STRAIN SOFTENING IN VISCOELASTICITY OF THE RATE TYPE

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Dedicated to John A. Nohel on his 65^{th} birthday

1. **Introduction**

The intent of this article is to study the behavior of solutions $(v(x,t), u(x,t))$ of the system of differential equations

$$v_t = \sigma_x \tag{1.1}$$

$$u_t = v_x \tag{1.2}$$

where

$$\sigma = \tau(u) v_x^n \tag{1.3}$$

with $\tau(u)$ a smooth function satisfying

$$\tau(u) > 0 \quad , \quad \tau'(u) < 0 \tag{1.4}$$

and n a positive parameter. Equations (1.1 - 1.3) give rise to a coupled system of partial differential equations in one space dimension. They are supplemented with initial conditions

$$v(x,0) = v_0(x) \quad , \quad u(x,0) = u_0(x)\,, \tag{1.5}$$

and, as a consequence,

$$\sigma(x,0) = \sigma_0(x) := \tau(u_0(x)) v_{0x}^n(x)\,, \tag{1.6}$$

and with boundary conditions that are discussed later.

To gain some perspective on the problem, note that, if $n = 0$, (1.1 - 1.3) leads to the pair of conservation laws

$$\begin{aligned} v_t &= \tau(u)_x \\ u_t &= v_x\,. \end{aligned} \tag{1.7}$$

If $\tau'(u) > 0$ then (1.7) is hyperbolic; however, under (1.4), the system (1.7) is elliptic and the initial value problem is ill-posed. Nevertheless, it admits an interesting class of special solutions

$$\begin{aligned} \bar{v}(x,t) &= x \\ \bar{u}(x,t) &= t + u_0\,, \end{aligned} \tag{1.8}$$

where u_0 is an arbitrary constant. Equations (1.1 - 1.3) with $n > 0$ can be thought as a particular regularization of (1.7).

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A motivation for studying this problem stems from a program of understanding the phenomenon of shear band formation at high strain rates. Shear bands are narrow regions of intensely concentrated shearing deformation that are observed during the plastic deformation of many materials. The occurrence of shear bands is typically associated with strain softening type response, past a critical strain, of the measured average shear stress $\Sigma(t)$ versus the measured average shear strain $U(t)$; that is, $\Sigma = \tau(U)$, where $\tau(\cdot)$ is increasing up to a certain critical strain and decreasing thereafter. Various mechanisms and associated continuum thermomechanics models, often depending on the particular context, have been proposed for the explanation of shear bands (see Shawki and Clifton [15] for an excellent survey of the related literature). An underlying common feature of several models is that they are regularizations of an ill-posed problem, or that the associated linearized problem exhibits growth of high frequency modes.

The model employed here describes the plastic shearing of an infinite plate of unit thickness subjected to either prescribed tractions or prescribed velocities at the boundaries. In this framework, $v(x,t)$ describes the velocity field in the shearing direction, $\sigma(x,t)$ stands for the shear stress and $u(x,t)$ for the plastic shear strain. Equation (1.1) describes the balance of linear momentum, while (1.2) is a kinematic compatibility relation (note that elastic effects are neglected); (1.1) and (1.2) are taken over $(x,t) \in [0,1] \times \{t > 0\}$ and are supplemented with the boundary conditions

$$\sigma(0,t) = 1 \,,\ \sigma(1,t) = 1 \quad , \quad t > 0 \,, \tag{1.9$_S$}$$

in case the shearing deformation is caused by prescribed tractions at the boundaries, or

$$v(0,t) = 0 \,,\ v(1,t) = 1 \quad , \quad t > 0 \,, \tag{1.9$_V$}$$

in the case of prescribed velocities. The constitutive law (1.3) is appropriate for a material exhibiting strain softening, as manifested in (1.4), and strain rate sensitivity, the strength of which is measured by the parameter n. Our objective is to use (1.1 - 1.6), (1.9) as a test problem to analyze the competition between the destabilizing effect of strain softening versus the stabilizing effect of strain rate dependence.

Technically, the model (1.1 - 1.3) belongs to the class of isothermal viscoelasticity of the rate type (for general information on the mathematical theory of viscoelasticity the reader is referred to Renardy, Hrusa and Nohel [14]). Metals, in general, exhibit strain hardening in isothermal deformations. However, an increase in temperature causes a decrease in the yield stress, so that in an adiabatic deformation the combined effect of strain hardening and thermal softening may deliver eventually a net softening. Thus, although (1.1 - 1.3) is a model in the framework of isothermal mechanical theories, thermal effects are implicitly taken into account through the hypothesis of strain softening . One of our goals is to reveal similarities in the structure and predictions of (1.1 - 1.3) as compared to related models incorporating thermal effects that have been studied recently in the mathematical literature [7, 17, 19, 2, 1].

We emphasize that the spirit of this study is not to recover solutions of (1.7)-(1.4) as $n \to 0$ limits of solutions of (1.1 - 1.4). Rather the rational here is the converse. Because of the inherent instability induced by strain softening , it has been postulated that higher order effects, such as strain-rate dependence, play an important role and cannot be ignored (cf. [11, 21]). Apart from some previous investigations using (1.3) for $n = 1$ [18, 2], other types of rate dependent constitutive relations have been used to analyze shear bands (e.g. Wu and Freund [21]), as well as strain-gradient dependent constitutive laws (e.g. Coleman and Hodgdon [5]). There is a very extensive mechanics literature on the subject and the reader is referred to [21, 11, 20, 15] and references therein.

From an analysis point of view, equations (1.1 - 1.3) give rise to a coupled system consisting of a parabolic equation in v coupled through the diffusion coeffient with (1.2) (cf. (3.1)). As the material is being sheared, under the effect of (1.9), the diffusion coefficient is decreasing. It is conceivable, that if the decrease is too rapid and/or nonuniform in the space variable, the diffusion may not be able to stabilize the process. To analyze this competition, it is helpful to recast (1.1 - 1.3) into an equivalent formulation of a reaction-diffusion system (cf. (3.4 - 3.5)).

In Section 2 we pursue an existence theory of classical solutions for a coupled system of partial differential equations (cf. (2.1 - 2.3)) that includes (1.1 - 1.3). This system also includes certain more general models in viscoelasticity with internal variables, as well as some models incorporating thermal effects that are used for the analysis of shear bands [15, 19]. Motivated by the problems under consideration, the main objective is to identify a minimal set of a-priori estimates sufficient for continuation of solutions. The existence theory is done in Schauder spaces and the main ingredient is an application of the Leray-Schauder fixed point theorem. The results are summarized in Theorems 2.4 and 2.5. For existence theories of weak solutions in structurally related systems the reader is referred to Charalambakis and Murat [3] and Nohel et al. [12].

In Section 3 we take up the problem $(\mathcal{P})_S$ consisting of (1.1 - 1.6) with stress boundary conditions $(1.9)_S$. Using the results of Section 2 together with the special structure of the system, it is shown in Theorem 3.2 that solutions of $(\mathcal{P})_S$ are globally defined if and only if the integral $\int_1^\infty \tau(\xi)^{\frac{1}{n}} d\xi$ diverges. Moreover, the evolution of solutions of $(\mathcal{P})_S$ is studied under various assumptions for the constitutive function $\tau(u)$. Below, we summarize the outcome of the analysis for the special case of a power law

$$\sigma = \frac{1}{u^m} v_x^n \tag{1.10}$$

with parameters m, n positive. The parameter region is decomposed into three distinct subregions $0 < \frac{m}{n} < \frac{1}{2}$, $\frac{1}{2} \le \frac{m}{n} \le 1$ and $\frac{m}{n} > 1$, across which the response changes drastically:

(i) In the region $0 < \frac{m}{n} < \frac{1}{2}$ solutions of $(\mathcal{P})_S$ are globally defined and, as $t \to \infty$, the shear stress $\sigma(x,t)$ is attracted to the constant state $\sigma \equiv 1$ while $u(x,t)$ behaves asymptotically as a function of time.

(ii) In the region $\frac{1}{2} \le \frac{m}{n} \le 1$ the constant state $\sigma \equiv 1$ loses its stability and nonuniformities in the strain persist for all times.

(iii) Finally, in the region $\frac{m}{n} > 1$, $u(x,t)$ becomes infinite in finite time.

For $\frac{m}{n} > \frac{1}{2}$, we exhibit initial data for which the corresponding $u(x,t)$ develops nonuniformities around $x = 0$ and $x = 1$ and looks like two shear bands located at the boundaries.

The analysis of Section 3 is effected by means of comparison principles for (3.4) and energy estimates for $(\mathcal{P})_S$.

For a power law (1.10) the system (1.1 - 1.2) is invariant under a family of scaling transformations. In Section 4 we take up the problem $(\mathcal{P})_V$ consisting of (1.1 - 1.2), (1.10) and velocity boundary conditions $(1.9)_V$, and introduce a change of variables motivated by the scaling property. The resulting system (4.20 - 4.22) admits positively invariant rectangles of arbitrary size. Using this observation together with energy estimates for $(\mathcal{P})_V$, it is shown in Theorem 4.1 that, if $m < min\{n, 1\}$, every solution of $(\mathcal{P})_V$ converges to the uniform shearing solution (1.8), as $t \to \infty$.

2. **Existence Theory and Regularizing Effect for a Coupled System**

We consider the initial-boundary value problem consisting of the system of quasilinear partial differential equations

$$\partial_t v = \partial_x w \tag{2.1}$$

$$\partial_t u = f(x, u, v_x) \tag{2.2}$$

for $(x, t) \,\epsilon\, Q_T := (0, 1) \times (0, T]$, $T > 0$, where

$$w = \varphi(x, u, v_x)\,, \tag{2.3}$$

with boundary conditions

$$v(0, t) = v(1, t) = 0, \qquad 0 < t \le T\,, \qquad (2.4)_V$$

or

$$w(0, t) = w(1, t) = 0, \qquad 0 < t \le T\,, \qquad (2.4)_S$$

and initial conditions

$$v(x, 0) = v_0(x)\,,\; u(x, 0) = u_0(x)\,, \qquad 0 \le x \le 1\,; \tag{2.5}$$

as a consequence of (2.3) and (2.5),

$$w(x, 0) = w_0(x) := \varphi(x, u_0(x),\, v_{0x}(x))\,. \tag{2.6}$$

The functions $v(x, t)$, $w(x, t)$ are real valued, while $u(x, t)$ stands for an R^N-valued function, all defined on $\bar{Q}_T = [0, 1] \times [0, T]$. The given functions $f(x, p, q) : [0, 1] \times R^N \times R \to R^N$ and $\varphi(x, p, q) : [0, 1] \times R^N \times R \to R$ are assumed to be smooth with respect to all their arguments (the hypothesis f and φ of class C^2 suffices for all that follows). In addition for each fixed (x, p) the function $\varphi(x, p, \cdot)$ is assumed to be strictly increasing and thus invertible. Let $\psi(x, p, r) : [0, 1] \times R^N \times R \to R$ be the inverse function. Inverting (2.3) yields

$$v_x = \psi(x, u, w)\,. \tag{2.7}$$

We seek solutions $(v(x, t), u(x, t))$ of (2.1 - 2.5) defined on $\bar{Q}_T$, $T > 0$. Our specific goals are to identify a minimal set of a-priori estimates that guarantee existence and continuation of solutions up to time $T > 0$, and to study the regularizing effect that the parabolic equation (2.1), (2.3) exerts on solutions.

To this end it is expedient to state an alternative formulation of the problem. The initial-boundary value problem (2.1 - 2.5) is formally equivalent to the system of reaction-diffusion equations

$$\partial_t w = a(x, u, w)\partial_x^2 w + b(x, u, w) \tag{2.8}$$

$$\partial_t u = g(x, u, w) \tag{2.9}$$

with boundary conditions

$$w_x(0,t) = w_x(1,t) = 0\,, \qquad 0 < t \le T\,, \tag{2.10$_V$}$$

or

$$w(0,t) = w(1,t) = 0\,, \qquad 0 < t \le T\,, \tag{2.10$_S$}$$

and initial conditions

$$w(x,0) = w_0(x)\,,\; u(x,0) = u_0(x)\,, \qquad 0 \le x \le 1\,; \tag{2.11}$$

where

$$\begin{aligned} a(x,u,w) &= \varphi_q(x,u,\psi(x,u,w)) \\ b(x,u,w) &= (\varphi_p \cdot f)(x,u,\psi(x,u,w)) \\ g(x,u,w) &= f(x,u,\psi(x,u,w))\,. \end{aligned} \tag{2.12}$$

Indeed, given any sufficiently smooth solution $(v(x,t), u(x,t))$ of (2.1 - 2.5) the pair $(w(x,t), u(x,t))$ satisfies (2.8 - 2.11) as follows: Differentiating (2.3) with respect to t and using (2.1), (2.2), (2.7) and (2.12) leads to (2.8); (2.2) and (2.7) yield (2.9); the rest are clear. Conversely, if $(w(x,t), u(x,t))$ is a classical solution of (2.8 - 2.12), define a function $v(x,t)$ on $\bar{Q}_T$ such that

$$\begin{aligned} v_x &= \psi(x,u,w) \\ v_t &= w_x \end{aligned} \tag{2.13}$$

and $v(x,0) = v_0(x)$, $0 \le x \le 1$. Since $\psi(x,p,\cdot)$ is the inverse function of $\varphi(x,p,\cdot)$, the list of relations

$$\begin{aligned} \varphi(x,p,\psi(x,p,r)) &= r \\ \varphi_q(x,p,\psi(x,p,r))\psi_r(x,p,r) &= 1 \\ \varphi_p(x,p,\psi(x,p,r)) + \varphi_q(x,p,\psi(x,p,r))\psi_p(x,p,r) &= 0 \end{aligned} \tag{2.14}$$

holds, and the compatibility of (2.13) amounts to (2.8) via (2.9) and (2.12). Moreover, $(v(x,t), u(x,t))$ satisfies (2.1 - 2.5).

