VISCOSITY AND RELAXATION APPROXIMATIONS FOR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

Athanasios E. Tzavaras

Abstract. These lecture notes deal with the approximation of conservation laws via viscosity or relaxation. The following topics are covered:

The general structure of viscosity and relaxation approximations is discussed, as suggested by the second law of thermodynamics, in its form of the Clausius-Duhem inequality. This is done by reviewing models of one dimensional thermoviscoelastic materials, for the case of viscous approximations, and thermomechanical theories with internal variables, for the case of relaxation.

The method of self-similar zero viscosity limits is an approach for constructing solutions to the Riemann problem, as zero-viscosity limits of an elliptic regularization of the Riemann operator. We present recent results on obtaining uniform BV estimates, in a context of strictly hyperbolic systems for Riemann data that are sufficiently close. The structure of the emerging solution, and the connection with shock admissibility criteria is discussed.

The problem of constructing entropy weak solutions for hyperbolic conservation laws via relaxation approximations is considered. We discuss compactness and convergence issues for relaxation approximations converging to the scalar conservation law, in a BV framework, and to the equations of isothermal elastodynamics, via compensated compactness.

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Department of Mathematics, University of Wisconsin, Madison, WI 53706, and Institute of Applied and Computational Mathematics, FORTH, 711 10 Heraklion, Crete. E-mail: tzavaras@math.wisc.edu

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1. Introduction

These lecture notes deal with the approximation of hyperbolic systems of conservation laws via viscosity or relaxation. Despite recent successes with analyzing these questions for systems of one-space dimension, their understanding remains incomplete and poses challenges to the theory of hyperbolic systems of conservation laws. A challenge amplified by the mere fact that theoretical understanding of such limiting processes reflects on the design and implementation of numerical algorithms for hyperbolic systems.

The interface between mechanical modeling and analytical theory has been a productive ground for the development of the theory of conservation laws. The problem of viscosity limits is intimately tied to the mechanical issue of passage from one continuum thermomechanical theory to another. In a similar fashion relaxation approximations, when viewed in the framework of continuum theories with internal variables, have analogous features. Therefore, we begin with the general structure of viscosity and relaxation approximations, as suggested by the second law of thermodynamics in its form of the Clausius-Duhem inequality. This presentation owes a lot to the point of view advocated by Dafermos [D_4]. Rather than stating the issues at an abstract level, we focus on the specific theories of thermoviscoelasticity, for the case of viscosity approximations, and thermomechanical theories with internal variables, for the case of relaxation.

We continue with a discussion of zero-viscosity limits for the scalar conservation law, in Section 3, and a presentation of self-similar viscosity limits in Section 4. The latter is an approach for constructing solutions of the Riemann problem, as zero-viscosity limits of an elliptic regularization of the Riemann operator. We present recent results on obtaining uniform BV estimates, in a context of strictly hyperbolic systems and for Riemann data that are sufficiently close [T_3]. The structure of the emerging solution, and the connection with shock admissibility criteria (in particular with the traveling wave criterion) is discussed.

In the last Section, we consider the problem of constructing entropy weak solutions for hyperbolic conservation laws via relaxation. Relaxation approximations exert a subtle dissipative effect on discontinuities as well as on oscillations, which is brought forth by analyzing their compactness and convergence properties. We present results of recent studies concerning relaxation limits to the scalar multi-d conservation law in a BV framework [KT_2], and to the system of isothermal elastodynamics via compensated compactness [T_4].

2. The Structure induced by Continuum Thermomechanics

Continuum physical theories are described by field equations that are called balance laws. A body occupying a reference configuration $\mathcal{R} \subset \mathbb{R}^d$ is deforming through the action of a map $y(\cdot, t) : \mathcal{R} \to \mathcal{R}_t$, $t > 0$, which carries the typical point $x \in \mathcal{R}$ to the point $y = y(x, t)$ in the current configuration $\mathcal{R}_t \subset \mathbb{R}^d$. The map y, called motion, is required to be a bi-Lipschitz homeomorphism. The integral balance laws,

$$\partial_t \int_\Omega g(x,t)dx + \int_{\partial\Omega} \sum_{\alpha=1}^{d} f_\alpha(x,t) n_\alpha dS = \int_\Omega h(x,t)dx \quad \text{for } \Omega \subset \mathcal{R},\ t > 0, \tag{2.1}$$

describe the rate of change of the vector-quantity $\int_\Omega g dx$, in a control volume Ω, due to the effect of flux through the boundary $\partial\Omega$ and production (or absorption) in Ω. The number of equations reflect the number N of balance laws in the continuum theory, the vector densities $g = (g^i)$ and $h = (h^i)$ express the balanced and produced quantities, respectively, while the flux terms are expressed through flux densities, $f^i \cdot n = \sum_{\alpha=1}^d f^i_\alpha n_\alpha$, where $f = (f^i_\alpha)$ takes values in $\mathbb{R}^{N\times d}$ and n is the outer normal to $\partial\Omega$. The balance laws may be expressed in a Lagrangean description, in terms of density fields g, f, h defined for $x \in \mathcal{R}$ and t, or in an Eulerian description, by density fields $\bar{g}$, $\bar{f}$, $\bar{h}$ defined for $y \in \mathcal{R}_t$ and t. The fields are connected through the formulas

$$g(x,t) = \bar{g}(y(x,t),t)\,, \quad f(x,t) = \bar{f}(y(x,t),t)\,, \quad h(x,t) = \bar{h}(y(x,t),t)\,. \tag{2.2}$$

If g, f, and h are smooth, the balance laws can be described through the local form

$$\partial_t g + \sum_{\alpha=1}^{d} \partial_\alpha f_\alpha = h\,, \tag{2.3}$$

obtained from the integral form by using the Gauss Theorem and averaging. (The local form is still valid for fields of bounded variation - whose distributional derivatives are locally finite Borel measures - in which case (2.3) is interpreted as an equality of measures.)

The balance laws are supplemented with constitutive relations, characterizing the material response, and yield evolution equations that describe the process. For instance, when the state of the material is described by the state vector $U \in \mathbb{R}^N$ and the material response is determined by the constitutive relations

$$g = G(U)\,, \quad f_\alpha = F_\alpha(U)\,, \quad h = H(U)\,, \tag{2.4}$$

with $G,\ F_\alpha,\ H : \mathbb{R}^N \to \mathbb{R}^N$, $\alpha = 1, \dots, d$, then (2.3) give rise to the first order system of conservation laws

$$\partial_t G(U) + \sum_{\alpha=1}^{d} \partial_\alpha F_\alpha(U) = H(U)\,, \tag{2.5}$$

where $x \in \mathbb{R}^d$, $t > 0$ and $U(x,t)$ takes values in $\mathbb{R}^N$. The constitutive relations (2.4) are the typical, abstract example of homogeneous elastic response. The system (2.5) comprises the equations of compressible gas flow, the equations describing dynamic deformations of nonlinear elastic materials and certain models of the equations of magnetohydrodynamics.

While the above derivation is appealing in its conciseness, it fails to address several mechanical considerations. One such consideration is that constitutive theories are required to be consistent with the second law of thermodynamics, to comply with the principle of material frame indifference, and to reflect existing material symmetries. In the sequel, we expand on the restrictions imposed on constitutive relations by the principle of consistency with the second law of thermodynamics and the ensuing structure of viscosity and relaxation approximations. For simplicity, the presentation is done in the context of one-dimensional thermomechanical theories.

2.a Thermomechanical theories in one-space dimension.

Thermomechanical theories seek to identify a pair of functions $(y(x,t), \theta(x,t))$ determining a thermomechanical process. The function $y(x,t)$ expresses the motion of the reference interval $[\alpha, \beta]$ while $\theta(x,t)$ stands for the temperature. The displacement $y(\cdot, t)$ is required, for each $t > 0$, to be a strictly increasing, bi-Lipschitz continuous map of the reference interval $[\alpha, \beta]$ onto the current configuration $[y(\alpha, t), y(\beta, t)]$ (see Fig. 1).

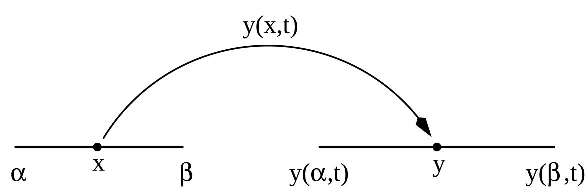

FIGURE 1.

The list of quantities entering in a Lagrangean description of the thermomechanical process are: $\rho_0(x)$ the mass density in the reference configuration, $\rho(y,t)$ the mass density in the current configuration, y the motion, $u = \frac{\partial y}{\partial x}$ the strain ($u > 0$), $v = \frac{\partial y}{\partial t}$ the velocity, τ the stress, f the body force per unit mass, θ the temperature ($\theta > 0$), e the specific internal energy, q the heat flux, r the radiating heat density and η the specific entropy. The equations

$$\rho(y,t)\frac{\partial y}{\partial x} = \rho_0(x) \tag{2.6}$$

$$\partial_t u - \partial_x v = 0 \tag{2.7}$$

$$\partial_t(\rho_0 v) - \partial_x \tau = \rho_0 f \tag{2.8}$$

$$\partial_t(\rho_0 \frac{1}{2} v^2 + \rho_0 e) = \partial_x(\tau v) + \partial_x q + \rho_0 f v + \rho_0 r \tag{2.9}$$

express the balance of mass, the kinematic compatibility relation, the balance of linear momentum, and the balance of energy (the first law of thermodynamics), respectively. They are supplemented with the Clausius-Duhem inequality, which reads, in integral form,

$$\frac{d}{dt}\int_a^b \rho_0 \eta \, dx \geq \frac{q}{\theta}(x,t)\Big|_{x=a}^{x=b} + \int_a^b \frac{\rho_0 r}{\theta} dx \qquad \text{for } [a,b] \subset [\alpha,\beta] \text{ and } t > 0, \tag{2.10}$$

or, in local form,

$$\rho_0 \partial_t \eta \geq \partial_x\big(\frac{q}{\theta}\big) + \frac{\rho_0 r}{\theta}\,. \tag{2.11}$$

The Clausius-Duhem inequality expresses that the net production of entropy per unit time, in any control volume $[a,b]$, is positive, and manifests (a form of) the second law of thermodynamics.

The thermomechanical variables are connected through constitutive relations that characterize the material response. A constitutive theory is determined by assigning a class of independent (prime) variables and a class of dependent variables, derived from the prime variables via constitutive relations. In this separation, the set of thermodynamic variables is implicitly divided into "causes" and "effects". From the phenomenological standpoint of continuum thermomechanics, there is no a-priori reason why a cause in one constitutive relation should not be a cause in another. Therefore, in determining the general form of constitutive theories, one imposes Truesdell's *principle of equipresence*, which states that a quantity present as an independent variable in one constitutive relation should be present in all, except if its presence contradicts some law of physics or material symmetry [TN]. Severe restrictions result from the second law of thermodynamics and the invariance under change of observers, called respectively *principle of consistency with the Clausius-Duhem inequality* and *principle of material frame indifference.*

The list of constitutive variables (prime and dependent) does not include the reference density ρ_0, the body force f, and the radiating heat transfer r, which are viewed as externally prescribed fields. Given a constitutive theory, the kinematic compatibility relation, and the balance laws of momentum and energy form a system of equations whose solution determines the thermomechanical process. In the Lagrangean description, the role of the balance of mass is to determine the current density ρ, once the process is identified. The role of the Clausius-Duhem inequality is subtler: For *smooth processes*, the Clausius-Duhem inequality is viewed as restricting the form of constitutive relations. By contrast for *non-smooth processes*[1], it becomes an additional constraint that weak solutions must satisfy. These points will be clarified in the context of specific constitutive theories.

For smooth processes the balance of energy, balance of linear momentum and Clausius-Duhem inequality imply the energy dissipation inequality

$$\rho_0\big(\partial_t e - \theta \partial_t \eta\big) - \tau u_t - \frac{q\theta_x}{\theta} \leq 0\,. \tag{2.12}$$

[1] The term non-smooth processes is used in a loose sense to signify processes containing shocks. It is a question of analysis to precise the smoothness class in each specific context.

Upon introducing the Helmholtz free energy $\psi = e - \theta\eta$, the latter takes the form

$$\rho_0 \partial_t \psi + \rho_0 \eta \partial_t \theta - \tau u_t - \frac{q\theta_x}{\theta} \le 0\,. \tag{2.13}$$

2.b The constitutive theory of thermoviscoelasticity.

In the constitutive theory of thermoviscoelasticity, the prime variables are the strain $u = y_x$, the strain rate $w = u_t = y_{xt}$, the temperature θ, and the temperature gradient $g = \theta_x$, while the dependent variables ψ, η, τ and q are determined through constitutive relations of the general form

$$\psi = \psi(u,w,\theta,g)\,, \quad \eta = \eta(u,w,\theta,g)\,, \quad \tau = \tau(u,w,\theta,g)\,, \quad q = q(u,w,\theta,g)\,, \tag{2.14}$$

following the principle of equipresence.

It is postulated that every smooth process is realizable and must be consistent with the Clausius-Duhem inequality. The postulate that smooth processes are realizable is compatible with both the balance laws and the constitutive relations, in the following sense. Given a smooth process $(y(x,t), \theta(x,t))$ and the referential density ρ_0, one computes u, w, θ, g and, in turn, ψ, η, τ, q and the internal energy $e = \psi + \theta\eta$. Then, the balance of mass determines the current density ρ, (2.7) is trivial, while the balance of momentum and balance of energy equations are satisfied by externally regulating the body force f and radiating heat r.

The energy dissipation inequality (2.13) implies that constitutive relations be constrained so that

$$\big(\rho_0 \psi_u - \tau\big)\dot u + (\rho_0 \psi_w)\dot w + \rho_0\big(\eta + \psi_\theta\big)\dot\theta + (\rho_0 \psi_g)\dot g - \big(\frac{qg}{\theta}\big) \le 0 \tag{2.15}$$

is satisfied for any smooth process. One constructs test processes, defined in the vicinity of (x_0, t_0), such that the local values of u, w, θ and g at (x_0, t_0) are assigned arbitrarily and, in addition, the local values of $\dot w$, $\dot\theta$ and $\dot g$ are assigned independently of the former. It follows that to comply with (2.15) the constitutive relations must be of the reduced form

$$\begin{aligned} \psi &= \psi(u,\theta) \\ \eta &= -\frac{\partial\psi}{\partial\theta}(u,\theta) \\ \tau &= \rho_0 \frac{\partial\psi}{\partial u} + Z(u,w,\theta,g) \\ q &= Q(u,w,\theta,g) \end{aligned} \tag{2.16}$$

where Q and Z are subject to the constraint

$$Zw + \frac{Qg}{\theta} \ge 0\,, \qquad \text{for any } u,\ w,\ \theta \text{ and } g. \tag{2.17}$$

Further analysis shows that $Z(u,0,\theta,0)=0$ and $Q(u,0,\theta,0)=0$. Hence,

$$\sigma(u,\theta) := \rho_0 \frac{\partial \psi}{\partial u} \tag{2.18}$$

is interpreted as the elastic part of the stress, Z as the viscous part of the stress, and the elastic part of the stress is derived from a potential. In practice, frequent use is made of the constitutive relations

$$Q = k(u,\theta) g\,, \quad Z = \mu(u,\theta) w\,, \tag{2.19}$$

where the viscous and heat conducting effects are decoupled. In that case, (2.17) dictates that the heat conductivity $k(u,\theta)$ and viscosity $\mu(u,\theta)$ coefficients are positive.

The constitutive relations of an ideal, viscous, heat conducting gas

$$\tau = -\rho_0 R \frac{\theta}{u} + \mu \frac{v_x}{u}\,, \quad e = c\theta\,, \quad q = k\frac{\theta_x}{u}\,, \qquad u, \theta > 0, \tag{2.20}$$

where R, c, μ and k are positive constants, are an example within the constitutive theory of thermoviscoelasticity. The free energy ψ and entropy η are given by

$$\psi(u,\theta) = -R\theta \ln u - c(\theta \ln \theta - \theta)\,, \quad \eta = -\frac{\partial \psi}{\partial \theta} = R \ln u + c \ln \theta\,, \tag{2.21}$$

and η is a concave function.

Constitutive theories should also comply with the principle of material frame indifference. In one-space dimension, the resulting restrictions are independence of the constitutive relations from the displacement y and the velocity v, and they have already been factored in (2.14). In several space dimensions, the restrictions are far more severe because of rotating frames, *c.f.* [TN].

The equations of one-dimensional thermoviscoelasticity take the form

$$\begin{aligned} &\partial_t u - \partial_x v = 0 \\ &\rho_0 \partial_t v - \partial_x \sigma(u,\theta) = (\mu v_x)_x + \rho_0 f \\ &\partial_t (\tfrac{1}{2}\rho_0 v^2 + \rho_0 e(u,\theta)) - \partial_x(\sigma(u,\theta) v) = (\mu v_x v)_x + (k\theta_x)_x + \rho_0 f v + \rho_0 r \end{aligned} \tag{2.22}$$

where we took Z, Q as in (2.19). The constitutive class is determined by the free energy function $\psi(u,\theta)$, in conjunction with the viscosity and conductivity coefficients $\mu = \mu(u,\theta) \geq 0$ and $k = k(u,\theta) \geq 0$. The remaining constitutive functions are determined by the thermodynamic relations

$$\sigma = \rho_0 \frac{\partial \psi}{\partial u}\,, \quad \eta = -\frac{\partial \psi}{\partial \theta}\,, \quad e = \psi + \theta\eta \tag{2.23}$$

Various derivative thermodynamic relations, like $\frac{\partial e}{\partial \theta} = \theta \frac{\partial \eta}{\partial \theta} = -\theta \frac{\partial^2 \psi}{\partial \theta^2}$, $\frac{\partial \sigma}{\partial \theta} = -\rho_0 \frac{\partial \eta}{\partial u}$, connect the thermodynamic functions.

By requirement, any smooth process is consistent with the Clausius-Duhem inequality. It is instructive to derive (2.11) directly from (2.22), (2.16) and (2.19). A calculation shows

$$\rho_0 \partial_t \eta(u,\theta) - \Big(\frac{k\theta_x}{\theta}\Big)_x = \frac{\mu v_x^2}{\theta} + \frac{k\theta_x^2}{\theta^2} + \frac{\rho_0 r}{\theta} \,. \tag{2.24}$$

The identity captures the dissipative structure of thermoviscoelastic materials, and is instrumental in global existence of smooth solutions for the system of thermoviscoelasticity [D3].

2.c A hierarchy of thermomechanical theories.

