CONTRACTIVE RELAXATION SYSTEMS AND THE SCALAR MULTIDIMENSIONAL CONSERVATION LAW

by

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1. Introduction

It is a classical result, Kruzhkov [16], that the Cauchy problem for the scalar multidimensional conservation law,

$$
(1.1) \qquad \begin{cases} \partial_t u + \sum_{i=1}^{n} \partial_{x_i} F_i(u) = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), & \end{cases}
$$

with $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ has a unique global weak solution $u(x,t)$ satisfying the Kruzhkov entropy conditions

$$
(1.2) \quad \partial_t |u-k| + \sum_{i=1}^{n} \partial_{x_i} \big[(F_i(u) - F_i(k)) \mathrm{sign}(u-k) \big] \le 0 \,, \qquad \text{in } \mathcal{D}', \text{ for all } k \in \mathbb{R}.
$$

Weak solutions of (1.1) are constructed via finite difference approximations, Conway and Smoller [4], or as zero-viscosity limits of parabolic regularizations, Volpert [27] and Kruzhkov [16], and the solution operator defines a contraction semigroup in $L^1(\mathbb{R}^n)$, Crandall [5].

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In this article, entropy weak solutions of (1.1) are constructed as the singular limit of associated relaxation systems. The relaxation models can be loosely interpreted as (and their selection was motivated by) discrete velocity kinetic equations. The relaxation parameter plays the role of the mean free path and the system models the macroscopic conservation law at a "mesoscopic" level. In that sense they are a discrete velocity analogue of the kinetic equations introduced by Perthame and Tadmor [24], Lions, Perthame and Tadmor [19], which also describe entropy solutions of (1.1) as the mean free path tends to zero and provide a kinetic formulation for the scalar conservation law. Furthermore, we may go back to the "microscopic" level and build a stochastic interacting particle system, which: (a) in mesoscopic scales approximates the relaxation model and (b) in the fluid-dynamic limit converges to the entropy solution of a scalar conservation law. This issue was addressed by Perthame and Pulvirenti [23] for interacting particle systems corresponding to the kinetic formulation of conservation laws in [19], and is undertaken in a companion article [14] for interacting particle systems induced by the relaxation schemes proposed here. These results as well as part of the work in the present article are announced in the note [13].

The presence of relaxation mechanisms is widespread in both the continuum mechanics as well as the kinetic theory contexts. Relaxation is known to provide a subtle "dissipative" mechanism for discontinuities against the destabilizing effect of nonlinear response [20], as well as a damping effect on oscillations [3] - at least when assisted by nonlinear response. We refer to Chen, Levermore and Liu [3] for a discussion of a general mathematical framework for studying relaxation processes, and to Jin and Xin [12] for a use of relaxation to construct nonoscillatory numerical schemes.

The objective of this work is to elucidate the "dissipative" effect of relaxation, by studying compactness properties and the associated dynamics - in the zero-relaxation limit - for a simple but revealing paradigm. We introduce a class of systems, on the physical space $\mathbb{R}^n \times [0, \infty)$, describing the dynamics in the state space $\mathbb{R}^{n+1}$ of the state vector (w, Z), with $Z = (z_1, \ldots, z_n)$. We call them contractive relaxation systems, and the main example is

$$
\begin{aligned}
&\partial_t w + \sum_{i=1}^{n} \omega_i V_i \partial_{x_i} w = \frac{1}{\varepsilon} \sum_{i=1}^{n} (h_i(w) - z_i) \\
&\partial_t z_i - V_i \partial_{x_i} z_i = \frac{1}{\varepsilon} (h_i(w) - z_i)\,, \qquad i = 1, ..., n.
\end{aligned}
\tag{1.3}
$$

In (1.3), the quantities z_i are convected with velocities $U_i = -V_i \hat{e}_i$ (V_i is a speed and $\hat{e}_i$ are the unit coordinate vectors), while the quantity w is convected with velocity $U_0 = (\omega_1 V_1, \ldots, \omega_n V_n)$ (ω_i are proportionality constants between velocity components). The functions $h_i(w)$ describing the interaction rates between w and z_i are smooth and strictly decreasing. The system (1.3) is equipped with a conservation law,

$$\partial_t(w - \sum_{i=1}^{n} z_i) + \sum_{i=1}^{n} \partial_{x_i} V_i(\omega_i w + z_i) = 0\,, \tag{1.4}$$

and may be interpreted as a discrete velocity model with a (strange) collision operator, or as a model for a chemical reaction between the quantity w and each of the components of Z.

As $\varepsilon \to 0$, it is expected that the local equilibrium, $z_i = h_i(w)$, $i = 1, \ldots, n$, is enforced and that the limiting dynamics of (1.3) is described by weak solutions of

$$\partial_t(w - \sum_{i=1}^{n} h_i(w)) + \sum_{i=1}^{n} \partial_{x_i} V_i(\omega_i w + h_i(w)) = 0\,. \tag{1.5}$$

The first task is to study the compactness properties for families of solutions to (1.3), in strong topologies - in connection to the effect of relaxation on discontinuities - as well as in weak topologies - with regard to the effect on oscillations. Accordingly, to establish convergence in the zero-relaxation limit to regular weak solutions, in the former case, and to measure-valued solutions, in the latter case, of (1.5).

There are two points of interest in the above question: First from a kinetic theory viewpoint, (1.3) may be thought as describing the dynamics of a state vector in a box of $\mathbb{R}^{n+1}$ which approaches, in the zero-relaxation limit, to a curve of local equilibria. The dynamics on the curve of local equilibria is formally described by the conservation law, and the relevant question is how the limiting process occurs. Second, the convergence process provides an example in the geometric theory of singular perturbations, in a case where lack of smoothness does not allow application of the conventional techniques of center manifold analysis.

The second task is to study whether any given multidimensional, scalar conservation law

$$\partial_t u + \sum_{i=1}^{n} \partial_{x_i} F_i(u) = 0\,, \tag{1.6}$$

can be realized as a zero-relaxation limit of solutions to (1.3). In view of the relation between (1.3) and (1.5), the question is rephrased whether (1.6) can be transformed to the form (1.5), with the functions h_i, describing the curve of local equilibria, being strictly decreasing. It turns out (Lemma 4.1) that this is possible whenever $\omega_i > 0$ and the fluxes $F_i(u)$ satisfy the conditions

$$
\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{V_i}\frac{dF_i}{du} > -1\,, \\
&\frac{1}{V_i}\frac{dF_i}{du} < \frac{\omega_i}{1+\Omega}\big(1 + \sum_{i=1}^{n} \frac{1}{V_i}\frac{dF_i}{du}\big)\,,
\end{aligned}
\tag{1.7}
$$

where $\Omega = \sum_{i=1}^{n} \omega_i$. Note that the second equation implies

$$
\sum_{i=1}^{n} \frac{1}{V_i}\frac{dF_i}{du} < \Omega
$$

and (1.7) is a multi-dimensional analogue of the subcharacteristic condition (*c.f.* Liu [20]). Lemma 4.1 gives a geometric interpretation to (1.7) in connection to our problem.

The key analytical property is that, under the hypotheses for the functions h_i:

$$
\frac{dh_i}{dw} < 0\,, \quad h_i(0) = 0\,, \quad \text{and} \quad \lim_{w\to\pm\infty} h_i(w) = \mp\infty\,,
\tag{H_1}
$$

(1.3) is an L^1-contraction. Then, in the spirit of Crandall and Tartar [6], the conservation law (1.4) induces comparison principles and associated maximum principles. The role of L^1-contraction in fluid-dynamic limit problems was pointed out by Kurtz [17] in his study of the fluid-dynamic limit for the Carleman model. Natalini [21] and Tveito and Winther [26] indicate the role of L^1-contraction in convergence theories from "nonequilibrium" relaxation models to the one-dimensional, scalar conservation law. It is shown in [21] that, if f satisfies the subcharacteristic conditions $-1 < \frac{1}{c} f'(u) < 1$, then

$$
\begin{aligned}
w_t + c w_x &= \frac{1}{\epsilon}\Big(f\big(\frac{w-z}{2c}\big) - \frac{w+z}{2}\Big) \\
z_t - c z_x &= \frac{1}{\epsilon}\Big(f\big(\frac{w-z}{2c}\big) - \frac{w+z}{2}\Big)
\end{aligned}
\tag{1.8}
$$

has the L^1-contraction property, and solutions of (1.8) converge to solutions of the scalar, one-dimensional conservation law. We refer to Friedman and Tzavaras [9] for a use of L^1-contraction on uniqueness questions for certain semilinear hyperbolic-diffusive systems in several space dimensions, and to Perthame and Tadmor [24] for the use of L^1-contraction in the fluid-dynamic limit of kinetic schemes.

We begin, in Section 2, by considering a class of relaxation systems, *c.f.* (2.1), that are equipped with a conservation law and are contractive in the L^1-norm. This class encompasses but is not limited to, (1.3). (For instance, the class includes (1.8), while (1.3) clearly does not). We show that, under structural conditions on the relaxation functions $G_i(w, z_i)$, the class generates an L^1-contraction, admits comparison principles, (uniform in ε) invariant regions and also a class of entropy inequalities. In Theorem 2.3, we state a global existence theory in L^1 and in BV and list the relevant properties. The contractive relaxation systems turn out to propagate BV-stability of initial data into BV-stability of solutions, and L^∞-stability of data into L^∞-stability of solutions. In Section 3, we discuss the compactness properties of BV-stable families, emanating from data that are "almost Maxwellian" (*c.f.* assumption (d_ε)). It is shown that the "moment"

$$u^\varepsilon = w^\varepsilon - \sum_{i=1}^{n} z_i^\varepsilon \tag{1.9}$$

converges to a weak solution of the conservation law (1.5), that satisfies the Kruzhkov entropy conditions. If also $\{u_0^\varepsilon\}_{\varepsilon>0}$ converges in L^1, the Kruzhkov uniqueness theorem [16] ensures convergence of $\{u^\varepsilon\}_{\varepsilon>0}$ along the entire family. The results are stated in Theorem 3.2. Essential to the proof is the existence of a Lyapunov function for the associated system of ordinary differential equations; this is guaranteed under Hypothesis (h_5) in Section 3.

In Sections 4 and 5, we restrict to the relaxation systems (1.3), under the sole structural hypothesis (H_1) on the functions h_i describing the curve of Maxwellian states,

$$\mathcal{M} = \{(w, h_1(w), \ldots, h_n(w)) : \ w \in \mathbb{R}\}\,.$$

This class is equipped with an additional entropy function

$$\partial_t\Big(\frac{1}{2}w^2 + \sum_{i=1}^{n} \Psi_i(z_i)\Big) + \operatorname{div}\Big(U_0\frac{1}{2}w^2 + \sum_{i=1}^{n} U_i\Psi_i(z_i)\Big) + \frac{1}{\varepsilon}\sum_{i=1}^{n}\phi_i(w, z_i) = 0\,, \tag{1.10}$$

where

$$\Psi_i(z_i) = -\int_0^{z_i} h_i^{-1}(\xi)\,d\xi\,,$$

is positive and strictly convex, while

$$\phi_i(w, z_i) = -(w - h_i^{-1}(z_i))(h_i(w) - z_i) \tag{1.11}$$

satisfies $\phi_i \geq 0$ and $\phi_i = 0$ if and only if $(w, Z) \in \mathcal{M}$. In view of the properties of ϕ_i, (1.10) suggests that even data that originate away from $\mathcal{M}$ will tend to approach $\mathcal{M}$, after an initial layer. As a first implication, this leads to an improved result on compactness of solutions to (1.3), in Section 4.a:

Theorem 1. *Let (1.7) be satisfied. For solutions to (1.3) emanating from BV-stable families of data, the family $\{u^\varepsilon\}_{\varepsilon>0}$ is compact in $L^1(\mathbb{R}^n \times [0,T])$. If also $u_0^\varepsilon \to u_0$ in L^1, then $u^\varepsilon \to u$ in L^1 and u is the entropy weak solution of the Cauchy problem (1.1).*

In Section 4.b we discuss the rate of convergence. (This part concerns again the general class (2.1)). Our approach is based on a new formulation and extension of Kruzhkov and Kuznetzov-type error estimates, due to Bouchut and Perthame [1]. In Theorem 4.5. we prove that if the initial data $(w_0^\varepsilon, Z_0^\varepsilon)$ are "almost Maxwellian", in the sense $\sum_{i=1}^n ||G_i(w_0^\varepsilon, z_{i0}^\varepsilon)||_{L^1} = O(\varepsilon)$, then for $t > 0$,

$$||u^\varepsilon(\cdot,t) - u(\cdot,t)||_{L^1} \leq ||u^\varepsilon(\cdot,0) - u(\cdot,0)||_{L^1} + O(\sqrt{\varepsilon})\,.$$

In Section 5, we study families of solutions stable in $L^1 \cap L^\infty$. The parametrized families of Young measures $\nu = \{\nu_{(x,t)}\}_{(x,t)}$ and $\mu = \{\mu_{(x,t)}\}_{(x,t)}$, describing the weak convergence properties of the sequences $(w^{\varepsilon_n}, (z_i^{\varepsilon_n})_i)$ and $(u^{\varepsilon_n}, (v_i^{\varepsilon_n})_i)$ respectively, are introduced. Note that the states $(u, (v_i)_i)$ correspond to the fluxes of the conservation law (1.4) and are obtained from the states $(w, (z_i)_i)$ via an invertible map $\mathcal{L} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. The transformation $\mathcal{L}$ maps the curve of Maxwellian states $\mathcal{M}$ to the manifold,

$$\mathcal{C} = \{(u, F_1(u), \ldots, F_n(u)) :\ u \in \mathbb{R}\}\,,$$

of states accessible by the conservation law (1.6). The uniform bounds imply that $\text{supp}\mu_{(x,t)}$ is contained in a parallelepiped in $\mathbb{R}^{n+1}$. Based on (1.10), we prove:

Theorem 2. *Let (1.7) be satisfied, and assume that the initial data of (1.3) lie in a bounded set of $L^1 \cap L^\infty$. Then $\text{supp}\mu_{(x,t)} \subset \mathcal{C}$ and $\mu_{(x,t)}$ can be viewed as a measure on the state space $\mathbb{R}$ for a.e. (x,t). The family of Young measures μ is a measure valued solution of (1.6), that is*

$$\partial_t < \mu, \rho > + \sum_{i=1}^n \partial_{x_i} < \mu, F_i(\rho) >= 0\,, \qquad \text{in } \mathcal{D}'\,. \tag{1.12}$$

In Section 6, we examine the implications of the developed theory in related models. In Section 6.a, we consider the case of one space dimension. One can then relax Hypothesis (h_5) in Theorem 3.2 and obtain a complete theory. We show how some previously known results on the fluid-dynamic limit for the Carleman model, Kurtz [17], and on strong convergence from (1.8) to the single conservation law, Natalini [21], fit in the present framework. In Section 6.b we discuss the implications on semi-discrete relaxation schemes. The class of contractive relaxation systems naturally suggests a corresponding class of total variation diminishing (TVD) schemes for approximating the single multi-dimensional conservation law. Such schemes are in the spirit of the relaxation schemes proposed by Jin and Xin [12], but since, for dimensions higher than one, the underlying relaxation mechanism is different than in [12], the resulting schemes also differ.

