A SELF-SIMILAR VISCOSITY APPROACH FOR THE RIEMANN PROBLEM IN ISENTROPIC GAS DYNAMICS AND THE STRUCTURE OF THE SOLUTIONS

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Abstract. We study the Riemann problem for the system of conservation laws of one dimensional isentropic gas dynamics in Eulerian coordinates. We construct solutions of the Riemann problem by the method of self-similar zero-viscosity limits, where the self-similar viscosity only appears in the equation for the conservation of momentum. No size restrictions on the data are imposed. The structure of the obtained solutions is also analyzed.

Key words. conservation laws, self-similar solutions, Riemann problem isentropic gas dynamics, zero-viscosity limits.

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1. Introduction. We consider the equations describing one dimensional isentropic motions of inviscid gases,

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x &= 0, \end{aligned} \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

in Eulerian coordinates. The functions $\rho = \rho(x,t)$, $u = u(x,t)$ and $p = p(\rho)$ represent density, velocity and pressure in that order. The density ρ takes nonnegative values and the pressure function $p(\rho)$ is smooth and defined for $\rho \geq 0$. We assume that the pressure function $p(\rho)$ satisfies the hypothesis

$$p'(\rho) > 0 \quad \text{for} \quad \rho > 0. \tag{H1}$$

Then (1.1) forms a strictly hyperbolic system with characteristic speeds $\lambda_\pm(\rho,u) = u \pm \sqrt{p'(\rho)}$ for $\rho > 0$. We do not assume strict hyperbolicity for $\rho = 0$ since it does not include the usual γ-laws, $p(\rho) = k\rho^\gamma, \quad \gamma \geq 1, \quad k > 0$. We see that it causes significant difficulties in the analysis when vacuum is considered. We also adopt a hypothesis

$$p(\rho) \to \infty \quad \text{as} \quad \rho \to \infty \quad \text{and} \quad p(\rho) \to 0 \quad \text{as} \quad \rho \to 0, \tag{H2}$$

which is natural for the pressure functions of gas dynamics.

We are interested in the Riemann problem: finding a weak solution for (1.1) with initial data

$$(\rho(x,0), u(x,0)) = \begin{cases} (\rho_-, u_-), & x < 0 \\ (\rho_+, u_+), & 0 < x \end{cases} \tag{1.2}$$

with $\rho_\pm > 0$. Since homogeneous conservation laws and Riemann initial data are invariant under the rescaling $(x,t) \to (\alpha x, \alpha t), \alpha > 0$, it is natural to expect that the solution of the Riemann problem should be a function of the scaling invariance variable $\xi = x/t$ which is called the self-similar variable of the Riemann problem. A

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simple computation shows that $(u,\rho)(x,t) = (u,\rho)(x/t)$ is a solution of (1.1),(1.2) if $(u,\rho)(\xi)$ is a solution of the boundary value problem $(\mathcal{P})$:

$$\begin{aligned} -\xi\rho' + (\rho u)' &= 0 \\ -\xi(\rho u)' + (\rho u^2 + p(\rho))' &= 0 \end{aligned} \tag{1.3}$$

$$\rho(\pm\infty) = \rho_\pm, \quad u(\pm\infty) = u_\pm, \tag{1.4}$$

where the ordinary differentiation is with respect to ξ.

It is well known that weak solutions are not unique and that the problem (1.3) with (1.4) should be studied by introducing admissibility criteria attempting to single out the physically admissible solution. We refer to [4], [5] for the solution of the Riemann problem for general strictly hyperbolic systems and to [2] for a discussion of the issue of admissibility for hyperbolic systems of conservation laws.

In this article we study the solution of the Riemann problem $(\mathcal{P})$ as $\varepsilon \to 0$ limit of the solutions of the perturbed problem $(\mathcal{P}_\varepsilon)$:

$$\begin{aligned} -\xi\rho' + (\rho u)' &= 0 \\ -\xi(\rho u)' + (\rho u^2 + p(\rho))' &= \varepsilon u'' \end{aligned} \tag{1.5}$$

$$\rho(\pm\infty) = \rho_\pm, \quad u(\pm\infty) = u_\pm. \tag{1.4}$$

The method of self-similar viscous limits is studied in [1], [3], [10], and the approximation of $(\mathcal{P})$ with full viscosity matrices is studied in [8]. Here, we are interested on the effect of singular diffusion matrices; in mechanical models viscosity appears in the equation of balance of momentum, but not in the equation of balance of mass.