The theory of thermoviscoelasticity is equipped with sufficiently strong dissipative structure to guarantee the persistence of smooth processes. On the other extreme, is the theory of thermoelastic non-conducting materials ($Z = 0$ and $Q = 0$), described by the system of equations

$$\begin{aligned}
\partial_t u - \partial_x v &= 0 \\
\rho_0 \partial_t v - \partial_x \sigma(u,\theta) &= \rho_0 f \\
\partial_t \Big(\frac{1}{2}\rho_0 v^2 + \rho_0 e(u,\theta)\Big) - \partial_x(\sigma(u,\theta)v) &= \rho_0 f v + \rho_0 r
\end{aligned} \tag{2.25}$$

with constitutive relations (2.23). If $\frac{\partial \eta}{\partial \theta} > 0$ and $\frac{\partial \sigma}{\partial u} > 0$, then (2.25) is hyperbolic with characteristic speeds $\lambda_\pm = \pm\big(\frac{\sigma_u}{\rho_0} + \frac{\sigma_\theta^2}{\rho_0^2 \eta_\theta}\big)^{1/2}$, $\lambda_0 = 0$. Under conditions of compression smooth processes can break down and develop shock waves. The theory of thermoelastic nonconductors of heat is regarded as a limiting theory of thermoviscoelasticity as the viscosity and heat conductivity tend to zero. Accordingly, non-smooth thermomechanical processes inherit the constraint

$$\rho_0 \partial_t \eta(u,\theta) \geq \frac{\rho_0 r}{\theta} \,, \tag{2.26}$$

and the Clausius-Duhem inequality becomes a restriction on admissible non-smooth processes.

The constitutive theory of thermoelasticity is an intermediate theory, appropriate for materials that the stress and the heat flux are independent of the strain rate. Thermoelastic materials are characterized by the constitutive relations

$$\begin{aligned}
&\psi = \psi(u,\theta) \\
&\eta = -\frac{\partial \psi}{\partial \theta}(u,\theta) \qquad \tau = \rho_0 \frac{\partial \psi}{\partial u}(u,\theta) \\
&q = Q(u,\theta,g)\,, \qquad \text{subject to} \;\; Qg \geq 0,
\end{aligned} \tag{2.27}$$

that are consistent for smooth processes with the Clausius-Duhem inequality (2.11). Processes of thermoelastic materials, with a Fourier law $Q = kg$, are described by the system (2.22) with $\mu = 0$. Again non-smooth processes inherit (2.11) as an admissibility restriction.

Isothermal motions of thermoelastic materials are processes $(y(x,t), \theta(x,t))$, where the temperature is kept constant, $\theta = \theta_0$, and accordingly $Q = Q(u, \theta_0, 0) = 0$. They are described by the equations

$$
\begin{aligned}
&\partial_t u - \partial_x v = 0 \\
&\rho_0 \partial_t v - \partial_x \sigma(u, \theta_0) = \rho_0 f \,,
\end{aligned}
\tag{2.28}
$$

that are pertinent to a purely mechanical process. When $\sigma_u > 0$ the system (2.28) is strictly hyperbolic, with characteristic speeds $\lambda_\pm = \pm\big(\frac{\sigma_u}{\rho_0}\big)^{1/2}$, and non-smooth processes can appear due to formation of shock waves. It is instructive to regard this situation as a limiting case of the theory of thermoviscoelasticity. From a mechanical viewpoint, isothermal processes are attained by externally controlling the radiation heat transfer r so that $\theta = \theta_0$ and $Q = 0$. The balance of energy $(2.22)_3$ and (2.24) imply

$$
\rho_0 \partial_t \big(\frac{1}{2} v^2 + [e(u,\theta_0) - \theta_0 \eta(u,\theta_0)]\big) - \partial_x \big(\sigma(u,\theta_0)\, v\big) + \mu v_x^2 = (\mu v_x v)_x + \rho_0 f v \,. \tag{2.29}
$$

In the zero-viscosity limit, non-smooth mechanical processes inherit from thermodynamics the admissibility constraint

$$
\rho_0 \partial_t \big(\frac{1}{2} v^2 + \psi(u,\theta_0)\big) - \partial_x \big(\sigma(u,\theta_0)\, v\big) \le \rho_0 f v \,. \tag{2.30}
$$

In gas dynamics, it is customary to express the pressure in terms of the specific volume u and the entropy η. This can be attained by assuming that $\frac{\partial \eta}{\partial \theta} > 0$, inverting the equation $\eta = -\frac{\partial \psi}{\partial \theta}$, and writing the constitutive theory (2.23) in the form

$$
e = e(u,\eta)\,, \quad \theta = \frac{\partial e}{\partial \eta}(u,\eta)\,, \quad \sigma = -p = \rho_0 \frac{\partial e}{\partial u}\,, \tag{2.31}
$$

where p is the pressure function. Non-smooth processes of thermoelastic nonconductors of heat have to comply with (2.26), which of course is still valid under the expression (2.31) of the constitutive relations. Isentropic motions of thermoelastic nonconductors ($\eta = \eta_0$ constant) are described by the system of equations

$$
\begin{aligned}
&\partial_t u - \partial_x v = 0 \\
&\rho_0 \partial_t v + \partial_x p(u, \eta_0) = \rho_0 f \,,
\end{aligned}
\tag{2.32}
$$

which is strictly hyperbolic system when $p_u < 0$. Non-smooth isentropic processes inherit from the expression (2.26) of the second law of thermodynamics the admissibility constraint

$$
\rho_0 \partial_t \big(\frac{1}{2} v^2 + e(u,\eta_0)\big) + \partial_x \big(p(u,\eta_0)\, v\big) \le \rho_0 f v \,. \tag{2.33}
$$

2.d Materials with internal variables.

Viscosity and heat conduction are one of the possible ways of prescribing dissipative mechanisms. Complementary descriptions of dissipation are supplied by the theory of simple materials with fading memory and the theory of materials with internal state variables. The class of *simple materials* consists of those materials for which the free energy, entropy, stress and heat flux at any point x and time t can be described in terms of the present value of the temperature gradient g at (x,t) and the history of strain u and temperature θ at the point x at all times prior to t. Under conditions of fading memory, simple materials are equipped with a subtle dissipative mechanism, brought forth by analyzing their thermodynamics [Co].

The class of materials with internal state variables is a subclass of materials with fading memory, which is appealing in its simplicity and encompasses some interesting models (like the ideal gas with vibrational relaxation). In a theory with internal variables, the thermomechanical process is described by a vector function $(y(x,t), \theta(x,t), \alpha(x,t))$, where y is the motion, θ the temperature, and the internal vector-variable α evolves according to a differential law

$$\partial_t \alpha = F(u, \theta, \alpha)\,. \tag{2.34}$$

The remaining thermodynamic quantities are determined by constitutive relations of the form

$$\psi = \Psi(u,\theta,g,\alpha)\,, \quad \eta = H(u,\theta,g,\alpha)\,, \quad \tau = S(u,\theta,g,\alpha)\,, \quad q = Q(u,\theta,g,\alpha)\,. \tag{2.35}$$

In rough terms, such models have fading memory when the differential equation (2.34) is exponentially dissipative.

We pursue the implications of the Clausius-Duhem inequality on the form of the constitutive functions. A remark is in order: while (2.35) satisfies the principle of equipresence, the differential constraint (2.34) does not. In fact, (2.34) is not viewed here as a constitutive relation but rather as defining the class of admissible processes. This simplifies somewhat the reduction process, while it is compatible with specific examples that motivate this theory. We refer to [CG] for the case that F also depends on g.

Consistency with the Clausius-Duhem inequality is tested against all admissible processes, that is all smooth processes that are compatible with the differential constraint (2.34). A count of equations and unknowns indicates that all admissible processes can be realized, by externally regulating f and r so as to fulfill the balance of momentum and energy. Then (2.13), (2.34) and (2.35) imply

$$(\rho_0 \Psi_u - S)\dot{u} + \rho_0(\Psi_\theta + H)\dot{\theta} + \rho_0 \Psi_g\, \dot{g} + \rho_0 \Psi_\alpha \cdot F(u,\theta,\alpha) - \frac{Qg}{\theta} \le 0 \tag{2.36}$$

for all admissible processes. Since the local values of u, θ, α, g, θ_t, u_t and g_t can be assigned independently, the constitutive relations have the reduced form

$$\begin{aligned}
\psi &= \Psi(u,\theta,\alpha) \\
\tau = S &= \rho_0 \frac{\partial \Psi}{\partial u} \\
\eta = H &= -\frac{\partial \Psi}{\partial \theta} \\
q &= Q(u,\theta,g,\alpha)
\end{aligned} \tag{2.37}$$

subject to the constraint

$$-\frac{\partial \Psi}{\partial \alpha} \cdot F(u,\theta,\alpha) + \frac{1}{\theta} Q(u,\theta,g,\alpha) g \geq 0 \quad \text{for all } u,\theta,g,\alpha. \tag{2.38}$$

It follows from (2.38) that

$$-\frac{\partial \Psi}{\partial \alpha} \cdot F(u,\theta,\alpha) \geq 0 \quad \text{for all } u,\theta,\alpha. \tag{2.39}$$

If Q is given by a Fourier law for heat conduction, $Q = k(u,\theta,\alpha)g$, then (2.38) is equivalent to (2.39) and $k \geq 0$.

The thermomechanical process $(y(x,t), \theta(x,t), \alpha(x,t))$ is described by (2.7-2.9) supplemented with (2.34) and the constitutive relations (2.37-2.39). For smooth processes with Fourier heat conduction, a direct computation yields

$$\rho_0 \partial_t H(u,\theta,\alpha) - \left(\frac{k\theta_x}{\theta}\right)_x = -\rho_0 \frac{1}{\theta} \Psi_\alpha \cdot F(u,\theta,\alpha) + \frac{k\theta_x^2}{\theta^2} + \frac{\rho_0 r}{\theta}. \tag{2.40}$$

Equation (2.40) captures the dissipative structure of a heat conducting thermoelastic material with internal variables.

2.e A thermomechanical model with stress relaxation.

Thermomechanical theories with internal variables provide a natural framework to consider the structure of relaxation approximations to conservation laws, in the continuum physics context. To develop the connections, consider a theory with one scalar internal variable α evolving according to the differential law

$$\partial_t \alpha = -\lambda(\alpha - h(u,\theta)). \tag{2.41}$$

This law is of exponential dissipative type with relaxation time $\frac{1}{\lambda}$ and equilibrium states $\alpha_{eq} = h(u,\theta)$. The internal variable theory is completed with constitutive relations $\psi = \Psi(u,\theta,\alpha)$ for the free energy, $\tau = S(u,\theta,\alpha)$ for the stress, $\eta = H(u,\theta,\alpha)$ for the entropy and a Fourier law, $q = Q = k(u,\theta,\alpha)g$, for the heat flux. The constitutive functions are required to satisfy (2.37-2.39), with $F = -\lambda(\alpha - h(u,\theta))$, so that the internal variable theory is consistent with the second law of thermodynamics and is equipped with the dissipation estimate (2.40). Then the function $-H(u,\theta,\alpha)$ provides, in the terminology of [CLL], a (possibly not convex) "entropy" function for the emerging relaxation process. We are interested to explore the relation of the thermomechanical model corresponding to $\lambda > 0$ with the model emerging in the small-relaxation time limit $\lambda \to \infty$.

In practice, one is often faced with the question: under what conditions is a given set of constitutive functions Ψ, S and H achieved from a theory consistent with the second law of thermodynamics. For example, suppose we are given a stress distribution $S(u,\theta,\alpha)$. Then the question becomes to investigate if there a free energy function $\Psi(u,\theta,\alpha)$ such that

$$\begin{aligned} \frac{\partial\Psi}{\partial u} &= \frac{1}{\rho_0} S(u,\theta,\alpha) \\ \text{subject to} \quad \frac{\partial\Psi}{\partial\alpha}(\alpha - h(u,\theta)) &\geq 0 \quad \text{for all } u,\theta,\alpha. \end{aligned} \tag{2.42}$$

Note that (2.42) implies in particular that Ψ satisfies

$$\begin{cases} \frac{\partial\Psi}{\partial\alpha} \geq 0 & \text{for } \alpha > h(u,\theta) \\ \frac{\partial\Psi}{\partial\alpha} = 0 & \text{for } \alpha = \alpha_{eq} = h(u,\theta) \\ \frac{\partial\Psi}{\partial\alpha} \leq 0 & \text{for } \alpha < h(u,\theta), \end{cases} \tag{2.43}$$

and that, since solutions of $(2.42)_1$ are given by

$$\rho_0 \Psi(u,\theta,\alpha) = G(\theta,\alpha) + \int_0^u S(\xi,\theta,\alpha)\, d\xi\,, \tag{2.44}$$

the inequality $(2.42)_2$ is satisfied if and only if there is a function $G(\theta,\alpha)$ such that

$$\Big(G_\alpha(\theta,\alpha) + \int_0^u S_\alpha(\xi,\theta,\alpha)\, d\xi\Big)\Big(\alpha - h(u,\theta)\Big) \geq 0 \quad \text{for all } u,\theta,\alpha. \tag{2.45}$$

We emphasize that solving (2.45) is equivalent to deciding whether the given model with internal variables is consistent with the second law of thermodynamics, and that, for (2.45) to admit solutions, conditions must be imposed on the functions S and h. For instance, (2.43) implies

$$G_\alpha(\theta, h(u,\theta)) = -\int_0^u S_\alpha(\xi,\theta,h(u,\theta))\, d\xi \tag{2.46}$$

Given a solution G, the associated free energy function is given by (2.44).

We next consider a special case, where the given stress distribution is

$$S(u,\theta,\alpha) = f(u,\theta) + \alpha\,. \tag{2.47}$$

This case is completely solvable. Indeed, (2.45) reads: is there a function $G(\theta, \alpha)$ such that $j(\theta, \alpha) := -G_\alpha(\theta, \alpha)$ satisfies

$$\Big(u - j(\theta, \alpha)\Big)\Big(\alpha - h(u, \theta)\Big) \geq 0 \quad \text{for all } u, \theta, \alpha. \tag{2.48}$$

It is easy to see that this happens if and only if $h(u, \theta)$ is strictly decreasing in u, $j(\theta, \alpha)$ is strictly decreasing in α, and $j = h^{-1}$ is the inverse function of h for θ fixed,

$$j(\theta, h(u, \theta)) = u\,, \quad h(j(\theta, \alpha), \theta) = \alpha\,.$$

For simplicity, we assume the slightly stronger condition $h_u(u, \theta) < 0$ and note that the associated G is given by the formula

$$G(\theta, \alpha) = -\int_0^\alpha j(\theta, \zeta) d\zeta - \int_1^\theta s(z)\, dz\,, \tag{2.49}$$

where s is an arbitrary function of θ. In turn, the constitutive functions of the internal variable theory are $S(u, \theta, \alpha) = f(u, \theta) + \alpha$, as requested, and

$$\begin{aligned} \rho_0 \psi = \rho_0 \Psi(u, \theta, \alpha) &= -\int_0^\alpha j(\theta, \zeta) d\zeta - \int_1^\theta s(z)\, dz + \alpha u + \int_0^u f(\xi, \theta) d\xi\,, \\ \rho_0 \eta = \rho_0 H(u, \theta, \alpha) &= \int_0^\alpha j_\theta(\theta, \zeta) d\zeta + s(\theta) - \int_0^u f_\theta(\xi, \theta) d\xi\,. \end{aligned} \tag{2.50}$$

As an application, consider a model for a viscoelastic material where the total stress τ is decomposed into a viscoelastic part, evolving according to stress relaxation, and a viscous part with Newtonian viscosity,

$$\begin{aligned} \tau = \sigma + \mu v_x\,, &\quad \mu \geq 0 \\ \partial_t(\sigma - f(u, \theta)) &= -\lambda(\sigma - g(u, \theta))\,. \end{aligned} \tag{2.51}$$

The viscoelastic part of the stress may be put into the integral form,

$$\sigma(\cdot, t) = f(u, \theta)(\cdot, t) + \int_{-\infty}^t \lambda e^{-\lambda(t-s)} \big(g(u, \theta) - f(u, \theta)\big)(\cdot, s)\, ds\,, \tag{2.52}$$

of a Maxwell type viscoelastic fluid with memory. The function $f(u, \theta)$ describes the instantaneous elastic stress-strain response, while $g(u, \theta)$ describes the equilibrium stress-strain response.

The inviscid version of (2.51) is formulated in the context of internal variables by setting

$$\begin{aligned} \sigma &= f(u, \theta) + \alpha \\ \partial_t \alpha &= -\lambda(\alpha - h(u, \theta)) \qquad \text{with } h(u, \theta) := g(u, \theta) - f(u, \theta). \end{aligned} \tag{2.53}$$

The model is achieved from a theory consistent with the second law of thermodynamics if and only if the functions f and g satisfy $(g-f)(u,\theta)$ is strictly decreasing in u. Henceforth, we focus on functions satisfying

$$g_u(u,\theta) < f_u(u,\theta) \tag{2.54}$$

while the free energy ψ and entropy η are determined by (2.50) for $\alpha = \sigma - f(u,\theta)$.

The thermomechanical process $\big(y(x,t), \theta(x,t), \sigma(x,t)\big)$, associated to the material model (2.51), is described by the system of equations

$$\begin{aligned}
\partial_t u - \partial_x v &= 0 \\
\rho_0 \partial_t v - \partial_x \sigma &= (\mu v_x)_x + \rho_0 f \\
\partial_t \Big(\frac{1}{2}\rho_0 v^2 + \rho_0 e\Big) - \partial_x(\sigma v) &= (\mu v_x v)_x + (k\theta_x)_x + \rho_0 f v + \rho_0 r \\
\partial_t \big(\sigma - f(u,\theta)\big) &= -\lambda(\sigma - g(u,\theta))
\end{aligned} \tag{2.55}$$

where the internal energy is determined by

$$\begin{aligned}
\rho_0 e =& \rho_0\big(\psi + \theta\eta\big) = \rho_0(\Psi + \theta H)(u,\theta,\sigma - f(u,\theta)) \\
=& \int_0^{\sigma - f(u,\theta)} \big(\theta j_\theta - j\big)(\theta,\zeta)d\zeta + \Big(\theta s(\theta) - \int_1^\theta s(z)\,dz\Big) \\
&+ (\sigma - f(u,\theta))u + \int_0^u \big(f - \theta f_\theta\big)(\xi,\theta)d\xi\,.
\end{aligned} \tag{2.56}$$

A direct computation using (2.55), in conjunction with (2.37), (2.47) and (2.50), shows that the thermomechanical process is equipped with the dissipation estimate

$$\begin{aligned}
\rho_0 \partial_t \Big(H(u,\theta,\sigma - f(u,\theta))\Big) - \Big(\frac{k\theta_x}{\theta}\Big)_x =& \lambda \frac{1}{\theta}(u - h^{-1}(\theta,\alpha))(\alpha - h(u,\theta))\Big|_{\alpha = \sigma - f(u,\theta)} \\
&+ \frac{k\theta_x^2}{\theta^2} + \frac{\mu v_x^2}{\theta} + \frac{\rho_0 r}{\theta}\,,
\end{aligned} \tag{2.57}$$

which, in view of (2.54) and (2.48), implies that smooth processes satisfy the Clausius-Duhem inequality, for all values of $\lambda > 0$ and μ, $k \geq 0$, and yields an estimate for the amount of dissipation.