We list in the Appendix some pertinent properties on functions of bounded variation and refer the reader to the standard references [8, 11, 28, 27] for further information on the subject.

After the completion of this work we received a preprint by Natalini [22], where a class of L^1-contractive relaxation systems is proposed and convergence to entropy solutions of the multidimensional conservation law is established.

2. **Contractive Relaxation Systems**

The objective of this section is to develop an existence theory and study the properties of the system of $n+1$ semilinear hyperbolic equations

$$\begin{aligned} \partial_t w + U_0 \cdot \nabla w &= \frac{1}{\varepsilon} \sum_i G_i(w, z_i) \\ \partial_t z_i + U_i \cdot \nabla z_i &= \frac{1}{\varepsilon} G_i(w, z_i)\,, \qquad i = 1, ..., n, \end{aligned} \tag{2.1}$$

in the variables (w, Z) with $Z = (z_1, \ldots, z_n) \in \mathbb{R}^n$. Here $U_0, U_1, \ldots, U_n \in \mathbb{R}^n$ are convective velocity vectors, $G_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are smooth functions satisfying for $i = 1, \ldots, n$

$$(h_1) \qquad \begin{cases} G_i(\cdot, z_i) \ \text{ is strictly decreasing in } w \text{ for fixed } z_i, \\ G_i(w, \cdot) \ \text{ is strictly decreasing in } z_i \text{ for fixed } w, \end{cases}$$

and $\varepsilon > 0$ is a small relaxation parameter. A key property of (2.1) is that its solutions satisfy the conservation law

$$\partial_t (w - \sum_i z_i) + \operatorname{div}(U_0 w - \sum_i U_i z_i) = 0\,. \tag{2.2}$$

We are interested in the initial value problem (2.1) with data

$$w(x,0) = w_0(x)\,,\ \ z_1(x,0) = z_{10}(x)\,, \ldots, z_n(x,0) = z_{n0}(x)\,, \quad x \in \mathbb{R}^n\,, \tag{2.3}$$

and seek solutions $(w, Z) = (w(x,t), Z(x,t)) : \mathbb{R}^n \times \{t > 0\} \to \mathbb{R}^{n+1}$ that approach at infinity a given equilibrium state (w_∞, Z_∞) of (2.1): $G_i(w_\infty, z_{i\infty}) = 0$ for all i, where $Z_\infty = (z_{1\infty}, \ldots, z_{n\infty})$. By performing a change of dependent variables, we may assume

$$(h_2) \qquad (w_\infty, Z_\infty) = (0,0) \quad \text{and} \quad G_i(0,0) = 0\,, \quad i = 1, \ldots, n\,.$$

In the sequel we proceed under this standing assumption.

The local existence theory for smooth solutions to the Cauchy problem (2.1)-(2.3) is classical. Given initial data satisfying

$$(a_1) \qquad \begin{cases} w_0\ ,\ z_{10}\ ,\ldots,\ z_{n0} \in C^1(\mathbb{R}^n) \\ w_0\ ,\ z_{10}\ ,\ldots,\ z_{n0} \ \text{ vanish outside the ball } B_M \text{ of radius } M, \end{cases}$$

there exists a unique classical solution (w, Z) defined on a maximal interval of existence $\mathbb{R}^n \times [0, T^*)$ and such that, if T^* is finite,

$$(2.4) \qquad \limsup_{t \to T^*} \sup_x \Big(|w(x,t)| + \sum_i |z_i(x,t)| \Big) = +\infty\,.$$

The solution $(w, Z) \in C^1(\mathbb{R}^n \times [0, T^*))$ and $w,\ Z$ vanish outside $\cup_{t\in[0,T^*)} B_{M+vt}$, where $v = \max\{|U_0|, \ldots, |U_n|\}$. (The proof of such a result is classical. It can be performed by using the characteristics to recast (2.1) into a system of parametric integral equations and applying the contraction mapping theorem to the resulting system). Additional regularity on the data implies additional smoothness for the solution. For instance, if the data $w_0,\ z_{10}\ ,\ldots,\ z_{n0} \in C^\infty(\mathbb{R}^n)$ and the functions G_i are infinitely differentiable then the solution is in $C^\infty_{x,t}$. The global in time existence of solutions to (2.1)-(2.3) given (h_1), is guaranteed by Lemma 2.2 below.

2.a L^1 contraction property. Our next objective is to establish that (2.1) is an L^1 contraction and satisfies certain comparison principles.

Let (w, Z) and $(\bar w, \bar Z)$ be two classical solutions that vanish outside the cone $\cup_t B_{M+vt}$. We first write the equation that the differences $(w - \bar w)$, $(z_i - \bar z_i)_i$ satisfy and multiply with the quantities $\operatorname{sign}(w - \bar w)$, $\operatorname{sign}(z_i - \bar z_i)$, respectively, to obtain

$$(2.5) \qquad \begin{aligned} \partial_t |w - \bar w| + U_0 \cdot \nabla |w - \bar w| &= \frac{1}{\varepsilon} \sum_i \operatorname{sign}(w - \bar w)(G_i(w, z_i) - G_i(\bar w, \bar z_i)) \\ \partial_t |z_i - \bar z_i| + U_i \cdot \nabla |z_i - \bar z_i| &= \frac{1}{\varepsilon} \operatorname{sign}(z_i - \bar z_i)(G_i(w, z_i) - G_i(\bar w, \bar z_i))\,. \end{aligned}$$

In deriving (2.5) we use relations of the type $\partial_t |w - \bar w| = \operatorname{sign}(w - \bar w)\partial_t (w - \bar w)$, which hold whenever, for instance, $(w - \bar w) \in W^{1,2}_{loc}$ (*c.f.* Gilbarg and Trudinger [10, Sec 7.4]).

Furthermore, (2.5) yields

$$\partial_t\Big(|w-\bar w| + \sum_i |z_i - \bar z_i|\Big) + \operatorname{div}\Big(U_0|w-\bar w| + \sum_i U_i|z_i-\bar z_i|\Big)$$
$$= \frac{1}{\varepsilon}\sum_i \Big(\operatorname{sign}(w-\bar w) + \operatorname{sign}(z_i-\bar z_i)\Big)\Big(G_i(w,z_i) - G_i(\bar w,\bar z_i)\Big) \le 0\,. \tag{2.6}$$

The last inequality is a consequence of the monotonicity Hypothesis (h_1) on the G_i, which implies that each of the terms

$$\big(\operatorname{sign}(w-\bar w) + \operatorname{sign}(z_i-\bar z_i)\big)\big(G_i(w,z_i) - G_i(\bar w,\bar z_i)\big) \le 0\,.$$

In turn, (2.6) yields the L^1 contraction property,

$$\int_{\mathbb{R}^n} |w(x,t)-\bar w(x,t)| + \sum_i |z_i(x,t)-\bar z_i(x,t)|\,dx$$
$$\le \int_{\mathbb{R}^n} |w(x,\tau)-\bar w(x,\tau)| + \sum_i |z_i(x,\tau)-\bar z_i(x,\tau)|\,dx\,, \quad 0\le\tau<t\,, \tag{2.7}$$

independently of ε, which plays a central role in the rest of the paper.

Remark 2.1. Using technical ideas from [9, 172-175], in conjunction with the process leading to (2.6), it is possible to extend the L^1-contraction property to the relaxation-diffusion type system

$$\begin{aligned} \partial_t w + U_0\cdot\nabla w - \nu\Delta w &= \frac{1}{\varepsilon}\sum_i G_i(w,z_i) \\ \partial_t z_i + U_i\cdot\nabla z_i - \nu\Delta z_i &= \frac{1}{\varepsilon}G_i(w,z_i)\,, \qquad i=1,\dots,n. \end{aligned} \tag{2.8}$$

If (w,Z), $(\bar w,\bar Z)$ are two solutions of (2.8) that decay sufficiently fast at infinity, then (w,Z), $(\bar w,\bar Z)$ satisfy (2.7), independently of ν and ε.

2.b Comparison principles. Crandall and Tartar [6] point out that any evolution operator which is an L^1 contraction and conservative is also order preserving. The solution operator of (2.1) is not conservative; however, a certain moment of the solution is conserved. This observation yields a comparison principle.

Let (w,Z), $(\bar w,\bar Z)$ be classical solutions that vanish outside $\cup_t B_{M+vt}$. They satisfy the conservation law

$$\partial_t\Big((w-\bar w) - \sum_i (z_i-\bar z_i)\Big) + \operatorname{div}\Big(U_0(w-\bar w) - \sum_i U_i(z_i-\bar z_i)\Big) = 0\,, \tag{2.9}$$

which implies

$$\int_{\mathbb{R}^n} (w(x,t) - \bar{w}(x,t)) - \sum_i (z_i(x,t) - \bar{z}_i(x,t))\, dx$$

$$= \int_{\mathbb{R}^n} (w_0(x) - \bar{w}_0(x)) - \sum_i (z_{i0}(x) - \bar{z}_{i0}(x))\, dx\,. \tag{2.10}$$

Recall that, given a function f, the positive f^+ and negative f^- parts of f are defined by

$$f^+ = \max\{f, 0\} = (f + |f|)/2\,, \quad f^- = -\min\{f, 0\} = (|f| - f)/2\,.$$

Then (2.7) and (2.10) lead to

$$\int_{\mathbb{R}^n} \big(w(x,t) - \bar{w}(x,t)\big)^+ + \sum_i \big(z_i(x,t) - \bar{z}_i(x,t)\big)^-\, dx$$

$$\leq \int_{\mathbb{R}^n} \big(w_0(x) - \bar{w}_0(x)\big)^+ + \sum_i \big(z_{i0}(x) - \bar{z}_{i0}(x)\big)^-\, dx\,, \quad t \geq 0\,, \tag{2.11}$$

and, in turn, (2.11) yields the following comparison lemma:

Lemma 2.1. *Let* (w, Z), $(\bar{w}, \bar{Z})$ *be solutions of* (2.1).

$$\textit{If} \begin{cases} w_0(x) \leq \bar{w}_0(x) \\ z_{i0}(x) \geq \bar{z}_{i0}(x), \quad i = 1, \ldots, n \end{cases}, \textit{ then} \begin{cases} w(x,t) \leq \bar{w}_0(x,t) \\ z_i(x,t) \geq \bar{z}_i(x,t), \quad i = 1, \ldots, n \end{cases}.$$

It is instructive to establish an alternative derivation of the comparison estimate (2.11). Let $\chi_{f>0}$ stand for the characteristic function of the set $\{(x,t) : f(x,t) > 0\}$, that is $\chi_{f>0} = 1$ if $f > 0$, and zero if $f \leq 0$. Consider the equations that the differences $(w - \bar{w})$, $(z_i - \bar{z}_i)$ satisfy and multiply the first equation by the quantity $\chi_{w-\bar{w}>0}$ and the remaining equations by the quantities $-\chi_{z_i - \bar{z}_i < 0}$. Using the formulas [10, Sec 7.4]

$$\begin{aligned} \partial_t (w - \bar{w})^+ &= \chi_{w-\bar{w}>0}\, \partial_t (w - \bar{w}) \qquad & \partial_t (z_i - \bar{z}_i)^- &= -\chi_{z_i - \bar{z}_i < 0}\, \partial_t (z_i - \bar{z}_i) \\ \partial_{x_j} (w - \bar{w})^+ &= \chi_{w-\bar{w}>0}\, \partial_{x_j} (w - \bar{w}) \qquad & \partial_{x_j} (z_i - \bar{z}_i)^- &= -\chi_{z_i - \bar{z}_i < 0}\, \partial_{x_j} (z_i - \bar{z}_i) \end{aligned}$$

and adding the resulting identities, we arrive at

$$\partial_t \Big((w - \bar{w})^+ + \sum_i (z_i - \bar{z}_i)^- \Big) + \operatorname{div} \Big(U_0 (w - \bar{w})^+ + \sum_i U_i (z_i - \bar{z}_i)^- \Big)$$

$$= \frac{1}{\varepsilon} \sum_i \Big(\chi_{w-\bar{w}>0} - \chi_{z_i - \bar{z}_i < 0} \Big) \Big(G_i(w, z_i) - G_i(\bar{w}, \bar{z}_i) \Big) \leq 0\,. \tag{2.12}$$

The last inequality in (2.12) holds because (h_1) dictates: $G_i(w,z_i) \geq G_i(\bar{w},\bar{z}_i)$ when $w \leq \bar{w}$, $z_i \leq \bar{z}_i$; $G_i(w,z_i) \leq G_i(\bar{w},\bar{z}_i)$ when $w \geq \bar{w}$, $z_i \geq \bar{z}_i$; thus each of the summands

$$I_i = \Big(\chi_{w-\bar{w}>0} - \chi_{z_i-\bar{z}_i<0}\Big)\Big(G_i(w,z_i) - G_i(\bar{w},\bar{z}_i)\Big) \leq 0\,.$$

An integration of (2.12) provides (2.11).

In the sequel, we place an additional hypothesis on the structure of equilibrium states of (2.1). Assume that, for each $i = 1,\dots,n$, there are smooth functions $h_i : \mathbb{R} \to \mathbb{R}$ such that

$$(h_3) \qquad \begin{cases} h_i \ \text{ strictly decreasing}\,, \quad h_i(0) = 0\,, \\ \lim\limits_{w\to-\infty} h_i(w) \to \infty\,, \quad \lim\limits_{w\to+\infty} h_i(w) \to -\infty\,, \\ G_i(w, h_i(w)) = 0 \ \text{ for } w \in \mathbb{R}\,. \end{cases}$$

Hypothesis (h_3) is compatible with (h_1). In fact, a strengthened version of (h_1) together with the implicit function theorem implies the existence of the functions h_i. It is the global structure of the equilibria that is the essence of Hypothesis (h_3). Due to (h_1), for each fixed w, $z_i = h_i(w)$ is the only solution of $G_i(w,z_i) = 0$. The curve $(w, h_1(w),\dots,h_n(w))$, $w \in \mathbb{R}$, is the manifold of equilibrium (or Maxwellian) states for (2.1).

Any constant state $(c, h_1(c),\dots,h_n(c))$ is an equilibrium solution for (2.1). Using such states as comparison functions in Lemma 2.1, we deduce that the flow generated by (2.1) admits rectangular invariant regions:

Lemma 2.2. *If (w_0, Z_0) satisfy, for some $a < 0 < b$, the bounds*

$$a \leq w_0(x) \leq b\,, \quad h_i(b) \leq z_{i0}(x) \leq h_i(a)\,,\ i = 1,\dots,n\,,$$

then the solution (w, Z) also satisfies

$$a \leq w(x,t) \leq b\,, \quad h_i(b) \leq z_i(x,t) \leq h_i(a)\,,\ i = 1,\dots,n\,.$$

Hence, the regions $\mathcal{R}_{a,b} := [a,b] \times \prod_{i=1}^{n}[h_i(b), h_i(a)]$ are positively invariant.