This approach is followed by Tzavaras in [9] for the system of one-dimensional isothermal elastic response in Lagrangian coordinates. Many of the ideas used here are based on this work. In both cases the interplay of hyperbolic and parabolic aspects of the problem must be analyzed. There are two differences in the present work. While in [9] the singularity lies at a fixed point $\xi = 0$(in Lagrangian coordinates), the singularity in (1.5) is located at moving points ξ with $u(\xi) = \xi$ thus depending on the solution. This leads to analyzing free boundary problem, as can be easily seen by considering the systems

$$\begin{aligned} (u-\xi)\rho' + \rho u' &= 0 \\ (u-\xi)\rho u' + p(\rho)' &= 0, \end{aligned} \tag{1.6}$$

$$\begin{aligned} (u-\xi)\rho' + \rho u' &= 0 \\ (u-\xi)\rho u' + p(\rho)' &= \varepsilon u'', \end{aligned} \tag{1.7}$$

that are equivalent to (1,3),(1.5) for smooth solutions. The second difference is associated with complications arising from the loss of strict hyperbolicity at vacuum. A substantial amount of the analytical effort is directed to resolving the complications associated with vacuum states.

The objectives of the article are (i) to show the existence of solutions to the problem $(\mathcal{P}_\varepsilon)$, (ii) to solve the Riemann problem $(\mathcal{P})$ as the $\varepsilon \to 0$ limit of solutions to $(\mathcal{P}_\varepsilon)$ and (iii) to study the structure of the emerging limit. We begin in Section 2 with an analysis of regularity properties and a-priori estimates for weak solutions (ρ, u) of $(\mathcal{P}_\varepsilon)$ with $\rho > 0$. It is shown that such solutions are smooth except for a unique singular point, induced by the singular diffusion matrix in (1.7), and located at $u(\xi) = \xi$. It turns out that at least one of ρ, u is monotone, while the other has at most one critical point.

For the existence of solutions of $(\mathcal{P}_\varepsilon)$ we consider a one-parameter family of boundary value problems $(\mathcal{P}_\varepsilon^\mu)$:

$$\begin{aligned}(u-\xi)\rho' + \rho u' &= 0 \\ (u-\xi)\rho u' + p(\rho)' &= \varepsilon u'',\end{aligned} \qquad -\infty < \xi < \infty, \tag{1.8}$$

$$\begin{aligned}\rho(\pm\infty) = \rho_\pm^\mu &:= \rho_- + \mu(\rho_\pm - \rho_-) \\ u(\pm\infty) = u_\pm^\mu &:= u_- + \mu(u_\pm - u_-),\end{aligned} \qquad 0 \le \mu \le 1, \tag{1.9}$$

which connect the solutions of $(\mathcal{P}_\varepsilon)$ to a trivial solution. In lemmas 2.4 and 2.5 we establish a-priori estimates on u and ρ that are used in Section 3 to construct solutions of $(\mathcal{P}_\varepsilon)$. In Lemma 2.4 an a-priori estimate

$$0 < \delta_\varepsilon < \rho(\xi) \tag{A}$$

is missing for the case that u is increasing on $\mathbb{R}$ (this is the case when vacuum appears in the Riemann problem). We have not been able, under the sole Hypotheses (H1) and (H2), to obtain the estimate (A) for general pressure laws. Nevertheless, in Section 5 the missing estimate is established for either special pressure functions or under restrictions on the Riemann data which prevent vacuum. (We remark that throughout the article we study solutions of $(\mathcal{P}_\varepsilon)$ with $\rho > 0$. It is however conceivable that the problem $(\mathcal{P}_\varepsilon)$ admits, for special pressure laws and boundary data, solutions that vanish in density. This would run counter to the well known fact that the solutions of the Cauchy problem for the momentum-viscosity approximation of (1.1) have the property that $\rho > 0$. On the other hand, the problem $(\mathcal{P}_\varepsilon)$ is essentially a boundary value problem, and the possibility that it has vanishing solutions in ρ cannot be a-priori excluded.)

In Section 3 we apply the Leray-Schauder degree theory to a construction scheme suggested by the a-priori estimates of Section 2. The obtained result on existence of the viscous problem $(\mathcal{P}_\varepsilon)$ is stated in Theorem 3.4.

