Shear Strain Localization in Plastic Deformations

Athanasios E. Tzavaras *

Abstract. Shear instabilities in the form of shear bands are often observed during high speed, plastic deformations of metals. According to one theory, their formation is attributed to effective strain-softening response, which results, at high strain rates, as the net outcome of the influence of thermal softening on the, normally, strain-hardening response of metals. In order to test the core instability mechanism set forth by such theories, we consider one-dimensional shear deformations of a material exhibiting strain softening and strain-rate sensitivity. The deformation is caused by either prescribed tractions or prescribed velocities. As it turns out, for moderate amounts of strain softening, strain-rate sensitivity exerts a dissipative effect and stabilizes the motion. However, once a threshold is exceeded, the response becomes unstable and shear strain localization can occur.

1. Introduction. In the framework of continuum mechanics, the motion of deforming bodies is described by systems of nonlinear evolution equations, that arise by combining the balance equations with, characteristic of the material, constitutive relations. The nonlinear character of the material response often induces a destabilizing mechanism that competes with dissipation. For instance, in a well studied situation, an amplification mechanism of compressible weak waves competes with dissipation induced by viscosity or thermal diffusion. Various models of such nature appear in the context of thermoviscoelasticity, leading to quasilinear systems of hyperbolic-parabolic partial differential equations. Roughly speaking, their study involves analyzing the interplay among hyperbolicity and viscosity. Depending on the strength of dissipation, weak waves may break down to form shock waves or they may be dissipated ([D1, D2, DH2]). On the other hand, when the end states differ, stable coherent structures of viscous shock waves can appear ([L]).

A different type of small disturbance amplification mechanism is deemed responsible for the phenomenon of shear band formation, at high speed deformations of metals. Shear bands are regions of intensely localized shear strain, that appear during the plastic deformation of many materials and often precede rupture. An underlying feature, in the core of most theories for their exlpanation, is the presence of some type of strain-softening mechanism (either net or effective). Strain softening enhances the plastic flow at regions of higher strains relative to regions of lower strains, thus having an autocatalytic effect. Opposed to this tendency lie internal dissipation properties of the material, triggered at high strain rates. As a consequence of the contest, the plastic strain may conceivably concentrate in a narrow region forming a shear band.

There is an extensive literature devoted to the study of shear bands, including experimental, numerical, linearized analysis (e.g. [N, SC, WW] and references therein), as well as nonlinear analysis investigations [BPV, C, CM, CDM, DH1, T1-T4]. Our objective is to review some of the insight offered by nonlinear analysis techniques on the ramifications of

* Department of Mathematics, University of Wisconsin, Madison, WI 57306.

the aforementioned destabilizing mechanism. The presentation is carried out for a simple test problem, intended as a paradigm, for which the analysis is rather conclusive.

We take up the plastic shearing of an infinite plate of unit thickness, subjected to either prescribed tractions or steady shearing at the boundaries. The deformation is described by the velocity $v(x,t)$ in the shearing direction, the plastic strain $u(x,t)$ and the shear stress $\sigma(x,t)$. We focus on the constitutive relation

$$\sigma = \tau(u) v_x^n \qquad \text{with} \quad n > 0 \tag{1.1}$$

and $\tau(u)$ a smooth function satisfying

$$\tau(u) > 0 \quad , \quad \tau'(u) < 0 \,. \tag{H_1}$$

This constitutive law is appropriate for materials exhibiting strain softening, manifested in (H_1), and strain-rate sensitivity, the strength of which is measured by the parameter n. Metals, in general, exhibit strain hardening in isothermal deformations. At high strain rates, though, the combined effect of thermal softening and strain hardening may deliver effective strain softening. It is precisely that effect that we opt to incorporate in our, otherwise, isothermal model.

The balance of linear momentum and kinematic compatibility considerations, supplemented with (1.1), give rise to the system of evolution equations

$$v_t = \sigma_x = (\tau(u) v_x^n)_x \,, \tag{1.2}$$

$$u_t = v_x \,, \tag{1.3}$$

for $(v(x,t), u(x,t))$, taken over $(x,t) \in [0,1] \times \{t > 0\}$. Initial conditions

$$v(x,0) = v_0(x) \quad , \quad u(x,0) = u_0(x) \,, \tag{1.4}$$

are provided and, as a consequence,

$$\sigma(x,0) = \sigma_0(x) := \tau(u_0(x)) v_{0x}^n(x) \,. \tag{1.5}$$

Also, boundary conditions:

$$\sigma(0,t) = 1 \,,\ \sigma(1,t) = 1 \quad , \quad t > 0 \,, \tag{1.6_S}$$

in case the shearing deformation is caused by prescribed tractions at the boundaries, or

$$v(0,t) = 0 \,,\ v(1,t) = 1 \quad , \quad t > 0 \,, \tag{1.6_V}$$

in the case of steady shearing by prescribed velocities. We refer to the initial-boundary value problem consisting of (1.2-1.6) as $(\mathcal{P})_S$ when the boundary conditions $(1.6)_S$ are used and as $(\mathcal{P})_V$ when $(1.6)_V$ are used. $(\mathcal{P})_S$ and $(\mathcal{P})_V$ serve as test problems to investigate the competition between strain softening and strain-rate sensitivity.

To gain some perspective on the problem, note that, if $n = 0$, (1.2-1.3) leads to the pair of conservation laws

$$\begin{aligned} v_t &= \tau(u)_x \\ u_t &= v_x \,. \end{aligned} \tag{1.7}$$

If $\tau'(u) > 0$ then (1.7) is hyperbolic; however, under (H_1), the system (1.7) is elliptic in the t-direction and the initial value problem is ill-posed. Nevertheless, it admits an interesting class of spatially uniform solutions

$$\begin{aligned} \bar{v}(x) &= x \\ \bar{u}(t) &= t + u_0 \,, \end{aligned} \tag{1.8}$$

where u_0 is an arbitrary constant. These solutions correspond to uniform shearing and are compatible with $(1.6)_V$. Equations (1.2-1.3) with $n > 0$ can be thought as a particular regularization of (1.7).

On the other hand, the system (1.2-1.3) belongs to what is formally classified as hyperbolic-parabolic systems, what appears in discord with the view as a regularization of an elliptic system. To reconcile the two points of view, consider a perturbation of the uniform shearing solution

$$\begin{aligned} v(x,t) &= x + V(x,t) \\ u(x,t) &= (t + u_0) + U(x,t) \,. \end{aligned} \tag{1.9}$$

The linearized equations for (V, U) are easily computed:

$$\begin{aligned} V_t &= n\,\tau(t + u_0)\,V_{xx} + \tau'(t + u_0)\,U_x \\ U_t &= V_x \end{aligned} \tag{1.10}$$

Under hypothesis (H_1), when $n = 0$, (1.10) becomes an elliptic system.

Regarding the effect of the boundary conditions, we remark that, for a strain softening material, the amount of external work required to prescribe the tractions is greater than the amount required to prescribe the velocities. It is thus expected that the shearing deformation is more intense in the former case, causing favorable conditions for unstable response. In the course of shearing, under either kind of loading, the material is strained and the diffusion coefficient in the parabolic equation (1.2) is decreasing. If the decrease is too rapid and/or spatially nonuniform, it is conceivable that the diffusion be unable to stabilize the motion, possibly resulting to spatially nonuniform structures. Investigating this question is greatly facilitated by recasting (1.2-1.3) into an equivalent formulation of a reaction-diffusion system (cf. (2.7-2.8)).

The analysis presented here is partly a review, emphasizing main points in [T3, T4], partly new, drawing on ideas developed in [BPV] for a related problem. Other similar in spirit studies concern: the case of a material that initially hardens but eventually softens [T2], and models that take into account explicit thermal effects [DH1, T1, C, T3, CDM, BPV]. The article is organized as follows.