Lemma 2.2 yields global existence for solutions of (2.1): Under Hypothesis (h_3), any initial data (w_0, Z_0) satisfying (a_1) can be enclosed in a region of the form $\mathcal{R}_{a,b}$, for some $a < 0 < b$. The continuation part (2.4) of the local existence theorem then implies that the (classical) solution (w, Z) is defined on $\mathbb{R}^n \times [0,\infty)$.

It is instructive to visualize the geometric features of the flow generated by (2.1). The invariant regions $\mathcal{R}_{a,b}$ are rectangular and the curve $(w, h_1(w), \ldots, h_n(w))$, $w \in [a,b]$, is a curve of Maxwellian states connecting two antidiagonal vertices of $\mathcal{R}_{a,b}$ (the vertices corresponding to $w = a$ and $w = b$). In the next section we will see that, under further assumptions on G_i, solutions of (2.1) approach the curve of Maxwellians as $\varepsilon \to 0$ and that the limiting dynamics is described by the multidimensional conservation law.

2.c L^1 existence theory. The L^1-contraction property, in conjunction with the invariant regions, leads to an existence theory of global weak solutions in L^1. We prove:

Theorem 2.3. *Let G_i satisfy $(h_1 - h_3)$ and assume the initial data w_0, $z_{10}, \ldots$, $z_{n0} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. There exists a unique globally defined weak solution of (2.1) such that*

$$w,\ z_1, \ldots,\ z_n \in C\big([0,T]; L^1(\mathbb{R}^n)\big)\,, \quad \text{for any } T > 0.$$

Furthermore, (2.1) enjoys the properties:

(i) Any two weak solutions (w, Z) and $(\bar{w}, \bar{Z})$ satisfy the L^1 contraction property,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |w(x,t) - \bar{w}(x,t)| + \sum_i |z_i(x,t) - \bar{z}_i(x,t)|\, dx \\
&\qquad \le \int_{\mathbb{R}^n} |w_0(x) - \bar{w}_0(x)| + \sum_i |z_{i0}(x) - \bar{z}_{i0}(x)|\, dx\,,
\end{aligned} \tag{2.13}$$

and the comparison estimates (2.11) and (2.12).

(ii) Solutions (w, Z) satisfy, for $\kappa \in \mathbb{R}$, the entropy-type inequalities

$$\begin{aligned}
&\partial_t\Big[(w-\kappa)^+ + \sum_i (z_i - h_i(\kappa))^-\Big] + \operatorname{div}\Big(U_0 (w-\kappa)^+ + \sum_i U_i (z_i - h_i(\kappa))^-\Big) \le 0\,, \\
&\partial_t\Big[(\kappa-w)^+ + \sum_i (h_i(\kappa) - z_i)^-\Big] + \operatorname{div}\Big(U_0 (\kappa-w)^+ + \sum_i U_i (h_i(\kappa) - z_i)^-\Big) \le 0\,,
\end{aligned} \tag{2.14}$$

(iii) For any $a < 0 < b$ the rectangular regions $\mathcal{R}_{a,b} := [a,b] \times \prod_{i=1}^n [h_i(b), h_i(a)]$ are positively invariant: If $(w_0(x), Z_0(x)) \in \mathcal{R}_{a,b}$ for a.e. $x \in \mathbb{R}^n$ then

$$(w(x,t), Z(x,t)) \in \mathcal{R}_{a,b} \quad \text{for a.e. } x \in \mathbb{R}^n,\ t > 0\,. \tag{2.15}$$

(iv) If w_0, $z_{i0} \in BV(\mathbb{R}^n)$ then $w(\cdot,t)$, $z_i(\cdot,t) \in BV(\mathbb{R}^n)$.

Proof. The proof is based on the derived estimates and on density arguments. We refer to the Appendix for the definition and properties of $BV(\mathbb{R}^n)$, as well as for approximation results of BV functions by smooth functions that are used in the course of the proof.

In view of hypothesis (h_3) we may assume that, for some $a < 0 < b$, the initial data are confined in a rectangle $\mathcal{R}_{a,b}$,

$$(a_2) \qquad \begin{cases} w_0 \ , \ z_{10} \ , \dots , \ z_{n0} \in L^1(\mathbb{R}^n) \\ (w_0(x) \ , Z_0(x)) \in \mathcal{R}_{a,b} \text{ for a.e. } x \in \mathbb{R}^n. \end{cases}$$

Step 1: We construct a sequence of approximating data $w_0^n, (z_{i0}^n)_i \in C_c^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned} &w_0^n, \ z_{i0}^n \ \text{ vanish outside a ball } B_{M_n} \text{ of radius } M_n \\ &\big(w_0^n(x), (z_{i0}^n(x))_i\big) \in \mathcal{R}_{a,b} \\ &\|w_0^n - w_0\|_{L^1} \to 0, \ \ \|z_{i0}^n - z_{i0}\|_{L^1} \to 0 \quad \text{as } n \to \infty \end{aligned}$$

The construction follows via standard density arguments: Let $f \in L^1(\mathbb{R}^n)$ with $a \le f \le b$. Fix an integer n and choose m_n so that the function f_1, defined by

$$f_1 = \quad \begin{cases} f & x \in B_{m_n} \\ 0 & \text{outside } B_{m_n}, \end{cases}$$

satisfies $\|f - f_1\|_{L^1} < \frac{1}{n}$. Clearly, $\|f_1\|_{L^1} \le \|f\|_{L^1}$ and $a \le f_1 \le b$. If $\{\rho_k\}$ is a sequence of positive symmetric mollifiers, then the convolution $\rho_k * f_1$ satisfies $\|\rho_k * f_1 - f_1\|_{L^1} \to 0$, as $k \to \infty$, $\mathrm{supp}(\rho_k * f_1) \subset B_{m_n + \frac{1}{k}}$ and $a \le \rho_k * f_1 \le b$ on $\mathbb{R}^n$. Define $f^n = \rho_{k_n} * f_1$, where k_n is such that $\|\rho_{k_n} * f_1 - f_1\|_{L^1} < \frac{1}{n}$. The sequence $\{f^n\}$ resulting from this selection process has all the asserted properties.

Step 2: Let w^n, $(z_i^n)_i \in C_{x,t}^\infty$ be the solutions of (2.1) emanating from the data $\big(w_0^n, (z_{i0}^n)_i\big)$. They vanish outside the cones $\cup_t B_{M_n + vt}$ and satisfy the uniform estimates

$$\begin{aligned} a \le w^n(x,t) \le b \\ h_i(b) \le z_i^n(x,t) \ \le h_i(a) \end{aligned} \tag{2.16}$$

Also, (2.7) implies

$$\begin{aligned} &\sup_{t\in[0,T]} \int_{\mathbb{R}^n} |w^n(x,t) - w^m(x,t)| + \sum_i |z_i^n(x,t) - z_i^m(x,t)| \, dx \\ &\qquad \le \|w_0^n - w_0^m\|_{L^1} + \sum_i \|z_{i0}^n - z_{i0}^m\|_{L^1} \, , \end{aligned} \tag{2.17}$$

and thus $\{w^n\}$ and $\{z_i^n\}$, $i = 1, \ldots, n$, are Cauchy in $L^\infty([0,T]; \mathrm{L}^1(\mathbb{R}^n))$ for any $T > 0$. Let w and $(z_i)_i$ be the corresponding limiting functions: w, z_i are in $C([0,T]; \mathrm{L}^1(\mathbb{R}^n))$,

$$w^n \to w, \quad z_i^n \to z, \qquad \text{in } L^\infty([0,T]; \mathrm{L}^1(\mathbb{R}^n))$$

and (along a subsequence)

$$w^n \to w, \quad z_i^n \to z, \qquad \text{a.e. } (x,t) \in \mathbb{R}^n \times [0,T] \, .$$

Using (2.16), we pass to the limit in (2.1) and obtain that (w, Z), $Z = (z_i)_i$, is a global weak solution of (2.1). Property (iii) follows by passing to the limit in (2.16).

Step 3: Next, we turn to property (i): Let (w, Z) and $(\bar{w}, \bar{Z})$ be two weak solutions emanating from initial data (w_0, Z_0) and $(\bar{w}_0, \bar{Z}_0)$ respectively. By approximating the data as in Step 1, we produce sequences of smooth solutions (w^n, Z^n) and $(\bar{w}^n, \bar{Z}^n)$ as in Step 2. In particular (2.7) yields

$$\begin{aligned}\int_{\mathbb{R}^n} |w^n(x,t) - \bar{w}^m(x,t)| + \sum_i |z_i^n(x,t) - \bar{z}_i^m(x,t)| \, dx \\ \leq \int_{\mathbb{R}^n} |w_0^n(x) - \bar{w}_0^m(x)| + \sum_i |z_{i0}^n(x) - \bar{z}_{i0}^m(x)| \, dx \, .\end{aligned}$$

Passing to the limit m, $n \to \infty$ we obtain (2.13). Note that (2.13) yields uniqueness of weak solutions and continuous dependence with respect to initial data. A similar argument yields (2.11) for (w, Z) and $(\bar{w}, \bar{Z})$.

Next, write the weak form of (2.12) for the smooth solutions (w^n, Z^n) and $(\bar{w}^n, \bar{Z}^n)$ and pass to the limit $n \to \infty$, using (2.16), to deduce that (2.12) holds in the sense of distributions, between any two weak solutions (w, Z) and $(\bar{w}, \bar{Z})$. Finally, testing (2.12) between a solution (w, Z) and an equilibrium solution $(\kappa, h_1(\kappa), \ldots, h_n(\kappa))$, $\kappa \in \mathbb{R}$, leads to the entropy-type inequalities (2.14).

Step 4. BV existence theory: We turn now to existence theories for solutions enjoying additional regularity properties. Specifically, we are interested in properties of solutions emanating from initial data w_0, $(z_{i0})_i \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We refer to the appendix for a quick reference on the material on functions of bounded variation needed here.

Let (w, Z) be the corresponding solution of (2.1). Using (2.13) and the translation invariance of (2.1), we see that (w, Z) satisfies, for $h \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |w(x+h,t) - w(x,t)| + \sum_i |z_i(x+h,t) - z_i(x,t)| \, dx$$

$$\leq \int_{\mathbb{R}^n} |w_0(x+h) - w_0(x)| + \sum_i |z_{i0}(x+h) - z_{i0}(x)| \, dx \, . \tag{2.18}$$

In turn, (f) and (g) in the Appendix imply $w(\cdot,t)$, $z_i(\cdot,t) \in BV(\mathbb{R}^n)$, for $t > 0$, and

$$\int |Dw(\cdot,t)| + \sum_i \int |Dz_i(\cdot,t)| \leq \int |Dw_0| + \sum_i \int |Dz_{i0}| \, . \tag{2.19}$$

Consider next a smooth approximation (w_0^n, Z_0^n) of the initial data (w_0, Z_0) with the properties

$$\begin{aligned}
&\big(w_0^n(x), (z_{i0}^n(x))_i\big) \in \mathcal{R}_{a,b} \\
&\|w_0^n - w_0\|_{L^1} \to 0, \ \ \|z_{i0}^n - z_{i0}\|_{L^1} \to 0 \quad \text{as } n \to \infty \\
&\int |Dw_0^n| \to \int |Dw_0|, \ \ \int |Dz_{i0}^n| \to \int |Dz_{i0}| \quad \text{as } n \to \infty \, ,
\end{aligned}$$

and let $(w^n, Z^n) \in C^\infty_{x,t}$ be the corresponding solutions. Then (w^n, Z^n) satisfy the uniform bounds (2.16), and the uniform estimates

$$\int_{\mathbb{R}^n} |w^n(x,t)| + \sum_i |z_i^n(x,t)| \, dx \leq \|w_0\|_{L^1} + \sum_i \|z_{i0}\|_{L^1}$$

and, for $h \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |w^n(x+h,t) - w^n(x,t)| + \sum_i |z_i^n(x+h,t) - z_i^n(x,t)| \, dx$$

$$\leq \int_{\mathbb{R}^n} |w_0^n(x+h) - w_0^n(x)| + \sum_i |z_{i0}^n(x+h) - z_{i0}^n(x)| \, dx \, . \tag{2.20}$$

Since $w_0^n \to w$ and $z_{i0}^n \to z_{i0}$ in $L^1(\mathbb{R}^n)$ it follows

$$w^n \to w \, , \ z_i^n \to z_i \, , \quad \text{in } \ L^\infty([0,T]; L^1(\mathbb{R}^n)) \, . \tag{2.21}$$

Finally, (2.20) together with (f) and (g) in the Appendix imply

$$\int_{\mathbb{R}^n} |\nabla w^n(x,t)| + \sum_i |\nabla z_i^n(x,t)| \, dx \leq K \, , \tag{2.22}$$

where K depends only on the BV-norm of w_0 and z_{0i}. $\square$

3. Convergence to local equilibria

Next, we study the limiting behavior of families $\{(w^\varepsilon, Z^\varepsilon)\}_{\varepsilon>0}$ of solutions to (2.1)-(2.3) as the relaxation parameter $\varepsilon \to 0$. We consider families emanating from initial data $(w_0^\varepsilon, Z_0^\varepsilon)$, that satisfy the uniform bounds

$$(a_\varepsilon) \qquad \begin{cases} \|w_0^\varepsilon\|_{L^1} + \sum_i \|z_{i0}^\varepsilon\|_{L^1} \le C \\ \quad a \le w_0^\varepsilon \le b \\ h_i(b) \le z_{i0}^\varepsilon \le h_i(a)\,, \quad i = 1, \dots, n\,, \end{cases}$$

$$(b_\varepsilon) \qquad \int |Dw_0^\varepsilon| + \sum_i \int |Dz_{i0}^\varepsilon| \le C\,,$$

and the assumption: for any $\delta > 0$ there exists a ball B_M such that

$$(c_\varepsilon) \qquad \int_{(\mathbb{R}^n - B_M)} |w_0^\varepsilon| + \sum_i |z_{i0}^\varepsilon|\, dx \le \delta \quad \text{for all } \varepsilon > 0\,.$$

The functions G_i satisfy the hypotheses $(h_1 - h_3)$. Note that (h_1) and (h_3) together imply that the curve of equilibrium states $(w, h_1(w), \dots, h_n(w))$ is unique:

$$(h_4) \qquad G_i(w, z_i) = 0 \quad \text{implies} \quad z_i = h_i(w)\,.$$

(h_1) dictates $\frac{\partial G_i}{\partial w} \le 0$ and $\frac{\partial G_i}{\partial z_i} \le 0$. We place the additional hypothesis on G_i: Given $\mathcal{R}_{a,b}$ there exists a constant $c = c(a,b) > 0$ such that

$$(h_5) \qquad \sum_i \Big(-\frac{\partial G_i}{\partial z_i} - \sum_j \Big| \frac{\partial G_j}{\partial w} \Big| \Big) |G_i| > c \sum_i |G_i| \quad \text{for states } (w, (z_i)_i) \in \mathcal{R}_{a,b}$$

This hypothesis is discussed later.