In Section 4 we consider a family of solutions $(\rho_\varepsilon, u_\varepsilon)$ to $(\mathcal{P}_\varepsilon)$ and study the limit $\varepsilon \to 0$. We show that the total variation of $(\rho_\varepsilon, u_\varepsilon)$ is uniformly bounded, and hence, by virtue of Helly's theorem, along a subsequence $\rho_{\varepsilon_n} \to \rho, u_{\varepsilon_n} \to u$ for some function $\rho \ge 0$ and u of bounded variation. The emerging limit (ρ, u) turns out in Theorem 4.1 to be a solution of the Riemann problem $(\mathcal{P})$ constructed through the method of self-similar zero-viscosity limits.

Then we study the structure of (ρ, u). For the case of convex pressure laws

$$p''(\rho) \ge 0 \quad \text{for} \quad \rho > 0, \tag{H3}$$

the structure of (ρ, u) is established in Theorem 4.7. The solution consists of two waves separated by a constant state. The waves are either a rarefaction or a shock. As an application, we consider the strictly hyperbolic case in Corollary 4.8 and show that vacuum can not appear in this case. Note that Hypothesis (H3) includes both cases of genuine nonlinearity and linear degeneracy.

In the last section we complete the a-priori estimates for the systems with convex pressure laws. First, if the system is strictly hyperbolic, i.e. there exists a constant $c > 0$ such that

$$p'(\rho) \ge c^2, \qquad \rho > 0, \tag{1.10}$$

then we obtain a lower bound for ρ and have a complete theory:

THEOREM. *(Strictly Hyperbolic Convex Laws) Suppose $p(\rho)$ satisfies (H1), (H2) and (H3). If the system (1.1) is strictly hyperbolic, then the boundary value problem $(\mathcal{P})$ has a solution (ρ, u) which is a $\varepsilon \to 0$ of solutions of $(\mathcal{P}_\varepsilon)$. The function (ρ, u) has the structure stated in Theorem 4.7 and does not contain vacuum.*

Second, for convex pressure laws (not necessarily strictly convex), we exhibit a sufficient condition on the data $(\rho_\pm, u_\pm)$ that prevents the appearance of vacuum and completes the theory. The construction of solutions of the Riemann problem (1.1) with (1.2) via self-similar viscous limits and the structure of the emerging solution of these two cases are stated in Theorem 5.3.

2. Weak Solutions and Regularity Properties. We consider the nonlinear boundary value problem $(\mathcal{P}_\varepsilon)$:

$$\begin{aligned} -\xi\rho' + (\rho u)' &= 0 \\ -\xi(\rho u)' + (\rho u^2 + p(\rho))' &= \varepsilon u'' \end{aligned} \qquad -\infty < \xi < \infty \tag{2.1}$$

$$\rho(\pm\infty) = \rho_\pm, \quad u(\pm\infty) = u_\pm, \tag{2.2}$$

with fixed boundary data $\rho_\pm > 0$, $u_\pm$ and $0 < \varepsilon < 1$. In this section we study the regularity for solutions of this problem and establish a-priori estimates which are used to show the existence of solutions to the problem $(\mathcal{P}_\varepsilon)$.

2.1 Regularity Properties. First, we give a definition of the solution of $(\mathcal{P}_\varepsilon)$ in the weak sense.

DEFINITION 2.1. A pair of functions $\rho > 0$ and u with $p(\rho), \rho \in L^\infty_{\text{loc}}(\mathbb{R})$ and $u \in W^1_{1\,\text{loc}}(\mathbb{R})$ is a solution of $(\mathcal{P}_\varepsilon)$ if (ρ, u) satisfies

$$\int (\zeta - u)\rho\varphi' d\zeta + \int \rho\varphi d\zeta = 0 \tag{2.3}$$

$$\int [(\zeta - u)\rho u - p(\rho) + \varepsilon u']\varphi' d\zeta + \int \rho u\varphi d\zeta = 0 \tag{2.4}$$

for all $\varphi \in C^1_c(\mathbb{R})$, continuously differentiable functions with compact support, and the essential limits $\rho(\pm\infty)$ and $u(\pm\infty)$ exist and satisfy (2.2).

It is clear that the integrals in (2.3) and (2.4) are well defined. The equations (2.3) and (2.4) imply that $\rho \in L^\infty_{\text{loc}}(\mathbb{R})$ and $\rho u \in L^1_{\text{loc}}(\mathbb{R})$ are the weak derivatives of $(\xi - u)\rho$ and $(\zeta - u)\rho u - p(\rho) + \varepsilon u'$ respectively. So $(\xi - u)\rho$, $(\zeta - u)\rho u - p(\rho) + \varepsilon u'$ are in $W^1_{1\,\text{loc}}(\mathbb{R})$, and hence absolutely continuous. So $-p(\rho) + \varepsilon u'$ is also continuous.