In Section 2 we record an existence theory for the initial-boundary value problems $(\mathcal{P})_S$ and $(\mathcal{P})_V$, in Schauder spaces. It includes local existence and continuation components, a characterization for global solvability of the problem $(\mathcal{P})_S$, in terms of the behavior at infinity of the function

$$\Phi(u) := \int_1^u \tau(\xi)^{\frac{1}{n}} d\xi \, , \tag{1.11}$$

and sufficient conditions for global existence of the problem $(\mathcal{P})_V$.

In Section 3 we analyze the behavior of solutions to the problem $(\mathcal{P})_S$, for functions

$$\tau(u) = \frac{1}{u^m} \, . \tag{H_2}$$

Other constitutive functions are treated in [T4], including functions that decay to a positive constant. The parameter space $\{m > 0, n > 0\}$ is then decomposed into three disjoint regions, divided by the lines $\frac{m}{n} = \frac{1}{2}$ and $\frac{m}{n} = 1$, across which the response changes drastically:

(i) In the region $0 < \frac{m}{n} < \frac{1}{2}$ solutions of $(\mathcal{P})_S$ are globally defined and, as $t \to \infty$, the shear stress $\sigma(x,t)$ is attracted to the constant state $\sigma \equiv 1$, while $u(x,t)$ behaves asymptotically as a function of time alone.

(ii) In the region $\frac{1}{2} \le \frac{m}{n} \le 1$ the constant state $\sigma \equiv 1$ loses its stability and spatial nonuniformities in the strain can develop and persist in time.

(iii) Finally, in the region $\frac{m}{n} > 1$, $u(x,t)$ becomes infinite in finite time.

Most interesting, when $\frac{m}{n} > \frac{1}{2}$, there are initial data for which the strain $u(x,t)$ develops nonuniformities around $x = 0$ and $x = 1$ and looks like two shear bands located at the boundaries. This response is concurrent with an inability for diffusion of the applied stresses.

In Section 4 we take up the problem $(\mathcal{P})_V$ and study the stability of the uniform shearing solutions (1.8). Since the basic solution grows with time, the meaning of stability needs to be qualified. For values of the parameters $m < min\{n, 1\}$, it turns out that solutions of $(\mathcal{P})_V$ converge to a uniform shear flow, as $t \to \infty$, in a fashion explicitly controlled by the time rates of the basic solution and the form of the initial data. (It is still possible that spatial nonuniformities appear in the higher orders of the expansions, evolving at slower time rates.) By contrast, for values of the parameters $m > n$ and $n < 1$, there are initial data, corresponding to strains concentrated in a region and a linear initial velocity profile, so that as time evolves either the strain becomes infinite in finite time, or the velocity asymptotically converges to a profile that looks like a shear band.

2. Existence Theory for $(\mathcal{P})_S$ and $(\mathcal{P})_V$. An existence theory for the initial-boundary value problems $(\mathcal{P})_S$ and $(\mathcal{P})_V$, in Schauder spaces, is outlined in this Section.

We assume that the initial data are smooth

$$v_0(x) \in C^{2+\alpha}[0,1] \, , \quad u_0(x) \in C^{1+\alpha}[0,1] \, , \tag{I_1}$$

and satisfy the sign restrictions

$$u_0(x) > 0, \qquad \sigma_0(x) > 0 \quad , \quad 0 \le x \le 1 \, , \tag{I_2}$$

and the compatibility conditions:

$$v_0(0) = 0\,,\; v_0(1) = 1 \quad , \quad \sigma_{0x}(0) = \sigma_{0x}(1) = 0\,, \tag{$(I_3)_V$}$$

in case $(1.6)_V$ applies, or

$$\sigma_0(0) = \sigma_0(1) = 1\,, \tag{$(I_3)_S$}$$

in case $(1.6)_S$ applies. We prove:

Theorem 2.1. *Under hypotheses (H_1) and $(I_1) - (I_3)$, there exists a unique classical solution $(v(x,t), u(x,t))$ of $(\mathcal{P})_S$ or $(\mathcal{P})_V$, defined on a maximal interval of existence $Q_{T^*} := [0,1] \times [0, T^*)$, such that the functions v, v_x, v_t, v_{xx}, u, u_t, u_x, u_{xt} are all in $C^{\alpha,\alpha/2}(\bar{Q}_T)$, for any $\bar{Q}_T := [0,1] \times [0,T]$ with $0 < T < T^*$. Furthermore:*

(i) *The solutions of $(\mathcal{P})_S$ or $(\mathcal{P})_V$ satisfy on Q_{T^*}*

$$u_t(x,t) = v_x(x,t) > 0 \quad , \quad u(x,t) \geq u_0(x) > 0 \tag{2.1}$$

and

$$0 < \sigma(x,t) \leq \max_{0 \leq x \leq 1} \sigma_0(x)\,. \tag{2.2}$$

(ii) *For either of the problems $(\mathcal{P})_S$ or $(\mathcal{P})_V$,*

$$if \quad T^* < \infty \quad then \quad \lim_{t \to T^*} \sup_{0 \leq x \leq 1} u(x,t) = \infty\,. \tag{2.3}$$

(iii) *In the case of the problem $(\mathcal{P})_S$,*

$$T^* = \infty \quad if\ and\ only\ if \quad \Phi(\infty) = \infty\,, \tag{2.4}$$

where

$$\Phi(\infty) = \int_1^\infty \tau(\xi)^{\frac{1}{n}}\, d\xi\,. \tag{2.5}$$

(iv) *In the case of the problem $(\mathcal{P})_V$,*

$$if \quad \Phi(\infty) = \infty \quad then \quad T^* = \infty\,. \tag{2.6}$$

Proof. We refer to [T4] for the proof of the local existence part of the theorem. It is also shown there that, given any compact subset $\mathcal{K}$ of $(0,\infty) \times R$, $(\sigma(x,t), u(x,t))$ escapes $\mathcal{K}$ as $t \uparrow T^*$, i.e., there are sequences $\{x_n\} \subset [0,1]$ and $\{t_n\}$ with $t_n \uparrow T^*$ such that $(\sigma(x_n,t_n), u(x_n,t_n)) \notin \mathcal{K}$. In particular, the constructed solution satisfies (2.1).

For the remainder of the proof, it is expedient to use an alternative formulation of the problem. Let $(v(x,t), u(x,t))$ be a solution of either $(\mathcal{P})_S$ or $(\mathcal{P})_V$ defined on a maximal interval of existence Q_{T^*} and satisfying (2.1). A simple calculation, using (1.1 - 1.3), shows that $(\sigma(x,t), u(x,t))$ is a positive solution of the reaction-diffusion system

$$\left(\sigma^{\frac{1}{n}}\right)_t = \tau(u)^{\frac{1}{n}} \sigma_{xx} + \frac{1}{n} \frac{\tau'(u)}{\tau(u)^{1+\frac{1}{n}}} \left(\sigma^{\frac{1}{n}}\right)^2 \tag{2.7}$$

$$u_t = \frac{\sigma^{\frac{1}{n}}}{\tau(u)^{\frac{1}{n}}}\,, \tag{2.8}$$

on Q_{T^*}. Integrating (2.8) and using (1.11) yields

$$\Phi(u(x,t)) = \Phi(u_0(x)) + \int_0^t \sigma^{\frac{1}{n}}(x,\tau)d\tau\,. \tag{2.9}$$

The following comparison principle (e.g. [PW, Ch.3]) is used to estimate $\sigma(x,t)$, viewed as a solution of the parabolic equation (2.7).