Our strategy is to first prove an existence theorem for classical solutions of (2.8 - 2.11), in Schauder spaces, using the Leray-Schauder fixed point theorem ([10]). This, in turn, yields an existence theorem for the equivalent system (2.1 - 2.5) provided the initial data are sufficiently smooth. The smoothness assumptions are then relaxed by means of density arguments. In the sequel $|\cdot|$ will stand for both the absolute value and the Euclidean norm in R^N. Also, $\|\cdot\|_{\beta,\beta/2}$ ($|\cdot|_\alpha$) will denote the usual Schauder norms (cf. [8,9]) in $C^{\beta,\beta/2}(\bar{Q}_T)$ ($C^\alpha[0,1]$) or $\left[C^{\beta,\beta/2}(\bar{Q}_T)\right]^N$ ($\left[C^\alpha[0,1]\right]^N$). The meaning of these symbols will be apparent from the context.

The possibility that (2.1 - 2.5) (or (2.8 -2.11)) admit globally defined solutions for general nonlinear functions can be ruled out by considering special cases when the system decouples. To ensure global solvability one could place certain growth restrictions on the functions f and φ (or a, b and g). Rather than doing this, we assume that solutions of (2.1 - 2.5) or (2.8 - 2.11) satisfy certain a-priori estimates, namely:

For fixed $T > 0$ there are positive constants μ and M, depending on norms of the initial data and T, such that any classical solution $\big(v(x,t), u(x,t)\big)$ of (2.1 - 2.5) on $\bar{Q}_T$ satisfies

$$|w(x,t)| \leq M \qquad , \qquad |u(x,t)| \leq M \tag{2.15}$$

and

$$\varphi_q(x, u(x,t), v_x(x,t)) \geq \mu > 0 \,, \tag{2.16}$$

for $(x,t) \in \bar{Q}_T$; correspondingly, if $\big(w(x,t), u(x,t)\big)$ is a classical solution of (2.8 - 2.11) on $\bar{Q}_T$, then (2.15) and

$$a(x, u(x,t), w(x,t)) \geq \mu > 0 \tag{2.17}$$

hold for $(x,t) \in \bar{Q}_T$.

The objective is to reveal (2.15 - 2.17) as a "minimal" set of a-priori estimates sufficient for continuation of solutions in some appropriate function classes. Although uniform parabolicity, embodied in (2.16) or (2.17), is not in general necessary for well-posedness, in light of the phenomena under consideration and for technical simplicity solutions will be continued up to the first time that uniform parabolicity fails. For the models at hand (2.15 - 2.17) are established in Sections 3 and 4. Finally, it is shown in Lemma 2.3 that, under natural restrictions on the initial data, (2.15 - 2.17) always hold provided T is sufficiently small.

The first goal is to prove an existence theorem for (2.8 - 2.11). For this the initial data are taken smooth

$$w_0(x) \in C^{2+\alpha}[0,1] \,,\; u_0(x) \in \left[C^{\alpha}[0,1]\right]^N , \tag{2.18}$$

for some $0 < \alpha < 1$, and compatible with the boundary conditions:

$$w_{0_x}(i) = 0 \,,\; i = 0, 1 \,, \tag*{(2.19)$_V$}$$

in case $(2.10)_V$ applies, or

$$w_0(i) = 0 \,,\; a(i, u_0(i), 0) w_{0xx}(i) + b(i, u_0(i), 0) = 0 \,,\; i = 0, 1 \,, \tag*{(2.19)$_S$}$$

in case $(2.10)_S$ applies. We prove:

Theorem 2.1. *Let $(w_0(x), u_0(x))$ satisfy (2.18), (2.19) and assume that the a-priori estimates (2.15) and (2.17) hold for some $T > 0$, with M and μ positive constants depending at most on $\|w_0\|_{2+\alpha}$, $\|u_0\|_\alpha$ and T. There exists a unique solution $(w(x,t), u(x,t))$ of (2.8 - 2.11) on $\bar{Q}_T$ such that w, w_t, w_x, w_{xx} are in $C^{\alpha,\alpha/2}(\bar{Q}_T)$ and u, u_t are in $\left[C^{\alpha,\alpha/2}(\bar{Q}_T)\right]^N$.*

Proof. The proof of uniqueness is lengthy but routine and it is omitted.

In view of (2.15) and (2.17), the triplet $(x, u(x,t), w(x,t))$ takes values in the set $E = \{(x,u,w) \in [0,1] \times R^N \times R \,:\, |u| \leq M,\, |w| \leq M,\, a(x,u,w) \geq \mu\}$. By modifying, if necessary, the functions a, b and g outside some open set containing E, it is assumed for the existence part of the proof that all the functions involved are bounded, globally Lipshitz and, wherever appropriate, with globally Lipshitz derivatives. Moreover, that

$$a(x,p,r) \geq \frac{\mu}{2} > 0 \tag{2.20}$$

for $(x,p,r)\,\epsilon\,[0,1]\times R^N\times R$. All bounds and Lipshitz constants depend only on M and μ. For the remainder of the proof K will stand for a generic constant that can be estimated solely in terms of M, μ and T.

We work with the boundary conditions $(2.10)_S$; the boundary conditions $(2.10)_V$ are treated similarly. Let $\mathcal{B}$ denote the Banach space

$$\mathcal{B}=\{w(x,t)\,\epsilon\, C^{\beta,\beta/2}(\bar{Q}_T): w(0,t)=w(1,t)=0,\ 0\le t\le T\} \tag{2.21}$$

and let $\mathcal{C}$ stand for the closed subset of $\left[C^{\beta,\beta/2}(\bar{Q}_T)\right]^N$

$$\mathcal{C}=\{u(x,t)\,\epsilon \left[C^{\beta,\beta/2}(\bar{Q}_T)\right]^N: u(x,0)=u_0(x),\ 0\le x\le 1\}\,. \tag{2.22}$$

For our purposes $\beta=min\{\alpha,\frac{1}{2}\}$. Define the map $\mathcal{T}:\mathcal{B}\to\mathcal{C}$ that carries $W(x,t)\,\epsilon\,\mathcal{B}$ to $U(x,t)$ the solution of the family of initial value problems

$$\begin{aligned} U_t=g(x,U,W(x,t)) \quad &, \quad 0\le x\le 1,\ 0\le t\le T\,,\\ U(x,0)=u_0(x) \quad &, \quad 0\le x\le 1\,. \end{aligned} \tag{2.23}$$

Also, for $\lambda\,\epsilon\,[0,1]$ define a second map $\mathcal{S}:[0,1]\times\mathcal{B}\times\mathcal{C}\to\mathcal{B}$ which takes $(\lambda, W(x,t),U(x,t))$ to $w(x,t)$ the solution of the initial-boundary value problem

$$\begin{aligned} &w_t=a_\lambda(x,U(x,t),W(x,t))w_{xx}+\lambda\, b(x,U(x,t),W(x,t))+(1-\lambda)F(x)\,,\\ &w(0,t)=w(1,t)=0 \quad , \quad 0\le t\le T\,,\\ &w(x,0)=w_0(x) \quad , \quad 0\le x\le 1\,, \end{aligned} \tag{2.24}$$

where $a_\lambda(x,p,r):=\lambda\, a(x,p,r)+(1-\lambda)\frac{\mu}{2}\ge\frac{\mu}{2}$, by (2.20), and $F(x):=-\frac{\mu}{2}w_{0xx}(x)$.

Given the maps $\mathcal{T}$ and $\mathcal{S}$, construct the composite map $P:[0,1]\times\mathcal{B}\to\mathcal{B}$ which carries $\lambda\,\epsilon\,[0,1]$, $W\,\epsilon\,\mathcal{B}$ to

$$w=P(\lambda,W):=\mathcal{S}(\lambda,W,\mathcal{T}(W))\,. \tag{2.25}$$

Observe that if $w(x,t)\,\epsilon\,\mathcal{B}$ is a fixed point of $P(1,\cdot)$ and $u(x,t)$ the corresponding solution of (2.23), then $(w(x,t),u(x,t))$ satisfies (2.8 - 2.11) on $\bar{Q}_T$. Our objective is to demonstrate that the map P fulfills the hypotheses of the Leray-Schauder fixed point theorem (for a formulation see [10], also [6]). To this end, certain properties of the maps $\mathcal{T}$ and $\mathcal{S}$ are recorded below.

First consider the map $\mathcal{T}:\mathcal{B}\to\mathcal{C}$. Since g is bounded and globally Lipshitz, the standard theory of ordinary differential equations implies that given any $W(x,t)\,\epsilon\,\mathcal{B}$ there is a unique solution $U(x,t)$ of (2.23) defined on $[0,1]\times[0,T]$ and such that

$$|U(x,t)|+|U_t(x,t)|\le K_1\,. \tag{2.26}$$

Moreover, by first integrating $(2.23)_1$ for two distinct points x_1, x_2 in $[0,1]$ over $[0,t]$, $0<t\le T$, and then estimating the difference using Gronwall's inequality, we deduce with the help of (2.26)

$$\|U\|_{\beta,\beta/2}\le K_2(|u_0|_\alpha+\|W\|_{\beta,\beta/2}+1)\,. \tag{2.27}$$

Next, consider the map $\mathcal{S} : [0,1] \times \mathcal{B} \times \mathcal{C} \to \mathcal{B}$. The classical Schauder theory for parabolic equations [8, 9] implies that, given any triplet $(\lambda, W, U) \epsilon [0,1] \times \mathcal{B} \times \mathcal{C}$, there is a unique solution $w(x,t)$ of (2.24) on $\bar{Q}_T$ belonging to $C^{2+\beta,1+\beta/2}(\bar{Q}_T)$ and satisfying

$$\|w\|_{2+\beta,\beta/2} \leq \Lambda_1(|w_0|_{2+\alpha} + \|W\|_{\beta,\beta/2} + \|U\|_{\beta,\beta/2} + 1)\,. \tag{2.28}$$

The constant Λ_1 can be estimated solely in terms of $\|W\|_{\beta,\beta/2}$, $\|U\|_{\beta,\beta/2}$, μ, M and T.

Finally, consider the map $P : [0,1] \times \mathcal{B} \to \mathcal{B}$. P is well defined by (2.25). Also:

(i) *For any fixed* $\lambda \,\epsilon\, [0,1]$, $P(\lambda,\cdot) : \mathcal{B} \to \mathcal{B}$ *is compact and continuous.*
Since the injection $C^{2+\beta,1+\beta/2}(\bar{Q}_T) \to C^{\beta,\beta/2}(\bar{Q}_T)$ is compact, (2.27) and (2.28) imply that $P(\lambda,\cdot)$ is a compact map. Let $\{W_n\}$ be a convergent sequence in $\mathcal{B}$, $W_n \to W$ in $C^{\beta,\beta/2}(\bar{Q}_T)$; consider $w_n = P(\lambda, W_n)$. Since $P(\lambda,\cdot)$ is a compact map, along a subsequence $w_n \to w$ in $C^{\beta,\beta/2}$ (in fact in $C^{2+\beta',1+\beta'/2}$, for any $\beta' < \beta$). One easily shows that $w = P(\lambda, W)$. Since $P(\lambda,\cdot)$ is single valued, $w_n \to w$ along the whole sequence and $P(\lambda,\cdot)$ is continuous.

(ii) *For any bounded subset* $\mathcal{K}$ *of* $\mathcal{B}$, *the family of maps* $P(\cdot, W) : [0,1] \to \mathcal{B}$, $W \epsilon \mathcal{K}$, *is uniformly equicontinuous.*
Let $\mathcal{K}$ be a bounded subset of $\mathcal{B}$. Fix $W \epsilon \mathcal{K}$. For $U = \mathcal{T}(W)$ and λ, ρ in $[0,1]$, let w_λ, w_ρ be the respective solutions of (2.24). Note that $w_\lambda = P(\lambda, W)$, $w_\rho = P(\rho, W)$. The difference $w_\lambda - w_\rho$ satisfies the parabolic equation

$$\begin{aligned}(w_\lambda - w_\rho)_t &= a_\lambda(x, U(x,t), W(x,t))(w_\lambda - w_\rho)_{xx} \\ &\quad + (\lambda - \rho)\big[a(x, U(x,t), W(x,t)) w_{\rho_{xx}} + b(x, U(x,t), W(x,t)) - \big(\frac{\mu}{2} w_{\rho_{xx}} + F(x)\big)\big]\,,\end{aligned} \tag{2.29}$$

with boundary conditions $(2.24)_2$ and initial condition $(w_\lambda - w_\rho)(x,0) = 0$. The Schauder estimates imply that

$$\|w_\lambda - w_\rho\|_{2+\beta,1+\beta/2} \leq \Lambda_2|\lambda - \rho|\Big\{\|w_\rho\|_{2+\beta,1+\beta/2} + \|W\|_{\beta,\beta/2} + \|U\|_{\beta,\beta/2} + |w_0|_{2+\alpha} + 1\Big\}\,, \tag{2.30}$$

with Λ_2 a constant as in (2.28). Combining (2.27), (2.28) and (2.30) we arrive at

$$\|w_\lambda - w_\rho\|_{2+\beta,1+\beta/2} \leq \Lambda_3|\lambda - \rho| \tag{2.31}$$

with Λ_3 depending only on $\mathcal{K}$, the initial data, μ, M and T. Thus $P(\cdot, W)$, $W \epsilon \mathcal{K}$ is a uniformly equicontinuous family of maps.

(iii) $P(0,\cdot)$ *has precisely one fixed point in* $\mathcal{B}$.
For $\lambda = 0$, $(2.24)_1$ becomes the heat equation, (2.23) and (2.24) decouple, and (2.24) has a unique solution in $\mathcal{B}$.