LEMMA 2.1. *Let (ρ, u) be a solution of $(\mathcal{P}_\varepsilon)$. (i) For $a, b \in \mathbb{R}$,*

$$\Big[(\xi - u(\xi))\rho(\xi)\Big]_a^b - \int_a^b \rho(\zeta)d\zeta = 0 \tag{2.5}$$

$$\Big[(\xi - u(\xi))\rho(\xi)u(\xi) - p(\rho(\xi)) + \varepsilon u'(\xi)\Big]_a^b - \int_a^b \rho(\zeta)u(\zeta)d\zeta = 0. \tag{2.6}$$

(ii) $u, (\xi - u)\rho$ and $-p(\rho) + \varepsilon u'$ are continuous on $\mathbb{R}$. If $p \in C^n(\mathbb{R}^+)$ for $n \geq 0$, then ρ and u are C^{n+1} for all ξ such that $\xi \neq u(\xi)$.

Proof. We already saw that $u, (\xi - u)\rho, -p(\rho) + \varepsilon u'$ and $(\xi - u)\rho u - p(\rho) + \varepsilon u'$ are continuous on $\mathbb{R}$. Fix $a, b \in \mathbb{R}$ with $a < b$ and consider

$$\psi_n(\xi) = \begin{cases} 0, & -\infty < \xi \le a - 1/n \\ n(\xi - a) + 1, & a - 1/n \le \xi \le a \\ 1, & a \le \xi \le b \\ -n(\xi - b) + 1, & b \le \xi \le b + 1/n \\ 0, & b + 1/n \le \xi < +\infty\,. \end{cases}$$

As $\psi_n \notin C_c^1(\mathbb{R})$, it cannot be directly used as a test function. However, since ψ_n is Lipschitz continuous, it can be approximated by C_c^1 functions. Let the sequence $\psi_n^k \in C_c^1(\mathbb{R})$ converge to ψ_n as $k \to \infty$. If we put ψ_n^k in the place of φ in (2.4), then we get

$$\int [(\zeta - u)\rho u - p(\rho) + \varepsilon u'](\psi_n^k)' d\zeta + \int \rho u \psi_n^k d\zeta = 0.$$

Taking the limit $k \to \infty$, we obtain

$$n \int_{a-1/n}^{a} [(\zeta - u)\rho u - p(\rho) + \varepsilon u'] d\zeta - n \int_{b}^{b+1/n} [(\zeta - u)\rho u - p(\rho) + \varepsilon u'] d\zeta \\ + \int_{a-1/n}^{b+1/n} \rho u \psi_n d\zeta = 0.$$

Since $(\zeta - u)\rho u - p(\rho) + \varepsilon u'$ is continuous and $|\rho u \psi_n| \le |\rho u| \in L^1_{\mathrm{loc}}$, Lebesgue Differentiation Theorem and Lebesgue Dominated Convergence Theorem imply that (2.6) holds in the limit $n \to \infty$. Similar statements show (2.5).

Let us consider (ii). Since $(\zeta - u)\rho$ is continuous, ρ is continuous at ξ if $\xi \ne u(\xi)$. From (2.6),

$$\varepsilon u'(\xi) = \int_a^\xi \rho(\zeta) u(\zeta) d\zeta - (\xi - u(\xi))\rho(\xi)u(\xi) + p(\rho(\xi)) \\ + (a - u(a))\rho(a)u(a) - p(\rho(a)) + \varepsilon u'(a). \tag{2.7}$$

If p is $C^0(\mathbb{R}^+)$, u is C^1 at $\xi \ne u(\xi)$. From (2.5),

$$(\xi - u(\xi))\rho(\xi) = \int_a^\xi \rho(\zeta) d\zeta + (a - u(a))\rho(a), \tag{2.8}$$

and ρ is C^1 at those points. If p is $C^n(\mathbb{R})$, we can consider (2.7) and (2.8) n more times to get C^{n+1} smoothness of ρ and u at $\xi \ne u(\xi)$. $\square$

Lemma 2.1 indicates that singularities may arise at s when $u(s) = s$, i.e. at the fixed points of u. Let s be a singular point. Since $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, (2.8) yields

$$\int_s^\xi \rho(\zeta) d\zeta = \rho(\xi)(\xi - u(\xi)). \tag{2.9}$$

Now we prove the uniqueness of the singularity. Note that the proof is closely related with the fact that Definition 2.1 does not accept zero density.