Lemma 2.2. *Suppose that $\sigma_1(x,t)$ and $\sigma_2(x,t)$ are both in $C^{2,1}(\bar{Q}_T)$ with $\sigma_1(x,t) > 0$ and $\sigma_2(x,t) > 0$, $A(x,t) \in C(\bar{Q}_T)$ with $A(x,t) > 0$ and $B(x,t) \in C(\bar{Q}_T)$, for any $0 < T < T^*$. If the inequalities*

$$\begin{gathered}(\sigma_1^{\frac{1}{n}})_t - A(x,t)\sigma_{1xx} + B(x,t)\big(\sigma_1^{\frac{1}{n}}\big)^2 \le \big(\sigma_2^{\frac{1}{n}}\big)_t - A(x,t)\sigma_{2xx} + B(x,t)\big(\sigma_2^{\frac{1}{n}}\big)^2, \quad on \quad \bar{Q}_T\,, \\ \sigma_1(x,0) \le \sigma_2(x,0) \quad , \quad 0 \le x \le 1\end{gathered} \tag{2.10}$$

and either

$$\sigma_1(i,t) \le \sigma_2(i,t) \qquad i = 0,1\,, \quad 0 \le t \le T\,, \tag{2.10$_S$}$$

or

$$\sigma_{1x}(0,t) \ge \sigma_{2x}(0,t) \quad , \quad \sigma_{1x}(1,t) \le \sigma_{2x}(1,t) \quad , \quad 0 \le t \le T\,, \tag{2.10$_V$}$$

hold for any $0 < T < T^*$, then

$$\sigma_1(x,t) \le \sigma_2(x,t) \qquad 0 \le x \le 1\,, \quad 0 \le t < T^*\,. \tag{2.11}$$

Under hypothesis (H_1), any positive semiconcave function $\Sigma(x)$ satisfies

$$-\tau(u)^{\frac{1}{n}}\Sigma_{xx} - \frac{1}{n}\frac{\tau'(u)}{\tau(u)^{1+\frac{1}{n}}}\Sigma^{\frac{2}{n}} > 0\,. \tag{2.12}$$

For both cases $(\mathcal{P})_S$ and $(\mathcal{P})_V$, an application of the Lemma 2.2 with comparison functions $\sigma(x,t)$ and $\Sigma(x) := \max_{0\le x\le 1} \sigma_0(x)$ yields (2.2).

Next, we turn to (ii). Assume that $T^* < \infty$ and at the same time $u(x,t)$ is bounded from above for $(x,t) \in Q_{T^*}$, i.e.,

$$\inf_{0\le x\le 1} u_0(x) =: u_{0-} \le u(x,t) \le U_+ < \infty\,. \tag{2.13}$$

Let $s(t)$ be the solution of the initial value problem

$$\begin{aligned}&\frac{d}{dt}s^{\frac{1}{n}} + B_0\big(s^{\frac{1}{n}}\big)^2 = 0\\ &s(0) = \sigma_{0-} := \inf_{0\le x\le 1} \sigma_0(x)\,,\end{aligned} \tag{2.14}$$

where $B_0 = \max\limits_{u_{0-} \le \xi \le U_+} \dfrac{1}{n} \dfrac{|\tau'(\xi)|}{\tau(\xi)^{1+\frac{1}{n}}}$. The solution of (2.14) is

$$s(t) = \frac{\sigma_{0-}}{\left(1 + \sigma_{0-}^{\frac{1}{n}} B_0 t\right)^n} > 0\,. \tag{2.15}$$

On account of (2.14) and (2.7), the comparison functions $s(t)$ and $\sigma(x,t)$ satisfy the parabolic differential inequalities (2.10) on $\bar{Q}_T$, for any $T < T^*$. Lemma 2.2 implies

$$s(t) \le \sigma(x,t)\,. \tag{2.16}$$

Relations (2.2), (2.13) and (2.16) contradict the assumption that T^* is maximal and finite. Since $u_t > 0$, (2.3) follows.

Finally, we proceed to show (iii) and (iv). The function $\Phi(u)$, defined in $(0,\infty)$ by (1.11), is increasing and invertible. The inverse function $\Phi^{-1}(\xi)$ is defined for $\xi \in (\Phi(0), \Phi(\infty))$ and is increasing. In case $\Phi(\infty) = \infty$, combining (2.9) and (2.2), we arrive at

$$u(x,t) \le \Phi^{-1}\big(\Phi(u_0(x)) + \Sigma^{\frac{1}{n}}(x)t\big)\,. \tag{2.17}$$

Then (2.13) holds for any Q_T with $T > 0$ and part (ii) implies that both $(\mathcal{P})_S$ and $(\mathcal{P})_V$ admit global solutions. By contrast, in case $\Phi(\infty) < \infty$ and for the problem $(\mathcal{P})_S$, evaluating (2.9) at $x = 0$ or $x = 1$ leads to

$$\Phi(u(i,t)) = \Phi(u_0(i)) + t \quad , \quad i = 0, 1\,. \tag{2.18}$$

In turn, (2.18) implies

$$u(i,t) \to \infty \quad \text{as} \quad t \to T_i\,, \tag{2.19}$$

where $T_i = \Phi(\infty) - \Phi(u_0(i)) < \infty$ for $i = 0, 1$. Therefore, $T^* \le min\{T_0, T_1\} < \infty$. ∎

3. On the Competition of Strain Softening and Strain-Rate Sensitivity . Our next objective is to study the behavior of solutions of the problem $(\mathcal{P})_S$. Throughout the section, $(v(x,t), u(x,t))$ is a classical solution of $(\mathcal{P})_S$ defined on a maximal interval of existence $Q_{T^*} := [0,1] \times [0, T^*)$.

According to Theorem 2.1, pending on the convergence of the integral (2.5), solutions of $(\mathcal{P})_S$ may blow up. The class of positive, decreasing constitutive functions $\tau(u)$ can be decomposed into two categories, depending on whether $\Phi(\infty)$ is finite or infinite. Roughly speaking, the dividing line consists of functions $\tau(u)$ that decay to zero like the power u^{-n}.

On the other end, for $\tau(u) \equiv \tau_0$ a constant, $\sigma(x,t)$ is a positive solution of

$$\left(\sigma^{\frac{1}{n}}\right)_t = \tau_0^{\frac{1}{n}} \sigma_{xx} \tag{3.1}$$

subject to the boundary conditions $(1.6)_S$. Thus $\sigma(x,t) \to 1$ uniformly for $x \in [0,1]$ as $t \to \infty$, and the response is stable.

In order to analyze the transition between these two extreme situations, we focus on functions $\tau(u)$ that are powers. Other constitutive functions are treated in [T4], including functions that decay to a positive constant. Under hypothesis (H_2), (2.7) and (2.8) take the form

$$
\begin{aligned}
\left(\sigma^{\frac{1}{n}}\right)_t &= u^{-\frac{m}{n}}\sigma_{xx} - \frac{m}{n}u^{\frac{m}{n}-1}\sigma^{\frac{2}{n}} \\
u_t &= u^{\frac{m}{n}}\sigma^{\frac{1}{n}}\,,
\end{aligned}
\tag{3.2}
$$

while (2.9) now reads:

$$
u(x,t)^{1-\frac{m}{n}} = u_0(x)^{1-\frac{m}{n}} + (1-\frac{m}{n})\int_0^t \sigma^{\frac{1}{n}}(x,\tau)d\tau\,, \tag{3.3}
$$

in case $\frac{m}{n} \neq 1$, and

$$
\ell n u(x,t) = \ell n u_0(x) + \int_0^t \sigma^{\frac{1}{n}}(x,\tau)d\tau \tag{3.4}
$$

in case $\frac{m}{n} = 1$.