Solutions of (2.1) satisfy a conservation law, (2.2), for the moment $w^\varepsilon - \sum_i z_i^\varepsilon$. We establish in this section that, under hypotheses $(h_1 - h_5)$, the limiting dynamics of solutions to (2.1), in the relaxation limit $\varepsilon \to 0$, is described by weak solutions of the conservation law

$$(3.1) \qquad \partial_t (w - \sum_i h_i(w)) + \operatorname{div}(U_0 w - \sum_i U_i h_i(w)) = 0\,.$$

Note that (3.1) emerges from (2.2) when relaxing z_i to their Maxwellian values $h_i(w)$.

Let $(w^\varepsilon, Z^\varepsilon)$ be solutions of (2.1) emanating from data subject to the bounds $(a_\varepsilon - c_\varepsilon)$. Theorem 2.3 and (2.19) imply that $(w^\varepsilon, Z^\varepsilon)$ satisfy the uniform bounds

$$\begin{cases} a \le w^\varepsilon(x,t) \le b \\ h_i(b) \le z_i^\varepsilon(x,t) \le h_i(a) \end{cases} \tag{3.2}$$

$$\int_{\mathbb{R}^n} |w^\varepsilon(x,t)| + \sum_i |z_i^\varepsilon(x,t)|\, dx \le \int_{\mathbb{R}^n} |w_0^\varepsilon| + \sum_i |z_{i0}^\varepsilon|\, dx \le C\,, \tag{3.3}$$

$$\int |Dw^\varepsilon(\cdot,t)| + \sum_i \int |Dz_i^\varepsilon(\cdot,t)| \le \int |Dw_0^\varepsilon| + \sum_i \int |Dz_{i0}^\varepsilon| \le C\,. \tag{3.4}$$

A key role for the convergence is played by the following lemma.

Lemma 3.1. *Under hypotheses $(h_1 - h_5)$, solutions $(w^\varepsilon, Z^\varepsilon)$ to (2.1) for $\varepsilon > 0$, emanating from data subject to the uniform bounds $(a_\varepsilon - b_\varepsilon)$, satisfy the (uniform in ε) estimates:*

(i) Distance from equilibrium:

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^n} \sum_i |G_i(w^\varepsilon(x,t), z_i^\varepsilon(x,t))|\, dx \le \frac{1}{\varepsilon} e^{-\frac{c}{\varepsilon}t} \int_{\mathbb{R}^n} \sum_i |G_i(w_0^\varepsilon, z_{i0}^\varepsilon)|\, dx + \frac{1}{c} K\,. \tag{3.5}$$

(ii) L^1-Lipshitz continuity in time: For $k > 0$, $i = 1, \dots, n$,

$$\frac{1}{k}\int_{\mathbb{R}^n} |w^\varepsilon(x,t+k) - w^\varepsilon(x,t)|\, dx \le \frac{1}{\varepsilon} e^{-\frac{c}{\varepsilon}t} \int_{\mathbb{R}^n} \sum_i |G_i(w_0^\varepsilon, z_{i0}^\varepsilon)|\, dx + K\,, \tag{3.6}$$

$$\frac{1}{k}\int_{\mathbb{R}^n} |z_i^\varepsilon(x,t+k) - z_i^\varepsilon(x,t)|\, dx \le \frac{1}{\varepsilon} e^{-\frac{c}{\varepsilon}t} \int_{\mathbb{R}^n} \sum_i |G_i(w_0^\varepsilon, z_{i0}^\varepsilon)|\, dx + K\,. \tag{3.7}$$

The constants K depend on $\mathcal{R}_{a,b}$ and the variation measure of the data w_0^ε, z_{i0}^ε.

Proof. The proof uses a density argument. Suppose first that $(w^\varepsilon, Z^\varepsilon)$ is a classical solution of (2.1), emanating from smooth data satisfying the uniform bounds $(a_\varepsilon - b_\varepsilon)$.

In view of $(h_3 - h_4)$, $|G_i(w, z_i)|$ can serve as a measure of the distance from equilibrium. A simple computation yields the identities

$$\partial_t G_i = \frac{\partial G_i}{\partial w}\Big(- U_0 \cdot \nabla w + \frac{1}{\varepsilon} \sum_j G_j \Big) + \frac{\partial G_i}{\partial z_i}\Big(- U_i \cdot \nabla z_i + \frac{1}{\varepsilon} G_i \Big)$$

and, upon multiplying by signG_i and adding the results,

$$\partial_t \sum_i |G_i| + \frac{1}{\varepsilon}\sum_i \Big(-\frac{\partial G_i}{\partial z_i}\Big)|G_i| = \frac{1}{\varepsilon}\sum_i \frac{\partial G_i}{\partial w}\mathrm{sign}G_i \sum_j G_j$$

$$+\sum_i \mathrm{sign}G_i\Big(\frac{\partial G_i}{\partial w}(-U_0\cdot\nabla w) + \frac{\partial G_i}{\partial z_i}(-U_i\cdot\nabla z_i)\Big)\,. \tag{3.8}$$

For states $(w,(z_i)_i)$ on the box $\mathcal{R}_{a,b}$ we have $|G_i(w,z)| \le L(|w|+|z|)$ and thus the functions $G_i(w^\varepsilon(\cdot,t), z_i^\varepsilon(\cdot,t))$ are integrable on $\mathbb{R}^n$. Integrating (3.8) over $\mathbb{R}^n$ and using (3.2) and (3.4), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^n}\sum_i |G_i|\,dx + \frac{1}{\varepsilon}\int_{\mathbb{R}^n}\sum_i \Big(-\frac{\partial G_i}{\partial z_i} - \sum_j \Big|\frac{\partial G_j}{\partial w}\Big|\Big)|G_i|\,dx$$

$$\le \int_{\mathbb{R}^n}\sum_i \Big|\frac{\partial G_i}{\partial w}\Big|\,\big|U_0\cdot\nabla w^\varepsilon\big| + \Big|\frac{\partial G_i}{\partial z_i}\Big|\,\big|U_i\cdot\nabla z_i^\varepsilon\big|\,dx$$

$$\le C\int_{\mathbb{R}^n} |Dw_0^\varepsilon| + \sum_i |Dz_{i0}^\varepsilon| \le K\,. \tag{3.9}$$

In view of Hypothesis (h_5), (3.9) yields the differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}^n}\sum_i |G_i|\,dx + \frac{1}{\varepsilon}c\int_{\mathbb{R}^n}\sum_i |G_i|\,dx \le K\,. \tag{3.10}$$

After integration, (3.10) gives

$$\int_{\mathbb{R}^n}\sum_i |G_i(w^\varepsilon(x,t), z_i^\varepsilon(x,t)|\,dx$$

$$\le e^{-\frac{c}{\varepsilon}t}\int_{\mathbb{R}^n}\sum_i |G_i(w_0^\varepsilon, z_{i0}^\varepsilon)|\,dx + \frac{\varepsilon}{c}K(1-e^{-\frac{c}{\varepsilon}t})\,, \tag{3.11}$$

and (3.5) follows.

We now turn to the proof of (3.6). For smooth solutions we have

$$|w^\varepsilon(x,t+k) - w^\varepsilon(x,t)| \le \int_t^{t+k} \frac{1}{\varepsilon}\sum_i |G_i(w^\varepsilon, z_i^\varepsilon)| + |U_0\cdot\nabla w^\varepsilon|\,ds \tag{3.12}$$

An integration of (3.12) over $\mathbb{R}^n$, in conjunction with (3.4) and (3.11), gives (3.6),

$$\int_{\mathbb{R}^n} |w^\varepsilon(x,t+k) - w^\varepsilon(x,t)|\,dx \le k\Big(\frac{1}{\varepsilon}e^{-\frac{c}{\varepsilon}t}\int_{\mathbb{R}^n}\sum_i |G_i(w_0^\varepsilon, z_{i0}^\varepsilon)|\,dx + K\Big)\,. \tag{3.13}$$

Similar arguments provide (3.7).

Suppose next that $(w^\varepsilon, Z^\varepsilon)$ is a family of weak solutions emanating from BV initial data. They can be approximated by smooth solutions $(w^{n,\varepsilon}, Z^{n,\varepsilon})$ as in Step 4 of the proof to Theorem 2.3. One can write (3.11) for $(w^{n,\varepsilon}, Z^{n,\varepsilon})$ and pass to the limit $n\to\infty$, using (2.16), (2.21) and (2.22), in order to deduce (3.11) for $(w^\varepsilon, Z^\varepsilon)$. In a similar fashion, passing to the limit $n\to\infty$ in (3.13) for $(w^{n,\varepsilon}, Z^{n,\varepsilon})$, leads to (3.6)-(3.7) for $(w^\varepsilon, Z^\varepsilon)$. $\square$

Remark 3.1. The function $V(w,Z)=\sum_i |G_i(w,z_i)|$ vanishes along the curve of equilibrium states but is strictly positive elsewhere. Hypothesis (h_5) guarantees that V is a Lyapunov function for the system of ordinary differential equations

$$\partial_t w = \frac{1}{\varepsilon}\sum_i G_i(w,z_i)\,, \quad \partial_t z_i = \frac{1}{\varepsilon} G_i(w,z_i)\,, \quad i=1,\dots,n\,, \tag{3.14}$$

on the box $\mathcal{R}_{a,b}$. A review of the proof indicates that any positive definite (off equilibria) Lyapunov function for (3.14) would produce an integral estimate of the type (3.5) for solutions of (2.1).

Next we turn to the main result of compactness and convergence for families of solutions. We use the notation $u^\varepsilon = w^\varepsilon - \sum_i z_i^\varepsilon$ and assume that u_0^ε converges to some function $u_0 \in L^1(\mathbb{R}^n)$. We prove:

Theorem 3.2. *Let $(w_0^\varepsilon, Z_0^\varepsilon)$ satisfy $(a_\varepsilon - c_\varepsilon)$ and G_i satisfy $(h_1 - h_5)$. If*

$$(d_\varepsilon) \qquad \|G_i(w_0^\varepsilon, z_{i0}^\varepsilon)\|_{L^1} = O(\varepsilon)\,, i=1,\dots,n\,, \quad \|u_0^\varepsilon - u_0\|_{L^1} = o(1)\,,$$

then $u^\varepsilon = w^\varepsilon - \sum_i z_i^\varepsilon \to u$ in $L^1(\mathbb{R}^n \times [0,T])$, where $u = w - \sum_i h_i(w)$ and u is the entropy weak solution of the Cauchy problem (3.1) with initial data $u(x,0) = u_0(x)$.

Proof. The family of solutions $\{(w^\varepsilon, Z^\varepsilon)\}$ satisfies the uniform estimates (3.2), (3.3) and (3.4). Also, (2.18) and (b_ε) imply

$$\int_{\mathbb{R}^n} |w^\varepsilon(x+h,t) - w^\varepsilon(x,t)| + \sum_i |z_i^\varepsilon(x+h,t) - z_i^\varepsilon(x,t)|\,dx \le C\,|h|\,. \tag{3.15}$$

Consider now the solution $(\hat w^\varepsilon, \hat Z^\varepsilon)$ emanating from initial data $(\hat w_0^\varepsilon, \hat Z_0^\varepsilon)$ such that

$$\hat w_0^\varepsilon = \begin{cases} w_0^\varepsilon & x \in B_M \\ 0 & x \notin B_M \end{cases}, \quad \hat z_{i0}^\varepsilon = \begin{cases} z_{i0}^\varepsilon & x \in B_M \\ 0 & x \notin B_M \end{cases}.$$

Clearly, $(\hat w^\varepsilon, \hat Z^\varepsilon)$ vanish outside the cone $\cup_t B_{M+vt}$. Then (c_ε), together with a comparison of $(w^\varepsilon, Z^\varepsilon)$ with $(\hat w^\varepsilon, \hat Z^\varepsilon)$ via (2.13), shows that given any $\delta > 0$ there is a ball B_M such that

$$\int_{(\mathbb{R}^n - B_{M+vt})} |w^\varepsilon(x,t)| + \sum_i |z_i^\varepsilon(x,t)|\,dx \le \int_{(\mathbb{R}^n - B_M)} |w_0^\varepsilon| + \sum_i |z_{i0}^\varepsilon|\,dx \le \delta\,, \tag{3.16}$$

for all $\varepsilon > 0$. Fix $T > 0$. Using a version of the M. Riesz-Fréchet-Kolmogorov compactness theorem [2, Corollary IV.26]), in conjunction with a diagonal argument, we deduce that there exists a subsequence ε_n such that, for any rational $\tau \in [0,T]$,

$$\begin{aligned} w^{\varepsilon_n}(\cdot,\tau) &\to w(\cdot,\tau) \quad \text{in } L^1(\mathbb{R}^n)\,, \\ z_i^{\varepsilon_n}(\cdot,\tau) &\to z_i(\cdot,\tau) \quad \text{in } L^1(\mathbb{R}^n)\,. \end{aligned} \tag{3.17}$$

We assert that $\{w^{\varepsilon_n}\}$, $\{z_i^{\varepsilon_n}\}$ are Cauchy in $L^1(\mathbb{R}^n \times [0,T])$. Indeed, fix $\eta > 0$ and, for $t \in [0,T]$, choose a rational τ so that $|t-\tau| < \eta$. Then (3.6) and (d_ε) give

$$
\begin{aligned}
\int_{\mathbb{R}^n} |w^{\varepsilon_n}(x,t) - w^{\varepsilon_m}(x,t)|\,dx \le & \int_{\mathbb{R}^n} |w^{\varepsilon_n}(x,t) - w^{\varepsilon_n}(x,\tau)|\,dx \\
& + \int_{\mathbb{R}^n} |w^{\varepsilon_n}(x,\tau) - w^{\varepsilon_m}(x,\tau)|\,dx + \int_{\mathbb{R}^n} |w^{\varepsilon_m}(x,\tau) - w^{\varepsilon_m}(x,t)|\,dx \\
& \le C\eta + \|w^{\varepsilon_n}(\cdot,\tau) - w^{\varepsilon_m}(\cdot,\tau)\|_{L^1}\,,
\end{aligned}
\tag{3.18}
$$

and (3.17) implies $\{w^{\varepsilon_n}\}$ is Cauchy in $L^1(\mathbb{R}^n \times [0,T])$. The assertion for the sequence $\{z_i^{\varepsilon_n}\}$ follows from (3.7) and (3.17).

