigma$ is $\ell$ instead of 1, one has the following scaling formulas for the traces $\Lambda_{\mathfrak{n}}(\ell,\omega)$ and $\Lambda_{\mathfrak{d}}(\ell,\omega)$ of "Neumann" and "Dirichlet" solutions of (1):

**Proposition 2.2.** *One has*

$$\Lambda_{\mathfrak{n}}(\ell,\omega) = \frac{1}{\ell}\Lambda_{\mathfrak{n}}(\ell\,\omega) \quad \textit{and} \quad \Lambda_{\mathfrak{d}}(\ell,\omega) = \frac{1}{\ell}\Lambda_{\mathfrak{d}}(\ell\,\omega)$$

So if we want to solve problem (1) on $\mathcal{T}^n$ instead on $\mathcal{T}$, one has to replace Helmholtz equation on each subtree by the DtN condition (3) written considering length of $\Sigma$ equal to $\ell_n$. Since $\ell_n \sim \alpha^n$, with $\alpha = \max(\alpha_i)$, for large $n$ it is sufficient to get a good approximation of $\Lambda(\omega)$ for small $\omega$ which can be done with Taylor expansions.

**Proposition 2.3.** *For $\omega$ small, one has*

$$\begin{aligned}
\Lambda_{\mathfrak{n}}(\omega) &= \frac{1}{1 - \sum \mu_i\alpha_i}\omega^2 + O(\omega^4) \\
\Lambda_{\mathfrak{d}}(\omega) &= \frac{1 - \sum \mu_i/\alpha_i}{\sum \mu_i/\alpha_i} + \frac{1 + \sum \mu_i/\alpha_i + (\sum \mu_i/\alpha_i)^2}{3\left((\sum \mu_i/\alpha_i)^2 - \sum \mu_i\alpha_i\right)}\omega^2 + O(\omega^4)
\end{aligned}$$

2.4. **Back to the time-domain wave equation.** Neglecting the $O(\omega^4)$ term in formulas of proposition 2.3 allows us to write $\Lambda_{\mathfrak{n}}(\omega)$ (resp. $\Lambda_{\mathfrak{d}}(\omega)$) under the form

$$\Lambda_{\mathfrak{n}}(\omega) = \lambda_{\mathfrak{n}}^0 + \lambda_{\mathfrak{n}}^2\,\omega^2 \qquad \big(\text{resp. } \Lambda_{\mathfrak{d}}(\omega) = \lambda_{\mathfrak{d}}^0 + \lambda_{\mathfrak{d}}^2\,\omega^2\big) \tag{6}$$

Injecting (6) in (3) and going back to time-domain leads to the following DtN operator

$$u'(t,\mathbf{0}) = \lambda^0\,u(t,\mathbf{0}) - \lambda^2\,\frac{\partial^2 u}{\partial t^2}(t,\mathbf{0}) \tag{7}$$

If we want to ensure stability for the time-domain wave equation with this condition, one has to check $\lambda^0 \leq 0$ and $\lambda^2 \geq 0$:

- for $\Lambda_{\mathfrak{d}}$, under hypothesis (2), one always has $\lambda_{\mathfrak{d}}^0 \leq 0$ and $\lambda_{\mathfrak{d}}^2 \geq 0$,
- for $\Lambda_{\mathfrak{n}}$, one has $\lambda_{\mathfrak{n}}^0 = 0$, and one has $\lambda_{\mathfrak{n}}^0 \geq 0$ if and only if one has $\sum \mu_i \alpha_i < 1$, i.e. if and only if the constant function $\mathbb{1}$ belongs to $\mathrm{L}^2_\mu(\mathcal{T})$.

## 3. Numerical results

To validate results of previous section, we solve time-domain wave equation on $\mathcal{T}_n$ for various values of $n$ with outgoing condition at $\mathbf{0}$ and different conditions at outer boundary of $n$ (with coefficients computed thanks to proposition 2.2):

- Dirichlet condition,
- First order impedance condition $u'(t,\cdot) = \lambda^0 u(t,\cdot)$,
- Second order impedance condition given by (7).

Numerical tests validate writing of condition (7) and show accuracy of this condition with respect to number of generations we consider. These results are in progress for general case.

## References


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