LEMMA 2.2. *The singular point of a solution (ρ, u) of $(\mathcal{P}_\varepsilon)$ is unique.*

Proof. Let s be a singular point and suppose there are no singular points in the interval $(s, s+\tau)$ for some $\tau > 0$ small. From (2.9), we have

$$\xi - u(\xi) = \int_s^\xi \frac{\rho(\zeta)}{\rho(\xi)} d\zeta > 0, \qquad \xi \in (s, s+\tau). \tag{2.10}$$

So $u(\xi) < \xi$ on $(s, s+\tau)$. Let us suppose there are no singular points in the interval $(s-\tau, s)$ for some $\tau > 0$ small. Then

$$\xi - u(\xi) = \int_s^\xi \frac{\rho(\zeta)}{\rho(\xi)} d\zeta < 0, \qquad \xi \in (s-\tau, s).$$

So $\xi < u(\xi)$ on $(s-\tau, s)$. From these facts, it is impossible that the graph of $y = u(\xi)$ meets the diagonal $y = \xi$ twice. Therefore, either the singular point is unique or the set of all singular points is a closed interval.

If the set of the singular points is a closed interval $[a, b]$ with $a \neq b$, then (2.5) implies $\int_c^d \rho(\zeta) d\zeta = 0$ for any $[c, d] \subset [a, b]$, and thus ρ vanishes on $[a, b]$. This contradicts Definition 2.1. $\square$

2.2 Monotonicity Properties. The monotonicity of solutions plays a key role in our problem. It can be easily verified that, at a point of smoothness, the solution (ρ, u) of $(\mathcal{P}_\varepsilon)$ satisfies

$$\begin{aligned} (u-\xi)\rho' + \rho u' &= 0 \\ (u-\xi)\rho u' + p(\rho)' &= \varepsilon u''. \end{aligned} \tag{2.11}$$

Next we analyze the behavior of (ρ, u) in a neighborhood of the singular point $\xi = s$ and $\xi = \pm\infty$. From (2.11) we obtain

$$\rho' = \frac{\rho u'}{(\xi - u)} \tag{2.12}$$

$$\varepsilon u'' + \frac{\{(\xi-u)^2 - p'(\rho)\}\rho}{\xi - u} u' = 0. \tag{2.13}$$

(2.13) can be written in a differential form

$$\frac{d}{d\xi}\Big[u'(\xi) \exp\Big\{\frac{1}{\varepsilon}\int^\xi \frac{\{(\zeta-u)^2 - p'(\rho)\}\rho}{\zeta - u} d\zeta\Big\}\Big] = 0, \tag{2.14}$$

and upon integrating (2.14) we get

$$u'(\xi) = \begin{cases} u'(\alpha_+) \exp\Big\{-\frac{1}{\varepsilon}\int_{\alpha_+}^\xi \frac{\{(\zeta-u)^2 - p'(\rho)\}\rho}{\zeta - u} d\zeta\Big\}, & s < \xi \\ u'(\alpha_-) \exp\Big\{-\frac{1}{\varepsilon}\int_{\alpha_-}^\xi \frac{\{(\zeta-u)^2 - p'(\rho)\}\rho}{\zeta - u} d\zeta\Big\}, & \xi < s \end{cases} \tag{2.15}$$

for any $\alpha_\pm$ such that $s < \alpha_+$ and $\alpha_- < s$, where s is the unique singular point.

Since $\exp\Big\{-\frac{1}{\varepsilon}\int_{\alpha_\pm}^\xi \frac{\{(\zeta-u)^2 - p'(\rho)\}\rho}{\zeta - u} d\zeta\Big\}$ is positive, (2.15) implies that either u is strictly monotone on (s, ∞) and $(-\infty, s)$ or identically constant on the intervals. It is clear from (2.12) that ρ has the same monotonicity as u on (s, ∞) and the opposite one on $(-\infty, s)$.

The monotonicity of the positive solution $\rho \in L^\infty$ implies that

$$0 < k \le \rho(\xi) \le K < \infty, \qquad \xi \in \mathbb{R}, \tag{2.16}$$

where k and K depend only on $\rho_\pm$ and $\rho(s\pm) = \lim_{\xi \to s\pm} \rho(\xi)$. Under Hypothesis (H1), $p'(\rho)$ is bounded by

$$0 < a_0 \le p'(\rho(\xi)) \le A_0, \qquad \xi \in \mathbb{R}, \tag{2.17}$$

where a_0 and A_0 may depend on k and K of (2.16).