We seek for steady super- and sub-solutions for the parabolic equation $(3.2)_1$. Set $-\gamma(u)$ to be the quotient of the coefficients of the reaction versus the diffusion term

$$
\gamma(u) := \frac{1}{n}\Big(-\frac{\tau'(u)}{\tau(u)}\Big)\frac{1}{\tau(u)^{2/n}} = \frac{m}{n}u^{2\frac{m}{n}-1} > 0\,. \tag{3.5}
$$

If we can find a positive function $S(x)$ defined on $[0,1]$ and satisfying for $(x,t) \in Q_{T^*}$ the differential inequalities

$$
\begin{aligned}
-S_{xx} + \gamma(u(x,t))S^{2/n} &\geq 0 \\
S(x) &\geq \sigma_0(x)\,,
\end{aligned}
\tag{3.6}
$$

then Lemma 2.2 implies

$$
\sigma(x,t) \leq S(x) \quad , \quad (x,t) \in Q_{T^*}\,. \tag{3.7}
$$

Similarly, if a positive function $s(x)$ satisfies on Q_{T^*}

$$
\begin{aligned}
-s_{xx} + \gamma(u(x,t))s^{2/n} &\leq 0 \\
s(x) &\leq \sigma_0(x)\,,
\end{aligned}
\tag{3.8}
$$

then

$$
s(x) \leq \sigma(x,t) \quad , \quad (x,t) \in Q_{T^*}\,. \tag{3.9}
$$

The coefficient $\gamma(u(x,t))$ is not *a priori* known; nevertheless, (2.1) induces the one sided bounds:

$$
\gamma(u(x,t)) \leq \gamma_0(x) := \frac{m}{n}u_0(x)^{2\frac{m}{n}-1} \qquad \text{for} \quad \frac{m}{n} < \frac{1}{2} \tag{3.10}
$$

$$
\gamma(u(x,t)) = \frac{m}{n} \qquad \text{for} \quad \frac{m}{n} = \frac{1}{2} \tag{3.11}
$$

$$
\gamma(u(x,t)) \geq \gamma_0(x) := \frac{m}{n}u_0(x)^{2\frac{m}{n}-1} \qquad \text{for} \quad \frac{m}{n} > \frac{1}{2} \tag{3.12}
$$

A new critical value for the ratio of the parameters is introduced at $\frac{m}{n} = \frac{1}{2}$, according to whether the diffusion or the reaction term dominates in $(3.2)_1$. Three separate regions in the parameter space $\{m > 0\,, n > 0\}$ are now considered:

(a) Region $\mathbf{0 < \frac{m}{n} < \frac{1}{2}}$. We test the parametric family $s_\beta(x) := \frac{\beta}{2}(x^2+1)$, with $\beta > 0$, as a candidate for subsolution. Indeed, if β can be chosen so that

$$\sigma_0(x) \geq s_\beta(x) = \frac{\beta}{2}(x^2+1) \quad , \quad u_0(x)^{1-2\frac{m}{n}} \geq \frac{m}{n}\beta^{\frac{2}{n}-1}\,, \tag{3.13}$$

then, by virtue of (3.10), s_β satisfies (3.8), hence also (3.9). That together with (2.2) gives

$$\frac{1}{c_1} \leq \sigma(x,t) \leq c_1 \quad , \quad (x,t) \in Q_\infty\,. \tag{3.14}$$

In turn, (3.3) and (2.1) imply

$$\frac{1}{c_2}(t+1)^{\frac{n}{n-m}} \leq u(x,t) \leq c_2(t+1)^{\frac{n}{n-m}}\,, \tag{3.15}$$

where the positive constants c_1 and c_2 can be explicitly computed in terms of the data.

Regarding the ability to choose β satisfying (3.13), we remark:

(i) If $n < 2$, given any $(\sigma_0(x), u_0(x))$ compatible with (I_2), we can select $\beta > 0$ sufficiently small so that (3.13) is satisfied.

(ii) If $n \geq 2$, restrictions on the data have to be placed. Given $\sigma_0(x)$, find the largest β satisfying $s_\beta(x) = \frac{\beta}{2}(x^2+1) \leq \sigma_0(x)$. With this β fixed, (3.13) imposes a restriction on the minimum of $u_0(x)$.

Once a solution of $(\mathcal{P})_S$, for $0 < \frac{m}{n} < \frac{1}{2}$, satisfies (3.14) and (3.15), it is possible to find its exact asymptotic behavior, leading to the following theorem (see [T4]):

Theorem 3.1. *Suppose the function $\tau(u)$ belongs to the class (H_2) with $0 < \frac{m}{n} < \frac{1}{2}$. Let $(v(x,t), u(x,t))$ be a classical solution of $(\mathcal{P})_S$ on Q_∞ as in Theorem 1.1. Then for any choice of the initial data, in case $0 < n < 2$, and under restrictions on the data outlined above, in case $n \geq 2$, the following hold: As $t \to \infty$,*

$$\sigma(x,t) = 1 + O(t^{-\delta})\,, \tag{3.16}$$

$$\int_{u_0(x)}^{u(x,t)} \xi^{-\frac{m}{n}}\,d\xi = t + O(\int_1^t s^{-\delta}ds) \tag{3.17}$$

uniformly on $[0,1]$, where $0 < \delta = \frac{n-2m}{n-m} < 1$.

The asymptotic behavior in this region is comparable to that of the equation (3.1). As $t \to \infty$, the stress $\sigma(x,t) \to 1$ uniformly for $x \in [0,1]$; integrating (3.17), we find for the strain

$$u(x,t) = (\frac{n-m}{n}\,t\,)^{\frac{n}{n-m}}\,[1 + O(t^{-\delta})]\,; \tag{3.18}$$

finally, the strain rate behaves like

$$u_t(x,t) = u^{\frac{m}{n}}\sigma^{\frac{1}{n}} = (\frac{n-m}{n}\,t\,)^{\frac{m}{n-m}}\,[1+O(t^{-\delta})]\,. \tag{3.19}$$

To conclude, note that the leading order terms in the expansions (3.16), (3.18) and (3.19) do not form a solution of $(\mathcal{P})_S$.

(b) Region $\mathbf{\frac{m}{n} > \frac{1}{2}}$. A summary of the analysis of steady supersolutions in this region, based on (3.6) and (3.12), is listed below:

Lemma 3.2. *Let $\frac{m}{n} > \frac{1}{2}$. If $S(x)$ is a smooth, positive function satisfying for $x \in [0,1]$*

$$\begin{aligned} u_0^{2\frac{m}{n}-1}(x)S^{\frac{2}{n}}(x) &\geq \frac{n}{m}S_{xx}(x) \\ S(x) &\geq \sigma_0(x)\,, \end{aligned} \tag{3.20}$$

then the solution of $(\mathcal{P})_S$ satisfies (3.9) on Q_{T^*}.

In particular, $\sigma_0(x)$ can be used as a choice for $S(x)$, provided $u_0(x)$ is restricted by $(3.20)_1$. Once (3.9) holds, (3.3) or (3.4) induce corresponding bounds for $u(x,t)$. We examine below the implications of some specific choices for $S(x)$.