In summary we have, along a further subsequence,

$$
\begin{aligned}
& w^{\varepsilon_n} \to w\,, \quad z_i^{\varepsilon_n} \to z_i\,, && \text{in } L^1(\mathbb{R}^n \times [0,T])\,, \\
& u^{\varepsilon_n} = w^{\varepsilon_n} - \sum_i z_i^{\varepsilon_n} \to u = w - \sum_i z_i\,, && \text{in } L^1(\mathbb{R}^n \times [0,T])\,, \\
& w^{\varepsilon_n} \to w\,, \quad z_i^{\varepsilon_n} \to z_i\,, \quad u^{\varepsilon_n} \to u\,, && \text{a.e. } (x,t) \in \mathbb{R}^n \times [0,T]\,,
\end{aligned}
\tag{3.19}
$$

and, passing to the limit in (3.5),

$$
\int_0^T \int_{\mathbb{R}^n} \sum_i |G_i(w(x,t), z_i(x,t))|\,dx = 0\,. \tag{3.20}
$$

By (h_4), $z_i = h_i(w)$ for $i = 1, \dots, n$, and

$$
u = w - \sum_i h_i(w) \qquad \text{a.e. in } \mathbb{R}^n \times [0,T]\,. \tag{3.21}
$$

Since h_i are strictly decreasing, (3.21) may be inverted to provide w as a function of u. Let $w = r(u)$ and define

$$
\begin{aligned}
H(w) &= U_0 w - \sum_i U_i h_i(w)\,, \\
F(u) &= (H \circ r)(u) = U_0 r(u) - \sum_i U_i h_i\big(r(u)\big)
\end{aligned}
\tag{3.22}
$$

Passing to the limit $\varepsilon_n \to 0$ in (2.2), in $\mathcal{D}'$, we conclude that w is a weak solution of (3.1) and, accordingly, u is a weak solution of

$$
\partial_t u + \sum_i \partial_{x_i} F_i(u) = 0\,, \quad u(\cdot,0) = u_0\,, \tag{3.23}
$$

where $F(u) = (F_1(u), \dots, F_n(u))$ is defined in (3.22).

Next, we show that u satisfies the Kruzhkov entropy conditions. Fix a state $k \in \mathbb{R}$ and let $\kappa = r(k)$, so that $k = \kappa - \sum_i h_i(\kappa)$. The functions $(w^\varepsilon, Z^\varepsilon)$ satisfy the entropy-type inequalities (2.14). We use (3.2), (3.19) and (3.21) to pass to the limit $\varepsilon_n \to 0$ in (2.14) and obtain

$$
\begin{aligned}
&\partial_t \Big[(w-\kappa)^+ + \sum_i (h_i(w) - h_i(\kappa))^- \Big] \\
&\qquad\qquad + \operatorname{div} \Big(U_0 (w-\kappa)^+ + \sum_i U_i (h_i(w) - h_i(\kappa))^- \Big) \le 0\,, \\
&\partial_t \Big[(\kappa - w)^+ + \sum_i (h_i(\kappa) - h_i(w))^- \Big] \\
&\qquad\qquad + \operatorname{div} \Big(U_0 (\kappa - w)^+ + \sum_i U_i (h_i(\kappa) - h_i(w))^- \Big) \le 0\,.
\end{aligned}
\tag{3.24}
$$

Observe that the Maxwellian states $(w, h_1(w), \dots, h_n(w))$ are ordered by w, and that $w > \kappa$ if and only if $u > k$ and $w < \kappa$ if and only if $u < k$. The monotonicity of the functions h_i, in conjunction with (3.21) and (3.22) and the fact that $F_i(k) = U_0 \kappa - \sum_i U_i h_i(\kappa)$, implies that (3.24) may be expressed in the form

$$
\partial_t |u - k| + \sum_i \partial_{x_i} \big(F_i(u) - F_i(k) \big) \operatorname{sign}(u-k) \le 0\,, \qquad \text{in } \mathcal{D}'\,. \tag{3.25}
$$

The entropy conditions (3.25) guarantee uniqueness of solutions for the Cauchy problem (3.23) [16]. As a consequence, there is only one possible limit point in L^1 for the family $\{u^\varepsilon\}$ and the whole family converges, $u^\varepsilon \to u$ in $L^1(\mathbb{R}^n \times [0,T])$ as $\varepsilon \to 0$. $\square$

4. Relaxation systems and the multidimensional conservation law

In Section 3 we showed that the limiting dynamics of solutions to (2.1), in the relaxation limit $\varepsilon \to 0$, is described by the conservation law (3.1) or (3.23). In this section we take up the converse question: Given an arbitrary multidimensional conservation law, is it possible to realize it as the relaxation limit of a class of semilinear hyperbolic systems?

We restrict to the class of relaxation systems

$$
\begin{aligned}
\partial_t w + \sum_i \omega_i V_i \partial_{x_i} w &= \frac{1}{\varepsilon} \sum_i (h_i(w) - z_i) \\
\partial_t z_i - V_i \partial_{x_i} z_i &= \frac{1}{\varepsilon} (h_i(w) - z_i)\,, \qquad i = 1, 2, \dots, n,
\end{aligned}
\tag{4.1}
$$

for initial data subject to the uniform bounds $(a_\varepsilon - c_\varepsilon)$. Assume that the functions $h_i \in C^1(\mathbb{R}\,;\mathbb{R})$, $i = 1, \dots, n$, satisfy

$$(H_1) \qquad \begin{cases} \frac{dh_i}{dw} < 0\,, \quad h_i(0) = 0\,, \\ \lim\limits_{w\to-\infty} h_i(w) = +\infty\,, \quad \lim\limits_{w\to+\infty} h_i(w) = -\infty\,. \end{cases}$$

At times we will require the additional hypothesis: given $\mathcal{R}_{a,b}$, there is $c = c(a,b) > 0$ such that

$$(H_2) \qquad 1 - \sum_i \left|\frac{dh_i}{dw}\right| > c\,, \qquad \text{for } w \in [a,b]\,.$$

The postulated behavior of $h_i(w)$ as $|w| \to \infty$ is only used to guarantee the existence of arbitrarily large invariant regions $\mathcal{R}_{a,b}$. Once these are established, we work within $\mathcal{R}_{a,b}$. Solutions of (4.1) satisfy the conservation law

$$\partial_t\Big(w - \sum_i z_i\Big) + \sum_i \partial_{x_i} V_i(\omega_i w + z_i) = 0\,. \tag{4.2}$$

Hypothesis (H_1) implies that the functions $G_i(w, z_i) = h_i(w) - z_i$ satisfy $(h_1 - h_4)$ in Sections 2 and 3; in turn, (H_2) implies (h_5). Theorem 3.2 is applicable and asserts that solutions to (4.1) will converge as $\varepsilon \to 0$ to the relaxed system (4.2) with $z_i = h_i(w)$.

A comparison of (4.1) with the system (2.1) studied in Sections 2 and 3 indicates two differences: in the form of the relaxation terms and in the convective velocities. The former is essential and in that sense (3.1) is a specialized case of (2.1). The latter though is inessential for the following reason. Suppose that the convective velocities $U_1, \dots, U_n$ in (2.1) are linearly independent. By performing a linear transformation of the coordinates x one can introduce new coordinate axes in the directions of the convective velocities U_i, $i = 1, \dots, n$. In the new coordinate system U_i are expressed in the form $(-V_i \hat{e}_i)$ for $i = 1, \dots, n$, where $\hat{e}_i$ are the unit coordinate vectors, while U_0 is expressed in the form $(\omega_1 V_1, \dots, \omega_n V_n)$. Hence, analyzing (4.1) has implications on general convective operators.

4.a Convergence to conservation laws. Let a scalar conservation law (1.6) be given with initial data $u_0(x)$ taking values in $[\alpha, \beta]$, and suppose the fluxes F_i are normalized so that $F_i(0) = 0$. The question arises: Is it possible to realize (1.6) as the relaxation limit of (4.1)?

Lemma 4.1. *Let $\omega_i > 0$ and suppose that V_i satisfy the conditions*

$$1 + \sum_i \frac{1}{V_i}\frac{dF_i}{du} > 0\,, \tag{4.3}$$

$$\frac{1}{\frac{1}{1+\Omega}\big(1 + \sum_i \frac{1}{V_i}\frac{dF_i}{du}\big)} \frac{1}{V_i}\frac{dF_i}{du} < \omega_i \tag{4.4}$$

where $\Omega = \sum_i \omega_i$. There is a map $r : \mathbb{R} \to \mathbb{R}$, strictly increasing and onto, defined by

$$w = \frac{1}{1+\Omega}\Big(u + \sum_i \frac{F_i(u)}{V_i}\Big)\,, \tag{4.5}$$

and functions $h_i : \mathbb{R} \to \mathbb{R}$, satisfying the conditions $\frac{dh_i}{dw} < 0$, $h_i(0) = 0$ and $\lim_{w\to\pm\infty} h_i(w) = \mp\infty$, such that

$$w - \sum_i h_i(w) = u\,, \tag{4.6}$$

$$\omega_i w + h_i(w) = \frac{F_i(u)}{V_i}\,; \tag{4.7}$$

thus (1.6) is put in the form (1.5). If also

$$\frac{1}{\frac{1}{1+\Omega}\big(1 + \sum_i \frac{1}{V_i}\frac{dF_i}{du}\big)} \sum_i \frac{1}{V_i}\frac{dF_i}{du} > -1 + \Omega + c\,, \tag{4.8}$$

then $(h_i)_i$ satisfy (H_2).

Proof. First, we address the question: Are there functions h_i with $\frac{dh_i}{dw} < 0$ so that (4.6-4.7) are satisfied. It is expedient to give a geometric interpretation to this question. Consider the linear transformation in $\mathcal{L} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, defined by

$$w - \sum_i z_i = u\,, \quad \omega_i w + z_i = \frac{v_i}{V_i}\,, \tag{4.9}$$

that maps $(w, z_1, \dots, z_n)$ to the fluxes $(u, v_1, \dots, v_n)$. If $1 + \Omega \neq 0$ then $\mathcal{L}$ is invertible and the inverse is given by the formulas

$$\begin{aligned} w &= \frac{1}{1+\Omega}\Big(u + \sum_i \frac{v_i}{V_i}\Big) \\ z_i &= \frac{v_i}{V_i} - \omega_i w = \frac{v_i}{V_i} - \frac{\omega_i}{1+\Omega}\Big(u + \sum_i \frac{v_i}{V_i}\Big)\,. \end{aligned} \tag{4.10}$$

The curve $\mathcal{C} = \{(u, F_1(u), \dots, F_n(u)) : \ u \in \mathbb{R}\}$ in the $(u, v_1, \dots, v_n)$-space describes the constitutive manifold accessible by the conservation law (1.6). It is mapped, via (4.10), to a curve γ in the $(w, z_1, \dots, z_n)$-space, parametrized by u and defined by

$$w = \frac{1}{1+\Omega}\Big(u + \sum_i \frac{F_i(u)}{V_i}\Big)\,, \tag{4.11}$$

$$z_i = \frac{F_i(u)}{V_i} - \frac{\omega_i}{1+\Omega}\Big(u + \sum_i \frac{F_i(u)}{V_i}\Big)\,. \tag{4.12}$$

We ask: when can γ be expressed as a graph $\mathcal{M} = \{(w, h_1(w), \ldots, h_n(w)) : w \in \mathbb{R}\}$, with h_i satisfying $\frac{dh_i}{dw} < 0$. A necessary and sufficient condition for γ to be expressed in the form of the graph $\mathcal{M}$ is that (4.3) and (4.4) are satisfied.

Indeed, if (4.6-4.7) hold for some h_i, with $\frac{dh_i}{dw} < 0$, the map $q : w \to u$ defined by (4.6) is invertible. The inverse map r is expressed by (4.11), r is strictly increasing, $\frac{dr}{du} > 0$, and $\frac{d}{du}(h_i \circ r) < 0$. Hence, (4.3) and (4.4) follow. Conversely, if (4.3-4.4) hold, the map $r : u \to w$ defined in (4.5) is strictly increasing and thus invertible. Let $u = q(w)$ be the inverse map, $q(0) = 0$. The functions $h_i = \frac{1}{V_i} F_i \circ q - \omega_i w$ are strictly decreasing and satisfy $h_i(0) = 0$ and (4.6-4.7).

The functions h_i will satisfy (H_2), $\sum_i \frac{dh_i}{dw} > -1 + c$, if and only if (4.8) is fulfilled. The common quantity appearing in (4.3), (4.4) and (4.8),

$$\frac{1}{1+\Omega}\left(1 + \sum_i \frac{1}{V_i}\frac{dF_i}{du}\right) = \frac{dr}{du}\,, \tag{4.13}$$

is the derivative of the map $r : u \to w$. $\square$

A given conservation law (1.6), with data u_0 taking values in the interval $[\alpha, \beta]$, is realized as the relaxation limit of (4.1) as follows. As the solution u is expected (by the maximum principle) to take values in $[\alpha, \beta]$, the fluxes may be altered outside $[\alpha, \beta]$, so that the new fluxes have linear growth for large u. Let $a = r(\alpha)$, $b = r(\beta)$. Given ω_i, with $\omega_i > 0$ and $\Omega = \sum_i \omega_i < 1$ (so that (4.8) is satisfied), $|V_i|$ are chosen sufficiently large so that (4.3), (4.4) and (4.8) are satisfied for $u \in \mathbb{R}$. There is a choice of h_i that satisfy $(H_1 - H_2)$, by Lemma 4.1, so that (1.6) is transformed to the form (1.5). Theorem 3.2 guarantees that, for data satisfying $(a_\varepsilon - d_\varepsilon)$, $u^\varepsilon = w^\varepsilon - \sum_i z_i^\varepsilon$ will converge in $L^1(\mathbb{R}^n \times [0,T])$ to u, the entropy weak solution of (1.6).

Next, we prove an improved version of the compactness-convergence theorem, that is specific to (4.1). We now retain Hypothesis (H_1), but relax (H_2): The parameters $\omega_i > 0$ and V_i are chosen so that (4.3) and (4.4) are satisfied. We assert:

Theorem 4.2. *Let (1.6) satisfy (4.3), (4.4), and let $u^\varepsilon = w^\varepsilon - \sum_i z_i^\varepsilon$. If $(w_0^\varepsilon, Z_0^\varepsilon)$ satisfy $(a_\varepsilon - c_\varepsilon)$ then the family $\{u^\varepsilon\}_{\varepsilon>0}$ is compact in $L^1(\mathbb{R}^n \times [0,T])$. If also $\|u_0^\varepsilon - u_0\|_{L^1} = o(1)$ then $u^\varepsilon \to u$ in $L^1(\mathbb{R}^n \times [0,T])$, and u is the entropy weak solution of the Cauchy problem (1.6) with initial data $u(x,0) = u_0(x)$.*

The system (4.1) is equipped with an entropy function. Let $h_i^{-1} : \mathbb{R} \to \mathbb{R}$ be the inverse function of h_i; h_i^{-1} is C^1 and strictly decreasing. Set

$$\Psi_i(z_i) = -\int_0^{z_i} h_i^{-1}(\xi)\, d\xi\,.$$

$\Psi_i(z_i)$ is positive, strictly convex and $|\Psi_i(z_i)| \leq K|z_i|$ on $\mathcal{R}_{a,b}$. Then (4.1) yields the identities

$$\partial_t \frac{1}{2} w^2 + \operatorname{div}(U_0 \frac{1}{2} w^2) = \frac{1}{\varepsilon} \sum_i w(h_i(w) - z_i)$$

$$\partial_t(-\Psi_i(z_i)) + \operatorname{div}(-U_i \Psi_i(z_i)) = \frac{1}{\varepsilon} h_i^{-1}(z_i)(h_i(w) - z_i), \quad i = 1, \ldots, n,$$

and, upon subtracting,

$$\partial_t \Big(\frac{1}{2} w^2 + \sum_i \Psi_i(z_i) \Big) + \operatorname{div} \Big(U_0 \frac{1}{2} w^2 + \sum_i U_i \Psi_i(z_i) \Big) = \frac{1}{\varepsilon} \sum_i (w - h_i^{-1}(z_i))(h_i(w) - z_i). \tag{4.14}$$

The function $\phi_i(w, z_i)$, defined by

$$\phi_i(w, z_i) = -(w - h_i^{-1}(z_i))(h_i(w) - z_i), \tag{4.15}$$

satisfies $\phi_i \geq 0$ and $\phi_i = 0$ if and only if $z_i = h_i(w)$.