LEMMA 2.3. *Let (ρ, u) be a solution of $(\mathcal{P}_\varepsilon)$ with a unique singular point $s \in \mathbb{R}$. (i) There exist two constants $\alpha_- < s, \alpha_+ > s$, depending on a_0, and a constant $\alpha > 0$, depending on a_0 and k, such that*

$$\begin{aligned} |u'(\xi)| &\le |u'(\alpha_+)| \Big| \frac{\xi - s}{\alpha_+ - s} \Big|^{\frac{\alpha}{\varepsilon}}, && s < \xi < \alpha_+, \\ |u'(\xi)| &\le |u'(\alpha_-)| \Big| \frac{\xi - s}{\alpha_- - s} \Big|^{\frac{\alpha}{\varepsilon}}, && \alpha_- < \xi < s. \end{aligned} \tag{2.18}$$

(ii) There exist two constants $\beta_- < s, \beta_+ > s$, depending on A_0, and a constant $\beta > 0$, depending on A_0 and k, such that

$$\begin{aligned} |u'(\xi)| &\le |u'(\beta_+)| \exp\Big\{ -\frac{\beta}{\varepsilon}\Big(\Big(\frac{\xi - s}{\beta_+ - s} \Big)^2 - 1 \Big) \Big\}, && \beta_+ < \xi, \\ |u'(\xi)| &\le |u'(\beta_-)| \exp\Big\{ -\frac{\beta}{\varepsilon}\Big(\Big(\frac{\xi - s}{\beta_- - s} \Big)^2 - 1 \Big) \Big\}, && \xi < \beta_-. \end{aligned} \tag{2.19}$$

(iii) $u'(s) = 0$ and, for the pressure $p \in C^n(\mathbb{R}^+), n \ge 1$, the solution (ρ, u) has the regularity

$$\rho \in C(\mathbb{R}) \cap C^{n+1}(\mathbb{R} - \{s\}) \quad ; \quad u \in C^1(\mathbb{R}) \cap C^{n+1}(\mathbb{R} - \{s\}). \tag{2.20}$$

Proof. $u(\xi) \to u_+$ as $\xi \to \infty$ and $u'(s+)$ is finite from (2.7). We have $u(\xi) < \xi$ on (s, ∞). Thus there is a positive constant b such that $-b(\xi - s) + s < u(\xi) < \xi$ on (s, ∞) (see Figure 1). Let α_+ be a constant such that $s < \alpha_+ < s + \frac{\theta}{b+1}$ with $\theta = \sqrt{a_0}$. Then for all $\zeta \in (s, \alpha_+)$,

$$\frac{\{(\zeta - u)^2 - p'(\rho)\}\rho}{\zeta - u} \le \{(1+b)(\alpha_+ - s)^2 - \frac{\theta^2}{(1+b)}\} \frac{\rho}{\zeta - s} \le -\alpha \frac{1}{\zeta - s} < 0$$

with

$$\alpha = \frac{(\theta^2 - (1+b)^2(\alpha_+ - s)^2)k}{(1+b)}. \tag{2.21}$$

Then α is positive and, from (2.14),

$$|u'(\xi)| \le |u'(\alpha_+)| \exp\Big\{ \frac{\alpha}{\varepsilon} \int_{\alpha_+}^{\xi} \frac{1}{\zeta - s} d\zeta \Big\} = |u'(\alpha_+)| \Big(\frac{\xi - s}{\alpha_+ - s} \Big)^{\frac{\alpha}{\varepsilon}}$$

for all $\xi \in (s, \alpha_+)$. The second statement of (i) can be proved similarly.

Now we prove (ii). Fix $\beta_+ > s + \max\{2(u_+ - u_-), 2\sqrt{2}\Theta\}$ with $\Theta = \sqrt{A_0}$. Then, for any $\xi \in (\beta_+, \infty)$, $\xi - u(\xi) \ge \frac{1}{2}(\xi - s)$ and

$$\frac{\{(\zeta - u)^2 - p'(\rho)\}\rho}{\zeta - u} \ge \{\frac{1}{2} - \frac{2\Theta^2}{(\zeta - s)^2}\} \rho(\zeta - s) \ge \frac{k}{4}(\zeta - s) > 0.$$