First, consider the one-parameter family of positive convex functions on $[0,1]$

$$S_\alpha(x) = 1 - \alpha x(1-x) \qquad \text{for} \quad 0 < \alpha < 4\,. \tag{3.21}$$

Lemma 3.2 implies that S_α is a supersolution, if the initial data are restricted by

$$\sigma_0(x) \leq S_\alpha(x) \quad , \quad u_0(x) \geq u_\alpha := \left[\frac{n}{m}\frac{2\alpha}{(1-\frac{\alpha}{4})^{\frac{2}{n}}}\right]^{\frac{n}{2m-n}} \quad , \quad 0 \leq x \leq 1\,. \tag{3.22}$$

Let $(\sigma_\alpha(x,t), u_\alpha(x,t))$ be the solution of $(\mathcal{P})_S$ corresponding to initial data $\sigma_0 = S_\alpha$ and $u_0 \geq u_\alpha$. It is defined on Q_{T^*} and satisfies

$$\sigma_\alpha(x,t) \leq S_\alpha(x) = 1 - \alpha x(1-x) \qquad \text{on} \quad Q_{T^*}\,. \tag{3.23}$$

A comparison of (3.16) and (3.23) reveals a striking difference in the behavior of solutions across the parameter value $\frac{m}{n} = \frac{1}{2}$. Observe that as $\alpha \to 0$, $u_\alpha \to 0$ and $S_\alpha \to 1$ uniformly on $[0,1]$. We conclude that in the region $\frac{m}{n} < \frac{1}{2}$ diffusion is too weak to uniformize the stress, and the state $\sigma \equiv 1$ loses its stability. We interpret that as indicating a collapse of the stress diffusion mechanism in this parameter region.

Next, consider a positive, convex, and symmetric with respect to $x = \frac{1}{2}$ function S_γ, that drops fast from $S_\gamma(0) = 1$ to approximately a constant level γ, stays there over almost the entire interval, and then increases fast to the value $S_\gamma(1) = 1$. An example of such a function is provided by

$$S_\gamma(x) = \gamma + (1-\gamma)[x^N + (1-x)^N]\,, \tag{3.24}$$

for N large positive integer and $0 < \gamma < 1$. Let u_0 be a symmetric, with respect to $x = \frac{1}{2}$, positive function, such that $(3.20)_1$ is satisfied for S_γ and $u_0(x) \le u_0(0) = u_0(1)$ for $x \in (0,1)$. Take (S_γ, u_0) as initial data and let $(\sigma_\gamma(x,t), u_\gamma(x,t))$ be the corresponding solution of $(\mathcal{P})_S$, defined on Q_{T^*}. Then σ_γ satisfies (3.7). Regarding the behavior of u_γ, we consider three separate regions:

(i) $\mathbf{\frac{1}{2} < \frac{m}{n} < 1}$: Here $T^* = \infty$. Moreover, (3.7) and (3.3) imply

$$u_\gamma(x,t) \le \left[u_0^{1-\frac{m}{n}}(x) + (1 - \frac{m}{n}) S_\gamma^{\frac{1}{n}}(x)\, t \right]^{\frac{n}{n-m}} \qquad \text{on} \quad Q_\infty . \tag{3.25}$$

It follows by the form of S_γ, $(1.6)_S$ and (3.3) that (3.25) is in fact an equality at $x = 0$ and $x = 1$. Since u_0 may be a constant function, (3.25) suggests that an initial nonuniformity in the stress, no matter how small, may induce spatial nonuniformities in the strain $u(x,t)$ that persist in time.

(ii) $\mathbf{\frac{m}{n} = 1}$: The situation is similar to part (i) with (3.25) replaced by

$$u_\gamma(x,t) \le u_0(x) exp\Big\{ S_\gamma^{\frac{1}{n}}(x) t \Big\} \qquad \text{on} \quad Q_\infty . \tag{3.26}$$

(iii) $\mathbf{\frac{m}{n} > 1}$: Here $T^* < \infty$. Also, (3.3) yields

$$u_\gamma(x,t) \le \left[u_0^{-(\frac{m}{n}-1)}(x) - (\frac{m}{n} - 1) S_\gamma^{\frac{1}{n}}(x)\, t \right]^{-\frac{n}{m-n}} . \tag{3.27}$$

By the choice of S_γ and u_0, the right hand side of (3.27) blows up for the first time at the boundary points $x = 0$ and $x = 1$, where (3.27) holds as an equality. Part (ii) of Theorem 2.1 then implies that (3.27) holds on Q_{T^*}. In blowing up, the right hand side appears like two shear bands located at the boundaries. Since (3.27) is an equality at $x = 0$ and $x = 1$, the same is true for $u_\gamma(x,t)$. As $t \to T^*$, σ_γ obeys the bound (3.7) while

$$\partial_t u_\gamma(x,t) = \partial_x v_\gamma(x,t) \le S_\gamma^{\frac{1}{n}}(x) \Big[u_0^{-(\frac{m}{n}-1)} - (\frac{m}{n} - 1) S_\gamma^{\frac{1}{n}}(x)\, t \Big]^{-\frac{m}{m-n}} \quad \text{on } Q_{T^*} \tag{3.28}$$

with an equality at $x = 0$ and $x = 1$.

(c) Region $\mathbf{\frac{m}{n} = \frac{1}{2}}$. The situation here is similar to that of the region $\frac{1}{2} < \frac{m}{n} < 1$. For, if α is sufficiently small so that $2\alpha \le \frac{1}{2}\big(1 - \frac{\alpha}{4}\big)^{\frac{2}{n}}$, then $S_\alpha(x)$, given by (3.21), satisfies (3.6) and thus (3.7). The uniform state $\sigma \equiv 1$ again loses stability.

4. Is the Uniform Shear Flow Stable? Under hypothesis (H_2), the constitutive equation for the stress becomes a power law

$$\sigma = \frac{1}{u^m} v_x^n \tag{4.1}$$

with m, n positive parameters. The equations that (u,v) satisfy now take the form

$$\begin{aligned} v_t &= \big(\frac{1}{u^m} v_x^n\big)_x \\ u_t &= v_x\,. \end{aligned} \tag{4.2}$$

Accordingly, (σ,u) satisfies the parabolic system (3.2). Let $(v(x,t),u(x,t))$ be a classical solution of $(\mathcal{P})_V$, defined on a maximal interval of existence $Q_{T^*} := [0,1]\times[0,T^*)$, as in Theorem 2.1. In case $\frac{m}{n}\le 1$, (2.6) implies that $T^*=\infty$; however, when $\frac{m}{n}>1$, it is not known whether T^* is finite or infinite.

The problem $(\mathcal{P})_V$ admits a special class of solutions describing uniform shearing

$$\bar v(x) = x \quad , \quad \bar u(t) = t + u_0 \quad , \quad \bar\sigma(t) = (t+u_0)^{-m}\,, \tag{4.3}$$

that correspond to initial data (x,u_0) with u_0 positive. We investigate the stability of the uniform shearing solutions.

4.1 The Case of Strong Dissipation. The systems (4.2) and (4.3) are invariant under the group of stretching transformations $T_{\alpha,\delta}$:

$$x\to\lambda x\,,\ t\to\lambda^{-\frac{1}{\alpha}}t\,,\ v\to\lambda^{\frac{\delta}{\alpha}}v\,,\ u\to\lambda^{\frac{\alpha+\delta+1}{\alpha}}u\,,\ \sigma\to\lambda^{\frac{\delta-\alpha-1}{\alpha}}\sigma\,;\ 0<\lambda<\infty \tag{4.4}$$

with α, δ constrained by

$$-m(\alpha+\delta+1) + n(\alpha+\delta) + \alpha - \delta + 1 = 0\,. \tag{4.5}$$

Note that (x,t) is a self similar solution under the transformation $T_{\alpha,\delta}$ with $\alpha=-\delta=\frac{m-1}{2}$.