(iv) *Any fixed point in* $\mathcal{B}$ *of* $P(\lambda,\cdot)$, $0 \leq \lambda \leq 1$, *is contained in some bounded subset* $\mathcal{K}$ *of* $\mathcal{B}$.

Let $w \,\epsilon\, \mathcal{B}$ be a fixed point of $P(\lambda,\cdot)$. Set $u = \mathcal{T}(w)$. Then (2.27) implies $u \,\epsilon \left[C^{\beta,\beta/2}(\bar{Q}_T)\right]^N$, and (2.26) now reads

$$|u(x,t)| + |u_t(x,t)| \leq K_3 \quad , \quad (x,t) \,\epsilon\, \bar{Q}_T\,. \tag{2.32}$$

By (2.27) and (2.28), $w \,\epsilon\, C^{2+\beta,1+\beta/2}(\bar{Q}_T)$ and satisfies

$$w_t = a_\lambda(x,u,w)w_{xx} + \lambda\, b(x,u,w) + (1-\lambda)F(x) \tag{2.33}$$

on $\bar{Q}_T$ with boundary and initial conditions as in (2.24). Since b is bounded, the maximum principle yields

$$|w(x,t)| \le \sup_{0\le x\le 1} |w_0(x)| + K_4 + (1-\lambda)T \sup_{0\le x\le 1} |F(x)|\,, \qquad (x,t)\,\epsilon\,\bar{Q}_T\,. \tag{2.34}$$

Next, we multiply (2.33) by $\frac{w_t}{a_\lambda}$, integrate by parts over $[0,1]\times[0,t]$, and use Schwarz's inequality to deduce

$$\int_0^t\int_0^1 w_t^2 dx d\tau + \int_0^1 w_x^2(x,t)dx \le K_5\big[1 + \int_0^1 w_{0x}^2(x)dx + (1-\lambda)^2 \sup_{0\le x\le 1}|F(x)|^2\big] =: C^2\,. \tag{2.35}$$

Using (2.35) we obtain

$$|w(x,t) - w(y,t)| \le C|x-y|^{1/2}\,. \tag{2.36}$$

Also, for fixed $\delta > 0$ the calculus inequality

$$\begin{aligned} 2\delta|w(x,t) - w(x,\tau)| \le &\int_{x-\delta}^{x+\delta} |w(x,t) - w(y,t)|dy + \int_{x-\delta}^{x+\delta} |\int_\tau^t w_s(y,s)ds|dy \\ &+ \int_{x-\delta}^{x+\delta} |w(y,\tau) - w(x,\tau)|dy \end{aligned} \tag{2.37}$$

holds. We estimate (2.37), using (2.36) and (2.35), and in the resulting inequality we set $\delta = |t-\tau|^{1/2}$ to arrive at

$$|w(x,t) - w(x,\tau)| \le 3C|t-\tau|^{1/4}\,. \tag{2.38}$$

On account of (2.34), (2.36) and (2.38), any fixed point $w\,\epsilon\,\mathcal{B}$ is contained in a bounded set of $C^{1/2,1/4}(\bar{Q}_T)$, and, since $\beta = min\{\alpha, \frac{1}{2}\}$, also in a bounded set of $\mathcal{B}$.

The map P fulfills the hypotheses of the Leray-Schauder fixed point theorem. Thus, the map $P(1,\cdot)$ has a fixed point in $\mathcal{B}$. If $w\,\epsilon\,\mathcal{B}$ is such a fixed point and $u = \mathcal{T}(w)$, then $(w(x,t), u(x,t))$ is a classical solution of (2.8 - 2.11) on $[0,1]\times[0,T]$. ∎

We collect in Lemma 2.2 certain a-priori estimates for solutions of (2.8 - 2.11) that serve as a starting point to develop an existence theory for the system (2.1 - 2.5). Estimates (2.41) capture the regularizing effect of the parabolic equation (2.8).

Lemma 2.2. *Let* $\big(w(x,t), u(x,t)\big)$ *be a classical solution of* (2.8 - 2.11) *on* $\bar{Q}_T$ *satisfying* (2.15) *and* (2.17). *Then*

$$\|w\|_{\frac{1}{2},\frac{1}{4}} \le C_1\big(1 + \int_0^1 w_{0x}^2(x)dx\big)\,, \tag{2.39}$$

$$\|u\|_{\beta,\beta/2} \leq C_2\big(1 + \int_0^1 w_{0x}^2(x)dx + |u_0|_\alpha\big)\,, \tag{2.40}$$

where $\beta = min\{\alpha, \frac{1}{2}\}$. *Moreover, for any* $x, y \,\epsilon\, [0,1]$, $s, \tau \,\epsilon\, [t, T]$ *with* $t > 0$

$$\begin{aligned} |w(x,\tau) - w(y,\tau)| &\leq \frac{C_3}{\sqrt{t}}|x-y|^{1/2} \\ |w(x,\tau) - w(x,s)| &\leq \frac{3C_3}{\sqrt{t}}|\tau - s|^{1/4}\,. \end{aligned} \tag{2.41}$$

The constants C_1, C_2 *and* C_3 *above depend only on* μ, M *and* T.

Proof. Estimate (2.39) is a direct consequence of (2.34) and (2.35) with $\lambda = 1$, together with (2.36) and (2.38); (2.40) follows by combining (2.27) with (2.39).

To show (2.41), first multiply (2.8) by $\frac{w}{a(x,u,w)}$ and write the resulting identity in the form

$$\partial_t \int_{w_0(x)}^{w} \frac{\xi}{a(x,u,\xi)} d\xi + w_x^2 = (ww_x)_x + \frac{wb(x,u,w)}{a(x,u,w)} - u_t \cdot \int_{w_0(x)}^{w} \frac{\xi a_u(x,u,\xi)}{a^2(x,u,\xi)} d\xi\,. \tag{2.42}$$

In view of (2.17), we may assume that (2.20) holds. Integrating (2.42) over $[0,1] \times [0,t]$, $0 < t \leq T$, and using (2.9), (2.10), (2.15) and (2.20) we obtain

$$\int_0^t \int_0^1 w_x^2 dx d\tau \leq K_1\,. \tag{2.43}$$

Next, multiply (2.8) by $\frac{tw_t}{a(x,u,w)}$ and integrate by parts over $[0,1] \times [0,t]$; by estimating the resulting identity via (2.15), (2.20), Schwarz's inequality and (2.43) we conclude that

$$\int_0^t \int_0^1 \tau w_t^2(x,\tau) dx d\tau + t \int_0^1 w_x^2(x,t) dx \leq K_2\,. \tag{2.44}$$

K_1 and K_2 depend only on μ, M and T. The derivation of (2.41) from (2.44) is similar to the derivation of (2.36) and (2.38) from (2.35); in (2.41) $C_3 = \sqrt{K_2}$. ∎

Lemma 2.3 guarantees that the a-priori estimates (2.15) and (2.17) required in the hypotheses of Theorem 2.1 are always valid for T sufficiently small.

Lemma 2.3. *Let* $(w(x,t), u(x,t))$ *be a classical solution of* (2.8 - 2.11) *defined on* $[0,1] \times [0, T^*)$, *for some* $T^* > 0$.

(a) *Suppose that*

$$m_1 := \min_{0 \leq x \leq 1} a(x, u_0(x), w_0(x)) > 0\,. \tag{2.45}$$

Then given any positive constants μ, M *with* $\mu < m_1$, *there is* $T < T^*$ *depending on* μ, M *and the* L^∞*-norm of* w_{0xx} *such that*

$$|w(x,t) - w_0(x)| \leq M\,, \tag{2.46}$$

$$|u(x,t) - u_0(x)| \leq M \tag{2.47}$$

and (2.17) *hold for* $(x,t) \,\epsilon\, \bar{Q}_T$.
(b) *Suppose that, in addition,*

$$m_2 := \min_{\substack{0 \leq x \leq 1 \\ w_{0-} \leq r \leq w_{0+}}} a(x, u_0(x), r) > 0\,, \tag{2.48}$$

where $w_{0-} = \inf\limits_{0 \leq x \leq 1} w_0(x)$, $w_{0+} = \sup\limits_{0 \leq x \leq 1} w_0(x)$. *Then given any* μ, M, M_-, M_+ *with* $0 < \mu < m_2$, $M > 0$, $M_- < w_{0-} \leq w_{0+} < M_+$, *there is* $T < T^*$ *depending solely on* μ, M, M_- *and* M_+ *such that* (2.17), (2.47) *and*

$$M_- \leq w(x,t) \leq M_+ \tag{2.49}$$

hold for $(x,t) \,\epsilon\, \bar{Q}_T$.
Proof. Let $W = w - w_0$, $U = u - u_0$. Then $(W(x,t), U(x,t))$ satisfy on $[0,1] \times [0, T^*)$ the differential equations

$$W_t - A(x,U,W)W_{xx} = A(x,U,W)w_{0xx}(x) + B(x,U,W) \tag{2.50}$$

$$U_t = G(x,U,W) \tag{2.51}$$

with boundary conditions (2.10) and initial conditions $W(x,0) = 0$, $U(x,0) = 0$, for $0 \leq x \leq 1$; the functions A, B and G relate to a, b and g through formulas of the format

$$A(x,U,W) = a(x, u_0(x) + U, w_0(x) + W)\,. \tag{2.52}$$

First consider part (a). Since a, w_0 and u_0 are continuous, (2.45) and (2.52) imply that there is $\rho > 0$ so that if $0 \leq x \leq 1$, $|U| \leq \rho$ and $|W| \leq \rho$ then $A(x,U,W) \geq \mu > 0$. Set $k = min\{\rho, M\}$. To complete the proof of part (a) it suffices to show: There is $T < T^*$ such that for $(x,t) \,\epsilon\, \bar{Q}_T$ the triplet

$$(x, U(x,t), W(x,t)) \,\epsilon\, E_1 := \{(x,U,W) \,\epsilon\, [0,1] \times R^N \times R : |W| \leq k,\ |U| \leq k\}\,. \tag{2.53}$$

Then (2.17), (2.46) and (2.47) follow from (2.52) and (2.53).

Clearly (2.53) holds on $\bar{Q}_\tau$ for some τ sufficiently small. Moreover so long as (2.53) holds, the maximum principle for the parabolic equation (2.50) gives the bound

$$|W(x,t)| \leq F_1 t \tag{2.54}$$

where $F_1 = \sup\limits_{E_1} |A(x,U,W)w_{0xx}(x) + B(x,U,W)|$. Also, if $G_1 = \sup\limits_{E_1} |G(x,U,W)|$, then (2.51) together with Gronwall's inequality yield

$$|U(x,t)| \leq G_1 t\,. \tag{2.55}$$

Finally, (2.54) and (2.55) imply that (2.53) holds on $\bar{Q}_T$, for any $0 < T < T^*$ with $T \le min\left\{\frac{k}{F_1}, \frac{k}{G_1}\right\}$. Since F_1 depends on the L^∞-norm of w_{0xx}, the resulting T will exhibit the same dependence. Under the stronger hypothesis (2.48) this dependence can be avoided.

Consider now part (b). By virtue of (2.48) and the continuity of a and u_0, there are ρ, ρ_-, ρ_+ with $\rho > 0$ and $\rho_- < w_{0-} \le w_{0+} < \rho_+$ such that $0 \le x \le 1$, $|U| \le \rho$ and $\rho_- \le w \le \rho_+$ imply $a(x, u_0(x) + U, w) \ge \mu > 0$. It now suffices to show that there is $T < T^*$ such that for $(x,t) \,\epsilon\, \bar{Q}_T$ the triplet

$$(x, U(x,t), w(x,t)) \epsilon E_2 := [0,1] \times \{U \,\epsilon\, R^N : |U| \le k\} \times [k_-, k_+]\,; \tag{2.56}$$

here $k = min\{\rho, M\}$, $k_- = max\{\rho_-, M_-\}$, $k_+ = min\{\rho_+, M_+\}$ and $k_- < w_{0-} \le w_{0+} < k_+$.

Let $B_\pm = \sup\limits_{E_2} max\{0, \pm b(x, u_0(x) + U, w)\}$ and consider the comparison functions $W_\pm(x,t) = \pm B_\pm t - w_0(x) + w_{0\pm}$. On account of (2.50), so long as (2.56) holds, the functions $W_\pm$ satisfy the differential inequalities

$$\begin{gathered} \partial_t W_- - A(x,U,W)\partial_x^2 W_- \le \partial_t W - A(x,U,W)\partial_x^2 W \le \partial_t W_+ - A(x,U,W)\partial_x^2 W_+ \\ W_-(x,0) \le W(x,0) = 0 \le W_+(x,0) \end{gathered} \tag{2.57}$$

and, by (2.19), corresponding inequalities at the boundaries. Using comparison principles for parabolic equations we obtain

$$-B_- t + w_{0-} \le w(x,t) \le B_+ t + w_{0+}\,. \tag{2.58}$$

Moreover, (2.51) yields

$$|U(x,t)| \le G_2 t\,, \tag{2.59}$$

where $G_2 = \sup\limits_{E_2} |g(x, u_0(x) + U, w)|$. Finally, (2.58) and (2.59) imply that (2.56) holds on $\bar{Q}_T$, for any $0 < T < T^*$ with $T \le min\{\frac{k_+ - w_{0+}}{B_+}, \frac{w_{0-} - k_-}{B_-}, \frac{k}{G_2}\}$. ∎

Theorem 2.1 in conjunction with Lemmas 2.2 and 2.3 give rise to the following local existence and continuation theorem for the initial-boundary value problem (2.8 - 2.11) in Schauder spaces.