Identity (4.14) leads to an estimate on the distance from equilibria.

Lemma 4.3. *Under Hypotheses (H_1) and (a_ε), weak solutions of (4.1) satisfy*

$$\int_{\mathbb{R}^n} \frac{1}{2} (w^\varepsilon)^2(x,t) + \sum_i \Psi_i(z_i^\varepsilon(x,t))\, dx + \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{R}^n} \sum_i \phi_i(w^\varepsilon, z_i^\varepsilon)\, dx d\tau \leq \int_{\mathbb{R}^n} \frac{1}{2} (w_0^\varepsilon)^2 + \sum_i \Psi_i(z_{i0}^\varepsilon)\, dx \leq C. \tag{4.16}$$

Moreover, for all $t > 0$,

$$\int_0^t \int_{\mathbb{R}^n} |h_i(w^\varepsilon) - z_i^\varepsilon|^2\, dx d\tau \leq C \varepsilon, \tag{4.17}$$

with the constant C independent of ε.

Proof. The proof of (4.16) follows from (4.14) via density arguments. We turn to (4.17). If g is a strictly increasing function defined on a rectangular box Q and satisfying $\frac{dg}{dx} \geq c > 0$ on Q, then $(g(y) - g(x))(y - x) \geq c(y - x)^2$ for $x, y \in Q$. This inequality, when applied to $g = -h_i^{-1}$, $y = h_i(w)$ and $x = z_i$, gives

$$\phi_i(w, z_i) = -(w - h_i^{-1}(z_i))(h_i(w) - z_i) \geq c(h_i(w) - z_i)^2,$$

where $\frac{1}{c}$ is an upper bound for $-\frac{dh_i}{dw}$ on the box $\mathcal{R}_{a,b}$. Then (4.17) follows from (4.16). □

Proof of Theorem 4.2. Let $\{(w^\varepsilon, Z^\varepsilon)\}_{\varepsilon>0}$ be a family of solutions to (4.1), subject to $(a_\varepsilon - c_\varepsilon)$. As in the proof of Theorem 3.2, there exist subsequences $\{w^{\varepsilon_n}\}$ and $\{z_i^{\varepsilon_n}\}$ so that (3.17) hold, and thus, for any rational $\tau \in [0,T]$,

$$u^{\varepsilon_n}(\cdot,\tau) \to u(\cdot,\tau) \quad \text{in } L^1(\mathbb{R}^n)\,. \tag{4.18}$$

Since u satisfies the conservation law (4.2), an argument as in (3.12-3.13) shows that

$$\int_{\mathbb{R}^n} |u^\varepsilon(x,t+k) - u^\varepsilon(x,t)|\,dx \le C\int_t^{t+k}\int_{\mathbb{R}^n} |\nabla w^\varepsilon| + \sum_i |\nabla z_i^\varepsilon|\,dx\,ds \le Ck \tag{4.19}$$

We deduce that $\{u^{\varepsilon_n}\}$ is Cauchy in $L^1(\mathbb{R}^n\times[0,T])$ and, along a further subsequence,

$$u^{\varepsilon_n} = w^{\varepsilon_n} - \sum_i z_i^{\varepsilon_n} \to u\,, \quad \text{in } L^1(\mathbb{R}^n\times[0,T]) \text{ and for a.e. } (x,t)\,. \tag{4.20}$$

Set $e_i^{\varepsilon_n} = z_i^{\varepsilon_n} - h_i(w^{\varepsilon_n})$. By (4.17),

$$e_i^{\varepsilon_n} \to 0\,, \quad \text{in } L^2(\mathbb{R}^n\times[0,T]) \text{ and for a.e. } (x,t)\in\mathbb{R}^n\times[0,T]\,. \tag{4.21}$$

Since

$$w^{\varepsilon_n} - \sum_i h_i(w^{\varepsilon_n}) = u^{\varepsilon_n} + \sum_i e_i^{\varepsilon_n} \tag{4.22}$$

and the function $w - \sum_i h_i(w)$ is strictly increasing, (4.20) and (4.21) imply $\{w^{\varepsilon_n}\}$ converges to some function w a.e. in $\mathbb{R}^n\times[0,T]$.

Passing to the limit in (4.22), we conclude $w - \sum_i h_i(w) = u$. Also, $z_i^{\varepsilon_n} = h_i(w^{\varepsilon_n}) + e_i^{\varepsilon_n} \to h_i(w)$ a.e. in $\mathbb{R}^n\times[0,T]$. We can now pass to the limit $\varepsilon_n \to 0$ in (4.2) to conclude that w satisfies (1.5) and thus, by Lemma 4.1, u is a weak solution of (1.6). As in Theorem 3.2, u satisfies the Kruzhkov entropy conditions. If $u_0^\varepsilon \to u_0$ in L^1 then u is the unique solution of the Cauchy problem with data u_0 and the whole family $\{u^\varepsilon\}$ converges in L^1 to u. □

4.b Rate of convergence. In this section we establish Kuznetzov–type error estimates for the approximation of the entropy solution of the multidimensional scalar conservation law by solutions $(w^\varepsilon, Z^\varepsilon)$ of the relaxation system. The analysis applies to system (2.1), under the additional assumption: Given $\mathcal{R}_{a,b}$, there is $\sigma = \sigma(a,b) > 0$ so that

$$|G_i(w,z_i)| \ge \sigma|h_i(w) - z_i|\,, \quad \text{for } i = 1,\dots,n\,,\ (w,Z)\in\mathcal{R}_{a,b}\,. \tag{h_6}$$

Note that, if $(h_1 - h_3)$ are satisfied with $\frac{\partial G_i}{\partial z_i} < 0$, then (h_6) is automatically satisfied.

Our analysis uses work by Bouchut and Perthame [1]. We first state (in a less general form) the central Lemma in [1]:

Lemma 4.4 (Bouchut and Perthame [1]). *Let $u, v \in L^\infty_{loc}([0,\infty), L^1_{loc}(\mathbb{R}^n))$ be right continuous with values in $L^1_{loc}(\mathbb{R}^n)$. Assume that u satisfies the entropy condition (1.2) and v satisfies, for all $k \in \mathbb{R}$,*

$$\partial_t |v-k| + \sum_{i=1}^n \partial_{x_i} \big[(F_i(v) - F_i(k)) \mathrm{sign}(v-k) \big] \le \partial_t J_k + \mathrm{div} H_k \,, \tag{4.23}$$

in $\mathcal{D}'$, where J_k, H_k^i are local Radon measures such that for some nonnegative k-independent Radon measures $(\alpha_H^i)_i$ and $\alpha_J \in L^\infty_{loc}([0,\infty), L^1_{loc}(\mathbb{R}^n))$, satisfy in the sense of measures

$$|J_k(x,t)| \le \alpha_J(x,t) \;\text{ and }\; |H_k^i(x,t)| \le \alpha_H^i(x,t) \;\;, i = 1,2,...n\,.$$

Then for any $T \ge 0, x_0 \in \mathbb{R}^n, R > 0, \delta > 0$ and $M = Lip(F), B_t = B(x_0, R + M(T-t) + \delta)$, we have

$$\int_{B(x_0,R)} |u(x,T) - v(x,T)| dx \le \int_{B_0} |u(x,0) - v(x,0)| dx + C(E^t + E^x + E^J + E^H)\,,$$

where C is a uniform constant and

$$0 \le E^t \le M\delta \int |Du_0|\,, \quad 0 \le E^x \le \delta \int |Du_0|\,,$$

$$E^H = \frac{1}{\delta} \sum_{i=1}^n \int_{0<t\le T} \int_{B_t} \alpha_H^i(x,t) dx dt\,,$$

$$E^J = (1 + \frac{(M+1)T}{\delta}) \sup_{(0,2T)} \int_{B_t} \alpha_J(x,t) dx\,.$$

Based on that we prove:

Theorem 4.5. *Let G_i satisfy $(h_1 - h_6)$. Let $(w^\varepsilon, Z^\varepsilon)$ be solutions of (2.1), emanating from BV-stable data, let u be the entropy solution of (3.23) and $u^\varepsilon = w^\varepsilon - \sum_i z_i^\varepsilon$. Then, for any fixed $T > 0$ and all $t \le T$,*

$$||u^\varepsilon(\cdot,t) - u(\cdot,t)||_{L^1} \le ||u^\varepsilon(\cdot,0) - u(\cdot,0)||_{L^1} + C_1\Big(I^\varepsilon + \sqrt{T I^\varepsilon \int |Du_0|} \Big)\,, \tag{4.24}$$

where $I^\varepsilon \ge 0$ satisfies

$$I^\varepsilon \le \max_{[0,T]} \Big\{ e^{-\frac{c}{\varepsilon}t} \sum_i ||G_i(w_0^\varepsilon, z_{i0}^\varepsilon)||_{L^1} + C_1 \frac{\varepsilon}{c}(1 - e^{-\frac{c}{\varepsilon}t}) \Big(\int |Dw_0^\varepsilon| + \sum_i \int |Dz_{i0}^\varepsilon| \Big) \Big\}\,.$$

Here, C_1 is a positive constant depending on the fluxes F^i, the L^∞-norm of the data $(w_0^\varepsilon, Z_0^\varepsilon)$ and $\frac{1}{\sigma}$ in (h_6), while c is given by (h_5).

If $(w_0^\varepsilon, Z_0^\varepsilon)$ are "almost Maxwellian", in the sense that $\sum_i ||G_i(w_0^\varepsilon, z_{i0}^\varepsilon)||_{L^1} = O(\varepsilon)$, then we immediately obtain the estimate

$$||u^\varepsilon(\cdot,t) - u(\cdot,t)||_{L^1} \le ||u^\varepsilon(\cdot,0) - u(\cdot,0)||_{L^1} + O(\sqrt{\varepsilon})\,.$$

Proof. Let $\hat{u}^\varepsilon = w^\varepsilon - \sum_i h_i(w^\varepsilon)$. We show that $\hat{u}^\varepsilon$ satisfies an approximate entropy condition (4.23). For any $k \in \mathbb{R}$, define $\kappa \in \mathbb{R}$ such that

$$k = \kappa - \sum_i h_i(\kappa)\,.$$

Since the functions h_i are decreasing for all i, we readily see that

$$|\hat{u}^\varepsilon - k| = |w^\varepsilon - \kappa| + \sum_i |h_i(w^\varepsilon) - h_i(\kappa)| = |w^\varepsilon - \kappa| + \sum_i |z_i^\varepsilon - h_i(\kappa)| + J^\varepsilon\,,$$

where $|J^\varepsilon(x,t)| \le \frac{1}{\sigma}\sum_i |G_i(w^\varepsilon(x,t), z_i^\varepsilon(x,t))|$. Similarly, by (3.22),

$$\begin{aligned}(F_j(\hat{u}^\varepsilon) - F_j(k))\mathrm{sign}(\hat{u}^\varepsilon - k) &= U_0^j|w^\varepsilon - \kappa| + \sum_i U_i^j|h_i(w^\varepsilon) - h_i(\kappa)| = \\ &= U_0^j|w^\varepsilon - \kappa| + \sum_i U_i^j|z_i^\varepsilon - h_i(\kappa)| + H_j^\varepsilon\,,\end{aligned}$$

where $|H_j^\varepsilon(x,t)| \le \frac{1}{\sigma}|G_j(w^\varepsilon(x,t), z_j^\varepsilon(x,t))|$. However $(w^\varepsilon, Z^\varepsilon)$ satisfy the entropy inequalities (2.14); consequently the previous strings of equalities yield,

$$\partial_t|\hat{u}^\varepsilon - k| + \sum_i \partial_{x_i}\big[(F_i(\hat{u}^\varepsilon) - F_i(k))\mathrm{sign}(\hat{u}^\varepsilon - k)\big] \le \partial_t J^\varepsilon + \sum_i \partial_{x_i} H_i^\varepsilon\,.$$

The functions $J^\varepsilon, H_i^\varepsilon$ are bounded by the L^1 function $\frac{1}{\sigma}\sum_i |G_i(w^\varepsilon(\cdot,t), z_i^\varepsilon(\cdot,t))|$. Since $u, w^\varepsilon, z_i^\varepsilon$ are all bounded, the hypothesis on F being Lipshitz, in Lemma 4.4, is satisfied. Sending $R \to \infty$, Lemma 4.4 delivers

$$\begin{aligned}||\hat{u}^\varepsilon(\cdot,T) - u(\cdot,T)||_{L^1(\mathbb{R}^n)} &\le ||\hat{u}^\varepsilon(\cdot,0) - u(\cdot,0)||_{L^1(\mathbb{R}^n)} \\ &+ C\Big(\delta\int|Du_0| + \frac{1}{\delta}\sum_i\int_0^T ||G_i(w^\varepsilon(\cdot,t), z_i^\varepsilon(\cdot,t))||_{L^1}dt \\ &+ \Big(1 + \frac{(M+1)T}{\delta}\Big)\sup_{[0,2T]}||G_i(w^\varepsilon(\cdot,t), z_i^\varepsilon(\cdot,t))||_{L^1}\Big)\end{aligned}$$

for all $\delta > 0$. Setting

$$I^\varepsilon = \sup_{[0,2T]}\sum_i ||G_i(w^\varepsilon(\cdot,t), z_i^\varepsilon(\cdot,t))||_{L^1}\,,$$

we obtain (here $C' = M + 2$)

$$||\hat{u}^\varepsilon(\cdot,T) - u(\cdot,T)||_{L^1(\mathbb{R}^n)} \le ||\hat{u}^\varepsilon(\cdot,0) - u(\cdot,0)||_{L^1(\mathbb{R}^n)} + C\Big(\delta\int|Du_0| + \frac{C'T}{\delta}I^\varepsilon + I^\varepsilon\Big)\,.$$

Picking a δ that minimizes the right hand side of the previous relation, yields for a new constant C_1,

$$||\hat{u}^\varepsilon(\cdot,T) - u(\cdot,T)||_{L^1(\mathbb{R}^n)} \le ||\hat{u}^\varepsilon(\cdot,0) - u(\cdot,0)||_{L^1(\mathbb{R}^n)} + C_1\Big(I^\varepsilon + \sqrt{TI^\varepsilon\int|Du_0|}\Big)\,.$$

The definition of $\hat{u}^\varepsilon$ implies that $|\hat{u}^\varepsilon - u^\varepsilon| = |\sum_i(h_i(w^\varepsilon) - z_i^\varepsilon)| \le \frac{1}{\sigma}\sum_i |G_i(w^\varepsilon, z_i^\varepsilon)|$. Hence, (4.24) follows with possibly different constants. The estimate on I^ε follows from Lemma 3.1. $\square$

5. Convergence to measure valued solutions

Next, we study families of solutions $\{(w^\varepsilon, Z^\varepsilon)\}_{\varepsilon>0}$ to (4.1) that are stable in $L^1 \cap \mathrm{L}^\infty$, emanating from initial data satisfying

$$(a_\varepsilon) \qquad \begin{cases} \|w_0^\varepsilon\|_{L^1} + \sum_i \|z_{i0}^\varepsilon\|_{L^1} \le C \\ \quad a \le w_0^\varepsilon \le b \\ h_i(b) \le z_{i0}^\varepsilon \le h_i(a)\,, \quad i = 1, \dots, n\,. \end{cases}$$

The Young measures associated with $(w_0^\varepsilon, Z_0^\varepsilon)$ are, in general, supported at any place of the box $\mathcal{R}_{a,b}$. We will show that the initial oscillations are killed off the curve of Maxwellian states, and the limiting dynamics, in the zero-relaxation limit, is governed by measure valued solutions of (1.6). The notion of measure valued solutions is introduced by DiPerna in [7], where the structure of measure valued solutions to the scalar multidimensional conservation law is studied. Chen, Levermore and Liu [3] study the convergence of L^∞ stable families of a class of relaxation systems, whose limit is the one-dimensional conservation law. It is proved in [3] that, under conditions of genuine nonlinearity, the initial oscillations are killed and strong convergence emerges.