The model (2.51), for materials with stress relaxation, gives rise to a hierarchy of thermomechanical theories as the parameters describing the viscosity μ and heat-conductivity k tend to zero, and to a second hierachy of theories as the relaxation parameter λ tends to infinity. In the limit $\lambda \to \infty$, one formally obtains the theory of thermoviscoelasticity presented in Section 2.b. As both $\lambda \to \infty$ and μ and/or k tend to zero one can obtain the various thermomechanical theories mentioned in Section 2.c. Any non-smooth limit processes inherit the limit form of the dissipation estimate (2.57).

We close by considering the case of isothermal motions, that is processes along which $\theta = \theta_0$ is kept constant and $Q = 0$. The process is described now by the equations

$$
\begin{aligned}
\partial_t u - \partial_x v &= 0 \\
\rho_0 \partial_t v - \partial_x \sigma &= (\mu v_x)_x + \rho_0 f \\
\partial_t \big(\sigma - f(u, \theta_0) \big) &= -\lambda (\sigma - g(u, \theta_0))
\end{aligned}
\tag{2.58}
$$

and inherits from thermodynamics the dissipative structure

$$
\begin{aligned}
&\partial_t \Big(\frac{1}{2} \rho_0 v^2 + \rho_0 \Psi(u, \theta_0, \sigma - f(u, \theta_0)) \Big) - \partial_x (\sigma v) \\
&\qquad + \mu v_x^2 + \lambda (u - h^{-1}(\theta_0, \alpha))(\alpha - h(u, \theta_0)) \Big|_{\alpha = \sigma - f(u, \theta_0)} = (\mu v_x v)_x + \rho_0 f v \,.
\end{aligned}
\tag{2.59}
$$

The limiting theory $\mu \to 0$ is described by

$$
\begin{aligned}
\partial_t u - \partial_x v &= 0 \\
\rho_0 \partial_t v - \partial_x \sigma &= \rho_0 f \\
\partial_t \big(\sigma - f(u, \theta_0) \big) &= -\lambda (\sigma - g(u, \theta_0)) \,.
\end{aligned}
\tag{2.60}
$$

It is known that the stress relaxation equation exerts a subtle dissipative effect on smooth processes, and as a result the system admits smooth solutions for initial data close to equilibrium. By contrast, for data away from equilibrium shock waves can develop in finite time, [D$_4$]. The inviscid theory inherits the dissipative structure

$$
\begin{aligned}
&\partial_t \Big(\frac{1}{2} \rho_0 v^2 + \rho_0 \Psi(u, \theta_0, \sigma - f(u, \theta_0)) \Big) - \partial_x (\sigma v) \\
&\qquad + \lambda (u - h^{-1}(\theta_0, \alpha))(\alpha - h(u, \theta_0)) \Big|_{\alpha = \sigma - f(u, \theta_0)} \le \rho_0 f v \,,
\end{aligned}
\tag{2.61}
$$

with equality for smooth isothermal processes.

In the limit $\lambda \to \infty$, the internal variable theory (2.60) yields the equations of one-dimensional isothermal elasticity,

$$
\begin{aligned}
\partial_t u - \partial_x v &= 0 \\
\rho_0 \partial_t v - \partial_x g(u, \theta_0) &= \rho_0 f \,,
\end{aligned}
\tag{2.62}
$$

a strictly hyperbolic system when $g_u > 0$. If $f_u > g_u$ the internal variable theory is consistent with the Clausius-Duhem inequality. (Remarkably, this is precisely the subcharacteristic condition for the associated relaxation process, *i.e.* consistency with the second law of thermodynamics implies, in this context, the subcharacteristic condition.) The function

$$\rho_0 \Psi(u, \theta_0, \alpha) = -\int_0^\alpha h^{-1}(\theta_0, \zeta) d\zeta + \alpha u + \int_0^u f(\xi, \theta_0) d\xi \tag{2.63}$$

provides an "entropy" function for the associated relaxation process, which is convex in (u, α) if $-\partial_\alpha h^{-1} \partial_u f \geq 1$ for all u and α.

Bibliographic remarks. We refer to books on Continuum Mechanics, *e.g.* Truesdell and Noll [TN], on the topics of consistency of constitutive relations with the second law of thermodynamics, the principle of material frame indifference, and the effect of material symmetries. The requirements imposed by consistency with the Clausius-Duhem inequality are developed in Coleman-Noll [CN] and Coleman-Mizel [CM] for the theory of thermoviscoelasticity, in Coleman [Co] for simple materials with fading memory, and in Coleman-Gurtin [CG] for materials with internal state variables. The thermodynamical structure of mechanical theories with internal variables has been extensively investigated in the mechanics literature, *c.f.* Coleman-Gurtin [CG], Gurtin-Williams-Suliciu [GWS], Faciu and Mihailescu-Suliciu [FM], Suliciu [Su] and references therein. Since consistency with the second law of thermodynamics leads to "entropy" functions for the relaxation process, the issue is important in both the design of numerical relaxation schemes, Coquel-Perthame [CPe], as well as for theoretical investigations of relaxation, Tzavaras [T_4]. There is an extensive literature on the classification of the strength of dissipation, for various mechanical theories, and the related issue of global existence of smooth processes. We refer to Dafermos [D_4] for a survey of results prior to 1985.

3. Zero-Viscosity Limits for the Scalar Conservation Law

The problem of zero-viscosity limits consists of constructing weak solutions of the hyperbolic system

$$\partial_t U + \partial_x F(U) = 0\,, \qquad x \in \mathbb{R} \quad t > 0\,, \tag{3.1}$$

as $\varepsilon \to 0$ limits of the viscous system

$$\partial_t U + \partial_x F(U) = \varepsilon\, \partial_x (B(U) U_x)\,, \tag{3.2}$$

where $U(x,t)$ takes values in $\mathbb{R}^N$ and $B(U)$ is a positive semidefinite diffusion matrix expressing the viscous structure.

Most of the analysis regarding this question is based on the notion of entropy-entropy flux pairs. A scalar-valued function $\eta(U)$ is called an entropy with corresponding entropy flux $q(U)$ if every smooth solution of the conservation law (3.1) satisfies the additional conservation law

$$\partial_t \eta(U) + \partial_x q(U) = 0\,. \tag{3.3}$$

Pairs $(\eta(U), q(U))$ are generated by solving the system of linear differential equations

$$\nabla q(U) = \nabla \eta(U) \cdot \nabla F(U)\,. \tag{3.4}$$

Trivial solutions are provided by $(c \cdot U\,,\, c \cdot F(U))$, with c any constant vector in $\mathbb{R}^N$. Since (3.4) is underdetermined of $N = 1$, determined for $N = 2$ and overdetermined for $N \geq 3$, for systems of two equations there exist many entropies, but for larger systems the existence of nontrivial entropies is the exception rather than the rule. Nevertheless, specific systems arising in applications are often endowed with some entropy-entropy flux pairs.

In the sequel we present the convergence of viscosity limits

$$u_t + f(u)_x = \varepsilon u_{xx} \tag{3.5}$$

to the scalar conservation law

$$u_t + f(u)_x = 0\,. \tag{3.6}$$

Let $\lambda(u) = f'(u)$. Consider a family of approximate solutions u^ε emanating from initial data u_0^ε that are stable in $L^2 \cap L^\infty$. By the maximum principle the family u^ε is stable in L^∞

$$|u^\varepsilon| \leq C\,, \tag{3.7}$$

and, by the representation theorem for Young measures [Ta], there exists a subsequence (denoted again by u^ε) and a measurable family of probability measures $\nu = \nu_{(x,t)}$ such that

$$f(u^\varepsilon) \rightharpoonup < \nu, f(k) >\,, \qquad \text{for any continuous } f. \tag{3.8}$$

Solutions of the viscosity problem satisfy the uniform bound

$$\int_{\mathbb{R}} \frac{1}{2} u^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 \leq \int_{\mathbb{R}} \frac{1}{2} {u_0^\varepsilon}^2 dx \leq O(1) \tag{3.9}$$

Let (η, q) be any C^2 entropy pair, $q'(u) = \lambda(u)\eta'(u)$. Along solutions of the viscosity problem

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) = \varepsilon \partial_x (\eta_u(u^\varepsilon) u_x^\varepsilon) - \varepsilon \eta_{uu}(u^\varepsilon) {u_x^\varepsilon}^2 = I_1 + I_2\,. \tag{3.10}$$

The term I_1 converges to zero in H^{-1}, the term I_2 is uniformly bounded in L^1, and the sum $I_1 + I_2$ is uniformly bounded in $W^{-1,\infty}$. It follows from [M$_2$] that $I_1 + I_2$ lies in a compact of H^{-1}_{loc}. One concludes with the following theorem due to Tartar [Ta]. The proof presented here is taken out of [PTz].

Theorem 3.1. *Suppose that*

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) \quad \textit{lies in a compact of } H^{-1}_{loc} \tag{3.11}$$

for any (η, q) *with* $\eta_u \in C^1_c(\mathbb{R})$. *Then either the support of the Young measure* ν *is a point or else it is contained in an interval where* f *is linear,*

$$supp\ \nu_{(x,t)} \subset \{\xi \in I : \lambda(\xi) = const.\} \tag{3.12}$$

Proof. Consider the following classes of entropy-entropy flux pairs (motivated by the kinetic formulation of the scalar conservation law):

$$\begin{aligned} \eta_1(u) &= \int_{-\infty}^{u} \phi(\xi) d\xi = \int_{\mathbb{R}} \mathbb{1}_{u>\xi} \phi(\xi) d\xi \\ q_1(u) &= \int_{-\infty}^{u} \lambda(\xi)\phi(\xi) d\xi = \int_{\mathbb{R}} \mathbb{1}_{u>\xi} \lambda(\xi)\phi(\xi) d\xi \end{aligned}$$

$$\begin{aligned} \eta_2(u) &= \int_{u}^{\infty} \psi(\theta) d\theta = \int_{\mathbb{R}} \mathbb{1}_{u<\theta} \psi(\theta) d\theta \\ q_2(u) &= \int_{u}^{\infty} \lambda(\theta)\psi(\theta) d\theta = \int_{\mathbb{R}} \mathbb{1}_{u<\theta} \lambda(\theta)\psi(\theta) d\theta \end{aligned}$$

where ϕ, $\psi \in C^1_c(\mathbb{R})$. Both pairs are constant near infinity, the first pair represents entropies that vanish at $-\infty$ and the second entropies vanishing at ∞.

Applying the usual compensated compactness bracket,

$$< \nu, \eta_1 q_2 - \eta_2 q_1 > = < \nu, \eta_1 >< \nu, q_2 > - < \nu, \eta_2 >< \nu, q_1 >, \tag{3.13}$$

to the pairs, we obtain, following an application of Fubini's Theorem,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \Big[[\lambda(\theta) - \lambda(\xi)] (\overline{\mathbb{1}_{u>\xi}\mathbb{1}_{u<\theta}} - \overline{\mathbb{1}_{u>\xi}}\ \overline{\mathbb{1}_{u<\theta}}) \Big] \phi(\xi)\psi(\theta) d\xi d\theta = 0\,, \tag{3.14}$$

where the notation

$$\overline{\mathbb{1}_{u>\xi}} = \int \mathbb{1}_{u>\xi} d\nu\,.$$

From (3.14) we deduce

$$[\lambda(\theta) - \lambda(\xi)]\,(\overline{\mathbb{1}_{u>\xi}\mathbb{1}_{u<\theta}} - \overline{\mathbb{1}_{u>\xi}}\ \overline{\mathbb{1}_{u<\theta}}) = 0 \quad \text{a.e } \xi, \theta \tag{3.15}$$

and in turn

$$[\lambda(\theta) - \lambda(\xi)]\, \overline{\mathbb{1}_{u>\xi}}\ \overline{\mathbb{1}_{u<\theta}} = 0 \quad \text{for a.e } \xi, \theta \text{ with } \xi > \theta\,. \tag{3.16}$$

Let $F(\xi)$ be the distribution function of ν, defined by $F(\xi) = \nu\big((-\infty, \xi]\big)$. Then F is right continuous, increasing and

$$F(\xi) = \int \mathbb{1}_{u<\xi} d\nu \quad \text{a.e } \xi.$$

Using (3.16), we conclude (upon taking limits)

$$[\lambda(\theta) - \lambda(\xi)]\,(1 - F(\xi))\,F(\theta) = 0 \quad \text{for all } \xi, \theta \text{ with } \xi > \theta. \tag{3.17}$$

If the supp ν is not a point, let ξ, θ be two points in supp ν with $\xi > \theta$. Then $0 < F(\theta) \leq F(\xi) < 1$ and (3.17) implies $\lambda(\xi) = \lambda(\theta)$ and concludes the proof. $\square$

Bibliographic Remarks. The notion of entropy-entropy flux pairs Lax [La$_2$] and the theory of compensated compactness of Murat [M$_1$] and Tartar [Ta] play an important role in the analysis of viscosity limits - in one space dimension - for the scalar conservation law, Tartar [Ta], for several systems of two equations, *e.g.* DiPerna [Dp$_1$, Dp$_2$], Serre [Se], Chen [Ch], Lin [Lin], Shearer [Sh], Serre-Shearer [SeSh], Lions-Perthame-Tadmor [LPT$_2$] and Lions-Perthame-Souganidis [LPS], and for systems containing rich families of entropies, *e.g.* Heibig [H].

4. The Riemann Problem and Self-Similar Viscosity Limits

We consider the strictly hyperbolic system of conservation laws

$$\partial_t U + \partial_x F(U) = 0\,, \qquad x \in \mathbb{R} \quad t > 0\,, \tag{4.1}$$

where $U(x,t)$ takes values in $\mathbb{R}^N$ and the Jacobian matrix $\nabla F(U)$ has real and distinct eigenvalues $\lambda_1(U) < \lambda_2(U) < \ldots < \lambda_N(U)$. The right and left eigenvectors $r_i(U)$ and $l_i(U)$ are linearly independent and are normalized,

$$\nabla F\, r_i = \lambda_i\, r_i\,, \quad l_i \cdot \nabla F = \lambda_i\, l_i\,, \quad l_i \cdot r_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}; \tag{4.2}$$

$\{r_i\}$ and $\{l_i\}$ form a pair of local bases in the state space $\mathbb{R}^N$.

The Riemann problem consists of solving (4.1) with initial data a single jump discontinuity

$$U(x,0) = \begin{cases} U_- & x < 0, \\ U_+ & x > 0. \end{cases} \tag{4.3}$$

Due to the invariance of (4.1), (4.3) under dilations of coordinates $(x,t) \mapsto (\alpha x\,, \alpha t)$, $\alpha > 0$, solutions of the Riemann problem are sought in the form of functions $U(\frac{x}{t})$ of the single variable $\xi = \frac{x}{t}$, where $U = U(\xi)$ is a weak solution of the boundary value problem

$$\begin{gathered} -\xi\, U' + F(U)' = 0\,, \\ U(\pm\infty) = U_\pm\,. \end{gathered} \tag{$\mathcal{P}$}$$

In solving $(\mathcal{P})$ one encounters lack of uniqueness that is accounted for by imposing admissibility restrictions on solutions. We refer to Dafermos [D_5] for a detailed discussion of the issue of admissibility together with historical references.

For weak waves in strictly hyperbolic systems it suffices to impose admissibility restrictions only at shocks. The classical solution of the Riemann problem proceeds in two steps: First, special solutions of rarefaction waves, shock waves or contact discontinuities are constructed and are in turn used for constructing the elementary wave curves. There is one elementary curve associated with each characteristic field with the parametrization of the curve serving as a measure of the strength of the associated wave. Second, it is shown that the compound curves emanating from a fixed left state U_- give rise to an invertible map that covers a full neighborhood of right end states U_+. The construction provides a unique solution for the Riemann problem, in the class of weak waves, for genuinely nonlinear systems (Lax [La_1]) as well as for a large class of non-genuinely nonlinear systems (Liu [Li_1, Li_2]).

4.a The problem of self-similar viscosity limits.

The method of self-similar viscosity limits, introduced in Dafermos [D_1], provides a complementary approach for solving the Riemann problem in the spirit of viscosity approximations. An elliptic regularization of the Riemann operator in $(\mathcal{P})$ is introduced

$$(\mathcal{P}_\varepsilon)_B \qquad \begin{aligned} -\xi\, U' + F(U)' &= \varepsilon\Big(B(U)U'\Big)' \\ U(\pm\infty) &= U_\pm\,, \end{aligned}$$

where $\varepsilon > 0$ and $B(U)$ is a positive matrix accounting for the viscous structure. The admissible solutions of $(\mathcal{P})$ are selected as $\varepsilon \searrow 0$ limit-points of solutions to the problem $(\mathcal{P}_\varepsilon)_B$. In contrast to shock admissibility criteria, self-similar viscosity limits penalize the whole wave-fan simultaneously, and the resulting admissibility criterion is called *viscous wave-fan admissibility criterion.*

In this section we review the method, in the framework of weak waves for strictly hyperbolic $N \times N$ systems with $B(U) = Id$,

$$(\mathcal{P}_\varepsilon) \qquad \begin{aligned} -\xi\, U' + F(U)' &= \varepsilon U'' \\ U(\pm\infty) &= U_\pm\,. \end{aligned}$$

We start with a summary of the result [T_3].