Theorem 2.4. *Let $w_0(x) \,\epsilon\, C^{1+\alpha}[0,1]$, $u_0(x) \,\epsilon\, \left[C^\alpha[0,1]\right]^N$ satisfy the compatibility conditions $w_{0x}(0) = w_{0x}(1) = 0$ in case $(2.10)_V$ applies, or $w_0(0) = w_0(1) = 0$ in case $(2.10)_S$ applies, and suppose that (2.48) holds. Then there exists a unique classical solution $(w(x,t), u(x,t))$ of (2.8 - 2.11) defined on a maximal interval of existence $[0,1] \times [0, T^*)$ such that, for any $0 < \tau < T < T^*$, w is in $C^{\alpha,\alpha/2}(\bar{Q}_T)$, u, u_t are in $\left[C^{\alpha,\alpha/2}(\bar{Q}_T)\right]^N$ and w_t, w_x, w_{xx} are in $C^{\alpha,\alpha/2}([0,1] \times [\tau, T])$. In case $T^* < \infty$, as $t \uparrow T^*$,*

$$\limsup_{t \uparrow T^*} \sup_{0 \le x \le 1} (|u(x,t)| + |w(x,t)|) = \infty \tag{2.60}$$

and/or

$$\liminf_{t\uparrow T^*}\inf_{0\le x\le 1} a(x,u(x,t),w(x,t)) = 0\,. \tag{2.61}$$

Furthermore, if $w_0(x)\,\epsilon\, C^{2+\alpha}[0,1]$ *and the compatibility conditions* (2.19) *hold, then* w_t, w_x, w_{xx} *are in* $C^{\alpha,\alpha/2}(\bar{Q}_T)$ *for any* $T<T^*$. *If, in addition,* $u_0(x)\,\epsilon\,\left[C^{1+\alpha}[0,1]\right]^N$, *then* u_x, u_{xt} *are in* $\left[C^{\alpha,\alpha/2}(\bar{Q}_T)\right]^N$ *for any* $T<T^*$.

Proof. We work with the boundary conditions $(2.10)_S$; the case of $(2.10)_V$ is treated similarly.

Let $w_0(x)\,\epsilon\, C^{1+\alpha}[0,1]$, $u_0(x)\,\epsilon\,\left[C^{\alpha}[0,1]\right]^N$ be given, satisfying the compatibility conditions $w_0(0)=w_0(1)=0$ as well as (2.48). We proceed to establish a local existence theorem for (2.8 - 2.11). First, construct approximating sequences $\{w_{0n}\}$ and $\{u_{0n}\}$ such that $w_{0n}(x)$ and $u_{0n}(x)$ are C^∞-functions on $[0,1]$ satisfying $(2.19)_S$, and, as $n\to\infty$,

$$w_{0n}\to w_0 \ \text{ in } \ C^1[0,1] \quad , \quad |w_{0n}|_{1+\alpha}\le K|w_0|_{1+\alpha} \tag{2.62}$$

$$u_{0n}\to u_0 \ \text{ in } \ \left[C[0,1]\right]^N \qquad , \quad |u_{0n}|_{\alpha}\le K|u_0|_{\alpha} \tag{2.63}$$

with K a fixed positive constant (for details of such a construction see [19]).

Consider the problem (2.8 - 2.11) with initial data $\big(w_{0n}(x),u_{0n}(x)\big)$. Referring to part (b) of Lemma 2.3, let

$$m_n = min\left\{a(x,u_{0n}(x),r) : 0\le x\le 1\,, \inf_{0\le x\le 1} w_{0n}(x)\le r\le \sup_{0\le x\le 1} w_{0n}(x)\right\}. \tag{2.64}$$

Since $\big(w_0(x),u_0(x)\big)$ satisfy (2.48) and $u_0(x)$ is continuous, $\liminf\limits_{n\to\infty} m_n>0$. By throwing away a finite number of terms, if needed, we may assume that $m_0 := \inf\limits_n m_n>0$. Fix μ, M, M_- and M_+ such that $M_-<\inf\limits_n\inf\limits_{0\le x\le 1} w_{0n}(x)\le \sup\limits_n\sup\limits_{0\le x\le 1} w_{0n}(x)<M_+$, $M>0$, $0<\mu<m_0$. Theorem 2.1 in conjunction with Lemma 2.3 imply that for each $n=1,2,...$ there is a classical solution $\big(w_n(x,t),u_n(x,t)\big)$ of (2.8 - 2.11) defined on $[0,1]\times[0,T_n]$, with smoothness as in Theorem 2.1, and corresponding to the initial data $\big(w_{0n}(x),u_{0n}(x)\big)$. Moreover, $T_0:=\inf\limits_n T_n>0$, and on the domain $\bar{Q}_{T_0}=[0,1]\times[0,T_0]$ the functions $\big(w_n(x,t),u_n(x,t)\big)$ satisfy the uniform bounds

$$|u_n(x,t)-u_{0n}(x)|\le M \quad , \quad M_-\le w_n(x,t)\le M_+ \tag{2.65}$$

and

$$a(x,u_n(x,t),w_n(x,t))\ge\mu>0\,. \tag{2.66}$$

Using (2.62), (2.63), (2.65) and (2.66), relations (2.39) and (2.40) in Lemma 2.2 together with (2.9) imply that on $\bar{Q}_{T_0}$

$$\|w_n\|_{\beta,\beta/2}\le K \quad , \quad \|u_n\|_{\beta,\beta/2}+\|\partial_t u_n\|_{\beta,\beta/2}\le K\,, \tag{2.67}$$

where $\beta=min\{\alpha,\frac{1}{2}\}$ and K is a constant independent of n.

Since the injection $C^{\beta,\beta/2}(\bar{Q}_{T_0}) \to C^{\beta',\beta'/2}(\bar{Q}_{T_0})$ is compact for $\beta' < \beta$, (2.67) implies that there are subsequences $\{w_{n'}\}$ and $\{u_{n'}\}$, as well as functions $w(x,t)$ and $u(x,t)$, with $w \,\epsilon\, C^{\beta,\beta/2}(\bar{Q}_{T_0})$ and u, $\partial_t u \,\epsilon\, \left[C^{\beta,\beta/2}(\bar{Q}_{T_0})\right]^N$, such that

$$\begin{aligned} w_{n'} &\to w && \text{in } \; C^{\beta',\beta'/2}(\bar{Q}_{T_0}), \\ u_{n'} \to u\,,\ \partial_t u_{n'} &\to \partial_t u && \text{in } \; \left[C^{\beta',\beta'/2}(\bar{Q}_{T_0})\right]^N. \end{aligned} \tag{2.68}$$

Clearly $\big(w(x,t), u(x,t)\big)$ satisfies (2.9), $(2.10)_S$ and (2.11). Using (2.68) together with results on families of solutions of parabolic equations (cf. Friedman [8, Sec. 3.6]), it follows that $\big(w(x,t), u(x,t)\big)$ is a classical solution of (2.8). The stated regularity of this solution is an outcome of the interior and boundary parabolic estimates [8, Sec. 4.7]. Uniqueness follows from a lengthy but routine argument that is omitted.

If $w_0(x) \,\epsilon\, C^{2+\alpha}[0,1]$ and satisfies $(2.19)_S$ then Theorem 2.1 implies that w_t, w_x and w_{xx} are in $C^{\alpha,\alpha/2}(\bar{Q}_{T_0})$. Suppose that, in addition, $u_0(x) \,\epsilon\, \left[C^{1+\alpha}[0,1]\right]^N$. Now $u(x,t)$ satisfies (2.9) with w, $w_x \,\epsilon\, C^{\alpha,\alpha/2}(\bar{Q}_{T_0})$. Using standard theorems on continuous dependence for ordinary differential equations, together with estimates in the spirit of the derivation of (2.27), leads to u_x, $u_{xt} \,\epsilon\, \left[C^{\alpha,\alpha/2}(\bar{Q}_{T_0})\right]^N$.

Finally, Theorem 2.1 implies that the solution $\big(w(x,t), u(x,t)\big)$ can be continued on a maximal interval of existence $[0,1] \times [0, T^*)$, such that either $T^* = \infty$, or at least one of (2.60) or (2.61) occurs. ∎

Next, we turn to the initial-boundary value problem (2.1 -2.5). We assume that the initial data satisfy

$$v_0(x) \,\epsilon\, C^{2+\alpha}[0,1]\,,\; u_0(x) \,\epsilon\, \left[C^{1+\alpha}[0,1]\right]^N, \tag{2.69}$$

$$m_0 = min\left\{ \varphi_q(x, u_0(x), \psi(x, u_0(x), r) : 0 \le x \le 1,\ \inf_{0 \le x \le 1} w_0(x) \le r \le \sup_{0 \le x \le 1} w_0(x) \right\} > 0\,, \tag{2.70}$$

where $w_0(x)$ is given by (2.6), and the compatibility conditions:

$$v_0(0) = v_0(1) = 0 \quad , \quad w_{0x}(0) = w_{0x}(1) = 0\,, \tag*{$(2.71)_V$}$$

in case $(2.4)_V$ applies, or

$$w_0(0) = w_0(1) = 0\,, \tag*{$(2.71)_S$}$$

in case $(2.4)_S$ applies. We prove:

Theorem 2.5. *Under the hypotheses* (2.69 - 2.71), *there exists a unique classical solution* $(v(x,t), u(x,t))$ *of* (2.1 - 2.5) *defined on a maximal interval of existence* $[0,1] \times [0, T^*)$ *such that, for any* $0 < T < T^*$, v, v_x, v_t, v_{xx} *are in* $C^{\alpha,\alpha/2}(\bar{Q}_T)$ *and* u, u_t, u_x, u_{xt} *are in* $\left[C^{\alpha,\alpha/2}(\bar{Q}_T)\right]^N$. *If* $T^* < \infty$, *then, as* $t \uparrow T^*$, *at least one out of* (2.60) *or*

$$\liminf_{t \uparrow T^*} \inf_{0 \le x \le 1} \varphi_q(x, u(x,t), v_x(x,t)) = 0 \tag{2.72}$$

occurs. Finally, $w(x,t) = \varphi(x, u(x,t), v_x(x,t))$ *satisfies* (2.41), *where* C_3 *only depends on* m_0 *and the sup-norms of* w_0 *and* u_0.

Proof. For concreteness, we treat the boundary conditions $(2.4)_S$. Let $(v_0(x), u_0(x))$ satisfying (2.69), (2.70) and $(2.71)_S$ be given and define $w_0(x)$ by (2.6). Consider the problem (2.8 - 2.11) with a, b and g defined by (2.12). Theorem 2.4 asserts that there is a unique solution $(w(x,t), u(x,t))$ of (2.8 - 2.11) defined on $[0,1]\times[0,T^*)$ and with regularity as stated there.

Our objective is to define $v(x,t)$ by (2.13) subject to the initial data $v(x,0) = v_0(x)$, $0 \le x \le 1$. Then $(v(x,t), u(x,t))$ is a solution of (2.1 - 2.5). For $v(x,t)$ to be well defined it is at least required that, for each fixed $x \in [0,1]$, $w_x(x,\cdot)$ is integrable. In view of Theorem 2.4, to complete the proof it suffices to show that, for some τ small, $w_x,\ v_t,\ v_{xx} \in C^{\alpha,\alpha/2}(\bar{Q}_\tau)$, $u_x,\ u_{xt} \in \left[C^{\alpha,\alpha/2}(\bar{Q}_\tau)\right]^N$ and also that (3.41) holds. This is accomplished by a density argument.

Consider approximating sequences $\{w_{0n}\}$ and $\{u_{0n}\}$, consisting of C^∞-functions on $[0,1]$, with $\{w_{0n}\}$ as in (2.62) and $\{u_{0n}\}$ satisfying

$$u_{0n} \to u_0 \ \text{ in } \ \left[C^1[0,1]\right]^N \quad , \quad |u_{0n}|_{1+\alpha} \le K_1 |u_0|_{1+\alpha}\,, \tag{2.73}$$

as $n \to \infty$. Set $v_{0n}(x) = v_0(0) + \int_0^x \psi(y, u_{0n}(y), w_{0n}(y))dy$ and observe that

$$v_{0n} \to v_0 \ \text{ in } \ C^2[0,1] \quad , \quad |v_{0n}|_{2+\alpha} \le K_2(|v_0|_{2+\alpha} + |u_0|_{1+\alpha})\,. \tag{2.74}$$

Let $(w_n(x,t), u_n(x,t))$ and $(v_n(x,t), u_n(x,t))$ be the corresponding solutions of (2.8 - 2.11) and (2.1 - 2.5), respectively; $v_n(x,t)$ is defined by solving (2.13) subject to $v_n(x,0) = v_{0n}(x)$. Note that by uniqueness for (2.8 - 2.11), (2.68) and (2.13) imply that

$$\begin{aligned} w_n \to w\,,\ \partial_x v_n \to \partial_x v \quad &\text{in} \quad C(\bar{Q}_T) \\ u_n \to u \qquad\qquad &\text{in} \quad \left[C(\bar{Q}_T)\right]^N \end{aligned} \tag{2.75}$$

for any fixed $T < T^*$.