Set
$$\beta = \frac{(\beta_+ - s)^2}{2}\frac{k}{4}.$$
Then β is positive and
$$\begin{aligned}|u'(\xi)| &\le |u'(\beta_+)| \exp\Big\{ - \frac{2\beta}{\varepsilon(\beta_+ - s)^2}\int_{\beta_+}^{\xi} \zeta - s d\zeta\Big\} \\ &= |u'(\beta_+)| \exp\Big\{ - \frac{\beta}{\varepsilon}\Big((\frac{\xi - s}{\beta_+ - s})^2 - 1\Big)\Big\}.\end{aligned}$$
The proof of the second statement of (ii) is similar. Part (i) implies regularity for u' near the singular point $\xi = s$ and especially that $u'(s) = 0$. Since $-p(\rho) + \varepsilon u'$ is continuous, ρ is also continuous (due to (H1)). So the regularity of Lemma 2.1 is improved to (2.20). □

The regularity of solution u will be given by the constant α in Lemma 2.3. If α can be chosen independently from ε, u can be assumed as smooth as we want by taking ε small enough. Note that α depends on the choice of α_+. For example, if we take $\alpha_+ = s + \frac{\theta}{\sqrt{2(b+1)}}$, we get $\alpha = \frac{\theta^2}{2(1+b)}$. Since $u'(s) = 0$, we get $b \to 0$ as $\alpha_+ \to s$ and we also get $\alpha \to p'(\rho(s))\rho(s)$. So u has C^2 regularity if $p'(\rho(s))\rho(s) > \varepsilon$.

2.3 A-priori Estimates. Throughout this section we consider a solution (ρ, u) of the family of boundary value problems $(\mathcal{P}_\varepsilon^\mu)$

$$\begin{aligned}(u - \xi)\rho' + \rho u' &= 0 \\ (u - \xi)\rho u' + p(\rho)' &= \varepsilon u''\end{aligned} \qquad -\infty < \xi < \infty, \tag{2.22}$$

$$\begin{aligned}\rho(\pm\infty) = \rho_\pm^\mu &:= \rho_- + \mu(\rho_\pm - \rho_-) \\ u(\pm\infty) = u_\pm^\mu &:= u_- + \mu(u_\pm - u_-)\end{aligned} \qquad 0 \le \mu \le 1, \tag{2.23}$$

which connect the solutions of $(\mathcal{P}_\varepsilon)$ to the trivial solution associated with $\mu = 0$.

Since $\rho_\pm > 0$, the boundary values $\rho_\pm^\mu$ in (2.23) are positive and the solutions of $(\mathcal{P}_\varepsilon^\mu)$ have the regularity derived from the previous sections. In this section we derive a-priori estimates of solutions (ρ, u) of $(\mathcal{P}_\varepsilon^\mu)$, which are used to establish the existence of solutions to $(\mathcal{P}_\varepsilon)$ in the next section. The a-priori estimates are:

$$0 < \delta < \rho(\xi) < M, \qquad \xi \in (-\infty, \infty) \tag{2.24}$$

$$|u(\xi)| < M, \qquad \xi \in (-\infty, \infty) \tag{2.25}$$

$$\begin{aligned}-b(\xi - s) + s < u(\xi) &< a(\xi - s) + s, \qquad \xi \in (s, \infty) \\ a(\xi - s) + s < u(\xi) &< -b(\xi - s) + s, \qquad \xi \in (-\infty, s),\end{aligned} \tag{2.26}$$

where positive constants δ and M are independent of μ, and constants a, b satisfy $0 < a < 1$ and $0 < b$ and depend only on $\delta, M, \rho_\pm$ and $u_\pm$. The point s in (2.26) is the singular point of the solution, and hence it should be bounded by (2.25) and it may depend on μ.

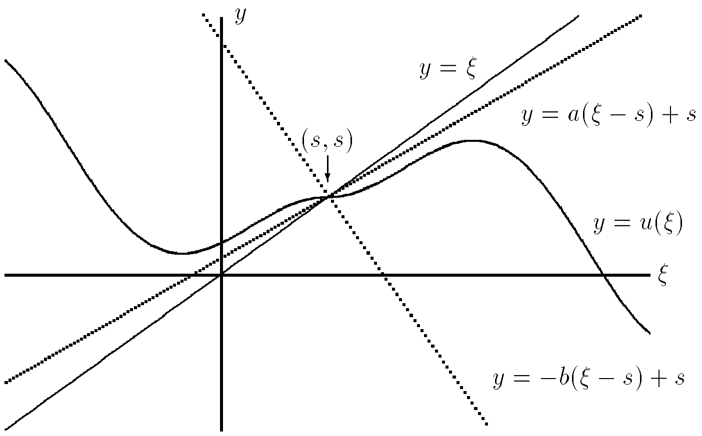

Figure 1

The main difference from the Lagrangian case is the estimate (2.26). It implies that the graph of the velocity $u(\xi)$ lies between two straight lines which have slopes less than 1 and pass through the point (s,s)(see Figure 1). We can easily check that (2.26) is equivalent to

$$A|\xi-s|<|u(\xi)-\xi|<B|\xi-s|,\qquad \xi\neq s, \tag{2.27}$$

with $0<A<1<B$.

