Computation of a Green's kernel in a bounded domain

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1 Computation of the Green kernel

Let Ω be a connex bounded domain in $\mathbb{R}^2$, and we are interested about studying the wave equation

$$\begin{cases} \dfrac{1}{c_0^2}\dfrac{\partial^2 u}{\partial t^2}(t,\mathbf{x}) - \Delta u(t,\mathbf{x}) = n(t,\mathbf{x}), \quad \text{for } (t,\mathbf{x}) \in \mathbb{R}_+^* \times \Omega \\ \dfrac{\partial u}{\partial \vec{n}}(t,\mathbf{x}) = 0, \quad \text{for } (t,\mathbf{x}) \in \mathbb{R}_+^* \times \partial\Omega \end{cases} \tag{1}$$

Let us denote G the Green's kernel, solution in sense of distribution of the equation, for a given $\mathbf{y} \in \Omega$:

$$\begin{cases} \dfrac{1}{c_0^2}\dfrac{\partial^2 G}{\partial t^2}(t,\mathbf{x},\mathbf{y}) - \Delta G(t,\mathbf{x},\mathbf{y}) = \delta(t)\delta(\mathbf{x}-\mathbf{y}), \quad \text{for } (t,\mathbf{x}) \in \mathbb{R}_+^* \times \Omega \\ \dfrac{\partial G}{\partial \vec{n}}(t,\mathbf{x},\mathbf{y}) = 0, \quad \text{for } (t,\mathbf{x}) \in \mathbb{R}_+^* \times \partial\Omega \end{cases} \tag{2}$$

Then, from (2), one can recompose solution of (1) by using the formula

$$u(t,\mathbf{x}) = \int_\Omega \int_0^t G(t-s,\mathbf{x},\mathbf{y}) n(s,\mathbf{y}) ds d\mathbf{y} \tag{3}$$

Now we focus on computation of Green's kernel. Let us denote $\hat{G}$ the Fourier transform of G with respect to time, defined by

$$\hat{G}(\omega,\mathbf{x},\mathbf{y}) = \int_\mathbb{R} G(t,\mathbf{x},\mathbf{y}) \exp(-\imath\omega t) dt \tag{4}$$

Equation (2) becomes, in Fourier domain,

$$\begin{cases} -\dfrac{\omega^2}{c_0^2}\hat{G}(\omega,\mathbf{x},\mathbf{y}) - \Delta \hat{G}(\omega,\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \quad \text{for } (\omega,\mathbf{x}) \in \mathbb{R} \times \Omega \\ \dfrac{\partial \hat{G}}{\partial \vec{n}}(\omega,\mathbf{x},\mathbf{y}) = 0, \quad \text{for } (t,\mathbf{x}) \in \mathbb{R} \times \partial\Omega \end{cases} \tag{5}$$

Now, assume that we know the eigenvalues and the eigenfunctions of the Laplacien, *i.e.* that we know the sequence $(\lambda_n, \Phi_n)_{n\geq 0}$ with λ_n sorted in ascend order such that

$$\begin{cases} -\Delta \Phi_n(\mathbf{x}) = \lambda_n \Phi_n(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega \\ \dfrac{\partial \Phi_n}{\partial \vec{n}} = 0, \quad \text{for } \mathbf{x} \in \partial\Omega \end{cases} \tag{6}$$

Moreover, we assume that the functions Φ_n are orthogonal in $\mathrm{L}^2(\Omega)$. We can decompose then G by using

$$\hat{G}(\omega, \mathbf{x}, \mathbf{y}) = \sum_{n \geq 0} \frac{\langle G(\omega, \cdot, \mathbf{y}), \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \Phi_n(\mathbf{x}) \tag{7}$$

We also have

$$\langle \Delta G(\omega, \cdot, \mathbf{y}), \Phi_n \rangle = \langle G(\omega, \cdot, \mathbf{y}), \Delta \Phi_n \rangle = -\lambda_n \langle G(\omega, \cdot, \mathbf{y}), \Phi_n \rangle \tag{8}$$

and

$$\langle \delta(\cdot - \mathbf{y}), \Phi_n \rangle = \Phi_n(\mathbf{y}) \tag{9}$$

Now, taking the scalar product in (5) with respect to Φ_n and using both (8, 9) leads to

$$\left(\lambda_n - \frac{\omega^2}{c_0^2} \right) \langle G(\omega, \cdot, \mathbf{y}), \Phi_n \rangle = \Phi_n(\mathbf{y}) \tag{10}$$