Motivated by the form of the uniform shearing solutions and the scaling invariance (4.4), we introduce the transformations

$$\begin{aligned} v(x,t) &= V(x,s(t)) \\ u(x,t) &= (t+1)\,U(x,s(t)) \\ \sigma(x,t) &= (t+1)^{-m}\,\Sigma(x,s(t)) \end{aligned} \tag{4.6}$$

with

$$s(t) = \ell n(t+1)\,. \tag{4.7}$$

A straightforward computation suggests that

$$\Sigma = \frac{1}{U^m} V_x^n \tag{4.8}$$

and that $(\Sigma(x,s),U(x,s))$ satisfy the system of reaction-diffusion equations

$$\begin{aligned} \Sigma_s &= n e^{(1-m)s} U^{-\frac{m}{n}} \Sigma^{1-\frac{1}{n}} \Sigma_{xx} - m\frac{\Sigma}{U^{1-\frac{m}{n}}}\big(\Sigma^{\frac{1}{n}} - U^{1-\frac{m}{n}}\big) \\ U_s &= U^{\frac{m}{n}}\big(\Sigma^{\frac{1}{n}} - U^{1-\frac{m}{n}}\big)\,, \end{aligned} \tag{4.9}$$

subject to boundary conditions

$$\Sigma_x(0,s) = \Sigma_x(1,s) = 0 \quad , \quad s > 0 \tag{4.10}$$

and initial conditions

$$\Sigma(x,0) = \sigma_0(x) \quad , \quad U(x,0) = u_0(x) \quad , \quad 0 \le x \le 1\,. \tag{4.11}$$

The vector field in the "reaction term" of (4.9) vanishes when $\Sigma = U^{n-m}$. The form of this line changes drastically across the critical parameter value $\frac{m}{n} = 1$. If $0 < \frac{m}{n} < 1$, the theory of Chueh, Conley and Smoller [CCS] guarantees that (4.9) admits positively invariant rectangles of arbitrary size in the first quadrant $\{\Sigma > 0, U > 0\}$, that are centered around the line $\Sigma = U^{n-m}$ (see Fig. 1).

Fig. 1: Invariant regions for (4.9) when $0 < \frac{m}{n} < 1$.

Given initial data $\sigma_0(x) > 0$, $u_0(x) > 0$, let U_-, U_+, Σ_- and Σ_+ be the defining coordinates of the smallest invariant rectangle containing the curve $(\sigma_0(x), u_0(x))$, $0 \le x \le 1$. Then

$$\Sigma_- \le \Sigma(x,s) \le \Sigma_+ \quad , \quad U_- \le U(x,s) \le U_+\,. \tag{4.12}$$

In turn, (4.6) and (4.7) in conjunction with (4.12) yield the relations:

$$\begin{aligned} U_-(t+1) &\le u(x,t) \le U_+(t+1) \\ \Sigma_-(t+1)^{-m} &\le \sigma(x,t) \le \Sigma_+(t+1)^{-m} \\ \Sigma_-^{\frac{1}{n}} U_-^{\frac{m}{n}} &\le v_x(x,t) \le \Sigma_+^{\frac{1}{n}} U_+^{\frac{m}{n}}\,. \end{aligned} \tag{4.13}$$

Relations (4.13) provide explicit bounds on perturbations of the uniform shear flow. Using them together with parabolic-type energy estimates, it turns out that, provided $m < min\{n, 1\}$, every solution is asymptotically attracted to a uniform shear flow (see [T4]).

Theorem 4.1. *Suppose that $m < min\{n, 1\}$. As $t \to \infty$,*

$$v_x(x,t) = 1 + O(t^{\beta-1}) \tag{4.14}$$

$$u(x,t) = t + O(t^{\beta}) \tag{4.15}$$

and

$$\sigma(x,t) = t^{-m}\left(1 + O(t^{\beta-1})\right), \tag{4.16}$$

uniformly on $[0,1]$, with $\beta = max\{\frac{m}{n}, m\} < 1$.

Remark. The relevance of the constraint $m < 1$ is illustrated by the following heuristic argument. For $1 < m < n$, relation $(4.6)_2$ implies that the diffusion coefficient in $(4.2)_1$ decays like $(t+1)^{-m}$. For $a(t) > 0$, consider the solution $v(x,t)$ of the diffusion problem

$$v_t = a(t)(v_x^n)_x\,, \tag{4.17}$$

subject to $(1.9)_V$ and $v(x,0) = v_0(x)$ with $v_{0x}(x) > 0$. The change of variables

$$V(x, s(t)) = v(x,t) \quad , \quad s(t) = \int_0^t a(\tau)d\tau \tag{4.18}$$

suggests that $V(x,s)$ satisfies $V_s = (V_x^n)_x$ subject to the same initial and boundary conditions. Moreover,

$$\lim_{t\to\infty} v(x,t) = \lim_{s\to s_\infty} V(x,s)\,. \tag{4.19}$$

When $a(t) = (t+1)^{-m}$ it turns out that $s_\infty := \lim\limits_{t\to\infty} s(t)$ is infinite for $m \le 1$, but finite for $m > 1$. If $s_\infty = +\infty$, then $\lim\limits_{t\to\infty} v(x,t) = x$. By contrast, this is in general false when $s_\infty < +\infty$.

4.2 The Case of Weak Dissipation. In the region $\frac{m}{n} > 1$, the behavior of solutions to $(\mathcal{P})_V$ is quite subtle. It is not known at present whether solutions blow up in finite time. On the other hand, the situation in the case of prescribed tractions suggests that unstable response be expected, at least for large values of the ratio $\frac{m}{n}$. This belief is confirmed in the case $n < 1$, by adapting an idea in [BPV] in conjunction with an estimate of [T1].

The analysis is carried out, under the restriction $n < 1$, for functions of class (H_1) that in addition satisfy:

$$\tau(u)^{-\frac{1}{n}} \quad \text{is convex} \qquad , \qquad \Phi(\infty) = \int_1^\infty \tau(\xi)^{\frac{1}{n}} d\xi \; < \infty\,. \tag{H_3}$$

For a power of class (H_2), the conditions in (H_3) amount to $m > n$. Let $(v(x,t), u(x,t))$ be a solution of $(\mathcal{P})_V$ on Q_{T^*} as in Theorem 2.1. The first lemma improves on (2.2).

Lemma 4.2. *If $\tau(u)^{-\frac{1}{n}}$ is convex, then*

$$0 < \sigma(x,t) \le \frac{\sigma_{0+}}{\left(1 + \sigma_{0+}^{\frac{1}{n}} A\, t\right)^n} \qquad \text{on} \quad Q_{T^*}\,, \tag{4.20}$$

where $\sigma_{0+} = \max\limits_{0\le x\le 1} \sigma_0(x)$ *and* A *is a positive constant depending on the data.*

Proof. The function $\sigma(x,t)$ satisfies on Q_{T^*} the parabolic equation (2.7). By the convexity of $\tau(u)^{-\frac{1}{n}}$ and (2.1), the coefficient of the reaction term is estimated by

$$-\frac{1}{n}\frac{\tau'\big(u(x,t)\big)}{\tau\big(u(x,t)\big)^{1+\frac{1}{n}}} \ge -\frac{1}{n}\frac{\tau'(u_{0-})}{\tau(u_{0-})^{1+\frac{1}{n}}} =: A\,, \tag{4.21}$$

where $u_{0-} = \min\limits_{0\le x\le 1} u_0(x)$.

Let $S(t)$ be the solution of the initial value problem

$$\begin{aligned} &\frac{d}{dt}S^{\frac{1}{n}} + A\left(S^{\frac{1}{n}}\right)^2 = 0 \\ &S(0) = \sigma_{0+} = \max_{0\le x\le 1}\sigma_0(x)\,, \end{aligned} \tag{4.22}$$

that is,

$$S(t) = \frac{\sigma_{0+}}{\left(1+\sigma_{0+}^{\frac{1}{n}}A\,t\right)^n}\,. \tag{4.23}$$

On account of (4.21) and (2.7), the comparison functions $S(t)$ and $\sigma(x,t)$ satisfy the parabolic differential inequalities (2.10), hence, Lemma 2.2 yields (4.20). ∎

If we set

$$\psi := \Phi(\infty) - \Phi(u) = \int_u^\infty \tau(\xi)^{\frac{1}{n}}d\xi\,, \tag{4.24}$$

then (2.8) implies that ψ satisfies, for each $x\in[0,1]$, the equation

$$\psi_t = -\sigma^{\frac{1}{n}} \tag{4.25}$$

with initial data $\psi(x,0) = \psi_0(x) = \Phi(\infty) - \Phi(u_0(x))$. Our next goal is to estimate the spatial oscillation of the function ψ.