The functions v_n and $\omega_n := \partial_x u_n$ satisfy the equations:

$$\partial_t v_n = A_n(x,t)\partial_x^2 v_n + B_n(x,t)\cdot\omega_n + C_n(x,t) \tag{2.76}$$

$$\partial_t \omega_n = D_n(x,t)\omega_n + E_n(x,t)\partial_x^2 v_n + F_n(x,t)\,, \tag{2.77}$$

where, on account of (2.66), (2.67) and $(2.13)_1$, the components of $A_n - F_n$ are uniformly bounded in $C^{\alpha,\alpha/2}(\bar{Q}_T)$, and for n large $A_n(x,t) \ge \mu$ for some $\mu > 0$. The Schauder estimates for (2.76), together with (2.74), yield

$$\|v_n\|_{2+\alpha,1+\alpha/2} \le K_3[|v_0|_{2+\alpha} + |u_0|_{1+\alpha} + \|\omega_n\|_{\alpha,\alpha/2} + 1]\,. \tag{2.78}$$

Proceeding as in the derivation of (2.27), (2.77) implies that on $\bar{Q}_\tau$ with $0 < \tau \le T$

$$\|\omega_n\|_{\alpha,\alpha/2} \le K_4[|u_0|_{1+\alpha} + (\tau + \tau^{1-\alpha})\|\partial_x^2 v_n\|_{\alpha,\alpha/2} + 1]\,. \tag{2.79}$$

The constants K_3 and K_4 are independent of n. Combining (2.78) and (2.79) we conclude that, provided $\tau + \tau^{1-\alpha} < \frac{1}{2K_3K_4}$, the estimate

$$\|v_n\|_{2+\alpha,1+\alpha/2} + \|\omega_n\|_{\alpha,\alpha/2} \le K_5(|v_0|_{2+\alpha} + |u_0|_{1+\alpha} + 1) \tag{2.80}$$

is valid on $\bar{Q}_\tau$. Also, by virtue of Lemma 2.2, $w_n(x,t)$ satisfies (2.41).

Relations (2.80) and (2.75) give $w_x,\ v_{xx} \in C^{\alpha,\alpha/2}(\bar{Q}_\tau)$ and $u_x \in \left[C^{\alpha,\alpha/2}(\bar{Q}_\tau)\right]^N$. Also, $v_t = w_x \in C^{\alpha,\alpha/2}(\bar{Q}_\tau)$ and $u_{xt} = f_x + f_p\cdot u_x + f_q\cdot v_{xx} \in \left[C^{\alpha,\alpha/2}(\bar{Q}_\tau)\right]^N$. Finally, $w(x,t)$ satisfies (2.41). ∎

3. On the Competition of Strain Softening and Strain Rate Dependence

The scope of this Section is to elucidate the competition between the destabilizing influence of strain softening and the stabilizing influence of strain rate sensitivity and their effect on the response of shearing motions. Also, to provide quantitative criteria that determine which one prevails.

We use as a test case, the initial-boundary value problem consisting of (1.1 - 1.3), namely

$$\begin{aligned} v_t &= (\tau(u)v_x^n)_x \\ & \qquad\qquad\qquad 0 \le x \le 1,\, t > 0 \\ u_t &= v_x \end{aligned} \tag{3.1}$$

with boundary conditions $(1.9)_S$ and initial conditions (1.5). The initial data are taken smooth: $v_0(x)\,\epsilon\, C^{2+\alpha}[0,1]$, $u_0(x)\,\epsilon\, C^{1+\alpha}[0,1]$, for some $0 < \alpha < 1$; compatible with the boundary conditions: $\sigma_0(0) = \sigma_0(1) = 1$; and satisfying the sign restrictions

$$u_0(x) > 0,\, \sigma_0(x) > 0 \quad , \quad 0 \le x \le 1\,. \tag{3.2}$$

Henceforth, we will refer to this problem as $(\mathcal{P})_S$. Recall that $\tau(u)$ is a smooth function satisfying (1.4) and n is a positive parameter.

The theory developed in Section 2 implies the existence of a unique solution $(v(x,t), u(x,t))$ of $(\mathcal{P})_S$, defined on a maximal interval of existence $[0,1]\times[0,T^*)$, such that v, v_t, v_x, v_{xx}, u, u_t and u_x are in $C^{\alpha,\alpha/2}(\bar{Q}_T)$, for any $T < T^*$. In addition, given any compact subset $\mathcal{K}$ of $(0,\infty)\times R$, $(\sigma(x,t), u(x,t))$ escapes $\mathcal{K}$ as $t \uparrow T^*$, i.e., there are sequences $\{x_n\} \subset [0,1]$ and $\{t_n\}$ with $t_n \uparrow T^*$ such that $(\sigma(x_n,t_n), u(x_n,t_n)) \notin \mathcal{K}$. The identification of $(\mathcal{P})_S$ with (2.1 - 2.6) is done by setting $w = \varphi(x,p,q) = \tau(p)q^n - 1$. (Although φ is not C^2 at $q = 0$, it is C^2 when restricted to compact subsets of $[0,1]\times R\times(0,\infty)$; this remark together with the results of Section 2 provide the above statements).

Our objectives are: (a) to characterize the class of functions $\tau(u)$ and parameters n that guarantee global solvability for $(\mathcal{P})_S$, and (b) to study the behavior of solutions $(v(x,t), u(x,t))$ of $(\mathcal{P})_S$.

To this end, it is expedient to use a different formulation of (3.1). Note that for $(x,t) \in [0,1]\times[0,T^*)$, a solution of $(\mathcal{P})_S$ satisfies

$$\sigma(x,t) > 0\,,\; u_t(x,t) = v_x(x,t) > 0\,,\; u(x,t) \ge u_0(x) > 0\,. \tag{3.3}$$

A simple calculation, using (1.1 - 1.3), shows that $(\sigma(x,t), u(x,t))$ is a positive solution of the reaction-diffusion system

$$\left(\sigma^{\frac{1}{n}}\right)_t = \tau(u)^{\frac{1}{n}}\sigma_{xx} + \frac{1}{n}\frac{\tau'(u)}{\tau(u)^{1+\frac{1}{n}}}\left(\sigma^{\frac{1}{n}}\right)^2 \tag{3.4}$$

$$u_t = \frac{\sigma^{\frac{1}{n}}}{\tau(u)^{\frac{1}{n}}}\,, \tag{3.5}$$

on $[0,1]\times[0,T^*)$. For this step and for the remainder of the Section we assume that $(v(x,t), u(x,t))$ enjoys some additional smoothness, namely, v_{xt}, v_{xxx} and u_{xx} are in

$C^{\alpha,\alpha/2}(\bar{Q}_T)$, for any $T < T^*$. Such solutions are generated if we take smoother initial data, a hypothesis that can later be relaxed using density arguments (cf. Section 2). Integrating (3.5), yields

$$\Phi(u(x,t)) = \Phi(u_0(x)) + \int_0^t \sigma^{\frac{1}{n}}(x,\tau)d\tau\,, \tag{3.6}$$

where

$$\Phi(u) = \int_1^u \tau(\xi)^{\frac{1}{n}} d\xi\,. \tag{3.7}$$

An important ingredient of the forthcoming analysis lies in estimating $\sigma(x,t)$ by means of comparison principles (e.g. [13, Ch. 3, Sec. 7]) for the parabolic equation (3.4); in turn, $u(x,t)$ is estimated using (3.6). We state the comparison principle used as a lemma for future reference.

Lemma 3.1. *Suppose that for any $T < T^*$, $\sigma_1(x,t)$ and $\sigma_2(x,t)$ are both in $C^{2,1}(\bar{Q}_T)$ with $\sigma_1(x,t) > 0$ and $\sigma_2(x,t) > 0$, $A(x,t) \in C(\bar{Q}_T)$ with $A(x,t) > 0$ and $B(x,t) \in C(\bar{Q}_T)$. If, for any $T < T^*$,*

$$\begin{aligned} (\sigma_1^{\frac{1}{n}})_t - A(x,t)\sigma_{1xx} + B(x,t)\big(\sigma_1^{\frac{1}{n}}\big)^2 \le \big(\sigma_2^{\frac{1}{n}}\big)_t - A(x,t)\sigma_{2xx} + B(x,t)\big(\sigma_2^{\frac{1}{n}}\big)^2, \quad &on \quad \bar{Q}_T\,,\\ \sigma_1(i,t) \le \sigma_2(i,t) \quad , \quad i = 0,1,\ 0 \le t \le T\,,&\\ \sigma_1(x,0) \le \sigma_2(x,0) \quad , \quad 0 \le x \le 1& \end{aligned} \tag{3.8}$$

then

$$\sigma_1(x,t) \le \sigma_2(x,t) \quad , \quad 0 \le x \le 1,\ 0 \le t < T^*\,. \tag{3.9}$$

In applications of the lemma, the functions A and B are taken respectively, $A = \tau(u)^{\frac{1}{n}} > 0$ and $B = -\frac{1}{n}\frac{\tau'(u)}{\tau(u)^{1+\frac{1}{n}}} > 0$.

Our first theorem characterizes global solvability of $(\mathcal{P})_S$ in terms of the behavior of $\Phi(u)$ as $u \to \infty$.

Theorem 3.2. *Let $(v(x,t), u(x,t))$ be a classical solution of $(\mathcal{P})_S$ defined on a maximal interval of existence $[0,1] \times [0,T^*)$. Suppose that $\tau(u)$ satisfies (1.4). Then:*

(i) *$T^* = \infty$ if and only if $\Phi(\infty) = \infty$.*

(ii) *If $T^* < \infty$, then*

$$\lim_{t\to T^*} \sup_{0\le x\le 1} u(x,t) = \infty\,. \tag{3.10}$$

Proof. Let $(v(x,t), u(x,t))$ be a solution of $(\mathcal{P})_S$ on $[0,1] \times [0,T^*)$, with T^* maximal; let $\sigma(x,t)$ be defined by (1.3). We estimate the solution in the interval of existence $[0,1] \times [0,T^*)$.

Under hypothesis (1.4), any positive, concave function $\Sigma(x)$ satisfies

$$-\tau(u)^{\frac{1}{n}}\Sigma_{xx} - \frac{1}{n}\frac{\tau'(u)}{\tau(u)^{1+\frac{1}{n}}}\Sigma^{\frac{2}{n}} > 0\,. \tag{3.11}$$

If, in addition,

$$\Sigma(x) \geq \sigma_0(x) \quad , \quad 0 \leq x \leq 1 \tag{3.12}$$

then applying Lemma 3.1, with comparison functions $\sigma(x,t)$ and $\Sigma(x)$, yields

$$\sigma(x,t) \leq \Sigma(x)\,. \tag{3.13}$$

First, we show (ii). Assume that $T^* < \infty$ and at the same time $u(x,t)$ is bounded from above on $[0,1] \times [0,T^*)$, i.e.,

$$u_{0-} \leq u(x,t) \leq U_+ < \infty\,; \tag{3.14}$$

here we used the fact that $u(x,t) \geq \inf\limits_{0\leq x\leq 1} u_0(x) =: u_{0-}$. Let $s(t)$ be the solution of the initial value problem

$$\begin{aligned} &\frac{d}{dt}s^{\frac{1}{n}} + B_0\big(s^{\frac{1}{n}}\big)^2 = 0 \\ &s(0) = \sigma_{0-} := \inf_{0\leq x\leq 1} \sigma_0(x)\,, \end{aligned} \tag{3.15}$$

where $B_0 = \max\limits_{u_{0-}\leq \xi\leq U_+} \dfrac{1}{n}\dfrac{|\tau'(\xi)|}{\tau(\xi)^{1+\frac{1}{n}}}$. Integrating (3.15), yields

$$s(t) = \frac{\sigma_{0-}}{\big(1+\sigma_{0-}^{\frac{1}{n}}B_0 t\big)^n} > 0\,. \tag{3.16}$$

On account of (3.15) and (3.4), the comparison functions $s(t)$ and $\sigma(x,t)$ satisfy the parabolic differential inequality (3.8) on $\bar{Q}_T$, for any $T < T^*$. Using Lemma 3.1, we deduce

$$s(t) \leq \sigma(x,t)\,. \tag{3.17}$$

Estimates (3.12), (3.14) and (3.17) imply that $(\sigma(x,t), u(x,t))$ remains in some compact subset of $(0,\infty)\times R$ as $t \uparrow T^*$. But then the solution can be continued past T^*, which contradicts the assumption that T^* is maximal and finite. Since $u_t > 0$, we conclude that if $T^* < \infty$ and maximal then (3.10) holds.

Next, we proceed to prove (i). Consider $\Phi(u)$ defined for $u \in (0,\infty)$ by (3.7). $\Phi(u)$ is increasing and invertible with an inverse function $\Phi^{-1}(\xi)$ defined for $\xi \in (\Phi(0),\Phi(\infty))$ and increasing. In case $\Phi(\infty) = \infty$, combining (3.6) and (3.13), we arrive at

$$u(x,t) \leq \Phi^{-1}\big(\Phi(u_0(x)) + \Sigma^{\frac{1}{n}}(x)t\big)\,. \tag{3.18}$$

Then (3.14) holds for any $\bar{Q}_T$ with $T > 0$ and, necessarily, if T^* is maximal then $T^* = \infty$. By contrast, in case $\Phi(\infty) < \infty$, (3.6) at $x = 0$ or $x = 1$ together with $(1.9)_S$ leads to

$$\Phi(u(i,t)) = \Phi(u_0(i)) + t \quad , \quad i = 0,1\,. \tag{3.19}$$

In turn, (3.19) implies

$$u(i,t) \to \infty \quad \text{as} \quad t \to T_i\,, \tag{3.20}$$

where $T_i = \Phi(\infty) - \Phi(u_0(i)) < \infty$ for $i = 0,1$. Therefore, in case $\Phi(\infty) < \infty$, $T^* \leq min\{T_0, T_1\} < \infty$. ∎

According to Theorem 3.2, the criterion for global solvability of $(\mathcal{P})_S$ is the divergence of the integral $\int_1^\infty \tau(\xi)^{\frac{1}{n}}d\xi =: \Phi(\infty)$. Therefore, it is the decay rate of $\tau(u)$ as $u \to \infty$ that determines global existence for $(\mathcal{P})_S$. The class of positive, decreasing constitutive functions $\tau(u)$ can be decomposed into two categories, depending on whether $\Phi(\infty)$ is finite or infinite. Roughly speaking, the dividing line consists of functions $\tau(u)$ that decay to zero like the power u^{-n}.