Theorem 4.1. *Let (4.1) be strictly hyperbolic, $B(U) = Id$ and suppose the jump of the Riemann data $|U_+ - U_-|$ is small.*

(i) There exists a family $\{U_\varepsilon\}$ of smooth solutions to $(\mathcal{P}_\varepsilon)$ such that U_ε satisfy the uniform bounds

$$(V) \qquad |U_\varepsilon| + TV U_\varepsilon \le C\,,$$

and $|U'_\varepsilon(\xi)| \le \frac{C}{\varepsilon} e^{-\frac{\alpha}{\varepsilon}\xi^2}$, $|\xi| \ge \Lambda$, *for some* α *and* Λ *independent of* ε.
(ii) Let U_{ε_n} *be a subsequence of* $\{U_\varepsilon\}$ *such that* $U_{\varepsilon_n} \to U(\xi)$ *pointwise for* $\xi \in \mathbb{R}$. *Then* U *is a* BV *function that satisfies*

$$-\xi\, U' + F(U)' = 0 \tag{4.4}$$

in the sense of measures and the Rankine Hugoniot conditions,

$$-\xi\,[U(\xi+) - U(\xi-)] + [F(U(\xi+)) - F(U(\xi-)] = 0\,, \tag{4.5}$$

at any point of discontinuity $\xi \in \mathcal{S}_U$.
(iii) The function U *consists of* N *wave fans separated by constant states. Each wave fan consists of an alternating sequence of shocks and rarefactions so that each shock adjacent to a rarefaction on one side is a contact on that side. At a shock* $\xi \in \mathcal{S}_U$ *belonging to the* k*-th wave fan, a weak form of the Lax shock conditions is satisfied*

$$\lambda_k(U(\xi+)) \le \xi \le \lambda_k(U(\xi-))\,. \tag{4.6}$$

Finally, if $\xi \in \mathcal{S}_U$ *then the sequence* U_{ε_n} *has at* ξ *the internal structure of a shock profile.*

The theorem provides an alternative route to the solution of the Riemann problem, requiring strict hyperbolicity but no geometric conditions on the wave curves. In that sense it provides a general theory within the class of strictly hyperbolic systems and weak waves. The proof is analytical, replacing the construction of the wave curves with the construction of a class of solutions to $(\mathcal{P}_\varepsilon)$, that we call approximate wave curves. The main issue towards proving (V) is to construct a framework for measuring the total variation of approximate solutions that persists in the $\varepsilon \to 0$ limit. This construction may provide insight to the understanding of the corresponding issue in the (harder) problem of viscosity limits.

The study of self-similar viscosity limits may be decomposed into three steps:
(i) Construction of smooth solutions to the problem $(\mathcal{P}_\varepsilon)_B$, $\varepsilon > 0$.
(ii) Performing the passage to the limit $\varepsilon \searrow 0$, from $(\mathcal{P}_\varepsilon)_B$ to $(\mathcal{P})$.
(iii) Study of the structure of the emerging solution.

Step (i) is technical but routine, and general results can be established under weak assumptions: If (4.1) is equipped with an L^p estimate then $(\mathcal{P}_\varepsilon)$ has smooth solutions for each $\varepsilon > 0$, [T$_3$]. This applies in particular to the class of *symmetric hyperbolic* systems.

The usual framework for Step (ii) is uniform stability in $BV\big([a,b];\mathbb{R}^N\big)$,

$$|U_\varepsilon| + TV U_\varepsilon \le C\,. \tag{V}$$

If (V) holds on an interval $[a,b]$ then Helly's Theorem implies that there exists a subsequence $\{U_{\varepsilon_n}\}$, with $\varepsilon_n \to 0$, and a function $U \in BV$ such that $U_{\varepsilon_n} \to U$ for $\xi \in [a,b]$.

Suppose now that the family $\{U_\varepsilon\}$ of solutions to $(\mathcal{P}_\varepsilon)_B$ satisfies the uniform BV-bound (V) and, for some C, $\alpha > 0$ and Λ independent of ε, the uniform estimates

$$|U'_\varepsilon(\xi)| \le \frac{C}{\varepsilon} e^{-\frac{\alpha}{\varepsilon}\xi^2}, \qquad \text{for } |\xi| \ge \Lambda, \tag{D}$$

$$\varepsilon \int_{\mathbb{R}} |U'|^2 \, d\xi \le C. \tag{M}$$

We show how to construct solutions of the Riemann problem. (Note that if (4.1) is strictly hyperbolic and the family $\{U_\varepsilon\}$ is bounded in L^∞, then (D) can be proved in both the case $B(U) = Id$ as well as in some cases with singular diffusion matrices [T$_2$, T$_3$, Ki]. Also, that (M) follows, if there is an entropy-entropy flux pair (η, q) such that $\nabla^2 \eta \cdot B \ge c\, Id$ for some $c > 0$.) Consider a subsequence $\{U_{\varepsilon_n}\}$ such that

$$U_{\varepsilon_n}(\xi) \to U(\xi) \quad \text{for } \xi \in \mathbb{R}. \tag{4.7}$$

From $(\mathcal{P}_\varepsilon)_B$ we obtain, for a test function ψ,

$$\int_{\mathbb{R}} U_\varepsilon \cdot \big(\xi\psi\big)' - F(U_\varepsilon) \cdot \psi' \, d\xi = -\varepsilon \int_{\mathbb{R}} B(U_\varepsilon) U'_\varepsilon \cdot \psi' \, d\xi. \tag{4.8}$$

Using (V), (M) and (4.7) we pass to the limit $\varepsilon_n \to 0$ and obtain

$$\int_{\mathbb{R}} U \cdot \big(\xi\psi\big)' - F(U) \cdot \psi' \, d\xi = 0. \tag{4.9}$$

As $U \in BV$, its domain can be decomposed into two disjoint sets : $\mathcal{C}_U$ the set of points of continuity of U and $\mathcal{S}_U$ the set of points of discontinuity, respectively. The set $\mathcal{S}_U$ is at most countable, and the right and left limits $U(\xi+)$, $U(\xi-)$ exist at each ξ. The equation (4.4) is satisfied in the sense of measures. In particular, at $\xi \in \mathcal{S}_U$, the Rankine-Hugoniot conditions (4.5) are satisfied. Finally, (D) implies that $U = U_-$ on the interval $(-\infty, -\Lambda)$ and $U = U_+$ on $(\Lambda, +\infty)$. The function $U(\frac{x}{t})$ is a weak solution of the Riemann problem (4.1), (4.3), and the set $\mathcal{S}_U$ is the set of shocks for this wave-fan solution.

In Sections 4.c and 4.d, we outline the derivation of uniform variation estimates for the problem $(\mathcal{P}_\varepsilon)$ with $B(U) = Id$, first for the single conservation law and then for a strictly hyperbolic system. In Section 4.b, we show that stable BV-families of solutions to $(\mathcal{P}_\varepsilon)$ have, near shocks, the internal structure of shock profiles.

4.b The connection with shock profiles.

First, we investigate the relation between self-similar viscosity limits and shock profiles. Let $\{U_\varepsilon\}$ be a family of solutions to $(\mathcal{P}_\varepsilon)$ satisfying (V), (D) and (4.7).

Fix a point of discontinuity ξ of U and note that $U(\xi\pm)$ satisfy the Rankine-Hugoniot conditions (4.5). Consider a sequence of points $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$. Define the functions

$$V_\varepsilon(\zeta) = U_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta), \qquad -\infty < \zeta < \infty\,. \tag{4.10}$$

This transformation introduces a stretching of the independent variable centered around ξ_ε; the point ξ_ε is a shift of the shock speed ξ. The functions V_ε are uniformly bounded in BV,

$$TV_\zeta\ V_\varepsilon(\cdot) = TV_\zeta\ U_\varepsilon(\xi_\varepsilon + \varepsilon\,\cdot) = TV_\xi\ U_\varepsilon(\cdot) \le C\,. \tag{4.11}$$

Using Helly's theorem and a diagonal argument we establish the existence of a subsequence and a function V such that

$$U_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) \to V(\zeta) \quad \text{pointwise for } -\infty < \zeta < \infty\,. \tag{4.12}$$

Proposition 4.2. *Let $\xi \in \mathcal{S}_U$ and suppose that $\{\xi_\varepsilon\}$ is a sequence of points with $\xi_\varepsilon \to \xi$. Then $V(\zeta)$, defined in* (4.12), *is continuously differentiable and satisfies on $(-\infty,\infty)$ the traveling wave equations*

$$-\xi\,[\,V - U(\xi-)\,] + [\,F(V) - F(U(\xi-))\,] = \frac{dV}{d\zeta} \tag{4.13}$$

with initial condition $V(0) = \lim_{\varepsilon\to 0} U_\varepsilon(\xi_\varepsilon)$. The limits $\lim\limits_{\zeta\to\pm\infty} V(\zeta) =: V_\pm$ exist, are finite, and V_+, V_- solve the algebraic equations

$$-\xi\,[\,V - U(\xi-)\,] + [\,F(V) - F(U(\xi-))\,] = 0\,. \tag{4.14}$$

Proof. We integrate $(\mathcal{P}_\varepsilon)$ between the points $\xi_\varepsilon + \varepsilon\zeta$ and θ and then integrate the resulting equation in θ between ξ and $\xi+\delta$, for some $\delta \ne 0$, to arrive at

$$\begin{aligned}
&[\,-(\xi_\varepsilon + \varepsilon\,\zeta)U_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) + F(U_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta))\,] - \frac{1}{\delta}\int_\xi^{\xi+\delta} [\,-\theta U_\varepsilon(\theta) + F(U_\varepsilon(\theta))\,]\,d\theta \\
&\qquad + \frac{1}{\delta}\int_\xi^{\xi+\delta}\int_\theta^{\xi_\varepsilon+\varepsilon\zeta} U_\varepsilon(\tau)\,d\tau d\theta = \frac{d}{d\zeta}\big(U_\varepsilon(\xi_\varepsilon+\varepsilon\zeta)\big) - \varepsilon\frac{1}{\delta}\int_\xi^{\xi+\delta} U_\varepsilon'(\theta)\,d\theta\,.
\end{aligned}$$

After an integration in ζ we get

$$\begin{aligned}
&\int_0^\zeta [\,-(\xi_\varepsilon+\varepsilon s)U_\varepsilon(\xi_\varepsilon+\varepsilon s) + F(U_\varepsilon(\xi_\varepsilon+\varepsilon s))\,]\,ds - \zeta\,\frac{1}{\delta}\int_\xi^{\xi+\delta}[-\theta U_\varepsilon(\theta) + F(U_\varepsilon(\theta))]\,d\theta \\
&\qquad + \frac{1}{\delta}\int_0^\zeta\int_\xi^{\xi+\delta}\int_\theta^{\xi_\varepsilon+\varepsilon s} U_\varepsilon(\tau)\,d\tau d\theta ds = U_\varepsilon(\xi_\varepsilon+\varepsilon\zeta) - U_\varepsilon(\xi_\varepsilon) - \frac{\varepsilon\zeta}{\delta}\int_\xi^{\xi+\delta} U_\varepsilon'(\theta)\,d\theta\,.
\end{aligned}$$

Letting $\varepsilon \to 0$ and using (4.7), (4.12) and (V), we deduce

$$\int_0^\zeta [-\xi\, V(s) + F(V(s))]\, ds - \zeta\, \frac{1}{\delta} \int_\xi^{\xi+\delta} [-\theta U(\theta) + F(U(\theta))]\, d\theta$$
$$+ \zeta \frac{1}{\delta} \int_\xi^{\xi+\delta} \int_\theta^\xi U(\tau)\, d\tau d\theta = V(\zeta) - V(0)\,.$$

Letting consecutively $\delta \to 0+$ and $\delta \to 0-$, we obtain

$$\int_0^\zeta \Big[-\xi\big(V(s) - U(\xi\pm)\big) + F(V(s)) - F(U(\xi\pm))\Big]\, ds = V(\zeta) - V(0)\,. \tag{4.15}$$

It follows that $V(\zeta)$ is a continuously differentiable function that satisfies the traveling wave equations (4.13). Since V is of bounded variation on $\mathbb{R}$, the limits $\lim\limits_{\zeta\to\pm\infty} V(\zeta) =: V_\pm$ exist and are finite. Also, for any integer n

$$\int_n^{n+1} \Big[-\xi\big(V(s) - U(\xi-)\big) + F(V(s)) - F(U(\xi-))\Big]\, ds = V(n+1) - V(n)\,.$$

Taking the i-th component and using the mean value theorem, we see hat there are t_n^i with $n \le t_n^i \le n+1$ such that

$$-\xi\big(V^i(t_n^i) - U^i(\xi-)\big) + F^i(V(t_n^i)) - F^i(U(\xi-)) = V^i(n+1) - V^i(n)\,, \qquad i = 1\,, \ldots\,, N\,.$$

Letting $n \to \infty$ shows that V_+ is an equilibrium of (4.13). Similarly, V_- satisfies (4.14). □

The function V as well as the limiting values $V_\pm$ depend on the choice of the sequence $\{\xi_\varepsilon\}$. For several choices of $\{\xi_\varepsilon\}$ it may happen that the traveling wave disintegrates to a constant solution. Two questions arise: (i) Is it always possible to choose $\{\xi_\varepsilon\}$ so that the resulting V does not disintegrate to a constant solution of (4.13). (ii) What is the relation of $U(\xi-)$, $U(\xi+)$ and nontrivial heteroclinic orbits. These questions are taken up in [T$_3$, Sec 9]. It turns out:

Proposition 4.3. *Let $\xi \in \mathcal{S}_U$ be fixed and suppose the set of solutions of* (4.14) *is not connected. There exists a sequence of shock shifts $\{\xi_\varepsilon\}$ such that the resulting V in* (4.12) *is a nontrivial heteroclinic* (*or homoclinic*) *orbit.*

The hypothesis in Proposition 4.3 is violated only for shocks associated with a linearly degenerate characteristic field : $\nabla\lambda_k(U) \cdot r_k(U) = 0$ for all U. Addressing (ii) is quite complicated at the full level of generality. We give one result indicating what can happen if there is a finite number of equilibria in B_C, the ball of radius C where the functions U_ε take values.

Proposition 4.4. *Let $\xi \in \mathcal{S}_U$ and suppose that (4.14) has a finite number of solutions in B_C. There exists a subsequence $\varepsilon_n \to 0$ and choices $\{\xi_{1\varepsilon_n}\}$, $\{\xi_{2\varepsilon_n}\}$ of the shock shifts such that $\xi_{1\varepsilon_n} \le \xi_{2\varepsilon_n}$, $\xi_{1\varepsilon_n} \to \xi$, $\xi_{2\varepsilon_n} \to \xi$,*

$$U_{\varepsilon_n}(\xi_{1\varepsilon_n} + \varepsilon_n \zeta) \to V_1(\zeta)\,, \quad U_{\varepsilon_n}(\xi_{2\varepsilon_n} + \varepsilon_n \zeta) \to V_2(\zeta)\,, \quad \text{pointwise for } -\infty < \zeta < \infty\,,$$

and V_1, V_2 are nontrivial solutions of (4.13) that satisfy $V_1(-\infty) = U(\xi-)$, $V_2(+\infty) = U(\xi+)$.

Associated to characteristic fields that are not linearly degenerate, there exists one heteroclinic orbit of (4.13) that emanates from $U(\xi-)$ and one that concludes at $U(\xi+)$. If more than two states in B_C satisfy the Rankine-Hugoniot conditions at a given $\xi \in \mathcal{S}_U$, or if multiple heteroclinic connections between two equilibria are possible, then the precise relation between self-similar limits and shock profiles requires a detailed analysis of the shock profiles. The structure of traveling wave solutions is well understood for weak shocks, even for general diffusion matrices (Majda and Pego [MP]). By contrast, relatively little is known for strong shocks. In general, it is possible that there are intermediate states V_j, $j = 1, ..., J$, finitely many or even countable, satisfying (4.14) and a chain of shock profiles, at the same shock speed ξ, that connect successively $U(\xi-)$ to V_1, each of the points V_j to the next, and V_J to $U(\xi+)$. The latter situation occurs for the equations of isothermal elasticity in the presence of multiple inflection points in the stress-strain relation, for specific positions of the Riemann data relative to the stress-strain curve [T$_2$].

4.c The scalar conservation law.

In this section we consider the problem of self-similar viscosity limits for the scalar conservation law, and discuss the proof of the uniform bounds (V) and the structure of the emerging solution. Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of scalar-valued functions satisfying

$$\begin{aligned} \varepsilon\, u_\varepsilon'' &= -\xi\, u_\varepsilon' + f(u_\varepsilon)' \\ u_\varepsilon(\pm\infty) &= u_\pm\,. \end{aligned} \tag{4.16}$$

It is easy to see that solutions of (4.1) satisfy the representation formula

$$u_\varepsilon'(\xi) = (u_+ - u_-)\, \frac{\exp\left\{-\frac{1}{\varepsilon}\int_\rho^\xi s - \lambda\big(u_\varepsilon(s)\big)\,ds\right\}}{\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\varepsilon}\int_\rho^\zeta s - \lambda\big(u_\varepsilon(s)\big)\,ds\right\} d\zeta} = \tau \varphi_\varepsilon(\xi)\,, \tag{4.17}$$

where $\lambda(u) = f'(u)$ denotes the characteristic speed and ρ is any real number. Above, we used the notations $\tau = (u_+ - u_-)$, as a measure of the strength of the wave, and

$$\varphi_\varepsilon(\xi) = \varphi_\varepsilon[u_\varepsilon](\xi) = \frac{e^{-\frac{1}{\varepsilon} g_\varepsilon(\xi)}}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon} g_\varepsilon(\zeta)} d\zeta} = \frac{1}{I_\varepsilon} e^{-\frac{1}{\varepsilon} g_\varepsilon(\xi)} \tag{4.18}$$

$$\text{where} \qquad g_\varepsilon(\xi) = g[u_\varepsilon](\xi) = \int_\rho^\xi s - \lambda\big(u_\varepsilon(s)\big)\,ds$$

$$\text{and } \ I_\varepsilon = \int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon} g_\varepsilon(\zeta)} d\zeta$$

Note that φ_ε and g_ε depend implicitly on the solution u_ε, through the dependence on the characteristic speed $\lambda(u_\varepsilon)$.

It follows from (4.17) that u'_ε has a sign, and thus

$$\min\{u_-, u_+\} \le u_\varepsilon \le \max\{u_-, u_+\}, \qquad TV u_\varepsilon = |u_+ - u_-|.$$

Another way to see (V) is to observe that φ_ε are positive functions and uniformly bounded in L^1, hence $\{u_\varepsilon\}$ is of uniformly bounded variation. Given the bound (V), we can pass to the $\varepsilon \to 0$ limit and obtain a solution of the problem $(\mathcal{P})$ for the scalar case.

In the sequel, we study the quantities φ_ε in (4.18), under various frameworks of uniform bounds. For a family of solutions $\{u_\varepsilon\}$ bounded in L^∞, we have

$$(A) \qquad \lambda_- \le \lambda(u_\varepsilon) \le \lambda_+ .$$

Lemma 4.5. *Under Hypothesis (A), as $\varepsilon \to 0$:*
(i) If $d = \lambda_+ - \lambda_- > 0$, then $\frac{1}{O(1)} \frac{\varepsilon}{d} \le I_\varepsilon \le d + \sqrt{2\pi\varepsilon}$, and

$$(4.19) \qquad 0 < \varphi_\varepsilon(\xi) \le \begin{cases} O(1)\, \frac{d}{\varepsilon}\, e^{-\frac{1}{2\varepsilon}(\xi-\lambda_-)^2} & \xi < \lambda_-, \\ O(1)\, \frac{d}{\varepsilon} & \xi \in \mathbb{R}, \\ O(1)\, \frac{d}{\varepsilon}\, e^{-\frac{1}{2\varepsilon}(\xi-\lambda_+)^2} & \xi > \lambda_+. \end{cases}$$

(ii) If $d = \lambda_+ - \lambda_- = 0$, then $I_\varepsilon = \sqrt{2\pi\varepsilon}$ and $\varphi_\varepsilon = \frac{1}{\sqrt{2\pi\varepsilon}}\, e^{-\frac{1}{2\varepsilon}(\xi-\lambda_-)^2}$.