Lemma 4.3. *Let* $n<1$. *The following estimates hold*

$$\frac{1}{4n^2}\int_0^t\int_0^1 \sigma^{\frac{1}{n}-2}v_t^2\,dx\,d\tau \le I_0 \tag{4.26}$$

$$(1+\frac{1}{r})\psi(y,t) - \psi(x,t) \le (1+\frac{1}{r})\psi_0(y) - \psi_0(x) + (1+r)\,I_0 \tag{4.27}$$

for any $(x,t)\,,(y,t)\in Q_{T^*}$ *and any* $r>0$. I_0 *is determined by the initial data through*

$$I_0 = \alpha\int_0^1 \tau(u_0(x))^{\frac{1-n}{n}}v_{0x}(x)^{2-n}\,dx \tag{4.28}$$

with $\alpha = \frac{1}{4n(1-n)(2-n)}$.

Proof. First, we show (4.26). To this end, multiply (1.2) by $\sigma^{\frac{1}{n}-2}v_t$, integrate by parts over $[0,1]$, at the same time using $(1.6)_V$ and (1.1), to arrive at

$$\int_0^1 \sigma^{\frac{1}{n}-2} v_t^2 dx + \frac{n}{1-n}\int_0^1 \tau(u)^{\frac{1-n}{n}} v_x^{1-n} v_{xt} dx = 0\,. \tag{4.29}$$

Integrating this over $[0,t]$ by parts and using (1.3), we obtain

$$\begin{aligned}\int_0^t\int_0^1 \sigma^{\frac{1}{n}-2} v_t^2 dx d\tau + \frac{n}{(1-n)(2-n)}\int_0^1 \tau(u(x,t))^{\frac{1-n}{n}} v_x(x,t)^{2-n} dx \\ - \frac{1}{2-n}\int_0^t\int_0^1 \tau(u)^{\frac{1-2n}{n}}\tau'(u) v_x^{3-n} dx d\tau = 4n^2 I_0\,,\end{aligned} \tag{4.30}$$

where I_0 is defined in (4.28). For functions of class (H_1), (4.30) leads to (4.26).

By virtue of the algebraic inequality $a^2 \le (1+r)(a-b)^2 + (1+\frac{1}{r})b^2$, any smooth function $f(x)$ satisfies the calculus inequalities

$$f^2(x) - (1+\frac{1}{r})f^2(y) \le (1+r)[f(x)-f(y)]^2 \le (1+r)\int_0^1 f_x^2(\xi)d\xi\,. \tag{4.31}$$

Applying (4.31), for $f = \sigma^{\frac{1}{2n}}$, and using (1.2), we conclude that

$$\sigma^{\frac{1}{n}}(x,t) - (1+\frac{1}{r})\sigma^{\frac{1}{n}}(y,t) \le (1+r)\frac{1}{4n^2}\int_0^1 \sigma^{\frac{1}{n}-2}(\xi,t)\, v_t^2(\xi,t)d\xi\,. \tag{4.32}$$

Next, we integrate (4.25) at two distinct points y and x, take a weighted difference of the resulting equations, and use (4.32) to obtain

$$\begin{aligned}(1+\frac{1}{r})\psi(y,t) - \psi(x,t) &= (1+\frac{1}{r})\psi_0(y) - \psi_0(x) - \int_0^t \big(\frac{1+r}{r}\sigma^{\frac{1}{n}}(y,\tau) - \sigma^{\frac{1}{n}}(x,\tau)\big)d\tau \\ &\le (1+\frac{1}{r})\psi_0(y) - \psi_0(x) + (1+r)\frac{1}{4n^2}\int_0^t\int_0^1 \sigma^{\frac{1}{n}-2}\, v_t^2 d\xi d\tau\,.\end{aligned} \tag{4.33}$$

Then (4.27) follows from (4.26). ∎

Substituting for ψ_0, (4.27) is rewritten in the form

$$\begin{aligned}(1+\frac{1}{r})\psi(y,t) &\le \psi(x,t) + \Big(-\int_{u_0(x)}^\infty \tau(\xi)^{\frac{1}{n}}d\xi + (1+\frac{1}{r})\int_{u_0(y)}^\infty \tau(\xi)^{\frac{1}{n}}d\xi + (1+r)\,I_0\Big) \\ &=: \psi(x,t) + a_0(x,y)\,.\end{aligned} \tag{4.34}$$

Fix some point y in $(0,1)$. In view of (2.3) and the definition of ψ in (4.24), it is $\psi(y,t) > 0$ in Q_{T^*}. If there is a choice of initial data so that $a_0(x,y)$ is negative at some x, then (4.34) yields a bound from below for $\psi(x,\cdot)$ valid for $0 \le t < T^*$, hence, by (4.24), also a bound from above for $u(x,\cdot)$.

With this objective, fix an interval of length 2δ centered around some fixed $y \in (0,1)$, and consider the initial data:

$$
\begin{aligned}
v_0(x) &= x \qquad 0 \le x \le 1 \\
u_0(x) &= \begin{cases} \bar{u} & 0 \le x \le y-\delta \quad , \quad y+\delta \le x \le 1 \\ \bar{U}(x) & y-\delta \le x \le y+\delta \end{cases}
\end{aligned}
\tag{4.35}
$$

We are thinking of $\bar{U}(x)$ as a bell shaped function that attains its maximum at y, say $\bar{U}(y) =: U_M$, and satisfies $\bar{U}(y-\delta) = \bar{U}(y+\delta) = \bar{u}$ and $\bar{U}(x) > \bar{u}$ for $x \in (y-\delta\,, y+\delta)$. In turn, (1.5) implies

$$
\sigma_0(x) = \begin{cases} \tau(\bar{u}) & 0 \le x \le y-\delta \quad , \quad y+\delta \le x \le 1 \\ \tau(\bar{U}(x)) & y-\delta \le x \le y+\delta \end{cases}
\tag{4.36}
$$

and $\tau(\bar{U}(x)) \le \tau(\bar{u})$. Finally, by (4.28),

$$
I_0 = \alpha \int_0^1 \tau(u_0(x))^{\frac{1-n}{n}} v_{0x}(x)^{2-n} dx \le \alpha\tau(\bar{u})^{\frac{1-n}{n}}\,. \tag{4.37}
$$

For such data, $a_0(x,y)$ is sharply estimated by

$$
a_0(x,y) \le -\int_{\bar{u}}^{\infty} \tau(\xi)^{\frac{1}{n}} d\xi + (1+\frac{1}{r}) \int_{U_M}^{\infty} \tau(\xi)^{\frac{1}{n}} d\xi + (1+r)\,\alpha\tau(\bar{u})^{\frac{1-n}{n}}\,, \quad x \notin (y-\delta\,, y+\delta). \tag{4.38}
$$

For the right hand side of (4.38) to be ever negative, a necessary condition is

$$
\alpha\tau(\bar{u})^{\frac{1-n}{n}} < \int_{\bar{u}}^{\infty} \tau(\xi)^{\frac{1}{n}} d\xi \qquad \text{for some} \quad \bar{u} > 0\,. \tag{H_4}
$$

It is viewed as a condition on the function $\tau(u)$. Any power compatible with (H_3) also satisfies (H_4) (except possibly for the case $m = 1$, where satisfying (H_4) becomes delicate depending on the particular coefficient of the power).