Next, we restrict attention to functions $\tau(u)$ such that $\Phi(\infty) = \infty$ (and thus $T^* = \infty$) and consider the asymptotic behavior of solutions $(v(x,t), u(x,t))$ of $(\mathcal{P})_S$ as $t \to \infty$. In case $\tau(u) \equiv \tau_0$ a constant, $\sigma(x,t)$ is a positive solution of

$$\left(\sigma^{\frac{1}{n}}\right)_t = \tau_0^{\frac{1}{n}}\sigma_{xx} \tag{3.21}$$

subject to the boundary conditions $(1.9)_S$; thus $\sigma(x,t) \to 1$ uniformly in $x \in [0,1]$ as $t \to \infty$. The question is whether this behavior persists for positive and decreasing functions $\tau(u)$.

Two representative classes of functions $\tau(u)$ are considered: Class (H1) consists of functions that decay to a positive constant $\tau(\infty)$ at a rate dominated by a power, i.e., for some $c > 0$ and $\alpha > 0$

$$\tau(u) > \tau(\infty) > 0 \quad , \quad 0 < -\tau'(u) \le \frac{c}{u^\alpha} \quad , \quad u > 0\,. \tag{$H1$}$$

Class (H2) consists functions that decay to zero like a power, up to first order derivatives, i.e., for some $c > 0$ and $m > 0$

$$\frac{1}{c}\frac{1}{u^m} \le \tau(u) \le \frac{c}{u^m} \quad , \quad 0 < -\frac{\tau'(u)}{\tau(u)} \le \frac{m}{u} \quad , \quad u > 0\,. \tag{$H2$}$$

Note that, to guarantee global existence for functions of class (H2), we assume $0 < m \le n$ so that $\Phi(\infty) = \infty$. Also, note that under hypothesis (H1),

$$\lim_{u\to\infty} \frac{\Phi(u)}{u} = \tau(\infty)^{\frac{1}{n}}\,, \tag{3.22}$$

while, under hypothesis (H2) for $0 < m < n$

$$\limsup_{u\to\infty} \frac{\Phi(u)}{u^{1-\frac{m}{n}}} < \infty \quad , \quad \liminf_{u\to\infty} \frac{\Phi(u)}{u^{1-\frac{m}{n}}} > 0\,, \tag{3.23}$$

and for $m = n$

$$\limsup_{u\to\infty} \frac{\Phi(u)}{\ell n u} < \infty \quad , \quad \liminf_{u\to\infty} \frac{\Phi(u)}{\ell n u} > 0\,. \tag{3.24}$$

We prove:

Theorem 3.3. *Suppose the function $\tau(u)$ belongs to either the class* (H1) *or the class* (H2) *with $0 < \frac{m}{n} < \frac{1}{2}$. Let $(v(x,t), u(x,t))$ be a classical solution of $(\mathcal{P})_S$ on $[0,1] \times [0,\infty)$, corresponding to initial data $(v_0(x), u_0(x))$ with $u_0(x) > 0$, $\sigma_0(x) > 0$ for $0 \le x \le 1$ and*

$\sigma_0(0) = \sigma_0(1) = 0$. *Then, for any choice of the initial data in case* $0 < n < 2$ *and under restrictions for the data that are outlined below in case* $n \geq 2$, *the following hold: As* $t \to \infty$,

$$\sigma(x,t) = 1 + O(t^{-\beta})\,, \tag{3.25}$$

$$\int_{u_0(x)}^{u(x,t)} \tau(\xi)^{\frac{1}{n}} d\xi = t + O(\int_1^t s^{-\beta} ds) \tag{3.26}$$

and

$$v(x,t) = \int_0^1 v_0(y)dy + \int_0^1 \int_y^x \frac{1}{\tau(u(\xi,t))^{\frac{1}{n}}} d\xi dy + O(t^{-\beta+\gamma}) \tag{3.27}$$

uniformly on $[0,1]$. *In case* (H1) *holds* $\beta = \alpha > 0$ *and* $\gamma = 0$, *while, in case* (H2) *holds* $0 < \beta = \frac{n-2m}{n-m} < 1$ *and* $\gamma = \frac{m}{n-m}$.

Proof. Let $(v(x,t), u(x,t))$ be a classical solution of $(\mathcal{P})_S$ defined on $\bar{Q}_\infty := [0,1] \times [0,\infty)$. Then (3.3) holds and $(\sigma(x,t), u(x,t))$ satisfies (3.4), (3.5) on $\bar{Q}_\infty$. We proceed to obtain an initial a-priori estimate, independent of t, for $\sigma(x,t)$ using comparison principles. In the sequel K will stand for a generic constant that can be estimated in terms of the initial data and properties of the function $\tau(u)$.

Set $-\gamma(u)$ to be the quotient of the coefficient of the reaction term over the coefficient of the diffusion term in (3.4)

$$\gamma(u) = \frac{1}{n}\Big(- \frac{\tau'(u)}{\tau(u)}\Big) \frac{1}{\tau(u)^{2/n}}\,. \tag{3.28}$$

If we can find positive functions $S(x)$ and $s(x)$ defined on $[0,1]$ and satisfying for $(x,t) \in \bar{Q}_\infty$ the differential inequalities

$$\begin{gathered} -S_{xx} + \gamma(u(x,t))S^{2/n} \geq 0 \\ S(x) \geq \sigma_0(x) \end{gathered} \tag{3.29}$$

and

$$\begin{gathered} -s_{xx} + \gamma(u(x,t))s^{2/n} \leq 0 \\ s(x) \leq \sigma_0(x)\,, \end{gathered} \tag{3.30}$$

respectively, then Lemma 3.1 implies

$$s(x) \leq \sigma(x,t) \leq S(x) \tag{3.31}$$

for $(x,t) \in \bar{Q}_\infty$.

Next, we examine the possibility of finding such functions $S(x)$ and $s(x)$. Observe that under hypothesis (H1),

$$0 < \gamma(u) \leq \frac{K_1}{u^\alpha} \leq K_2 \quad , \quad u \geq \inf_{0 \leq x \leq 1} u_0(x) \tag{3.32}$$

while, under hypothesis (H2) for $0 \le \frac{m}{n} \le \frac{1}{2}$,

$$0 < \gamma(u) \le \frac{K_3}{u^{1-2\frac{m}{n}}} \le K_4 \quad , \quad u \ge \inf_{0\le x\le 1} u_0(x) . \tag{3.33}$$

Under either of (H1) or (H2), any concave, positive function $S(x)$ satisfies $(3.29)_1$. Moreover, for any choice of the initial function $\sigma_0(x) > 0$ there is a concave function $S(x)$ such that $S(x) \ge \sigma_0(x)$, $0 \le x \le 1$. Therefore, the right hand inequality of (3.31) is valid for all values of the parameters and choices of the initial data.

Turn now to (3.30). If a positive function $s(x)$ satisfies

$$-s_{xx} + \gamma_0 s^{\frac{2}{n}} \le 0 \quad , \quad 0 \le x \le 1 , \tag{3.34}$$

where $\gamma_0 = K_2$ in case (H1) holds or $\gamma_0 = K_4$ in case (H2) holds ($0 \le \frac{m}{n} \le \frac{1}{2}$), then $s(x)$ also satisfies $(3.30)_1$. Consider two cases $n < 2$ and $n \ge 2$.

(i) If $n < 2$, the parametric family $s_\alpha(x) = \frac{\alpha}{2}(x^2+1)$ satisfies the inequality (3.34) provided $\alpha > 0$ and $\alpha^{\frac{2}{n}-1} \le \frac{1}{\gamma_0}$. Given any $(\sigma_0(x), u_0(x))$ both positive on $[0,1]$, we fix γ_0 (which depends on $u_0(x)$) and choose α sufficiently small so that $s_\alpha(x) \le \sigma_0(x)$ and (3.34) is fulfilled for $x \in [0,1]$. Then $s_\alpha(x)$ satisfies (3.30), and the left hand side of (3.31) is established.

(ii) If $n \ge 2$, a function $s(x)$ satisfying (3.34) and $(3.30)_2$ can only be found for restricted choices of the data $(\sigma_0(x), u_0(x))$. For instance, given $\sigma_0(x)$ find the largest $\alpha > 0$ such that $s_\alpha(x) = \frac{\alpha}{2}(x^2+1) \le \sigma_0(x)$. With this α fixed, $s_\alpha(x)$ satisfies (3.34) provided $\gamma_0 \le \alpha^{1-\frac{2}{n}}$. In view of the choice of γ_0 and (3.32), (3.33), this imposes a restriction on $u_0(x)$ and/or the function $\tau(u)$. If these restrictions are satisfied then the left hand inequality of (3.31) holds.

Henceforth, we restrict attention to the cases when (3.31) holds. Using (3.31), (3.6) yields

$$\frac{1}{K_5} t \le \Phi(u(x,t)) - \Phi(u_0(x)) \le K_5 t \quad , \quad t > 0 . \tag{3.35}$$

Combining (3.3) and (3.35) with (3.22) or (3.23) we conclude that, under hypothesis (H1),

$$\frac{1}{K_6}(t+1) \le u(x,t) \le K_6(t+1) , \tag{3.36}$$

while, under hypothesis (H2) with $0 < \frac{m}{n} \le \frac{1}{2}$,

$$\frac{1}{K_7}(t+1)^{\frac{n}{n-m}} \le u(x,t) \le K_7(t+1)^{\frac{n}{n-m}} . \tag{3.37}$$

In view of (1.1) and $(1.9)_S$ we have the following identity:

$$\frac{1}{2}\frac{d}{dt}\int_0^1 v_t^2 dx + \int_0^1 \sigma_t v_{xt} dx = 0 . \tag{3.38}$$

Using (3.38), together with (3.4) and (1.1),

$$\frac{1}{2}\frac{d}{dt}\int_0^1 v_t^2 dx + n\int_0^1 \tau(u)^{\frac{1}{n}}\sigma^{1-\frac{1}{n}}v_{xt}^2 dx = \int_0^1 \big(-\frac{\tau'(u)}{\tau(u)}\big)\frac{\sigma^{1+\frac{1}{n}}}{\tau(u)^{\frac{1}{n}}}v_{xt}dx\,, \tag{3.39}$$

which, together with Schwarz's inequality, implies

$$\frac{1}{2}\frac{d}{dt}\int_0^1 v_t^2 dx + \frac{n}{2}\int_0^1 \tau(u)^{\frac{1}{n}}\sigma^{1-\frac{1}{n}}v_{xt}^2 dx \le \frac{1}{2n}\int_0^1 \big(\frac{\tau'(u)}{\tau(u)}\big)^2\frac{\sigma^{1+\frac{3}{n}}}{\tau(u)^{\frac{3}{n}}}dx\,. \tag{3.40}$$

Also, the calculus identity

$$\sigma_x(x,t) = \int_0^1 \sigma_x(y,t)dy + \int_0^1\int_y^x \sigma_{xx}(\xi,t)d\xi\,, \tag{3.41}$$

together with (1.1), $(1.9)_S$ and Schwarz's inequality lead to

$$v_t^2(x,t) \le \int_0^1 v_{xt}^2(\xi,t)d\xi\,. \tag{3.42}$$

Suppose first that (H2) holds and $0 < \frac{m}{n} \le \frac{1}{2}$. Then, combining (3.31), (3.37) and the inequalities in (H2) with (3.40) and (3.42) we arrive at the differential inequality

$$\frac{d}{dt}\int_0^1 v_t^2 dx + \frac{1}{K_8}(t+1)^{-\frac{m}{n-m}}\int_0^1 v_t^2 dx \le K_9(t+1)^{\frac{3m-2n}{n-m}}\,. \tag{3.43}$$

If $0 < \frac{m}{n} < \frac{1}{2}$, then integrating (3.43) we obtain

$$\int_0^1 v_t^2(x,t)dx \le K_{10}(t+1)^{\frac{4m-2n}{n-m}}\,; \tag{3.44}$$

however, if $\frac{m}{n} = \frac{1}{2}$, then (3.43) does not provide any decay. Relation (3.44), in conjunction with the Poincarè inequality

$$(\sigma(x,t)-1)^2 \le \int_0^1 \sigma_x^2(\xi,t)d\xi \tag{3.45}$$

and (1.1), yields (3.25) when (H2) holds.

When (H1) holds, the same sequence of steps using (3.36) and (H1) in the place of (3.37) and (H2) lead to the differential inequality

$$\frac{d}{dt}\int_0^1 v_t^2 dx + \frac{1}{K_{11}}\int_0^1 v_t^2 dx \le K_{12}(t+1)^{-2\alpha}\,, \tag{3.46}$$

which, once integrated, yields

$$\int_0^1 v_t^2(x,t)dx \le K_{13}(t+1)^{-2\alpha}\,. \tag{3.47}$$

Combining (3.45) and (3.47) we arrive at (3.25), in case (H1) holds.

To show (3.26), observe that (3.6), (3.7), Poincarè's inequality, (3.31) and (1.1) yield

$$\begin{aligned}\Big|\int_{u_0(x)}^{u(x,t)} \tau(\xi)^{\frac{1}{n}}d\xi - t\Big| &\le \int_0^t |\sigma^{\frac{1}{n}}(x,\tau)-1|d\tau \\ &\le K_{14}\int_0^t \Big(\int_0^1 v_t^2(x,\tau)dx\Big)^{1/2}d\tau\,.\end{aligned} \tag{3.48}$$

Using (3.48) together with (3.44) or (3.47) in cases (H2) or (H1), respectively, we deduce (3.26).