Proof. The estimates for φ_ε reflect the property that, under Hypothesis (A), g_ε has the form of a potential-well function (*cf.* Figure 2). We select ρ as the point where g_ε achieves its global minimum. Then ρ satisfies $\lambda_- \le \rho = \lambda(u_\varepsilon(\rho)) \le \lambda_+$, and $g_\varepsilon(\xi) \ge g_\varepsilon(\rho) = 0$.

Assume first that $d > 0$. Then

$$g_\varepsilon(\zeta) = \int_\rho^\zeta s - \lambda(u_\varepsilon(s))ds \le \begin{cases} \frac{1}{2}(\zeta-\rho)^2 + d(\zeta-\rho) & \text{for } \zeta > \rho \\ \frac{1}{2}(\zeta-\rho)^2 - d(\zeta-\rho) & \text{for } \zeta < \rho \end{cases}$$

In turn,

$$\begin{aligned} I_\varepsilon &= \int_{-\infty}^{\rho} e^{-\frac{1}{\varepsilon}g_\varepsilon} d\zeta + \int_{\rho}^{+\infty} e^{-\frac{1}{\varepsilon}g_\varepsilon} d\zeta \\ &\ge \sqrt{\varepsilon}\, e^{\frac{d^2}{2\varepsilon}} \int_{-\infty}^{0} e^{-\frac{1}{2}\left(\eta - \frac{d}{\sqrt{\varepsilon}}\right)^2} d\eta + \sqrt{\varepsilon}\, e^{\frac{d^2}{2\varepsilon}} \int_{0}^{\infty} e^{-\frac{1}{2}\left(\eta + \frac{d}{\sqrt{\varepsilon}}\right)^2} d\eta \\ &\ge \frac{1}{O(1)} \frac{\varepsilon}{d}, \qquad \text{for } \varepsilon \text{ small.} \end{aligned}$$

On the other hand, estimating g_ε from below yields

$$g_\varepsilon(\zeta) \geq \begin{cases} \frac{1}{2}(\zeta - \lambda_-)^2 & \text{for } \zeta < \lambda_- \\ 0 & \text{for } \lambda_- < \zeta < \lambda_+ \\ \frac{1}{2}(\zeta - \lambda_+)^2 & \text{for } \zeta > \lambda_+ \end{cases}$$

whence

$$I_\varepsilon \leq \int_{-\infty}^{\lambda_-} e^{-\frac{1}{2\varepsilon}(\zeta-\lambda_-)^2}\, d\zeta + d + \int_{\lambda_+}^{\infty} e^{-\frac{1}{2\varepsilon}(\zeta-\lambda_+)^2}\, d\zeta = d + \sqrt{2\pi\varepsilon}\,.$$

The proof of (4.19) now follows from (4.18). Finally, if $d = 0$ then $\lambda(u_\varepsilon)$ remains constant, say λ_-, and part (ii) follows from a direct calculation. □

The family $\{u_\varepsilon\}$ is uniformly bounded in BV, while $\{\varphi_\varepsilon\}$ is uniformly bounded in L^1. There is a subsequence u_{ε_n}, φ_{ε_n} and a finite positive Borel measure ϕ such that

$$\begin{aligned} &u_{\varepsilon_n} \to u\,, && \text{pointwise in } \mathbb{R}, \\ &\varphi_{\varepsilon_n} \rightharpoonup \phi\,, && \text{weak-}\star \text{ in measures.} \end{aligned} \tag{4.20}$$

By (4.19) no mass escapes at infinity and the total mass of the measure ϕ is one. The distribution function of ϕ is the right continuous function $\frac{1}{\tau}(u(\xi+) - u_-)$. Along the same sequence

$$g_{\varepsilon_n}(\xi) = \int_{\rho_{\varepsilon_n}}^{\xi} s - \lambda(u_{\varepsilon_n}(s))\, ds \to \int_{\rho}^{\xi} s - \lambda(u(s))\, ds =: g(\xi) \tag{4.21}$$

uniformly on compact subsets of $(-\infty, \infty)$. We show that points in the support of ϕ are global minima for the function g.

Proposition 4.6. *If $\xi \in \operatorname{supp} \phi$ then $g(\zeta) \geq g(\xi)$ for all $\zeta \in (-\infty, \infty)$.*

Proof. Fix $\xi \in \mathbb{R}$, $\alpha > 0$ and consider the set

$$\mathcal{A} = \{\zeta \in \mathbb{R} \,:\, g(\zeta) - g(\xi) < -\alpha < 0\}\,. \tag{4.22}$$

Step 1 : If the Lebesgue measure $m(\mathcal{A}) > 0$, then $\xi \notin \operatorname{supp} \phi$.

Since $g(\zeta) \to \infty$ as $|\zeta| \to \infty$, $\mathcal{A}$ is contained in some compact interval $[a, b]$. By (4.21) and the continuity of g there are δ and ε_0 such that

$$g_{\varepsilon_n}(\zeta) - g_{\varepsilon_n}(\theta) < -\frac{\alpha}{2}$$

for $\varepsilon < \varepsilon_0$, $\zeta \in \mathcal{A}$ and $\theta \in J = (\xi - \delta, \xi + \delta)$. Hence,

$$0 < \varphi_{\varepsilon_n}(\theta) \leq \frac{1}{\int_{\mathcal{A}} \exp\{-\frac{1}{\varepsilon_n}\big(g_{\varepsilon_n}(\zeta) - g_{\varepsilon_n}(\theta)\big)\}\, d\zeta} \leq \frac{e^{-\frac{\alpha}{2\varepsilon_n}}}{m(\mathcal{A})}\,. \tag{4.23}$$

Let $\chi \in C_c(J)$. Then (4.23) and (4.20) give

$$\int_{(\xi-\delta,\xi+\delta)} \varphi_{\varepsilon_n}(\theta)\,\chi(\theta)\,d\theta \to 0\,, \qquad \text{as } \varepsilon_n \to 0\,,$$

and thus $\xi \notin \operatorname{supp} \phi$.

Step 2 : If $\xi \in \operatorname{supp} \phi$, then $m(\mathcal{A}) = 0$ and thus $\mathcal{A}$ is empty for any $\alpha > 0$. Hence, $g(\zeta) \geq g(\xi)$ for $\zeta \in \mathbb{R}$. □

The minimization property for g provides information on the structure of the BV-function u. In particular, a weak form of the Lax shock conditions is induced at points of discontinuity.

Corollary 4.7. *Let ξ, $\xi' \in$ supp $\phi \subset [\lambda_-, \lambda_+]$ with $\xi < \xi'$.*
(a) If $\xi \in \mathcal{C}_u$, then $\xi = \lambda(u(\xi))$.
(b) If $\xi \in \mathcal{S}_u$, then u satisfies at ξ the jump conditions (4.5) and the inequalities

$$\lambda(u(\xi+)) \leq \xi \leq \lambda(u(\xi-))\,. \tag{4.24}$$

(c) If ξ, $\xi' \in$ supp ϕ then $\lambda(u(\xi+)) = \xi$, $\lambda(u(\xi'-)) = \xi'$. Moreover, at any $\theta \in (\xi, \xi')$,

$$\begin{aligned} \theta = \lambda(u(\theta)) \quad &\textit{if} \quad \theta \in \mathcal{C}_u\,,\\ \lambda(u(\theta+)) = \theta = \lambda(u(\theta-)) \quad &\textit{if} \quad \theta \in \mathcal{S}_u\,. \end{aligned} \tag{4.25}$$

Proof. The function g is continuous, satisfies $g(\xi) \to \infty$ as $|\xi| \to \infty$, and the limits

$$\lim_{\zeta\to\xi\pm} \frac{g(\zeta) - g(\xi)}{\zeta - \xi} = \lim_{\zeta\to\xi\pm} \frac{1}{\zeta - \xi} \int_{\xi}^{\zeta} s - \lambda(u(s))\,ds = \xi - \lambda(u(\xi\pm))$$

exist. Proposition 4.6 implies that if $\xi \in \operatorname{supp} \phi$ then $\xi - \lambda(u(\xi+)) \geq 0$ and $\xi - \lambda(u(\xi-)) \leq 0$. In turn, this implies (a) if $\xi \in \mathcal{C}_u$ and (b) if $\xi \in \mathcal{S}_u$.

It remains to show (*c*). Let ξ, $\xi' \in \operatorname{supp} \phi$ with $\xi < \xi'$. Then ξ, ξ' are both global minima for g with $g(\xi) = g(\xi')$. We claim

$$g(\theta) = g(\xi) \;\text{ for any } \theta \in (\xi, \xi')\,. \tag{4.26}$$

If (4.26) is violated, there exist a, b with $\xi \leq a < b \leq \xi'$ such that

$$g(a) = g(b) = g(\xi)\,, \quad g(\theta) > g(\xi) \quad \text{for } a < \theta < b\,.$$

At the points a, b we have

$$\begin{aligned} \lambda(u(a+)) \leq a \leq \lambda(u(a-))\\ \lambda(u(b+)) \leq b \leq \lambda(u(b-))\,. \end{aligned} \tag{4.27}$$

On the other hand, at any $\theta \in (a,b)$ the set $\mathcal{A} = \{\zeta \in \mathbb{R} : \ g(\zeta) - g(\theta) < -\alpha\}$ is nonempty for some $\alpha > 0$. Proposition 4.6 implies that $\theta \notin \operatorname{supp}\phi$ and the function $u(\xi)$ remains constant on the interval (a,b). Hence $\lambda(u(a+)) = \lambda(u(b-))$ and the inequalities (4.27) yield $b \leq a$. This contradicts $a < b$ and (4.26) follows. □

In summary, the region where u is nonconstant consists of one closed interval I_λ (which could degenerate to one single point). The solution u takes the values u_- and u_+ on the complement of I_λ and looks like a wave-fan consisting of rarefactions, shocks and contacts at points of I_λ.

4.d BV stability for self-similar viscosity limits.

Next, we outline the derivation of the uniform BV bounds, for weak waves in $N \times N$ strictly hyperbolic systems:

Theorem 4.8. *Let* (4.1) *be strictly hyperbolic and* U_- *be fixed. If* $|U_+ - U_-|$ *is sufficiently small, the problem* $(\mathcal{P}_\varepsilon)$ *admits a smooth solution* U_ε *for each* $\varepsilon > 0$. *Moreover, the family of solutions* $\{U_\varepsilon\}_{\varepsilon>0}$ *is of uniformly bounded (and small) oscillation and total variation.*

Sketch of Proof. First, $(\mathcal{P}_\varepsilon)$ is recast into an alternative formulation. Let U_ε be a solution to $(\mathcal{P}_\varepsilon)$ connecting U_- to U_+ and consider the decomposition of U_ε' in the basis $\{r_k(U_\varepsilon)\}$,

$$U_\varepsilon'(\xi) = \sum_{k=1}^{N} a_{k\varepsilon}(\xi)\, r_k\big(U_\varepsilon(\xi)\big)\,. \tag{4.28}$$

The amplitudes $a_{k\varepsilon}$ can be recovered from the formula

$$a_{k\varepsilon}(\xi) = l_k(U_\varepsilon(\xi)) \cdot U_\varepsilon'(\xi)\,, \tag{4.29}$$

and a simple calculation, taking the inner product of the system in $(\mathcal{P}_\varepsilon)$ with $l_k(U_\varepsilon)$, shows that $a_{k\varepsilon}$ satisfy the equations

$$\varepsilon\, a_{k\varepsilon}' + \big[\xi - \lambda_k(U_\varepsilon(\xi))\big]\, a_{k\varepsilon} = \varepsilon \sum_{m,n=1}^{N} \big[\nabla l_k(U_\varepsilon(\xi))\, r_m(U_\varepsilon(\xi)) \cdot r_n(U_\varepsilon(\xi))\big]\, a_{m\varepsilon} a_{n\varepsilon}\,. \tag{4.30}$$

Integrating (4.28) over $(-\infty,\infty)$, we have

$$U_+ - U_- = \sum_{k=1}^{N} \int_{-\infty}^{\infty} a_{k\varepsilon}(\zeta)\, r_k\big(U_\varepsilon(\zeta)\big)\, d\zeta\,. \tag{4.31}$$

Equations (4.30)-(4.31) provide an equivalent formulation of the problem $(\mathcal{P}_\varepsilon)$. Henceforth we suppress the ε-dependence of functions and introduce the notation

$$\begin{aligned} \lambda_k &= \lambda_k(U_\varepsilon(\xi)) \\ \beta_{k,mn} &= \beta_{k,mn}(U_\varepsilon(\xi)) = \nabla l_k(U_\varepsilon(\xi))\, r_m(U_\varepsilon(\xi)) \cdot r_n(U_\varepsilon(\xi))\,. \end{aligned} \tag{4.32}$$

The functions a_k satisfy the coupled system of ordinary differential equations with variable coefficients

$$\varepsilon\, a_k' + (\xi - \lambda_k)\, a_k = \varepsilon \sum_{m,n=1}^{N} \beta_{k,mn}\, a_m\, a_n\,. \tag{4.33}$$

We consider the following question: Assume we are given a family $\{U_\varepsilon\}_{\varepsilon>0}$ of solutions that are of uniformly bounded, small oscillation

$$\sup_{-\infty<\xi<+\infty} |U_\varepsilon(\xi) - U_-| \le \mu. \tag{C_o}$$

Examine under what conditions the given family is of uniformly bounded variation

$$TV_{(-\infty,+\infty)}\,(U_\varepsilon)\; \le\; C. \tag{S}$$

Note that (C_o) imposes the restriction $|U_+ - U_-|$ small on the Riemann data, and dictates that U_ε satisfy the uniform L^∞-bound, $\sup_{-\infty<\xi<+\infty} |U_\varepsilon(\xi)| \le M$, with the constants M and μ independent of ε and μ also small. Along the family $\{U_\varepsilon\}$, each wave speed is bounded

$$\lambda_{k-} \le \lambda_k\big(U_\varepsilon(\xi)\big) \le \lambda_{k+} \tag{4.34}$$

by constants λ_{k-}, λ_{k+} independent of ε. If the oscillation of U_ε is sufficiently small, then the wave speeds are *totally separated*, that is

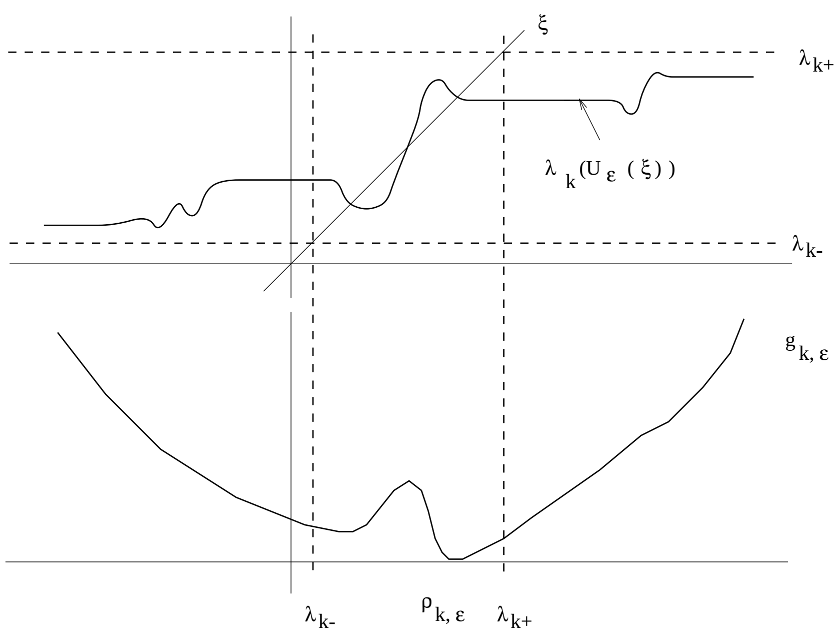

FIGURE 2.

$$
\begin{aligned}
\lambda_{1-} \le \lambda_1\big(U_\varepsilon(\xi)\big) \le \lambda_{1+} < \lambda_{2-} \le \lambda_2\big(U_\varepsilon(\xi)\big) \le \lambda_{2+} < \dots \\
< \lambda_{(N-1)-} \le \lambda_{N-1}\big(U_\varepsilon(\xi)\big) \le \lambda_{(N-1)+} < \lambda_{N-} \le \lambda_N\big(U_\varepsilon(\xi)\big) \le \lambda_{N+}.
\end{aligned}
\tag{4.35}
$$

Finally, the coefficients $\beta_{k,mn}$ are uniformly bounded, $|\beta_{k,mn}| \le B$, by a constant B depending on μ but not on ε.

The L^1 norm of the function $\sum_{k=1}^N |a_k|$ provides a natural measure of the variation of U_ε. Hence, in order to prove (S) it suffices to estimate in L^1 the solutions $a_{k\varepsilon}$ of the system (4.33), under the hypotheses that the wave speeds λ_k are totally separated and the coefficients $\beta_{k,mn}$ are bounded. The quadratic terms in (4.33) represent the effect induced on the k-family by interactions of waves of all the families, and $\beta_{k,mn}$ measure the weights of such contributions. There are three problems to be resolved : First, to find a natural framework for measuring the L^1 norm of $\sum_{k=1}^N |a_k|$. Second, to understand the effect of the quadratic terms. Third, differential systems like (4.33) are best amenable to analysis under pointwise conditions. On the other hand the existing information connecting a_k with the data is of integral type. Therefore, a scheme is needed that connects pointwise to integral information.

Let g_k be an antiderivative of $g_k' = \xi - \lambda_k(U_\varepsilon(\xi))$. By (4.34), $g_k' > 0$ for $\xi > \lambda_{k+}$, $g_k' < 0$ for $\xi < \lambda_{k-}$, and thus g_k looks like a potential-well function (see Fig. 2). Let $\rho_{k\varepsilon}$ be a point where g_k attains its global minimum. If we set

$$
g_k(\xi) = g_k[U_\varepsilon](\xi) := \int_{\rho_{k\varepsilon}}^{\xi} s - \lambda_k(U_\varepsilon(s))\, ds
\tag{4.36}
$$

then $\lambda_{k-} \le \rho_{k\varepsilon} = \lambda_k(U_\varepsilon(\rho_{k\varepsilon})) \le \lambda_{k+}$, $g_k(\xi) \ge g_k(\rho_{k\varepsilon}) = 0$, and $g_k(\xi) = O\big(|\xi|^2\big)$ as $|\xi| \to \infty$.