We prove:

Theorem 4.4. *Assume that $n < 1$ and that the function $\tau(u)$ of class (H_1) satisfies (H_3) and (H_4) for some $\bar{u} > 0$. Fix any $\delta > 0$ and suppose the initial data are as in (4.35), with U_M sufficiently large. Let $(v(x,t), u(x,t))$ be the corresponding solution of $(\mathcal{P})_V$, defined on a maximal Q_{T^*}. Then,*

(i) *either T^* is finite, in which case*

$$
\lim_{t \to T^*} \sup_{0 \le x \le 1} u(x,t) = \infty\,, \tag{4.39}
$$

(ii) *or, as $t \to \infty$,*

$$
v_x(x,t) = O(t^{-1})\,, \tag{4.40}
$$

uniformly for $x \in [0, y-\delta]$ *and* $x \in [y+\delta, 1]$,

$$v(x,t) = \begin{cases} O(t^{-1}) & x \in [0, y-\delta] \\ 1 + O(t^{-1}) & x \in [y+\delta, 1] \end{cases} \tag{4.41}$$

and $u(x,t)$ *converges monotonically, for* $x \notin (y-\delta, y+\delta)$, *to some bounded limiting function.*

Proof. On account of (H_4),

$$\lambda = \frac{\alpha \tau(\bar{u})^{\frac{1-n}{n}}}{\int_{\bar{u}}^{\infty} \tau(\xi)^{\frac{1}{n}} d\xi} < 1 \,. \tag{4.42}$$

With $\bar{u}$ as above, let (v_0, u_0) be as in (4.35) and let (v, u) be the corresponding solutions of $(\mathcal{P})_V$.

Suppose that T^* is infinite. Then $\psi(y,t) > 0$, for $0 \le t < \infty$, and, for any $x \notin (y-\delta, y+\delta)$, (4.34), (4.38) and (4.42) give

$$\frac{1+r}{r}\psi(y,t) \le \psi(x,t) + (-1 + \lambda(1+r)) \int_{\bar{u}}^{\infty} \tau(\xi)^{\frac{1}{n}} d\xi + (1 + \frac{1}{r}) \int_{U_M}^{\infty} \tau(\xi)^{\frac{1}{n}} d\xi \,. \tag{4.43}$$

Choose first r so that $(1+r)\lambda = \frac{1+\lambda}{2}$, and then choose U_M so that

$$(1 + \frac{1}{r}) \int_{U_M}^{\infty} \tau(\xi)^{\frac{1}{n}} d\xi \le \frac{1-\lambda}{4} \int_{\bar{u}}^{\infty} \tau(\xi)^{\frac{1}{n}} d\xi \,. \tag{4.44}$$

With these choices of λ and U_M,

$$0 < \frac{1+r}{r}\psi(y,t) \le \psi(x,t) - \frac{1-\lambda}{4} \int_{\bar{u}}^{\infty} \tau(\xi)^{\frac{1}{n}} d\xi \,. \tag{4.45}$$

Hence, for $x \in [0, y-\delta]$ or $x \in [y+\delta, 1]$,

$$\int_{u(x,t)}^{\infty} \tau(\xi)^{\frac{1}{n}} d\xi \ge \frac{1-\lambda}{4} \int_{\bar{u}}^{\infty} \tau(\xi)^{\frac{1}{n}} d\xi \,, \qquad 0 \le t < \infty \,, \tag{4.46}$$

and $u(x,\cdot)$ is bounded. Since $u_t > 0$, $u(x,t)$ converges as $t \to \infty$ to some bounded limiting configuration, for $x \notin (y-\delta, y+\delta)$. Finally, (4.40) follows from (1.1), (4.20) and (4.46), while (4.41) is derived by using (4.40) and the identities

$$v(x,t) = \int_0^x v_x(\xi,t) d\xi = 1 - \int_x^1 v_x(\xi,t) d\xi \,. \tag{4.48}$$

This completes the proof of the theorem. ∎

Acknowledgement. This research was partially supported by NSF under DMS-8716132 and ARO under DAALO3-88-K-0185.

References

[BPV] M. Bertsch, L. A. Peletier & S. M. Verduyn Lunel, The effect of temperature dependent viscosity on shear flow of incompressible fluids, SIAM J. Math. Anal. **22** (1991), 328-343.

[C] N. Charalambakis, Stabilité asymptotique lorsque $t \to \infty$ en thermoviscoelasticité et thermoplasticité, in "Nonlinear partial differential equations and their applications: Collège de France Seminar Vol IX", (H. Brezis and J.-L. Lions, eds.), Pitman Res. Notes in Math. No 181, Longman Scientific & Technical, Essex, England, 1988.

[CM] N. Charalambakis & F. Murat, Weak solutions to the initial-boundary value problem for the shearing of non-homogeneous thermoviscoplastic materials, Proc. Royal Soc. Edinburgh **113A** (1989), 257-265.

[CDM] H.T. Chen, A.S. Douglas & R. Malek-Madani, An asymptotic stability condition for inhomogeneous for inhomogeneous simple shear, Quart. Appl. Math. **47** (1989), 247-262.

[CCS] K. Chueh, C. Conley & J. Smoller, Positively invariant regions for systems of nonlinear diffusion equations, Indiana Univ. Math. J. **26** (1977), 372 - 411.

[D1] C. M. Dafermos, Global smooth solutions to the initial boundary value problem of one-dimensional nonlinear thermoviscoelasticity, SIAM J. Math. Analysis **13** (1982), 397-408.

[D2] C. M. Dafermos, Contemporary Issues in the Dynamic Behavior of Continuous Media, LCDS Lecture Notes # 85-1, 1985.

[DH1] C. M. Dafermos & L. Hsiao, Adiabatic shearing of incompressible fluids with temperature dependent viscosity, Quart. Appl. Math. **41** (1983), 45 - 58.

[DH2] C. M. Dafermos & L. Hsiao, Development of singularities in solutions of the equations of nonlinear thermoelasticity, Quart. Appl. Math. **44** (1986), 463-474.

[L] T.-P. Liu, Nonlinear stability of shock waves for viscous conservation laws, Amer. Math. Soc. Mem. **328** (1985).

[N] A. Needleman, Material rate dependence and mesh sensitivity in localization problems, Comp. Methods in Appl. Mech. and Engineering **67** (1988), 69 - 85.

[PW] M. H. Protter & H. F. Weinberger, *Maximum Principles in Differential Equations,* Prentice Hall, Englewood Cliffs, New Jersey, 1967.

[SC] T. G. Shawki & R. J. Clifton, Shear band formation in thermal viscoplastic materials, Mech. Mater. **8** (1989), 13 - 43.

[T1] A. E. Tzavaras, Shearing of materials exhibiting thermal softening or temperature dependent viscosity, Quart. Appl. Math. **44** (1986), 1 - 12.

[T2] A. E. Tzavaras, Plastic shearing of materials exhibiting strain hardening or strain softening, Arch. Rational Mech. Analysis **94** (1986), 39 - 58.

[T3] A. E. Tzavaras, Effect of thermal softening in shearing of strain-rate dependent materials, Arch. Rational Mech. Analysis **99** (1987), 349 - 374.

[T4] A. E. Tzavaras, Strain softening in viscoelasticity of the rate type, J. Integral Equations Appl. **3** (1991), 195-238.

[WW] T. W. Wright & J. W. Walter, On stress collapse in adiabatic shear bands, J. Mech. Phys. Solids **35** (1988), 701-720.