Throughout the section, it is assumed that the fluxes $F_i(u)$ of (1.6) satisfy the subcharacteristic conditions (4.3)-(4.4), and the convective velocities in (4.1) are selected $U_0 = (\omega_1 V_1, \dots, \omega_n V_n)$, and $U_i = -V_i \hat{e}_i$, with $\hat{e}_i$ the unit coordinate vectors. Lemma 4.1 asserts that the transformation $w = r(u)$ in (4.5) puts the conservation law (1.6) to the form (1.5), with the functions h_i satisfying (H_1).

Theorem 2.1 asserts that the family $\{(w^\varepsilon, Z^\varepsilon)\}_{\varepsilon>0}$, emanating from data subject to (a_ε), satisfies the uniform bounds (3.2-3.3) and is thus stable in $L^1 \cap L^\infty$. There exists a subsequence $\{(w^{\varepsilon_n}, Z^{\varepsilon_n})\}$ and a parametrized family of probability measures $\{\nu_{(x,t)}\}_{(x,t)}$ with support in $\mathcal{R}_{a,b}$ such that

$$(5.1) \qquad \begin{aligned} &f(w^{\varepsilon_n}, Z^{\varepsilon_n}) \rightharpoonup \bar{f} &&\text{in } L^\infty(\mathbb{R}^n \times [0,T]) \text{ weak-}\star, \\ &\bar{f} = < \nu_{(x,t)}, f(\lambda) > &&\text{a.e. } (x,t)\ , \end{aligned}$$

for any continuous function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ (Tartar [25]).

For (x,t) fixed, $\nu_{(x,t)}$ is a measure over the state space $\mathbb{R}^{n+1}$ and $\operatorname{supp} \nu_{(x,t)} \subset \mathcal{R}_{a,b}$. We use the notation $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_n)$ and note that

$$(5.2) \qquad w^{\varepsilon_n} \rightharpoonup \bar{w} = < \nu_{(x,t)}, \lambda_0 >, \quad z_i^{\varepsilon_n} \rightharpoonup \bar{z}_i = < \nu_{(x,t)}, \lambda_i >, \quad i = 1, \dots, n\,,$$

$$(5.3) \qquad h_i(w^{\varepsilon_n}) \rightharpoonup \bar{h}_i = < \nu_{(x,t)}, h_i(\lambda_0) >\,,$$

in $L^\infty(\mathbb{R}^n \times [0,T])$ weak-$\star$.

Let $e^{\varepsilon_n} = z_i^{\varepsilon_n} - h_i(w^{\varepsilon_n})$. Then $e^{\varepsilon_n} \rightharpoonup \bar{z}_i - \bar{h}_i$ in $L^\infty(\mathbb{R}^n \times [0,T])$ weak-$\star$. Lemma 4.3 yields

$$\int_0^t \int_{\mathbb{R}^n} |e^{\varepsilon_n}|^2 \, dx d\tau \leq C \, \varepsilon_n \to 0 \, . \tag{5.4}$$

We now obtain:

Lemma 5.1. *For any $f : \mathbb{R}^{n+1} \to \mathbb{R}$ continuous*

$$< \nu_{(x,t)} \, , \, f(\lambda_0, \lambda_1, \dots, \lambda_n) > = < \nu_{(x,t)} \, , \, f(\lambda_0, h_1(\lambda_0), \dots, h_n(\lambda_n)) > ; \tag{5.5}$$

hence, $\operatorname{supp} \nu_{(x,t)} \subset \mathcal{M} = \{(\lambda_0, h_1(\lambda_0), \dots, h_n(\lambda_n)) : \lambda_0 \in [a,b]\}$ *for* a.e. $(x,t) \in \mathbb{R}^n \times [0,T]$.

Proof. By (5.5), along subsequences, $e^{\varepsilon_n} \to 0$ in L^2, and also $e^{\varepsilon_n} \rightharpoonup \bar{z}_i - \bar{h}_i$. Thus

$$< \nu_{(x,t)}, \lambda_i > = < \nu_{(x,t)}, h_i(\lambda_0) > \qquad \text{a.e. } (x,t) \in \mathbb{R}^n \times (0,\infty) \, . \tag{5.6}$$

If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a Lipshitz continuous function and φ a C^∞ compactly supported function, then (3.2) and (5.4) give

$$\Big| \int_0^t \int_{\mathbb{R}^n} \big(f(w^{\varepsilon_n}, Z^{\varepsilon_n}) - f(w^{\varepsilon_n}, h_1(w^{\varepsilon_n}), \dots, h_n(w^{\varepsilon_n})) \big) \, \varphi \, dx d\tau \Big| \leq L \int \int_{\operatorname{supp} \varphi} |e^{\varepsilon_n}| \, |\varphi| \, dx d\tau \to 0 \, .$$

Hence, for any test function φ,

$$\int_0^t \int_{\mathbb{R}^n} \Big(< \nu_{(x,t)} \, , f(\lambda_0, \lambda_1, \dots, \lambda_n) > - < \nu_{(x,t)} \, , \, f(\lambda_0, h_1(\lambda_0), \dots, h_n(\lambda_n)) > \Big) \, \varphi \, dx d\tau = 0,$$

and (5.5) holds for Lipshitz functions. Since the Lipshitz functions are dense in the continuous functions, (5.5) follows. $\square$

We turn now to establishing convergence to measure valued solutions. The weak form of (4.2) for a smooth test function φ reads

$$\int_0^t \int_{\mathbb{R}^n} \Big((w^{\varepsilon_n} - \sum_i z_i^{\varepsilon_n}) \partial_t \varphi + \sum_i V_i(\omega_i w^{\varepsilon_n} + z_i^{\varepsilon_n}) \partial_{x_i} \varphi \Big) dx d\tau + \int_{\mathbb{R}^n} u_0^{\varepsilon_n}(x) \varphi(x,0) \, dx = 0 \, .$$

If $u_0^{\varepsilon_n} \rightharpoonup \bar{u}_0$, then passing to the limit we obtain

$$\int_0^t \int_{\mathbb{R}^n} \Big(< \nu\,,\, \lambda_0 - \sum_i h_i(\lambda_0) > \partial_t \varphi + < \nu\,,\, \sum_i V_i(\omega_i \lambda_0 + h_i(\lambda_0)) > \partial_{x_i} \varphi \Big) dx d\tau$$

$$+ \int_{\mathbb{R}^n} \bar{u}_0(x) \varphi(x, 0)\, dx = 0\,. \tag{5.7}$$

Because $\nu_{(x,t)}$ is supported on the one-dimensional set $\mathcal{M}$, it may be visualized as a measure on continuous functions $f : \mathcal{M} \to \mathbb{R}$. In the terminology of DiPerna [7], the parametrized family $\nu = \{\nu_{(x,t)}\}_{(x,t)}$ is a measure valued solution for (1.5).

Consider next the family $(u^\varepsilon, (v_i^\varepsilon)_i)$. Let $\mu_{(x,t)}$ be the Young measure associated with a weakly convergent sequence $(u^{\varepsilon_n}, (v_i^{\varepsilon_n})_i)$, and $\nu_{(x,t)}$ the Young measure associated with $(w^{\varepsilon_n}, (z_i^{\varepsilon_n})_i)$. The states $(u, (v_i)_i)$ and $(w, (z_i)_i)$ are related through the linear transformation $\mathcal{L} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ via relations (4.9) and (4.10). Any continuous function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ can be expressed as

$$g(u^{\varepsilon_n}, (v_i^{\varepsilon_n})_i) = g\big(w^{\varepsilon_n} - \sum_i z_i^{\varepsilon_n}, V_1(\omega_1 w^{\varepsilon_n} + z_1^{\varepsilon_n}), \ldots, V_n(\omega_n w^{\varepsilon_n} + z_n^{\varepsilon_n})\big)$$

Therefore, μ and ν (we suppress the (x,t) dependence) are related through

$$\begin{aligned} &< \mu, g(\rho_0, \rho_1, \ldots, \rho_n) > \\ &= < \nu, g(\lambda_0 - \sum_i \lambda_i, V_1(\omega_1 \lambda_0 + \lambda_1), \ldots, V_n(\omega_n \lambda_0 + \lambda_n) > \end{aligned} \tag{5.8}$$

A priori, $\mathrm{supp}\mu \subset \mathcal{L}(\mathcal{R}_{a,b})$. However, since $\mathrm{supp}\nu \subset \mathcal{M}$, (5.8) implies $\mathrm{supp}\mu \subset \mathcal{L}(\mathcal{M}) = \mathcal{C}$, where

$$\mathcal{C} = \{(\rho_0, F_1(\rho_0), \ldots, F_n(\rho_0)) : \ \rho_0 \in [\alpha, \beta]\}$$

and $r(\alpha) = a$, $r(\beta) = b$ as in Lemma 4.1. The Young measure μ gives rise to a measure $\bar{\mu}$ acting on the one-dimensional state space $\mathcal{C}$. This is done as follows: Given a continuous function $\bar{g} : \mathcal{C} \to \mathbb{R}$ it is extended to $g : \mathbb{R}^{n+1} \to \mathbb{R}$ continuous, and we define

$$\begin{aligned} < \bar{\mu}, \bar{g} > &= < \mu, g(\rho_0, F_1(\rho_0), \ldots, F_n(\rho_0)) > \\ &= < \nu, g(\lambda_0 - \sum_i \lambda_i, V_1(\omega_1 \lambda_0 + h_1(\lambda_0)), \ldots, V_n(\omega_n \lambda_0 + h_n(\lambda_n)) >\,, \end{aligned}$$

by (5.8) and (5.5). The definition is independent of the extension, and henceforth μ and $\bar{\mu}$ are identified. In particular

$$\begin{aligned} < \mu, \rho_0 > &= < \nu, \lambda_0 - \sum_i h_i(\lambda_0) >\,, \\ < \mu, F_i(\rho_0) > &= < \nu, V_i(\omega_i \lambda_0 + h_i(\lambda_0)) > \ . \end{aligned}$$

We conclude that the parametrized family $\mu = \{\mu_{(x,t)}\}_{(x,t)}$ is a measure valued solution for (1.6). We thus have.

Theorem 5.2. *Let (1.6) satisfy (4.3), (4.4), and suppose* $(w_0^\varepsilon, Z_0^\varepsilon)$ *satisfy the uniform bounds* (a_ε)*. There exists a subsequence and a parametrized family of Young measures* $\{\mu_{(x,t)}\}_{(x,t)}$ *such that* $\operatorname{supp}\mu_{(x,t)} \subset \mathcal{C}$ *and* $\mu_{(x,t)}$ *can be viewed as a measure on the state space* $\mathbb{R}$ *for a.e.* $(x,t) \in \mathbb{R}^n \times [0,T]$*. The family* $\mu = \{\mu_{(x,t)}\}_{(x,t)}$ *satisfies*

$$\partial_t < \mu, \rho > + \sum_i \partial_{x_i} < \mu, F_i(\rho) >= 0\,, \tag{5.9}$$

in the sense of distributions.

6. Other systems under the same framework

Next, we study some other systems that fit under the developed framework. In Section 6.a we review the one space-dimensional case and list the implications on two specific models. In Section 6.b we consider certain semidiscrete relaxation schemes. Results on such schemes have implications on convergence for numerical algorithms [12], and are used as a building block in studies of convergence for interacting particle systems [14].

6.a One space-dimensional relaxation systems. Consider the one space dimensional version of the relaxation system (2.1),

$$\begin{aligned} w_t + c w_x &= \frac{1}{\varepsilon} G(w,z) \\ z_t - v z_x &= \frac{1}{\varepsilon} G(w,z)\,, \end{aligned} \tag{6.1}$$

where c and v are two convective speeds, with initial data $w(x,0) = w_0(x)$, $z(x,0) = z_0(x)$, $x \in \mathbb{R}$. (6.1) is equipped with the conservation law

$$\frac{\partial}{\partial t}(w - z) + \frac{\partial}{\partial x}(cw + vz) = 0\,. \tag{6.2}$$

We assume that $G \in C^1(\mathbb{R}^2;\mathbb{R})$ satisfies:

(a) $\quad G(\cdot, z)$ is strictly decreasing, $\quad G(w,\cdot)$ is strictly decreasing.

(b) $\quad$ There is a unique C^1 curve $(w, h(w))$ of equilibria, with h strictly decreasing.

(c) $\quad$ There is $\alpha \geq 0$ such that $-\big(G_w(w,z) + G_z(w,z)\big) \geq \alpha$.