Consider the linearization of the system (4.18), consisting of the decoupled system of equations

$$
\varepsilon\, \varphi_k' + (\xi - \lambda_k)\, \varphi_k = 0\,.
\tag{4.37}
$$

The solutions of (4.37) are constant multiples of

$$
\varphi_k = \frac{\exp\big\{ -\frac{1}{\varepsilon}\int_{\rho_{k\varepsilon}}^{\xi} s - \lambda_k(U_\varepsilon(s))\, ds \big\}}{\int_{-\infty}^{\infty} \exp\big\{ -\frac{1}{\varepsilon}\int_{\rho_{k\varepsilon}}^{\zeta} s - \lambda_k(U_\varepsilon(s))\, ds \big\}\, d\zeta} = \frac{e^{-\frac{1}{\varepsilon} g_k}}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon} g_k}\, d\zeta} = \frac{1}{I_k} e^{-\frac{1}{\varepsilon} g_k}
\tag{4.38}
$$

The functions $\{\varphi_{k\varepsilon}\}$ are strictly positive and uniformly bounded in L^1, independently of ε. Due to (4.34) and Lemma 4.5, $\varphi_{k\varepsilon}$ satisfy, in the case $d_k = \lambda_{k+} - \lambda_{k-} > 0$, the estimates

$$
0 < \varphi_{k\varepsilon}(\xi) \le \begin{cases} O(1)\, \frac{d_k}{\varepsilon}\, e^{-\frac{1}{2\varepsilon}(\xi - \lambda_{k-})^2} & \xi < \lambda_{k-}, \\ O(1)\, \frac{d_k}{\varepsilon} & \xi \in \mathbb{R}, \\ O(1)\, \frac{d_k}{\varepsilon}\, e^{-\frac{1}{2\varepsilon}(\xi - \lambda_{k+})^2} & \xi > \lambda_{k+}, \end{cases}
\tag{4.39}
$$

and can be evaluated as in Lemma 4.5 (ii), in case $d_k = 0$.

The functions φ_k serve as a yardstick to estimate the amplitudes a_k, as follows. First, one constructs solutions of (4.33) that admit the representation

$$a_k = \tau_k \, \varphi_k + \theta_k(\cdot\,;\tau), \tag{4.40a}$$

where $\tau = (\tau_1, \tau_2, \ldots, \tau_N)$ is a vector-parameter in $\mathbb{R}^N$ and $\theta_k(\xi;\,\tau)$ is of second order in τ in the sense that, for some constant C independent of ε, it satisfies the estimate

$$|\theta_k(\cdot\,;\,\tau)| \le C\,|\tau|^2 \sum_{j=1}^{N} \varphi_j . \tag{4.40b}$$

The decomposition (4.40) can be thought as an asymptotic expansion of the amplitudes a_k in a parameter τ representing the strength of elementary waves. Since "most" of the ε-dependence is carried by the φ_k's, the expansion is uniform in ε in the L^1-norm.

The key step in validating the expansion (4.40), concerns the pointwise behavior of the integrals

$$F_{k,mn} = e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k} \, \varphi_m \, \varphi_n \, d\zeta \,,$$

which express the contributions on the k-th family effected by interactions between elementary waves of the m-th and n-th families. As $\varepsilon \to 0$, the terms $F_{k,mn}$ behave as follows [T_3, Lemmas 4.3, 4.4]: $F_{k,mk}$ and $F_{k,km}$ $F_{k,kk}$ have non-zero limiting contributions supported on the k-th wave speed. $F_{k,mn} \to 0$ as $\varepsilon \to 0$ when $m \neq n$, $m \neq k$ and $n \neq k$, which suggests that diffusion induced interactions of two distinct families have no contribution as $\varepsilon \to 0$ on a third family. (Recall that we are dealing with Riemann data solutions.) By contrast, the terms $F_{k,mm}$, $m \neq k$, accounting for the effect of two interacting waves of the m-th family on the k-th family, have a non-zero contribution in the $\varepsilon \to 0$ limit which is supported on the m-th wave speed.

The second problem is to connect the parameters τ with the data U_-, U_+ in order to fulfill (4.31). To this end, for U_- fixed, one considers a map S_ε that takes τ in a neighborhood of $0 \in \mathbb{R}^N$ to the vector

$$S_\varepsilon(\tau) = U_- + \sum_{k=1}^{N} \int_{-\infty}^{+\infty} \left[\tau_k \, \varphi_{k\varepsilon}(\zeta) + \theta_{k\varepsilon}(\zeta;\,\tau)\right] r_k(U_\varepsilon(\zeta))\, d\zeta \,. \tag{4.41}$$

It is shown that S is locally invertible in a neighborhood of $\tau = 0$ and that the inverse map S_ε^{-1} is uniformly bounded, independently of ε, for ε small.

Finally, the formulation (4.30-4.31) suggests a construction scheme for proving existence of solutions U_ε of $(\mathcal{P}_\varepsilon)$ in weighted spaces, so that the constructed solutions satisfy the asymptotic expansion (4.40). We refer to [T_3] for details and state the final result:

Theorem 4.9. *Assume* (4.1) *is strictly hyperbolic and let* U_- *be fixed. There exists a (sufficiently small)* r *such that, for* $\varepsilon > 0$ *and for any* U_+ *satisfying* $|U_+ - U_-| \le r$, *the problem* $(\mathcal{P}_\varepsilon)$ *admits a solution* U_ε *with the following properties:*
(i) The family $\{U_\varepsilon\}_{\varepsilon>0}$ *satisfies* (C_o) *with some* μ *independent of* ε.
(ii) The solutions U_ε *satisfy the representation formula*

$$U'_\varepsilon = \sum_{k=1}^{N} \left[\tau_{k,\varepsilon}\varphi_{k\varepsilon} + \theta_{k\varepsilon}(\cdot;\tau_\varepsilon)\right] r_k(U_\varepsilon), \tag{4.42}$$

where $\varphi_{k\varepsilon}$ *is given by* (4.38), $a_{k\varepsilon}(\cdot;\tau)$ *of the form* (4.40a) *satisfy* (4.40b), *and* τ_ε *solves* $S(\tau_\varepsilon) = U_+$.
(iii) The family $\{U'_\varepsilon\}_{\varepsilon>0}$ *is uniformly bounded in* $L^1(\mathbb{R})$ *and* $\{U_\varepsilon\}_{\varepsilon>0}$ *is of uniformly bounded (and small) total variation.*

We conclude by indicating the proof of the variation bounds from the representation formula (4.42). Let $\{U_\varepsilon\}_{\varepsilon>0}$ be a family of solutions to $(\mathcal{P}_\varepsilon)$, of uniformly bounded oscillation (C_o) and satisfying (4.42). By the construction process,

$$U'_\varepsilon(\xi) = \sum_k a_{k\varepsilon}(\xi;\tau_\varepsilon) r_k(U_\varepsilon(\xi))$$
$$a_{k\varepsilon}(\cdot;\tau_\varepsilon) \ \text{ satisfies the asymptotic expansion (4.40)}$$
$$S_\varepsilon(\tau_\varepsilon) = U_+ \quad \text{and there exists } C \text{ such that } |\tau_\varepsilon| \le C|U_+ - U_-|$$

Using (4.40),

$$|a_{k\varepsilon}(\xi;\,\tau_\varepsilon)| \le |\tau_{k,\varepsilon}|\varphi_{k\varepsilon} + C|\tau_\varepsilon|^2 \sum_j \varphi_{j\varepsilon} \le C\,|U_+ - U_-|\Big(\varphi_{k\varepsilon} + \sum_j \varphi_{j\varepsilon}\Big)$$

and thus

$$|U'_\varepsilon(\xi)| \le K \sum_{j=1}^{N} \varphi_{j\varepsilon}. \tag{4.43}$$

where the constant K is of order $O(|U_+ - U_-|)$ and independent of ε. As $\{\varphi_{j\varepsilon}\}$ are uniformly bounded in $L^1(\mathbb{R})$, we deduce $\{U'_\varepsilon\}$ is uniformly bounded in $L^1(\mathbb{R})$.

4.e The relation with the problem of viscosity limits.

It is interesting to see how the the problem of self-similar viscosity limits relates to viscosity approximations for Riemann data solutions. For the system of viscous conservation laws

$$\partial_t U + \partial_x F(U) = \varepsilon\, \partial_x^2 U \tag{4.44}$$

subject to Riemann data, the invariance under dilations $(x,t) \mapsto (\alpha x, \alpha t)$, $\alpha > 0$, no longer holds. Due to uniqueness results for parabolic systems, the solution U^ε of (4.44)-(4.3) can be expressed as

$$U^\varepsilon(x,t) = V(\frac{x}{t}, -\frac{\varepsilon}{t}) \tag{4.45}$$

where $V(\xi, s)$ is independent of ε and satisfies

$$V_s - V_{\xi\xi} = \frac{1}{s}\big(- \xi V_\xi + F(V)_\xi \big) \tag{4.46}$$

for $-\infty < \xi < \infty$, $-\infty < s < 0$. We see that the zero-viscosity limits problem for Riemann data is a two parameter problem and that studying the limit of U^ε as $\varepsilon \downarrow 0$ amounts to studying the limit of $V(\xi, s)$ as $s \uparrow 0-$. The problem $(\mathcal{P}_\varepsilon)$ arises when replacing the parabolic operator in (4.46) by an elliptic operator and solving on the collapsed domain $\xi \in \mathbb{R}$.

Bibliographic remarks. Elliptic regularizations of the Riemann problem operator appear in [Ka, Tu, D$_1$]. Tupciev [Tu] uses $(\mathcal{P}_\varepsilon)$ as a starting point to motivate that admissible shocks should have an associated viscous shock profile. Dafermos [D$_1$] proposed this regularization as a devise to select the admissible solutions of the Riemann problem. The procedure is carried out in [D$_1$, D$_2$, DDp, ST$_1$] for strictly hyperbolic 2×2 systems, and in [T$_3$] for weak waves of $N \times N$ systems. These studies concern the case $B(U) = Id$. As the equations of continuum thermomechanics involve singular diffusion matrices, there are investigations of the systems of isothermal elasticity [T$_2$] and isentropic gas dynamics [Ki] with singular diffusion matrices. A comparison of self-similar viscosity limits with viscosity approximations is carried out in [S$_2$] for Burgers's equation. Self-similar viscosity limits serve as a tool for investigating wave admissibility in situations involving loss of strict hyperbolicity, or when "exotic" phenomena are at play. There are a number of such investigations concerning: large shocks or even delta shock waves [KK$_1$, KK$_2$, TZZ, E], mixed hyperbolic-elliptic systems [S$_1$, F$_1$, F$_2$], Riemann type solutions for fully nonlinear systems [SS], and fluid dynamic limits for the Broadwell model [ST$_2$, T$_1$]. We point out that self-similar limits provide a notion of solution and an existence theory for the Riemann problem in the class of non-conservative, strictly-hyperbolic systems [LT$_2$, LT$_3$].

5. Relaxation Approximations of Hyperbolic Conservation Laws

The presence of relaxation mechanisms is widespread in both the continuum mechanics as well as the kinetic theory contexts. Relaxation provides a subtle "dissipative" mechanism against the destabilizing effect of nonlinear response, as well as a damping effect on oscillations (at least when assisted by nonlinear response). The objective of this section is to bring up these properties, by examining the zero-relaxation limit in two examples, concerning respectively a single conservation law in several space dimensions and a system of two conservation laws in one space dimension.

5.a The structure of relaxation approximations.

We begin with an outline of the general structure of relaxation approximations. For $\varepsilon > 0$, a system of semilinear hyperbolic equations,

$$\partial_t U + \sum_{i=1}^{d} A_i \partial_{x_i} U = \frac{1}{\varepsilon} R(U)\,, \tag{5.1}$$

governs the dynamics of a function $U = U(x,t)$, $x \in \mathbb{R}^d$, $t > 0$. The state variable U takes values in $\mathbb{R}^N$ and will be called the mesoscopic variable. The matrices A_i are assumed constant $N \times N$ matrices such that (5.1) is hyperbolic. (All examples considered in this section are semilinear systems. Relaxation of quasilinear systems is also of interest for applications, but we will not pursue it here).

It is assumed that (5.1) is equipped with m conservation laws, *i.e.* there are linearly independent vectors $q_j \in \mathbb{R}^N$, $j = 1, \ldots, m$, such that the variables $u_j = q_j \cdot U$ satisfy the conservation laws

$$\partial_t \big(q_j \cdot U \big) + \sum_{i=1}^{d} \partial_{x_i} \big(q_j \cdot A_i U \big) = 0\,. \tag{5.2}$$

The variables u_i are called macroscopic variables, and $u = (u_1, \ldots, u_m)$ stands for the vector of all the macroscopic variables.

For the system of ordinary differential equations

$$U_t = \frac{1}{\varepsilon} R(U) \tag{5.3}$$

it is assumed : (i) It is equipped with m conservation laws for the variables $u_j = q_j \cdot U$, that is $q_j \cdot R(U) = 0$, $j = 1, \ldots, m$. (ii) There is an m-dimensional manifold of equilibria $\mathcal{M}$ parametrized by the m macroscopic variables u_j, the set of equilibria $\mathcal{M}$ is described in the form $U = \mathcal{E}(u)$. (iii) The flow of the system of ordinary differential equations (5.3) is attracted to $\mathcal{M}$. This is a minimum set of hypotheses. Additional hypotheses are imposed in the examples.

Finally, we assume that the system (5.1) is equipped with an entropy function $\Psi(U)$, *i.e.* there is a multiplier Ψ_U such that $\Psi_U \cdot R(U) \le 0$ and

$$\partial_t \Psi(U) + \sum_{i=1}^{d} \partial_{x_i} \Phi_i(U) = \frac{1}{\varepsilon} \Psi_U \cdot R(U) \le 0\,. \tag{5.4}$$

This structure is common in several systems of relaxation type, whose origin is in the kinetic theory of gases [CLL], and is induced by the second law of thermodynamics for models with internal variables originating in the continuum physics context.

Under these hypotheses, it is conceivable that $u_\varepsilon = q \cdot U_\varepsilon \to u$ as $\varepsilon \to 0$, where u is a solution of the system of m conservation laws

$$u_t + \sum_{i=1}^{d} \partial_{x_i}\big(q \cdot A_i \mathcal{E}(u)\big) = 0 \tag{5.5}$$

In the sequel, we discuss two examples concerning convergence of relaxation systems to a scalar conservation law in several space dimensions ($m = 1$ with $d = n$) and to the system of isothermal elastodynamics in one space dimension ($m = 2$ with $d = 1$).

5.b The scalar multi-d conservation law via relaxation.

It is a classical result that the Cauchy problem for the scalar conservation law,

$$\begin{cases} \partial_t u + \sum_{i=1}^{n} \partial_{x_i} F_i(u) = 0, \quad x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{5.6}$$

with $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ admits a unique global weak solution $u(x,t)$ satisfying the Kruzhkov entropy conditions [Kr],

$$\partial_t |u-k| + \sum_{i=1}^{n} \partial_{x_i}\big[\,(F_i(u) - F_i(k))\text{sign }(u-k)\,\big] \le 0\,, \qquad \text{in } \mathcal{D}', \text{ for all } k \in \mathbb{R}. \tag{5.7}$$

Entropy weak solutions of (5.6) are constructed as viscosity limits for parabolic regularizations, [V], [Kr], or as the small mean-free-path limit for kinetic equations, [PT]. Here, we review the construction of entropy solutions for (5.6) via relaxation approximations, [KT_2].

This problem is decomposed in two steps. First, we consider the semilinear hyperbolic system of relaxation type,

$$\begin{aligned} \partial_t w + U_0 \cdot \nabla w &= \frac{1}{\varepsilon}\sum_{j=1}^{n}(z_j - h_j(w)) \\ \partial_t z_i + U_i \cdot \nabla z_i &= -\frac{1}{\varepsilon}(z_i - h_i(w))\,, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{5.8}$$

The system governs the dynamics of the state vector (w, Z), $Z = (z_1, \ldots, z_n)$, $U_0, U_1, \ldots, U_n$ are given convective velocities, and $h_i(w)$ are strictly increasing smooth functions with $h_i(0) = 0$. It may be interpreted as a discrete velocity system with an unconventional collision operator, or as a model in chemical kinetics. Solutions of (5.8) satisfy the conservation law

$$\partial_t (w + \sum_i z_i) + \text{div}\big(U_0 w + \sum_i U_i z_i\big) = 0\,. \tag{5.9}$$

As $\varepsilon \to 0$, the local equilibria $z_i = h_i(w)$ are enforced and the limiting dynamics is described by the conservation law

$$\partial_t(w + \sum_i h_i(w)) + \operatorname{div}\big(U_0 w + \sum_i U_i h_i(w)\big) = 0\,. \tag{5.10}$$

This convergence is justified provided that $h_i(w)$ are strictly increasing.

The question arises under what circumstances a given conservation law (5.6) can be realized as a relaxation limit. In view of the convergence of (5.8) to (5.10), the question becomes whether (5.6) can be mapped into the form (5.10). Suppose first that the velocities $U_1, \dots U_n$ are in the coordinate directions, $U_i = V_i \hat{e}_i$, $i = 1, \dots, n$, and that U_0 is expressed as $U_0 = \sum_i \omega_i V_i \hat{e}_i$. Then mapping (5.6) into (5.10) leads to the algebraic problem: Given a curve $(u, F_1(u), ..., F_n(u))$, is it possible to find w and strictly increasing functions $h_i(w)$ such that

$$w + \sum_{i=1}^n h_i(w) = u\,, \qquad \omega_i w + h_i(w) = \frac{1}{V_i} F_i(u)\,. \tag{5.11}$$

Solving (5.11) explicitly, we see that this happens if and only if the following multidimensional analog of the subcharacteristic condition,

$$1 + \sum_i \frac{1}{V_i}\frac{dF_i}{du} > 0\,, \qquad \frac{1}{V_i}\frac{dF_i}{du} < \frac{\omega_i}{1 + \sum_i \omega_i}\big(1 + \sum_i \frac{1}{V_i}\frac{dF_i}{du}\big)\,, \tag{5.12}$$

is satisfied. Clearly, (5.12) holds if $\omega_i > 0$ and the speeds V_i are selected sufficiently large. The general problem, when $U_1, \dots, U_n$ are linearly independent, can be transformed into the above special case, by performing a linear transformation of coordinates.