We replace (10) in (7), and we get

$$\hat{G}(\omega, \mathbf{x}, \mathbf{y}) = \sum_{n \geq 0} \frac{1}{\left(\lambda_n - \frac{\omega^2}{c_0^2} \right)} \frac{\Phi_n(\mathbf{x}) \Phi_n(\mathbf{y})}{\langle \Phi_n, \Phi_n \rangle} \tag{11}$$

2 Computation of eigenvalues / eigenfunctions

2.1 General case

We consider problem (6) in $\mathrm{H}^1(\Omega)$, and we multiply by a function test $V \in \mathrm{H}^1(\Omega)$. Using Neumann condition, we can write variational formulation of this problem as follow:

$$\int_\Omega \nabla \Phi_n \cdot \nabla V = \lambda_n \int_\Omega \Phi_n V \tag{12}$$

We assume that we've build matrices K and M corresponding to rigidity and mass matrix, then we compute N first eigenvalues and eigenvectors as

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[V,D]=eigs(K,M,N,'SM');
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D is a $N \times N$ diagonal matrix containing smallest eigenvalues, and V is a $N_{\mathrm{ddl}} \times N$ matrix containing eigenvectors.

$$\mathrm{K}\,\mathrm{V} = \mathrm{M}\,\mathrm{V}\,\mathrm{D}$$

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mesh = meshcircle(1,0.001);
[~,ddl,~,lblddl,Diag,Phi,PhiX,PhiY]=mef(mesh,1,2);
Rigidity = PhiX'*Diag*PhiX+PhiY'*Diag*PhiY;
Mass = Phi'*Diag*Phi;
[V,D]=eigs(Rigidity, Mass, 7, 'SM');
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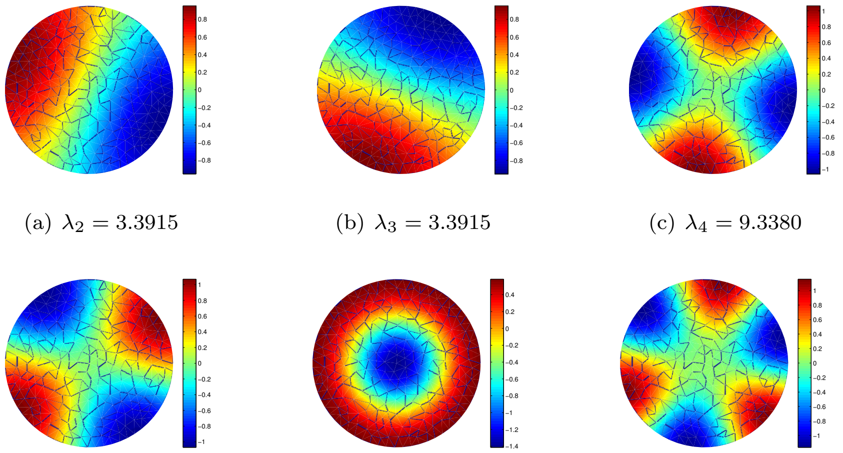

(a) $\lambda_2 = 3.3915$ (b) $\lambda_3 = 3.3915$ (c) $\lambda_4 = 9.3380$

(d) $\lambda_5 = 9.3381$ (e) $\lambda_6 = 14.7107$ (f) $\lambda_7 = 17.6828$

Figure 1: Computation of six first nonconstant eigenvectors when $\Omega = B(0,1)$

2.2 Rectangular domain

We consider domain $\Omega = (0, L_x) \times (0, L_y)$, and we consider problem (6). Since variables are separable in this domain, one can look at eigenproblem of Laplacian in $(0, L)$ with Neumann condition: function is given by

$$\phi_p(x, L) = \cos\left(\frac{p\pi x}{L}\right), \quad p \geq 0 \tag{13}$$

Then, one define, for a couple of integers (p, q), one defines $\Phi_{p,q}(x, y)$ as

$$\Phi_{p,q}(x, y) = \phi_p(x, L_x)\phi_q(y, L_y) = \cos\left(\frac{p\pi x}{L_x}\right)\cos\left(\frac{q\pi y}{L_y}\right) \tag{14}$$

and computation of Lapacian gives

$$\Delta\Phi_{p,q}(x, y) = -\left(\left(\frac{p\pi}{L_x}\right)^2 + \left(\frac{q\pi}{L_y}\right)^2\right)\Phi_{p,q}(x, y) \tag{15}$$

so that we can define eigenvalues $\lambda_{p,q}$ as

$$\lambda_{p,q} = \left(\frac{p\pi}{L_x}\right)^2 + \left(\frac{q\pi}{L_y}\right)^2 \tag{16}$$