The system (6.1) has the L^1 contraction property and admits invariant regions. There are two improvements in the theory for the one-dimensional case. It is no longer necessary to pursue a theory for solutions that approach one single equilibrium value at infinity. Instead, a cumbersome but straightforward extension shows global existence for solutions of (6.1) that approach distinct equilibrium values $(w_-, h(w_-))$ as $x \to -\infty$, and $(w_+, h(w_+))$ as $x \to +\infty$. Of course one has to place appropriate assumptions on the data, of the type (w_0, z_0) are such that $(w_0 - w_-), (z_0 - h(w_-)) \in L^1(-\infty, 0]$ and $(w_0 - w+), (z_0 - h(w_+)) \in L^1[0, +\infty)$. The second improvement is that (c) is weaker than hypothesis (h_5). If we pursue the formal calculation leading to Lemma 3.1, we obtain the analogue of (3.5),

$$\int_{\mathbb{R}} |G(w^\varepsilon(x,t), z^\varepsilon(x,t)|\, dx \le e^{-\frac{\alpha}{\varepsilon}t} \int_{\mathbb{R}} |G(w_0^\varepsilon, z_{i0}^\varepsilon)|\, dx + \frac{\varepsilon}{\alpha} K(1 - e^{-\frac{\varepsilon}{\varepsilon}t}).$$

When $\alpha = 0$ then

$$\int_{\mathbb{R}} |G(w^\varepsilon(x,t), z^\varepsilon(x,t)|\, dx \le \int_{\mathbb{R}} |G(w_0^\varepsilon, z_{i0}^\varepsilon)|\, dx + Kt\,.$$

In either case there is an estimate on the "distance from equilibria" and one can prove an analogue of Theorem 3.2: As $\varepsilon \to 0$, solutions of the relaxation system (6.1) converge to the entropy weak solution of the conservation law

$$\frac{\partial}{\partial t}(w - h(w)) + \frac{\partial}{\partial x}(cw + vh(w)) = 0\,. \tag{6.3}$$

We turn now to applications on specific models:

1. The Carleman model. The Carleman model of the kinetic theory of gases is the system of semilinear equations

$$\begin{aligned} u_t + u_x &= \frac{1}{\varepsilon}(v^2 - u^2) \\ v_t - v_x &= \frac{1}{\varepsilon}(u^2 - v^2) \end{aligned} \tag{6.4}$$

Here u is the density of particles moving to the right with speed 1 and v the density of particles moving to the left with speed 1. It enjoys the conservation law $\partial_t(u+v) + \partial_x(u-v) = 0$, corresponding to conservation of the total mass. Its formal fluid-dynamic limit $\varepsilon \to 0$ is the conservation law $\partial_t u = 0$. Kurtz [17] uses the L^1 contraction property of (6.4) to establish the limit in a diffusive scaling, that also provides the next term of the expansion and leads to a nonlinear diffusion equation.

If we set $(w, z) = (u, -v)$, then (6.4) is written in the form (6.1) with $G(w, z) = z^2 - w^2$. The domain of interest is $J = \{w \ge 0\} \times \{z \le 0\}$. The curve of equilibria in this case is $(w, -w)$ and G satisfies all of the hypotheses (a), (b) and (c) on J. Note that as an implication of the comparison principles, J is an invariant region.

2. Relaxation approximations of the single conservation law. The relaxation system

$$
\begin{aligned}
u_t + v_x &= 0 \\
v_t + c^2 u_x &= \frac{1}{\varepsilon}(f(u) - v)
\end{aligned}
\tag{6.5}
$$

has as formal relaxation limit the single one-dimensional conservation law

$$
\partial_t u + \partial_x f(u) = 0\,. \tag{6.6}
$$

Such approximations motivate a class of nonoscillatory numerical schemes, for shock wave computations in systems of conservation laws of one or several space variables (Jin and Xin [12]). The convergence of (6.5) to (6.6) is established by Natalini [21]. The Riemann invariants, $w = v + cu$, $z = v - cu$, are introduced, and the system (6.5) is recast to the form (6.1) with

$$
G(w,z) = f\Big(\frac{w-z}{2c}\Big) - \frac{w+z}{2}\,. \tag{6.7}
$$

If $f(u)$ satisfies the subcharacteristic conditions

$$
-1 < \frac{1}{c} f'(u) < 1\,, \tag{6.8}
$$

then $G_w, G_z < 0$ and (a), (c) are clear. The equation $G(w,z) = 0$ can then be solved and yields a unique curve $(w, h(w))$ of equilibria, with $h'(w) = -(G_w/G_z) < 0$, and (b) follows. In this case the constant α in Hypothesis (c) is strictly positive. An analogue of Theorem 3.2 yields convergence as $\varepsilon \to 0$ of (6.1) to (6.6) and, hence, of (6.5) to (6.6) and Theorem 4.5 provides a rate of convergence.

6.b Semi-discrete relaxation schemes. In this subsection we briefly present the stability properties of semidiscrete schemes for the relaxation system (2.1). A detailed study of semidiscrete schemes and their convergence to the entropy solution of scalar conservation laws will appear in [15]. Such schemes provide a crucial ingredient in the convergence of certain interacting particle systems with relaxation to scalar conservation laws [13], [14].

A straightforward upwind space-discretization of the system (4.1) is as follows: Let $\hat{e}_i$ be the unit coordinate vectors in $\mathbb{R}^n$ and $q \in \mathbb{Z}^n$ and γ denote the grid size.

Let $\omega_i > 0$, and consider the case that $V_1, \ldots, V_n > 0$. Then the scheme reads

$$
\begin{aligned}
&\frac{\partial}{\partial t} w(q,t) + \frac{1}{\gamma} \sum_i \omega_i V_i \Big(w(q,t) - w(q - \gamma \hat{e}_i, t) \Big) \\
&\qquad = \frac{1}{\varepsilon} \sum_i G_i \Big(w(q,t), z_i(q,t) \Big) \\
&\frac{\partial}{\partial t} z_i(q,t) - \frac{1}{\gamma} V_i \Big(z_i(q + \gamma \hat{e}_i, t) - z(q,t) \Big) \\
&\qquad = \frac{1}{\varepsilon} G_i \Big(w(q,t), z_i(q,t) \Big) , \qquad i = 1, ..., n,
\end{aligned}
\tag{6.9}
$$

where $(q,t) \in \mathbb{Z}^n \times (0, \infty)$ and γ denotes the grid size. (If some V_i were negative, then the corresponding forward differences in the scheme are replaced by backward differences and vice versa).

Let l^1 be the space of summable functions defined on $\mathbb{Z}^n$. We prove in [15] that the semi-discrete scheme (6.10) admits invariant regions and satisfies an l^1 contraction property:

$$
\begin{aligned}
&\sum_{q \in \mathbb{Z}^n} \Big(|w(q,t) - \bar{w}(q,t)| + \sum_i |z_i(q,t) - \bar{z}_i(q,t)| \Big) \\
&\qquad \le \sum_{q \in \mathbb{Z}^n} \Big(|w(q,0) - \bar{w}(q,0)| + \sum_i |z_i(q,0) - \bar{z}_i(q,0)| \Big) .
\end{aligned}
$$

In addition the scheme is TVD:

$$
\begin{aligned}
&\sum_{q \in \mathbb{Z}^n} \Big(|w(q + \hat{e}_k, t) - w(q,t)| + \sum_i |z_i(q + \hat{e}_k, t) - z_i(q,t)| \Big) \\
&\qquad \le \sum_{q \in \mathbb{Z}^n} \Big(|w(q + \hat{e}_k, 0)) - w(q,0)| + \sum_i |z_i(q + \hat{e}_k, 0) - z_i(q,0)| \Big) ,
\end{aligned}
$$

for all directions $\hat{e}_k$, $k = 1, ..., n$.

This analysis applies directly to the semidiscrete upwind relaxation scheme introduced by Jin and Xin [12] for the scalar one dimensional conservation law, after rewritting the scheme in Riemann invariants, in the spirit of the discussion in 6a.2.

Appendix. On functions of bounded variation

The space BV has long been identified as an appropriate space to study existence and regularity properties for the scalar multidimensional conservation law [4, 16, 18, 27], and the article of Volpert [27] is the standard reference on BV

theory in the field. As the amount of information from BV theory required for the purposes of existence is minimal, we list the pertinent properties in order to make the article more accessible and in order to put it in the language of recent books by Giusti [11], Ziemer [28, Ch 5], and Evans and Gariepy [8, Ch 5] that are becoming standard references in the subject of functions of bounded variation. We are specifically interested in $BV(\mathbb{R}^n)$; we refer to [8, 11, 28] for definitions of $BV(\Omega)$ with Ω an open set and for the proofs of statements in the Appendix.

A function $f \in L^1(\mathbb{R}^n)$ has bounded variation in $\mathbb{R}^n$ if

$$\int |Df| = \sup \{ \int f \mathrm{div} g \, dx \, : \, g \in C^1_c(\mathbb{R}^n;\mathbb{R}^n) \ \text{ with } \ |g(x)| \leq 1 \} < \infty \, .$$

We write $BV(\mathbb{R}^n)$ the space of functions of bounded variation and note:

(i) If $f \in W^{1,1}(\mathbb{R}^n)$ then $\int |Df| = \int_{\mathbb{R}^n} |\nabla f| dx$

(ii) If $f \in BV(\mathbb{R}^n)$ and Df is the gradient of f in the sense of distributions, then Df is a (vector valued) finite Radon measure on $\mathbb{R}^n$ and $\int |Df|$ is the total variation of Df on $\mathbb{R}^n$.

(iii) If $\{f_j\}$ is a sequence of functions in $BV(\mathbb{R}^n)$, then $f_j \to f$ in $L^1_{loc}(\mathbb{R}^n)$ implies

$$\int |Df| \leq \liminf_{j \to \infty} \int |Df_j| \, .$$

(iv) Under the norm $\|f\|_{BV} = \|f\|_{L^1} + \int |Df|$, $BV(\mathbb{R}^n)$ is a Banach space.

We are interested in approximation theorems in $BV(\mathbb{R}^n)$. Let $\rho(x)$ be a positive, symmetric kernel with $\int \rho = 1$, and consider the sequence $\{\rho_k\}$ of mollifiers generated by ρ:

$$\rho_k \in C_c^\infty(\mathbb{R}^n) \, , \quad \mathrm{supp} \rho_k \subset B_{\frac{1}{k}} \, , \quad \int \rho_k dx = 1 \, , \quad \rho_k \geq 0 \, .$$

Given $f \in L^1_{loc}(\mathbb{R}^n)$ the convolution $f_k = \rho_k * f$ satisfies the properties:

(a) $f_k \in C^\infty(\mathbb{R}^n)$, $f_k \to f$ in $L^1_{loc}(\mathbb{R}^n)$; if $f \in L^1(\mathbb{R}^n)$ then $f_k \to f$ in $L^1(\mathbb{R}^n)$.

(b) $a \leq f(x) \leq b$ implies $a \leq f_k(x) \leq b$.

(c) $\mathrm{supp} f_k \subset \mathrm{supp} f + B_{\frac{1}{k}}$.

(d) If $f \in C^1(\mathbb{R}^n)$ then $\partial_{x_i}(f_k) = (\partial_{x_i} f)_k$.

(e) If $f \in BV(\mathbb{R}^n)$ then $f_k = \rho_k * f$ satisfy

$$f_k \to f \quad \text{in } L^1(\mathbb{R}^n) \, , \quad \lim_{k \to \infty} \int |Df_k| = \int |Df| \, .$$

Property (e) yields a theorem on approximation of functions of bounded variation by smooth functions. There exists a (quite nontrivial) analogue of this result for functions in $BV(\Omega)$.

For $f \in L^1(\mathbb{R}^n)$ define the L^1-modulus of continuity

$$\lambda(h, f) = \int_{\mathbb{R}^n} |f(x+h) - f(x)|\, dx\,, \quad h \in \mathbb{R}^n\,,$$

and coordinate-wise

$$\lambda_i(\delta, f) := \lambda(\delta \hat{e}_i, f) = \int_{\mathbb{R}^n} |f(x + \delta\hat{e}_i) - f(x)|\, dx\,,$$

where $\delta \in \mathbb{R}$ and $\hat{e}_i \in \mathbb{R}^n$ is the unit vector in the x_i-direction. Regarding, the relation of the variation measure $\int |Df|$ and the L^1-modulus of continuity, we have

(f) If $\int |Df| < +\infty$ then

$$\limsup_{\delta \to 0} \sum_i \frac{\lambda_i(\delta, f)}{|\delta|} \leq \int |Df|$$

(g) If $\sup_{0<|\delta|\leq 1} \frac{\lambda_i(\delta,f)}{|\delta|} < +\infty$ for each $i = 1, \ldots, n$, then

$$\int |Df| \leq \liminf_{\delta \to 0} \sum_i \frac{\lambda_i(\delta, f)}{|\delta|}\,.$$

Therefore, $BV(\mathbb{R}^n)$ may be identified with the class of L^1 functions with a Lipschitz L^1-modulus of continuity, and we have

$$\int |Df| = \lim_{\delta \to 0} \sum_i \frac{\lambda_i(\delta, f)}{|\delta|}\,.$$

(See [18].) Using the identification we may deduce compactness theorems for families of functions bounded in the BV-norm, from a starting point of the M. Riesz-Fréchet-Kolmogorov theorem on compactness of bounded families of uniformly L^p-equicontinuous ($p \geq 1$) functions (*c.f.* Brezis [2, Ch IV.5]).

Proof of (f). Let $f \in BV(\mathbb{R}^n)$ and consider an approximating smooth sequence $\{f_k\}$ as in (a), (e). Then

$$\begin{aligned}\int_{\mathbb{R}^n} |f_k(x+\delta\hat{e}_i) - f_k(x)|\, dx &= \int_{\mathbb{R}^n} \Big| \int_0^1 \nabla f_k(x + s\delta\hat{e}_i)\cdot \delta\hat{e}_i\, ds\Big|\, dx \\ &\leq |\delta| \int_{\mathbb{R}^n} |\partial_{x_i} f_k(x)|\, dx\end{aligned}$$

and

$$\sum_i \frac{1}{|\delta|} \int_{\mathbb{R}^n} |f_k(x+\delta\hat{e}_i) - f_k(x)|\, dx \le \int_{\mathbb{R}^n} |\nabla f_k|\, dx$$

As $k \to \infty$, (a) and (e) give

$$\sum_i \frac{1}{|\delta|} \int_{\mathbb{R}^n} |f(x+\delta\hat{e}_i) - f(x)|\, dx \le \int_{\mathbb{R}^n} |Df|$$

whence (f) follows.

Proof of (g). Suppose that $f \in L^1(\mathbb{R}^n)$ with $\sup_{0<|\delta|\le 1} \frac{\lambda_i(\delta,f)}{|\delta|} < +\infty$ for all i, and fix $g \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ with $|g(x)| \le 1$. Then, for $\delta \neq 0$,

$$\frac{1}{\delta} \int_{\mathbb{R}^n} \big(f(x+\delta\hat{e}_i) - f(x)\big) g(x) dx = \frac{1}{\delta} \int_{\mathbb{R}^n} f(x)\big(g(x-\delta\hat{e}_i) - g(x)\big)\, dx$$

and, letting $\delta \to 0$,

$$\int_{\mathbb{R}^n} f\, \partial_{x_i} g\, dx = \lim_{\delta\to 0} \frac{1}{\delta} \int_{\mathbb{R}^n} \big(f(x) - f(x+\delta\hat{e}_i)\big) g(x) dx \le \liminf_{\delta\to 0} \frac{\lambda_i(\delta,f)}{|\delta|}$$

We conclude

$$\int_{\mathbb{R}^n} f \operatorname{div} g\, dx \le \liminf_{\delta\to 0} \sum_i \frac{\lambda_i(\delta,f)}{|\delta|},$$

whence (g) follows.

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