Theorem 5.1. *Let ω_i, V_i be such that (5.12) is satisfied, let $U_i = V_i \hat{e}_i$, $i = 1, \dots, n$ and $U_0 = \sum_i \omega_i V_i \hat{e}_i$. Suppose that the initial data $w_{0\varepsilon}, z_{i0\varepsilon}$ lie in a bounded set of $BV \cap L^\infty(\mathbb{R}^n)$ and are tight in $L^1(\mathbb{R}^n)$. If $\|u_{0\varepsilon} - u_0\|_{L^1} = o(1)$, then there exists a function*

$$u \in L^\infty([0,T]; BV \cap L^\infty(\mathbb{R}^n)) \cap Lip([0,T]; L^1(\mathbb{R}^n))$$

such that

$$u_\varepsilon = w_\varepsilon + \sum_{i=1}^n z_{i\varepsilon} \to u \quad \text{in} \quad L^1(\mathbb{R}^n \times \mathbb{R}^+)\,, \tag{5.13}$$

and u is a weak solution of (5.6) satisfying the Kruzhkov entropy conditions (5.7).

The main step is to show that, for any $U_0, \dots, U_n$ and for h_i strictly increasing, solutions of (5.8) converge to an entropy solution of (5.10). We present an outline of this proof and refer to

[KT$_2$] for the details as well as for further properties of rates of convergence, and convergence to measure-valued solutions under weaker hypotheses on the data.

First, if (w, z_i) and $(\bar{w}, \bar{z}_i)$ are two solutions of (5.8), then they satisfy

$$\begin{aligned}
&\partial_t\Big(|w-\bar{w}| + \sum_i |z_i - \bar{z}_i|\Big) + \operatorname{div}\Big(U_0|w-\bar{w}| + \sum_i U_i|z_i-\bar{z}_i|\Big) \\
&\quad = \frac{1}{\varepsilon}\sum_i \Big(\text{sign }(w-\bar{w}) - \text{sign }(z_i-\bar{z}_i)\Big)\Big((z_i-\bar{z}_i) - (h_i(w)-h_i(\bar{w}))\Big) \le 0
\end{aligned} \tag{5.14}$$

In particular, if $(\bar{w}, \bar{z}_i)$ is the equilibrium solution $(\kappa, h_i(\kappa))$ then (5.14) yields the inequalities

$$\partial_t\Big(|w-\kappa| + \sum_i |z_i - h_i(\kappa)|\Big) + \operatorname{div}\Big(U_0|w-\kappa| + \sum_i U_i|z_i-h_i(\kappa)|\Big) \le 0 \quad \text{for } \kappa \in \mathbb{R}, \tag{5.15}$$

which is a version of the Kruzhkov entropy inequalities for the relaxation system, and turn out to provide the Kruzhkov entropy conditions for the conservation law (5.10) in the limit $\varepsilon \to 0$.

Using (5.14) and the conservation law (5.9) as key ingredients, we have the following theorem.

Theorem 5.2. *Let h_i be strictly increasing. If $w_0, z_{i0} \in L^1 \cap L^\infty(\mathbb{R}^n)$ then there exists a unique globally defined weak solution (w, Z) of (5.8), which satisfies:*

(i) If (w, Z) and $(\hat{w}, \hat{Z})$ are two solutions then

$$\int |w(x,t) - \hat{w}(x,t)| + \sum_{i=1}^n |z_i(x,t) - \hat{z}_i(x,t)|\, dx \le \int |w_0 - \hat{w}_0| + \sum_{i=1}^n |z_{i0} - \hat{z}_{i0}|\, dx\,.$$

(ii) For any $a < b$ the sets $\mathcal{R}_{a,b} := [a,b] \times \prod_{i=1}^n \big[h_i(a), h_i(b)\big]$ are positively invariant.

(iii) If w_0, $z_{i0} \in BV(\mathbb{R}^n)$ then $w(\cdot,t)$, $z_i(\cdot,t) \in BV(\mathbb{R}^n)$.

For h_i strictly increasing, the relaxation system (5.8) is equipped with a globally defined entropy function

$$\partial_t\Big(\frac{1}{2}w^2 + \sum_{i=1}^n \Psi_i(z_i)\Big) + \operatorname{div}\Big(U_0\frac{1}{2}w^2 + \sum_{i=1}^n U_i\Psi_i(z_i)\Big) + \frac{1}{\varepsilon}\sum_{i=1}^n \phi_i(w, z_i) = 0\,, \tag{5.16}$$

where

$$\Psi_i(z_i) = \int_0^{z_i} h_i^{-1}(\xi)\, d\xi\,,$$

is positive and strictly convex, while

$$\phi_i(w, z_i) = (w - h_i^{-1}(z_i))(h_i(w) - z_i)$$

satisfies $\phi_i \geq 0$ and $\phi_i = 0$ if and only if $(w, Z) \in \mathcal{M}$. The "dissipation" estimate (5.16) provides control of the distance of solutions from the equilibrium curve $\mathcal{M}$. Since $\phi_i \geq c(h_i(w) - z_i)^2$, it leads to

$$\int_0^\infty \int_{\mathbb{R}^n} (h_i(w) - z_i)^2 \, dxdt \leq C\varepsilon \,. \tag{5.17}$$

Let now (w, z_i) be a solution of (5.8) and $(\kappa, h_i(\kappa))$, $\kappa \in \mathbb{R}$, be an equilibrium. Then, (5.15) gives

$$\begin{aligned} \partial_t \Big(|w - \kappa| + \sum_i |h_i(w) - h_i(\kappa)| \Big) + \mathrm{div} \Big(U_0 |w - \kappa| + \sum_i U_i |h_i(w) - h_i(\kappa)| \Big) \\ \leq \partial_t \sum_i g_i + \mathrm{div} \sum_i U_i g_i \end{aligned} \tag{5.18}$$

where g_i can be estimated in terms of the distance of each solution from the Maxwellian values,

$$|g_i| = \Big| |h_i(w) - h_i(\kappa)| - |z_i - h_i(\kappa)| \Big| \leq |h_i(w) - z_i| \,. \tag{5.19}$$

If we set

$$u = w + \sum_i h_i(w) \,, \quad k = \kappa + \sum_i h_i(\kappa) \tag{5.20}$$

we see that $u > k$ if and only if $w > \kappa$. Letting

$$F(u) = U_0 w + \sum_i U_i h_i(w) \,, \quad F(k) = U_0 \kappa + \sum_i U_i h_i(\kappa) \,, \tag{5.21}$$

be the fluxes of (5.10), it follows that the the right hand side of (5.17) is written

$$\partial_t |u - k| + \mathrm{div} \Big((F(u) - F(k)) \mathrm{sign}\,(u - k) \Big) \leq \partial_t \sum_i g_i + \mathrm{div} \sum_i U_i g_i \,. \tag{5.22}$$

The formula indicates that u in (5.20) is an approximate solution of (5.10).

To complete the proof, consider a family of solutions $(w^\varepsilon, z_i^\varepsilon)$ of the relaxation system. The L^1 contraction property together with the conservation law (5.9) enables us to deduce precompactness of $w^\varepsilon + \sum_i z_i^\varepsilon$ in $L^1(\mathbb{R}^n \times [0, T])$. There exists a subsequence w^{ε_n} and $z_i^{\varepsilon_n}$ and a function u such that

$$u^{\varepsilon_n} = w^{\varepsilon_n} + \sum_i z_i^{\varepsilon_n} \to u \,, \quad \text{a.e. } (x, t).$$

Since $g_i^{\varepsilon_n} \to 0$ and the functions h_i are strictly increasing, it follows that w^{ε_n} also converges a.e. to some function w, with $u = w + \sum_i h_i(w)$. Passing to the limit $\varepsilon \to 0$ in (5.22) we deduce that the limiting u is an entropy solution of (5.10).

5.c A relaxation limit to the equations of isothermal elastodynamics.

In this section, we address the problem of constructing weak solutions of the equations of isothermal elasticity with $g_u > 0$,

$$\begin{aligned} \partial_t u - \partial_x v &= 0 \\ \partial_t v - \partial_x g(u) &= 0\,, \end{aligned} \tag{5.23}$$

as $\varepsilon \to 0$ limits of the relaxation system

$$\begin{aligned} \partial_t u - \partial_x v &= 0 \\ \partial_t v - \partial_x \sigma &= 0 \\ \partial_t(\sigma - Eu) &= -\frac{1}{\varepsilon}(\sigma - g(u))\,. \end{aligned} \tag{5.24}$$

The model (5.24) is suggested as an approximating model for the equations of isothermal elastodynamics in [FM].

We work under the standing hypotheses $g(0) = 0$ and $0 < g_u < E$, in which case (5.24) admits globally defined smooth solutions, if the initial data are smooth. The hypothesis $g_u < E$ can be motivated in two ways. On the one hand, it guarantees that the internal variable theory described by (5.24) is consistent with the Clausius-Duhem inequality (see Sec. 1). On the other hand, it can be motivated by the analog of the Champan-Enskog expansion for the relaxation process.

In the Chapman-Enskog expansion one seeks to identify the effective response of the relaxation process as it approaches the surface of local equilibria. It is postulated that the relaxing variable σ^ε can be described in an asymptotic expansion that involves *only* the local macroscopic values u^ε, v^ε and their derivatives, *i.e.*

$$\sigma^\varepsilon = g(u^\varepsilon) + \varepsilon S(u^\varepsilon, v^\varepsilon, u^\varepsilon_x, v^\varepsilon_x, \dots) + O(\varepsilon^2) \tag{5.25}$$

To calculate the form of S, we use (5.24),

$$\begin{aligned} \partial_t u^\varepsilon - \partial_x v^\varepsilon &= 0 \\ \partial_t v^\varepsilon - \partial_x g(u^\varepsilon) &= \varepsilon S_x + O(\varepsilon^2) \\ \partial_t(g(u^\varepsilon) - Eu^\varepsilon) + O(\varepsilon) &= -S + O(\varepsilon)\,, \end{aligned} \tag{5.26}$$

whence we obtain

$$S = [E - g_u(u^\varepsilon)]v^\varepsilon_x + O(\varepsilon)\,, \tag{5.27}$$

and we conclude that the effective equations describing the process are

$$\begin{aligned} \partial_t u^\varepsilon - \partial_x v^\varepsilon &= 0 \\ \partial_t v^\varepsilon - \partial_x g(u^\varepsilon) &= \varepsilon \partial_x\big([E - g_u(u^\varepsilon)]v^\varepsilon_x\big) + O(\varepsilon^2)\,. \end{aligned} \tag{5.28}$$

This is a stable parabolic system provided the condition $g_u < E$ is satisfied.

According to Section 2.e, when $g_u < E$ the system (5.24) describes a theory with internal variables that is consistent with the second law of thermodynamics. Smooth solutions (u, v, σ) satisfy the energy dissipation identity

$$\partial_t\Big(\frac{1}{2}v^2 + \Psi(u, \sigma - Eu)\Big) - \partial_x(\sigma v) + \frac{1}{\varepsilon}(u - h^{-1}(\alpha))(\alpha - h(u))\Big|_{\alpha=\sigma-Eu} = 0 \tag{5.29}$$

where

$$\Psi(u, \alpha) = -\int_0^\alpha h^{-1}(\zeta)d\zeta + \alpha u + \int_0^u E\xi d\xi \tag{5.30}$$

$h(u) = g(u) - Eu$ and h^{-1} is the inverse function of h. The function Ψ provides an "entropy" function for the associated relaxation process, which is convex in (u, α) if $-\partial_\alpha h^{-1}\partial_u f \geq 1$ for all u and α.

Henceforth, we assume that the initial data (u_0, v_0, σ_0) are smooth (of compact support or decaying fast at infinity) and the function $g(u) \in C^3$ satisfies

$$0 < \gamma \leq g_u(u) \leq \Gamma < E\,. \tag{h}$$

for some positive constants γ and Γ. It is easy to check that (5.24) admits global smooth solutions, and we proceed to study the $\varepsilon \to 0$ relaxation process. Equation (5.29) provides stability in L^2 for the relaxation process.

Lemma 5.3. *Under hypothesis (h),*

$$\int_{\mathbb{R}} (u^2 + v^2 + \sigma^2)dx + \frac{1}{\varepsilon C}\int_0^t \int_{\mathbb{R}} \big(\sigma - g(u)\big)^2 dxdt \leq C \int_{\mathbb{R}} (u_0^2 + v_0^2 + \sigma_0^2)dx \tag{5.31}$$

for some C independent of ε and t.

Proof. From (5.30) we have

$$\begin{aligned}\Psi(u, \sigma - Eu) &= -\int_0^{\sigma-Eu} h^{-1}(\zeta)d\zeta + \frac{\sigma^2}{2E} - \frac{1}{2E}(\sigma - Eu)^2 \\ &= \int_0^{\sigma-Eu} \kappa(\zeta)d\zeta + \frac{\sigma^2}{2E}\end{aligned} \tag{5.32}$$

where $\kappa(\alpha) = -\frac{1}{E}\alpha - h^{-1}(\alpha)$. Hypothesis (h) implies

$$\frac{\gamma}{E(E-\gamma)} \leq \frac{d\kappa}{d\alpha} = \frac{g_u}{E(E - g_u)} \leq \frac{\Gamma}{E(E-\Gamma)}$$

and thus there is a constant C, depending only on γ, Γ and E, so that

$$\frac{1}{C}\big((\sigma - Eu)^2 + \sigma^2\big) \le \Psi(u, \sigma - Eu) \le C\big((\sigma - Eu)^2 + \sigma^2\big) \tag{5.33}$$

Furthermore, since $-\frac{d}{d\alpha}h^{-1}(\alpha) = \frac{1}{E - g_u} \ge \frac{1}{E}$, we have

$$(u - h^{-1}(\alpha))(\alpha - h(u)) \ge \frac{1}{E}(\alpha - h(u))^2 \tag{5.34}$$

The result now follows from (5.29), upon using (5.33) and (5.34). □

We proceed with some estimations that capture the dissipative structure of the relaxation process. In preparation, note that solutions of (5.24) satisfy

$$\begin{aligned} &\partial_t u - \partial_x v = 0 \\ &\partial_t v - \partial_x g(u) = \partial_x(\sigma - g(u)) = \varepsilon(E v_{xx} - v_{tt}) \end{aligned} \tag{5.35}$$

The problem under consideration is thus approximation of (5.23) via the wave equation.

Lemma 5.4. *Suppose that the initial data satisfy*

$$\begin{aligned} \int_{\mathbb{R}} v_0^2 + u_0^2 + \sigma_0^2 dx \le O(1)\,, \\ \varepsilon^2 \int_{\mathbb{R}} u_{0x}^2 + v_{0x}^2 + \sigma_{0x}^2 dx \le O(1)\,. \end{aligned} \tag{a}$$

Under hypothesis (h), solutions (u, v, σ) of (5.24) satisfy the ε independent estimates

$$\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 + v_x^2 + \sigma_x^2 dx dt \le O(1)\,. \tag{5.36}$$

Proof. We multiply $(5.35)_1$ by $g(u)$ and $(5.35)_2$ by v. Adding and rearranging the terms we obtain the energy identity

$$\partial_t\big(\frac{1}{2}v^2 + W(u) + \varepsilon v v_t\big) - \partial_x\big(v g(u)\big) + \varepsilon(E v_x^2 - v_t^2) = \varepsilon \partial_x(E v v_x)\,, \tag{5.37}$$

where the stored energy function $W(u)$ is given by

$$W(u) = \int_0^u g(\xi) d\xi\,. \tag{5.38}$$

The problem is that the term $Ev_x^2 - v_t^2$ is not positive definite. To compensate for that, we first multiply $(5.35)_2$ by v_t to obtain

$$v_t^2 - g_u u_x v_t = \varepsilon\Big[(E v_t v_x)_x - \partial_t\big(\frac{1}{2}E v_x^2 + \frac{1}{2}v_t^2\big)\Big]$$

and, in turn

$$\varepsilon^2 \partial_t \big(E v_x^2 + v_t^2 \big) + \varepsilon (2 v_t^2 - 2 g_u u_x v_t) = 2\varepsilon^2 \partial_x (E v_t v_x) . \tag{5.39}$$

Using once again $(5.35)_2$ and the identity $a_x b_t - a_t b_x = \partial_t (a_x b) - \partial_x (a_t b)$, we have

$$\begin{aligned} g_u u_x^2 &= u_x \partial_t (v + \varepsilon v_t) - \varepsilon E u_x v_{xx} \\ &= \Big[u_t \partial_x (v + \varepsilon v_t) + \partial_t \big(u_x (v + \varepsilon v_t) \big) - \partial_x \big(u_t (v + \varepsilon v_t) \big) \Big] - \varepsilon \partial_t \big(\frac{1}{2} E u_x^2 \big) , \end{aligned}$$

which in turn yields

$$\varepsilon^2 \partial_t \big(\frac{1}{2} E^2 u_x^2 - \frac{1}{2} E v_x^2 \big) - \varepsilon \partial_t \big(E u_x (v + \varepsilon v_t) \big) + \varepsilon (E g_u u_x^2 - E v_x^2) = - \varepsilon \partial_x \big(E u_t (v + \varepsilon v_t) \big) . \tag{5.40}$$

Adding (5.37), (5.39) and (5.40), we arrive at

$$\begin{aligned} &\partial_t \Big(\frac{1}{2} (v + \varepsilon v_t - \varepsilon E u_x)^2 + \frac{1}{2} \varepsilon^2 (v_t^2 + E v_x^2) + W(u) \Big) - \partial_x (v g(u)) \\ &\qquad + \varepsilon \big[v_t^2 - 2 g_u u_x v_t + E g_u u_x^2 \big] = \varepsilon^2 (E v_t v_x)_x \end{aligned} \tag{5.41}$$

Because of (h) the second term in (5.41) is positive definite

$$\varepsilon \big[v_t^2 - 2 g_u u_x v_t + E g_u u_x^2 \big] \geq \varepsilon g_u (E - g_u) u_x^2 \geq 0 . \tag{5.42}$$

Therefore, we conclude

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{2} (v + \varepsilon v_t - \varepsilon E u_x)^2 + \frac{1}{2} \varepsilon^2 (v_t^2 + E v_x^2) + W(u) dx \\ &\qquad + \varepsilon \int_0^t \int_{\mathbb{R}} g_u (E - g_u) u_x^2 dx dt \\ &\qquad \leq \int_{\mathbb{R}} \frac{1}{2} (v_0 + \varepsilon \sigma_{0x} - \varepsilon E u_{0x})^2 + \frac{1}{2} \varepsilon^2 (\sigma_{0x}^2 + E v_{0x}^2) + W(u_0) dx \leq O(1) \end{aligned} \tag{5.43}$$

and, due to (h) and (a),

$$\varepsilon \int_0^t \int_{\mathbb{R}} g_u (E - g_u) u_x^2 dx dt \leq O(1)$$

In turn, (5.39) and (5.37) imply

$$\begin{aligned} \varepsilon \int_0^t \int_{\mathbb{R}} \sigma_x^2 dx dt &\leq O(1) \\ \varepsilon \int_0^t \int_{\mathbb{R}} v_x^2 dx dt &\leq O(1) \end{aligned}$$

and (5.36) follows. □

We come next to the convergence Theorem.