Finally, the identities

$$v(x,t) = \int_0^1 v(y,t)dy + \int_0^1\int_y^x \frac{\sigma^{\frac{1}{n}}(\xi,t)}{\tau(u(\xi,t))^{\frac{1}{n}}}d\xi dy \tag{3.49}$$

and

$$\int_0^1 v(y,t)dy = \int_0^1 v_0(y)dy \tag{3.50}$$

(by (1.1) and $(1.9)_S$), together with Poincarè's inequality and (3.31) imply

$$\begin{aligned}\Big|v(x,t) - \int_0^1 v_0(y)dy - \int_0^1\int_y^x \frac{1}{\tau(u(\xi,t))^{\frac{1}{n}}}d\xi dy\Big| &\le \int_0^1 \frac{|\sigma^{\frac{1}{n}}-1|}{\tau(u)^{\frac{1}{n}}}dx \\ &\le K_{15}\Big(\int_0^1 v_t^2dx\Big)^{1/2}\Big(\int_0^1 \frac{1}{\tau(u)^{\frac{1}{n}}}dx\Big)\,.\end{aligned} \tag{3.51}$$

Using (3.47) in case (H1) holds, or (3.44) and (3.37) in case (H2) holds we arrive at (3.27).

■

Between the class of functions $\tau(u)$ satisfying $\Phi(\infty) < \infty$, for which solutions of $(\mathcal{P})_S$ blow up in finite time, and the class of $\tau(u)$ satisfying (H2) with $0 < \frac{m}{n} < \frac{1}{2}$, for which solutions of $(\mathcal{P})_S$ behave asymptotically like in (3.21) (cf. (3.25)), there remains a gap to be analyzed. For a power law

$$\tau(u) = u^{-m} \tag{3.52}$$

the gap corresponds to the powers $\frac{1}{2} \le \frac{m}{n} \le 1$.

In the sequel we consider the power law (3.52) with $\frac{m}{n} \ge \frac{1}{2}$ and analyze the behavior of solutions of $(\mathcal{P})_S$. Let $(v(x,t), u(x,t))$ be such a solution defined on $[0,1]\times[0,T^*)$; here $T^* = +\infty$ if $\frac{1}{2} \le \frac{m}{n} \le 1$ and $T^* < +\infty$ if $\frac{m}{n} > 1$. The function $\sigma(x,t)$ satisfies (3.4) with $\tau(u)$ as in (3.52), while (3.6) yields

$$u(x,t)^{1-\frac{m}{n}} = u_0(x)^{1-\frac{m}{n}} + (1-\frac{m}{n})\int_0^t \sigma^{\frac{1}{n}}(x,\tau)d\tau\,, \tag{3.53}$$

in case $\frac{m}{n} \neq 1$, and

$$\ell n u(x,t) = \ell n u_0(x) + \int_0^t \sigma^{\frac{1}{n}}(x,\tau)d\tau \tag{3.54}$$

in case $\frac{m}{n} = 1$.

We examine the class of solutions of the differential inequality (3.29). Note that, by virtue of (3.28), (3.52) and (3.3), for $\frac{m}{n} \geq \frac{1}{2}$

$$\gamma(u(x,t)) = \frac{m}{n} u(x,t)^{2\frac{m}{n}-1} \geq \frac{m}{n} u_0(x)^{2\frac{m}{n}-1} . \tag{3.55}$$

Then, Lemma 3.1 implies:

Lemma 3.4. *Let $\frac{m}{n} \geq \frac{1}{2}$. If $S(x)$ is a smooth, positive function satisfying for $x \in [0,1]$*

$$\begin{gathered} -S_{xx}(x) + \gamma_0(x) S^{\frac{2}{n}}(x) \geq 0 \\ S(x) \geq \sigma_0(x) , \end{gathered} \tag{3.56}$$

where $\gamma_0(x) := \frac{m}{n} u_0(x)^{2\frac{m}{n}-1}$, then

$$\sigma(x,t) \leq S(x) \quad , \quad 0 \leq x \leq 1,\, 0 \leq t < T^* . \tag{3.57}$$

If $\frac{m}{n} > \frac{1}{2}$, given any $S(x) \geq \sigma_0(x) > 0$, (3.56) can be always satisfied by choosing $u_0(x) > 0$ appropriately. Namely ,

$$\frac{m}{n} u_0(x)^{2\frac{m}{n}-1} \geq \frac{S_{xx}(x)}{S^{\frac{2}{n}}(x)} . \tag{3.58}$$

In particular, $\sigma_0(x)$ can be used as a choice for $S(x)$ for restricted choices of $u_0(x)$.

If $\frac{m}{n} = \frac{1}{2}$, then $S(x) \geq \sigma_0(x) > 0$ will satisfy (3.56) provided

$$\frac{1}{2} \geq \frac{S_{xx}(x)}{S^{\frac{2}{n}}(x)} . \tag{3.59}$$

In light of the above remark, we study the family of potential choices

$$S_\alpha(x) = 1 + \alpha x(x-1) \quad , \quad 0 \leq x \leq 1 \tag{3.60}$$

where $0 < \alpha < 4$. Note that $S_\alpha(x)$ is convex and attains its minimum $S_\alpha\left(\frac{1}{2}\right) = 1 - \frac{\alpha}{4} > 0$. Also $S_\alpha(x) \to 1$ uniformly on $[0,1]$ as $\alpha \to 0$.

(a) Case $\mathbf{\frac{m}{n} > \frac{1}{2}}$

If the initial data $(u_0(x), \sigma_0(x))$ are restricted so as to satisfy

$$\sigma_0(x) \leq S_\alpha(x) \quad , \quad u_0(x) \geq u_\alpha := \left[\frac{n}{m}\frac{2\alpha}{\left(1-\frac{\alpha}{4}\right)^{\frac{2}{n}}}\right]^{\frac{n}{2m-n}} , \, 0 \leq x \leq 1 \tag{3.61}$$

then Lemma 3.4, in conjunction with (3.58), implies

$$\sigma(x,t) \le S_\alpha(x) \quad , \quad 0 \le x \le 1\,,\, 0 \le t < T^*\,. \tag{3.62}$$

Observe that $u_\alpha \to 0$ as $\alpha \to 0$.

Consider now the problem $(\mathcal{P})_S$ with initial data $(\sigma_0(x), u_0(x)) = (S_\alpha(x), u_\alpha)$, for some $0 < \alpha < 4$. Let $(\sigma_\alpha(x,t), u_\alpha(x,t))$ be the corresponding solution; it is defined on $[0,1] \times [0, T^*)$. Also,

$$\sigma_\alpha(x,t) \le S_\alpha(x) = 1 + \alpha x(x-1) \tag{3.63}$$

for $(x,t) \in [0,1] \times [0,T^*)$.

We consider three separate regions:

(i) $\mathbf{\frac{1}{2} < \frac{m}{n} < 1}$: Here $T^* = +\infty$. In addition, (3.63) and (3.53) yield

$$u_\alpha(x,t) \le \left[u_\alpha^{1-\frac{m}{n}} + (1 - \frac{m}{n}) S_\alpha^{\frac{1}{n}}(x) t \right]^{\frac{n}{n-m}} \tag{3.64}$$

for $0 \le x \le 1$, $0 \le t < \infty$. Note that, since $S_\alpha(0) = S_\alpha(1) = 1$, the boundary condition $(1.9)_S$ together with (3.53) imply that (3.64) is in fact an equality at $x = 0$ and $x = 1$. A comparison of (3.25) and (3.26) with (3.63) and (3.64) reveals the drastic difference in the behavior of solutions across the parameter values $\frac{m}{n} = \frac{1}{2}$. In particular $\sigma_\alpha(x,t)$ does not converge to 1 as $t \to \infty$, and spatial nonuniformities of the strain $u(x,t)$ develop and persist in time. This is the case no matter how close to the constant function 1 the initial state $S_\alpha(x)$ is. The diffusion is in this case too weak to uniformize the solution.

(ii) $\mathbf{\frac{m}{n} = 1}$: The situation is similar to part (i) with (3.64) replaced by

$$u_\alpha(x,t) \le u_\alpha exp\Big\{ S_\alpha^{\frac{1}{n}}(x) t \Big\} \tag{3.65}$$

by virtue of (3.54).

(iii) $\mathbf{\frac{m}{n} > 1}$: Here $T^* < +\infty$. The solution $(\sigma_\alpha(x,t), u_\alpha(x,t))$ satisfies on $[0,1] \times [0,T^*)$ the bounds (3.63) and

$$u_\alpha(x,t) \le \left[u_\alpha^{1-\frac{m}{n}} - \left|1 - \frac{m}{n}\right| S_\alpha^{\frac{1}{n}}(x) t \right]^{-\frac{n}{m-n}}\,. \tag{3.66}$$

The bound on the right hand side of (3.66) blows up for the first time at the boundary points $x = 0$ and $x = 1$ as $t \to T_{cr} := \frac{1}{\left|1-\frac{m}{n}\right|} u_\alpha^{1-\frac{m}{n}}$. Then (3.10) together with (3.66) imply that $T^* \ge T_{cr}$. However, since $S_\alpha(0) = S_\alpha(1) = 1$, (3.66) is in fact an equality at $x = 0$ and $x = 1$ and thus $T^* = T_{cr}$. The function $u_\alpha(x,t)$ blows up exactly at the boundary points $x = 0$ and $x = 1$ as $t \to T_{cr} = T^*$; in blowing up it satisfies (3.66) and appears like two shear bands located at the boundaries $x = 0$ and $x = 1$. Moreover, as $t \to T^*$, the function $\sigma_\alpha(x,t)$ obeys the bound (3.63), while

$$\partial_t u_\alpha(x,t) = \partial_x v_\alpha(x,t) \le S_\alpha^{\frac{1}{n}}(x) \Big[u_\alpha^{1-\frac{m}{n}} - \left|1 - \frac{m}{n}\right| S_\alpha^{\frac{1}{n}}(x) t \Big]^{-\frac{m}{m-n}} \tag{3.67}$$

with an equality at $x = 0$ and $x = 1$.

Note that in the above cases any $0 < \alpha < 4$ can be chosen. Also, by choosing other types of functions $S(x)$ in the place of (3.60), the nonuniformities that develop near the boundary can be made very strong, at the expense of restrictions on the initial data (cf. (3.58)).

(b) Case $\frac{\mathbf{m}}{\mathbf{n}} = \frac{\mathbf{1}}{\mathbf{2}}$

If α small enough so that $2\alpha \leq \frac{1}{2}\left(1 - \frac{\alpha}{4}\right)^{\frac{2}{n}}$, then $S_\alpha(x)$ satisfies (3.59). Consider initial data $(S_\alpha(x), u_0(x))$, with α small and $u_0(x) > 0$ but otherwise unrestricted. Let $(\sigma_\alpha(x,t), u_\alpha(x,t))$ be the corresponding solution of $(\mathcal{P})_S$; it is defined on $[0,1] \times [0,\infty)$. On account of (3.60) and Lemma 3.4, $\sigma_\alpha(x,t)$ satisfies (3.63) and the uniform state $\sigma \equiv 1$ again loses stability, as is the situation in the case $\frac{m}{n} > \frac{1}{2}$.

4. **The Power Law and Scale Invariance**

For the particular choice $\tau(u) = \frac{1}{u^m}$, the constitutive relation (1.3) takes the form of the power law

$$\sigma = \frac{1}{u^m} v_x^n \tag{4.1}$$

where m, n are positive parameters. Under (4.1), (1.1 - 1.2) read

$$\begin{aligned} v_t &= \big(\frac{1}{u^m} v_x^n\big)_x \\ u_t &= v_x \end{aligned} \tag{4.2}$$

and, correspondingly, (3.4 - 3.5) take the form

$$\begin{aligned} \big(\sigma^{\frac{1}{n}}\big)_t &= u^{-\frac{m}{n}} \sigma_{xx} - \frac{m}{n} u^{\frac{m}{n}-1} \sigma^{\frac{2}{n}} \\ u_t &= u^{\frac{m}{n}} \sigma^{\frac{1}{n}} \,. \end{aligned} \tag{4.3}$$

The power structure of the system (4.2) induces the following scaling property: If $(v(x,t), u(x,t))$ is a solution of (4.2) on $R \times (0,\infty)$ then $(v_\lambda(x,t), u_\lambda(x,t))$, defined by

$$v_\lambda(x,t) = \lambda^{\frac{\delta}{\alpha}} v(\lambda x, \lambda^{-\frac{1}{\alpha}} t) \tag{4.4}$$

$$u_\lambda(x,t) = \lambda^{\frac{\alpha+\delta+1}{\alpha}} u(\lambda x, \lambda^{-\frac{1}{\alpha}} t)\,, \tag{4.5}$$

where $\lambda > 0$ and δ, α are any constants with

$$-m(\alpha+\delta+1) + n(\alpha+\delta) + \alpha - \delta + 1 = 0\,, \tag{4.6}$$

is also a solution of (4.2) on $R \times (0,\infty)$ ([16]). As a consequence

$$\sigma_\lambda(x,t) = \lambda^{\frac{\delta-\alpha-1}{\alpha}} \sigma(\lambda x, \lambda^{-\frac{1}{\alpha}} t)\,, \tag{4.7}$$

and $(\sigma_\lambda(x,t), u_\lambda(x,t))$ satisfies (4.3) on $R \times (0,\infty)$ for any $\lambda > 0$. Therefore the systems (4.2) and (4.3) are invariant under the group of stretching transformations $T_{\alpha,\delta}$:

$$x \to \lambda x\,,\ t \to \lambda^{-\frac{1}{\alpha}} t\,,\ v \to \lambda^{\frac{\delta}{\alpha}} v\,,\ u \to \lambda^{\frac{\alpha+\delta+1}{\alpha}} u\,,\ \sigma \to \lambda^{\frac{\delta-\alpha-1}{\alpha}} \sigma\ ;\ 0 < \lambda < \infty \tag{4.8}$$

with α, δ constrained by (4.6).