Theorem 5.5. *Let $g(u)$ be a smooth function satisfying (h) such that g_{uu} vanishes at exactly one point. If $(u^\varepsilon, v^\varepsilon, \sigma^\varepsilon)$ is a family of smooth solutions of (5.24) that are uniformly stable in L^∞,*

$$
|u^\varepsilon| + |v^\varepsilon| + |\sigma^\varepsilon| \le C\,, \tag{H}
$$

and emanate from initial data satisfying the uniform bounds (a), then, along a subsequence if necessary,

$$
u^\varepsilon \to u\,, \quad v^\varepsilon \to v\,, \quad a.e.\ (x,t)\,. \tag{5.44}
$$

If in addition $u_0^\varepsilon \to u_0$, $v_0^\varepsilon \to v_0$ a.e x, and

$$
\varepsilon^2 \int_{\mathbb{R}} u_{0x}^2 + v_{0x}^2 + \sigma_{0x}^2 dx = o(1)\,, \quad as\ \varepsilon \to 0, \tag{b}
$$

then (u,v) is a weak solution of (5.23) and

$$
\partial_t\big(\frac{1}{2}v^2 + W(u)\big) - \partial_x\big(g(u)v\big) \le 0\,, \quad in\ \mathcal{D}'. \tag{5.45}
$$

The hypothesis of uniform L^∞ stability is artificial. The convergence in the natural framework of L^2 stability will be pursued in [T$_4$]. It is worth noting that while Hypothesis (a) is sufficient to establish (5.44), Hypothesis (b) is necessary to justify the energy dissipation (5.45) relative to the initial data (u_0, v_0).

Proof. Let $\eta(u,v)$, $q(u,v)$ be an entropy pair for the equations of isothermal elasticity. Using (5.35) we obtain

$$
\begin{aligned}
\partial_t\eta(u^\varepsilon,v^\varepsilon) + \partial_x q(u^\varepsilon,v^\varepsilon) &= \eta_v\partial_x(\sigma - g(u)) \\
&= \partial_x(\eta_v(\sigma - g(u))) - (\eta_{vu}\varepsilon^{\frac12}u_x + \eta_{vv}\varepsilon^{\frac12}v_x)\frac{\sigma - g(u)}{\varepsilon^{\frac12}} \\
&= I_1 + I_2
\end{aligned} \tag{5.46}
$$

In view of (5.31), (5.36) and (H), the term I_1 lies in a compact of H^{-1}, the term I_2 is uniformly bounded in L^1, and the sum $I_1 + I_2$ is uniformly bounded in $W^{-1,\infty}$. One concludes from a lemma of Murat [M$_2$] that $I_1 + I_2$ lies in a compact of H^{-1}_{loc}. Then from a theorem of DiPerna [Dp$_1$] we obtain, along a subsequence, $u^\varepsilon \to u$ and $v^\varepsilon \to v$ a.e. (x,t).

It remains to prove (5.45). Let φ be a positive test function with compact support in $[0,T)\times\mathbb{R}$. From (5.41) we have

$$
\begin{aligned}
&-\int_0^T \int_{\mathbb{R}} \varphi_t \big[\frac{1}{2}(v^\varepsilon + \varepsilon v_t^\varepsilon - \varepsilon E u_x^\varepsilon)^2 + \frac{1}{2}\varepsilon^2({v_t^\varepsilon}^2 + E{v_x^\varepsilon}^2) + W(u^\varepsilon)\big] - \varphi_x(v^\varepsilon g(u^\varepsilon))\,dxdt \\
&\qquad + \varepsilon \int_0^T \int_{\mathbb{R}} \varphi\big[{v_t^\varepsilon}^2 - 2g_u u_x^\varepsilon v_t^\varepsilon + E g_u {u_x^\varepsilon}^2\big]\,dxdt \\
&\qquad - \int_{\mathbb{R}} \varphi(x,0)\big[\frac{1}{2}(v_0^\varepsilon + \varepsilon\sigma_{0x}^\varepsilon - \varepsilon E u_{0x}^\varepsilon)^2 + \frac{1}{2}\varepsilon^2({\sigma_{0x}^\varepsilon}^2 + E{v_{0x}^\varepsilon}^2) + W(u_0^\varepsilon)\big]\,dx \\
&\qquad = -\varepsilon^2 \int_0^T \int_{\mathbb{R}} \varphi_x(E v_t^\varepsilon v_x^\varepsilon)\,dxdt\,.
\end{aligned}
\tag{5.47}
$$

We use $u_{0\varepsilon} \to 0$, $v_{0\varepsilon} \to 0$ a.e, Hypotheses (a) and (b) for the data, together with (5.36) and (5.42), to conclude

$$
-\int_0^T \int_{\mathbb{R}} \varphi_t\big[\frac{1}{2}v^2 + W(u)\big] - \varphi_x v g(u)\,dxdt \le \int_{\mathbb{R}} \varphi(x,0)\big[\frac{1}{2}v_0^2 + W(u_0)\big]\,dx\,.
\tag{5.48}
$$

The convergence of (5.24) to (5.23) follows from a similar argument, passing to the limit in (5.35) and using (5.31). □

Bibliographic remarks. Weak solutions of the scalar multidimensional conservation law can be constructed as zero-viscosity limits of parabolic regularizations, Volpert [V], Kruzhkov [Kr], via fluid-dynamic limits for BGK models Perthame-Tadmor [PT], or via relaxation approximations Katsoulakis-Tzavaras [KT_1, KT_2], Natalini [N_2]. There are two equivalent notions of solution, the Kruzhkov entropy solution [Kr], and the kinetic formulation of Lions-Perthame-Tadmor [LPT_1]; the solution operator defines a contraction semigroup in L^1.

The importance of the Chapman-Enskog expansion and the subcharacteristic condition was recognized in early studies of relaxation phenomena, Liu [Li_3]. The Hilbert expansion is very useful for studying relaxation to smooth solutions and initial layers, Caflisch-Papanicolaou [CP], Yong [Yo]. A general framework for investigating relaxation to processes containing shocks is proposed in Chen-Levermore-Liu [CLL], and the mechanism is exploited in Jin-Xin [JX] to construct a class of nonoscillatory numerical schemes. There are a number of studies establishing convergence to scalar conservation laws in one-space dimension, [CLL], [N_1], [TW], and relaxation can be used as an intermediate step to establish convergence of stochastic interacting particle systems to scalar equations [KT_3]. Concerning relaxation limits to systems, we refer to Perthame-Coquel [CPe], Brenier-Corrias-Natalini [BCN], and Tzavaras [T_4] (from where the material of Sections 2.d, 2.e and 5.c is taken).

References

[BCN] Brenier Y., Corrias L. Natalini R., Relaxation limits for a class of balance laws with kinetic formulation, (1997) (preprint).

[CP] Caflisch R.E and Papanicolaou G.C., The fluid dynamic limit of a nonlinear model of the Boltzmann equation, *Comm. Pure Appl. Math.* **32** (1979), 589-616.

[Ch] Chen G.-Q., Propagation and cancellation of oscillations for hyperbolic systems of conservation laws, *Comm. Pure Appl. Math.* **44** (1991), 121-139.

[CLL] Chen G.-Q,, Levermore C.D. and Liu T.-P., Hyperbolic conservation laws with stiff relaxation terms and entropy, *Comm. Pure Appl. Math.* **47** (1994), 789-830.

[CN] Coleman B.D. and Noll W., The thermodynamics of elastic materials with heat conduction and viscosity, *Arch. Rational Mech. Anal.* **13** (1963), 167-178.

[Co] Coleman B.D., Thermodynamics of materials with memory, *Arch. Rational Mech. Anal.* **17** (1964), 1-46.

[CM] Coleman B.D. and Mizel V.J., Existence of caloric equations of state in thermodynamics, *J. Chem. Phys.* **40** (1964), 1116-1125.

[CG] Coleman B.D. and Gurtin M.E., Thermodynamics with internal state variables, *J. Chem. Physics* **47** (1967), 597-613.

[CPe] Coquel F. and Perthame B., Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics, *SIAM Num. Anal.* (to appear).

[Cr] Crandall M., The semigroup approach for first order quasilinear equations in several space variables, *Israel J. Math.*, **12** (1972), 108-132.

[D_1] Dafermos C.M., Solution of the Riemann problem for a class of hyperbolic conservation laws by the viscosity method, *Arch. Rational Mech. Analysis* **52** (1973), 1-9.

[D_2] Dafermos C.M., Structure of solutions of the Riemann problem for hyperbolic systems of conservations laws, *Arch. Rational Mech. Analysis* **53** (1974), 203-217.

[D_3] Dafermos, C.M., Global smooth solutions to the initial-boundary value problem for the equations of one-dimensional nonlinear thermoviscoelasticity. *SIAM J. Math. Anal.* **13** (1982), 397-408.

[D_4] Dafermos C.M., Contemporary issues in the dynamic behavior of continuous media, Lecture Notes, Brown University, 1985.

[D_5] Dafermos C.M., Admissible wave fans in nonlinear hyperbolic systems, *Arch. Rational Mech. Analysis* **106** (1989), 243-260.

[DDp] Dafermos C.M. and DiPerna R.J., The Riemann problem for certain classes of hyperbolic systems of conservation laws, *J. Diff. Equations* **20** (1976), 90-114.

[Dp_1] DiPerna R., Convergence of approximate solutions to conservation laws, *Arch. Rational Mech. Analysis* **60** (1983), 75-100.

[Dp_2] DiPerna R., Measure-valued solutions to conservation laws, *Arch. Rational Mech. Analysis* **88** (1985), 223-270.

[E] Ercole G., Delta-shock waves as self-similar viscosity limits, (1997) (preprint).

[FM] Faciu C. and Mihailescu-Suliciu M., The energy in one-dimensional rate-type semilinear viscoelasticity, *Int. J. Solids Structures* **23** (1987), 1505-1520.

[F_1] Fan H.-T., A limiting "viscosity" approach to the Riemann problem for materials exhibiting change of phase (II), *Arch. Rational Mech. Analysis* **116** (1992) 317-338.

[F_2] Fan H.-T., One-phase Riemann problem and wave interactions in systems of conservation laws of mixed type *SIAM J. Math. Anal.* **24** (1993) 840-865.

[GWS] Gurtin M.E., Williams W.O. and Suliciu I., On rate type constitutive equations and the energy of viscoelastic and viscoplastic materials, *Int. J. Solids Structures* **16** (1980), 607-617.

[H] Heibig A., Existence and uniqueness of solutions for some hyperbolic systems of conservation laws. *Arch. Rational Mech. Anal.* **126** (1994), 79-101.

[JX] Jin S. and Xin Z., The relaxing schemes for systems of conservation laws in arbitrary space dimensions, *Comm. Pure Appl. Math.* **48** (1995), 235-277.

[Ka] Kalasnikov A.S., Construction of generalized solutions of quasi-linear equations of first order without convexity conditions as limits of solutions of parabolic equations with a small parameter, *Dokl. Akad. Nauk SSSR* **127** (1959), 27-30 (in Russian).

[KT_1] Katsoulakis M.A. and Tzavaras A.E., Contractive relaxation systems and interacting particles for scalar conservation laws *C. R. Acad. Sci. Paris, Sér. I Math.* **323** (1996), 865-870.

[KT_2] Katsoulakis M.A. and Tzavaras A.E., Contractive relaxation systems and the scalar multidimensional conservation law, *Comm. Partial Differential Equations* **22** (1997), 195-233.

[KT_3] Katsoulakis M.A. and Tzavaras A.E., Multiscale analysis of interacting particles: Relaxation schemes and scalar conservation laws, (1998) (submitted)

[KK_1] Keyfitz B. and Kranzer H., A viscosity approximation to a system of conservation laws with no classical Riemann solution, in " Proceedings of International Conference on Hyperbolic Problems", Bordeaux, 1988.

[KK_2] Keyfitz B. and Kranzer H., Spaces of weighted measures for conservation laws with singular shock solutions, *J. Differential Equations* **118** (1995), 420–451.

[Ki] Kim Y.-J., A self-similar viscosity approach for the Riemann problem in isentropic gas dynamics and the structure of its solutions, (1998) (preprint).

[Kr] Kruzhkov S.N., First order quasilinear equations with several independent variables, *Math. USSR Sbornik* **10** (1970), 217-243.

[La_1] Lax P.D., Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* **10** (1957), 537-566.

[La_2] Lax P.D., Shock waves and entropy, in: "Contributions to Nonlinear Functional Analysis." E.H. Zarantonello, ed. New York: Academic Press, 1971, pp. 603-634.

[LT_1] LeFloch P.G. and Tzavaras A.E., Existence theory for the Riemann problem for non-conservative hyperbolic systems, *C.R. Acad. Sci., Paris, Série I* **323** (1996), 347-352.

[LT_2] LeFloch P.G. and Tzavaras A.E., Representation of weak limits and definition of nonconservative products, (1997) (preprint).

[LT_3] LeFloch P.G. and Tzavaras A.E., (in preparation).

[Lin] Lin P., Young measures and an application of compensated compactness to one-dimensional nonlinear elastodynamics. *Trans. Amer. Math. Soc.* **329** (1992), 377-413.

[LPT_1] Lions P.L., Perthame B. and Tadmor E., A kinetic formulation of scalar multidimensional conservation laws, *J. AMS* **7** (1994), 169-191.

[LPT_2] Lions P.L., Perthame B. and Tadmor E., Kinetic formulation of the isentropic gas dynamics and p-systems, *Comm. Math. Physics* **163** (1994), 415-431.

[LPS] Lions P.L., Perthame B. and Souganidis P.E., Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, *Comm. Pure Appl. Math.* **49** (1996), 599-638.

[Li_1] Liu T.-P., The Riemann problem for general 2×2 conservation laws, *Trans. Amer. Math. Society* **199** (1974), 89-112.

[Li_2] Liu T.-P., The Riemann problem for general systems of conservation laws, *J. Diff. Equations* **18** (1975), 218-234.

[Li_3] Liu T.-P., Hyperbolic conservation laws with relaxation, *Comm. Math. Phys.* **108** (1987), 153-175.

[MP] Majda A. and Pego R.L., Stable viscosity matrices for systems of conservation laws, *J. Diff. Equations* **56** (1985), 229-262.

[M_1] Murat F., Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **5** (1978), 489-507.

[M_2] Murat F., L'injection du cone positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$. *J. Math. Pures Appl.* **60** (1981), 309-322.

[N_1] Natalini R., Convergence to equilibrium for the relaxation approximations of conservation laws, *Comm. Pure Appl. Math.* **49** (1996), 795-823.

[N$_2$] Natalini R., A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws, (1996), (preprint).

[PT] Perthame B. and Tadmor E., A kinetic equation with kinetic entropy functions for scalar conservation laws, *Comm. Math. Phys.* **136** (1991), 501-517.

[PTz] Perthame B. and Tzavaras A.E., (in preparation).

[Se] Serre D., La compacitè par compensation pour les systèmes hyperboliques non linéaires de deux équations a une dimension d' espace. *J. Maths. Pures et Appl.* **65** (1986), 423-468.

[SeSh] Serre D. and Shearer J., Convergence with physical viscosity for nonlinear elasticity, (1993) (preprint).

[SS] Shearer, M. and Schaeffer, D.G., Fully nonlinear hyperbolic systems of partial differential equations related to plasticity. *Comm. Partial Differential Equations* **20** (1995), 1133-1153.

[Sh] Shearer J.W., Global existence and compactness in L^p for the quasi-linear wave equation. *Comm. Partial Differential Equations* **19** (1994), 1829-1877.

[S$_1$] Slemrod M., A limiting "viscosity" approach to the Riemann problem for materials exhibiting change of phase, *Arch. Rational Mech. Analysis* **105** (1989), 327-365.

[S$_2$] Slemrod M., A comparison of two viscous regularizations of the Riemann problem for Burgers's equation, *SIAM J. Math. Analysis* **26** (1995), 1415-1424.

[ST$_1$] Slemrod M. and Tzavaras A.E., A limiting viscosity approach for the Riemann problem in isentropic gas dynamics, *Indiana Univ. Math. J.* **38** (1989), 1047-1074.

[ST$_2$] Slemrod M. and Tzavaras A.E., Self-similar fluid-dynamic limits for the Broadwell system, *Arch. Rational Mech. Anal.* **122** (1993), 353-392.

[Su] Suliciu I., On the thermodynamics of rate-type fluids and phase transitions. I-Rate-type fluids and II-Phase transitions, (1997) (preprint).

[TZZ] Tan D., Zhang T. and Zheng Y., Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws. *J. Differential Equations* **112** (1994), 1-32.

[Ta] Tartar L., Compensated compactness and applications to partial differential equations, In *Nonlinear Analysis and Mechanics, Heriot Watt Symposium, Vol. IV*, R.J. Knops, ed., Pitman Research Notes in Math., New York, 1979, pp. 136–192.

[TN] Truesdell C.A. and Noll W., *The Nonlinear Field Theories of Mechanics*, Handbuch der Physik III/3, Springer-Verlag, Berlin, 1965.

[TW] Tveito A. and Winther R., On the rate of convergence to equilibrium for a system of conservation laws with a relaxation term, *SIAM J. Math. Anal.* **28** (1997), 136-161.

[T$_1$] Tzavaras A.E., Wave structure induced by fluid dynamic limits in the Broadwell model, *Arch. Rational Mech. Anal.* **127** (1994), 361-387.

[T$_2$] Tzavaras A.E., Elastic as limit of viscoelastic response, in a context of self-similar viscous limits, *J. Diff. Equations*, **123** (1995), 305-341.

[T$_3$] Tzavaras A.E., Wave interactions and variation estimates for self-similar zero-viscosity limits in systems of conservation laws, *Arch. Rational Mech. Anal.* **135** (1996), 1-60.

[T$_4$] Tzavaras A.E., Materials with internal variables and relaxation to conservation laws, (1998) (preprint).

[Tu] Tupciev V.A., On the method of introducing viscosity in the study of problems involving the decay of a discontinuity, *Dokl. Akad. Nauk SSSR* **211** (1973), 55-58. English translation: *Soviet Math. Dokl.* **14** (1973), 978-982.

[V] Volpert A.I., The space BV and quasilinear equations *Math. Sbornik* **73(115)** (2) (1967), 225–267.

[Yo] Yong W.A., Singular perturbations of first-order hyperbolic systems, in "Nonlinear Hyperbolic Problems: Theoretical, Applied and Computational Aspects", Notes on Numerical Fluid Mechanics, Vol. 43, Vieweg, Braunschweig 1993; also Ph.D. Thesis, Univ. of heidelberg, 1992.