The system (4.2) admits a special class of solutions describing uniform shearing

$$\bar{v}(x,t) = x \quad , \quad \bar{u}(x,t) = t + u_0\,. \tag{4.9}$$

They correspond to initial data $\bar{v}_0(x) = x$ and $\bar{u}_0(x) = u_0$, where u_0 is an arbitrary positive constant. For the special choice $u_0 = 0$, (x,t) is a self similar solution under the transformation $T_{\alpha,\delta}$ with $\alpha = -\delta = \frac{m-1}{2}$.

Consider now the initial-boundary value problem consisting of (4.2) on $[0,1] \times \{t > 0\}$ with boundary conditions $(1.9)_V$ and initial conditions (1.5). Suppose that the initial data are smooth $v_0(x) \in C^{2+\alpha}[0,1]$, $u_0(x) \in C^{1+\alpha}[0,1]$, for some $0 < \alpha < 1$, they are compatible with the boundary data, and satisfy the sign restrictions $\sigma_0(x) > 0$, $u_0(x) > 0$, $0 \le x \le 1$. We will refer to this problem as $(\mathcal{P})_V$ (including the assumptions on the initial data).

The existence theory developed in Section 2 implies that $(\mathcal{P})_V$ admits a unique classical solution defined on a maximal interval of existence $[0,1] \times [0,T^*)$. Moreover, if $T^* < +\infty$, given any compact subset $\mathcal{K}$ of $(0,\infty) \times (0,\infty)$, $(\sigma(x,t), u(x,t))$ escapes $\mathcal{K}$ as $t \uparrow T^*$. Also, for $(x,t) \in [0,1] \times [0,T^*)$,

$$\sigma(x,t) > 0 \quad , \quad u_t(x,t) > 0 \quad , \quad u(x,t) \ge u_0(x)\,. \tag{4.10}$$

The uniform shearing solution (4.9) is a special solution of $(\mathcal{P})_V$ for initial data (x, u_0). Our objective is to study the stability of this solution. To this end we use a transformation, motivated by the scaling properties of (4.2), to obtain a system that admits invariant regions [4]. A similar idea has been independently pursued by Bertsch, Peletier and Verduyn Lunel [1] for a related system.

We prove

Theorem 4.1. *Suppose $m < n$. There exists a unique classical solution $(v(x,t), u(x,t))$ of $(\mathcal{P})_V$ defined on $[0,1] \times [0,\infty)$ such that v, v_x, v_t, v_{xx}, u, u_x and u_t are in $C^{\alpha,\alpha/2}(\bar{Q}_T)$, for any $T > 0$. Moreover, if $m < min\{n,1\}$, as $t \to \infty$,*

$$v_x(x,t) = 1 + O(t^{\beta-1}) \tag{4.11}$$

$$u(x,t) = t + O(t^{\beta}) \tag{4.12}$$

and

$$\sigma(x,t) = t^{-m}\big(1 + O(t^{\beta-1})\big)\,, \tag{4.13}$$

uniformly on $[0,1]$, with $\beta = max\{\frac{m}{n}, m\} < 1$.

Proof. Introduce the transformations

$$v(x,t) = V(x, s(t)) \tag{4.14}$$

$$u(x,t) = (t+1)U(x, s(t)) \tag{4.15}$$

$$\sigma(x,t) = (t+1)^{-m}\Sigma(x, s(t)) \tag{4.16}$$

with

$$s(t) = \ell n(t+1)\,, \tag{4.17}$$

which are motivated by the form of the uniform shearing solutions and the scaling invariance (4.8). Relations (4.14 - 4.16) and (4.1) induce

$$\Sigma = \frac{1}{U^m} V_x^n\,. \tag{4.18}$$

Moreover, since $(\sigma(x,t), u(x,t))$ satisfies (4.3) and

$$\sigma_x(0,t) = \sigma_x(1,t) = 0 \quad , \quad t > 0\,, \tag{4.19}$$

it follows that $(\Sigma(x,s), U(x,s))$ solve the system of reaction-diffusion equations

$$\begin{aligned} \Sigma_s &= n e^{(1-m)s} U^{-\frac{m}{n}} \Sigma^{1-\frac{1}{n}} \Sigma_{xx} - m \frac{\Sigma}{U^{1-\frac{m}{n}}} \left(\Sigma^{\frac{1}{n}} - U^{1-\frac{m}{n}} \right) \\ U_s &= U^{\frac{m}{n}} \left(\Sigma^{\frac{1}{n}} - U^{1-\frac{m}{n}} \right), \end{aligned} \tag{4.20}$$

subject to boundary conditions

$$\Sigma_x(0,s) = \Sigma_x(1,s) = 0 \quad , \quad s > 0 \tag{4.21}$$

and initial conditions

$$\Sigma(x,0) = \sigma_0(x) \quad , \quad U(x,0) = u_0(x) \quad , \quad 0 \le x \le 1\,. \tag{4.22}$$

If $0 < \frac{m}{n} < 1$, the theory of Chueh, Conley and Smoller [4] guarantees that (4.20) admits positively invariant rectangles of arbitrary size in the first quadrant $\{(\Sigma, U) \in R^2 : \Sigma > 0, U > 0\}$. They are centered around the line $\Sigma = U^{n-m}$ and look like in Fig. 1.

Fig. 1: Invariant regions for (4.20)

Given initial data $\sigma_0(x) > 0$, $u_0(x) > 0$, let U_-, U_+, Σ_- and Σ_+ be the defining coordinates of the smallest invariant rectangle containing $(\sigma_0(x), u_0(x))$, $0 \le x \le 1$. Then

$$\Sigma_- \le \Sigma(x,s) \le \Sigma_+ \quad , \quad U_- \le U(x,s) \le U_+\,. \tag{4.23}$$

In turn, (4.15), (4.16), (4.14) and (4.18) in conjunction with (4.23) yield

$$U_-(t+1) \le u(x,t) \le U_+(t+1)\,, \tag{4.24}$$

$$\Sigma_-(t+1)^{-m} \le \sigma(x,t) \le \Sigma_+(t+1)^{-m} \tag{4.25}$$

$$\Sigma_-^{\frac{1}{n}} U_-^{\frac{m}{n}} \le v_x(x,t) \le \Sigma_+^{\frac{1}{n}} U_+^{\frac{m}{n}} \,. \tag{4.26}$$

The first implication of (4.24 - 4.26) is: For $0 < \frac{m}{n} < 1$ the functions $(\sigma(x,t), u(x,t))$ remain in a compact subset of $(0,\infty) \times (0,\infty)$ for any finite time. Thus $T^* = +\infty$, that is solutions $(v(x,t), u(x,t))$ of $(\mathcal{P})_V$ are globally defined. In addition (4.24 - 4.26) provide preliminary information on the time evolution of solutions. They are supplemented below with parabolic-type energy estimates to establish the stated asymptotic behavior. In what follows K will stand for a generic constant that depends only on the data and the parameters m and n.

Our first goal is to estimate the L^2-norm of v_t. To this end, differentiate (1.1) with respect to t and use (4.1) and (1.2) to obtain

$$v_{tt} = \Big(\frac{1}{u^m} n v_x^{n-1} v_{xt} - \frac{m}{u^{m+1}} v_x^{n+1} \Big)_x \,. \tag{4.27}$$

We multiply (4.27) by v_t and integrate by parts over $[0,1]$, using $(1.9)_V$, to arrive at

$$\frac{1}{2}\frac{d}{dt}\int_0^1 v_t^2 dx + n\int_0^1 \frac{v_x^{n-1}}{u^m} v_{xt}^2 dx = m\int_0^1 \frac{v_x^{n+1}}{u^{m+1}} v_{xt} dx \,. \tag{4.28}$$

On account of (4.24), (4.26) and Schwarz's inequality, (4.28) yields

$$\frac{d}{dt}\int_0^1 v_t^2 dx + \frac{1}{K_1}(t+1)^{-m}\int_0^1 v_{xt}^2 dx \le K_1 (t+1)^{-m-2} \,. \tag{4.29}$$

Finally, combining (4.29) with the Poincarè inequality

$$v_t^2(x,t) \le \int_0^1 v_{xt}^2(x,t) dx \tag{4.30}$$

we arrive at the differential inequality

$$\frac{d}{dt}\int_0^1 v_t^2 dx + \frac{1}{K_1}(t+1)^{-m}\int_0^1 v_t^2 dx \le K_1 (t+1)^{-m-2} \,. \tag{4.31}$$

Integrating (4.31), we deduce

$$\begin{aligned} \int_0^1 v_t^2(x,t)dx \le \Big(\int_0^1 v_t^2(x,0)dx\Big) exp\{-\frac{1}{K_1}\int_0^t (s+1)^{-m} ds\} \\ + K_1 \int_0^t (s+1)^{-m-2} exp\{-\frac{1}{K_1}\int_s^t (\tau+1)^{-m} d\tau\} ds \,. \end{aligned} \tag{4.32}$$

In case $m < 1$, L'Hopital's rule implies

$$\lim_{t\to\infty} \frac{\int_0^t (s+1)^{-m-2} exp\{\frac{1}{K_1}\int_0^s (\tau+1)^{-m} d\tau\} ds}{(t+1)^{-2} exp\{\frac{1}{K_1}\int_0^t (\tau+1)^{-m} d\tau\}} = K_1 \,. \tag{4.33}$$

In view of (4.33), (4.31) yields for $0 < m < 1$

$$\int_0^t v_t^2(x,t)dx \le K_2(t+1)^{-2}\,. \tag{4.34}$$

By contrast, if $m > 1$, (4.32) does not provide decay for the L^2-norm of v_t. Finally, if $m = 1$ the decay rate depends on the coefficient K_1 in (4.31).

Equations $(4.3)_2$ and (1.1) readily imply

$$u^{-\frac{m}{n}}(x,t)u_x(x,t) = u_0^{-\frac{m}{n}}(x)u_{0x}(x) + \frac{1}{n}\int_0^t \sigma^{\frac{1}{n}-1}(x,\tau)v_t(x,\tau)d\tau\,. \tag{4.35}$$

Two cases are considered: (i) If $n \neq 1$, then, by virtue of (4.24), (4.25) and (4.34), (4.35) yields

$$\begin{aligned}\int_0^1 |u_x(x,t)|dx &\le K_3(t+1)^{\frac{m}{n}} + K_4(t+1)^{\frac{m}{n}}\int_0^t (\tau+1)^{-\frac{m}{n}+m}\Big(\int_0^1 v_t^2(x,\tau)dx\Big)^{1/2}d\tau \\ &\le K_5(t+1)^{\frac{m}{n}} + K_6(t+1)^m\,.\end{aligned} \tag{4.36}$$

(ii) If $n = 1$, then (4.35) reads

$$u^{-m}(x,t)u_x(x,t) = u_0^{-m}(x)u_{0x}(x) + v(x,t) - v_0(x)\,. \tag{4.37}$$

Using (4.24) together with the maximum principle for $(4.2)_1$ we arrive again at (4.36). Combining (4.36) with the identities

$$\begin{aligned} u(x,t) - \int_0^1 u(y,t)dy &= \int_0^1\int_y^x u_x(\xi,t)d\xi dy \\ \int_0^1 u(y,t)dy &= t + \int_0^1 u_0(y)dy \end{aligned} \tag{4.38}$$

we obtain (4.12).

Next, use the identity

$$nv_x^{n-1}v_{xx} = u^m v_t + \frac{m}{u}u_x v_x^n \tag{4.39}$$

in conjunction with (4.26), (4.24), (4.34) and (4.36) to deduce

$$\begin{aligned}\int_0^1 |v_{xx}(x,t)|dx &\le K_7(t+1)^m\Big(\int_0^1 v_t^2(x,t)dx\Big)^{1/2} + \frac{K_8}{t+1}\int_0^1 |u_x(x,t)|dx \\ &\le K_9(t+1)^{m-1} + K_{10}(t+1)^{\frac{m}{n}-1}\,.\end{aligned} \tag{4.40}$$

Then (4.11) follows by virtue of (4.40) and the Poincarè inequality

$$|v_x(x,t) - 1| \le \int_0^1 |v_{xx}(x,t)|dx\,. \tag{4.41}$$

Finally, to show (4.13), note that on account of (4.11), (4.12), as $t \to \infty$

$$v_x^n(x,t) = 1 + O(t^{\beta-1}) \tag{4.42}$$

$$u^{-m}(x,t) = t^{-m}\big(1 + O(t^{\beta-1})\big)\,, \tag{4.43}$$

where $\beta = max\{\frac{m}{n}, m\} < 1$. Combining (4.1) with (4.42) and (4.43) gives (4.13). ∎

To shed some light on the relevance of the constraint $m < 1$, consider the case $1 < m < n$ and observe that (4.24) implies

$$U_+^{-m}(t+1)^{-m} \le u^{-m}(x,t) \le U_-^{-m}(t+1)^{-m}\,. \tag{4.44}$$

Thus, the diffusion coefficient in $(4.2)_1$ decays like $(t+1)^{-m}$.

Consider the problem

$$v_t = a(t)(v_x^n)_x \tag{4.45}$$

on $[0,1] \times [0,\infty)$, subject to $(1.9)_V$ and $v(x,0) = v_0(x)$ with $v_{0x}(x) > 0$. The change of variables

$$V(x, s(t)) = v(x,t) \tag{4.46}$$

$$s(t) = \int_0^t a(\tau)d\tau \tag{4.47}$$

suggests that $V(x,s)$ satisfies

$$V_s = (V_x^n)_x \tag{4.48}$$

subject to the same initial and boundary conditions. For $a(t) = (t+1)^{-m}$, we have: $s_\infty := \lim\limits_{t\to\infty} s(t)$ is infinite for $m \le 1$, but finite for $m > 1$. Also,

$$\lim_{t\to\infty} v(x,t) = \lim_{s\to s_\infty} V(x,s)\,. \tag{4.49}$$

If $s_\infty = +\infty$, then $\lim\limits_{t\to\infty} v(x,t) = x$; however, if $s_\infty < +\infty$, in general, this will no longer be true.

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