The proof of these a-priori estimates strongly depends on the shape of solutions (ρ,u). From the monotonicity property of the previous section, we can classify solutions into 4 categories:
C_1: ρ is increasing on $(-\infty,\infty)$, u is decreasing on $(-\infty,s)$ and increasing on (s,∞).
C_2: ρ is decreasing on $(-\infty,\infty)$, u is increasing on $(-\infty,s)$ and decreasing on (s,∞).
C_3: ρ is increasing on $(-\infty,s)$ and decreasing on (s,∞), u is decreasing on $(-\infty,\infty)$.
C_4: ρ is decreasing on $(-\infty,s)$ and increasing on (s,∞), u is increasing on $(-\infty,\infty)$.
Furthermore, we may assume that the monotonicities are all strict. If not, the solution is constant on $(-\infty,s)$ or (s,∞) and the above estimates are trivial.

To establish those a-priori estimates we accept the second hypothesis on the pressure function:

$$p(\rho)\to\infty\quad\text{as}\quad\rho\to\infty\quad\text{and}\quad p(\rho)\to 0\quad\text{as}\quad\rho\to 0. \tag{H2}$$

Lemma 2.4. *Let (ρ,u) be a solution of $(\mathcal{P}^{\mu}_{\varepsilon})$. If (ρ,u) belongs to the classes C_1,C_2 or C_3, there exist positive constants M and δ which are independent of μ and ε so that (ρ,u) satisfies* (2.24) *and* (2.25). *If (ρ,u) belongs to the class of C_4, there exists a constant M which is independent of μ and ε so that (ρ,u) satisfies* (2.24) *and* (2.25).

Proof. We consider (2.24) first. If (ρ,u) is of class C_1 or C_2, we can take $\delta=\min\{\rho_-,\rho_+\}$ and $M=\max\{\rho_-,\rho_+\}$ for the estimations in (2.24). We can also take $\delta=\min\{\rho_-,\rho_+\}$ for the case of C_3 and $M=\max\{\rho_-,\rho_+\}$ for the case of C_4. So (2.24) is completed, if we prove the existence of an upper bound of ρ for class C_3.

Let $\xi > s$. Then,

$$\rho(\xi) = \rho_+^\mu - \int_\xi^\infty \rho' d\zeta \le \rho_+^\mu - \frac{1}{\xi - s}\int_\xi^\infty (\zeta - s)\rho' d\zeta = \rho_+^\mu - \frac{1}{\xi - s}\int_\xi^\infty (\rho u)' - s\rho' d\zeta$$
$$= \rho_+^\mu + \frac{1}{\xi - s}(\rho_\xi u_\xi - \rho_+ u_+ + s\rho_+ - s\rho_\xi) \le \max\{\rho_-, \rho_+\} + \frac{1}{\xi - s}\rho_+(u_- - u_+).$$

So $\rho(\xi)$ is bounded by a constant which is independent of μ and ε for any fixed $\xi \ne 0$. Let $\tau \in [s+1, s+2]$ satisfy $u'(\tau) = u(s+2) - u(s+1) > u_+ - u_-$. If we integrate $(2.1)_2$ from $\xi > s$ to τ, we get

$$\rho(\xi)u^2(\xi) + p(\rho(\xi)) - \varepsilon u'(\xi) - \rho(\tau)u^2(\tau) - p(\rho(\tau)) + \varepsilon u'(\tau) = -\int_\xi^\tau \zeta(\rho u)' d\zeta$$
$$= \int_\xi^\tau \zeta(\rho(s-u))' d\zeta - s\int_\xi^\tau \zeta\rho' d\zeta = \Big[\zeta\rho(s-u)\Big]_\xi^\tau - \int_\xi^\tau \rho(s-u)d\zeta - s\int_\xi^\tau (\rho u)' d\zeta$$
$$= \tau(\rho(\tau)(s - u(\tau)) - \xi(\rho(\xi)(s - u(\xi)) - \int_\xi^\tau \rho(s-u)d\zeta - s\rho(\tau)u(\tau) + s\rho(\xi)u(\xi)$$
$$\le \tau(\rho(\tau)(s - u(\tau)) - s\rho(\tau)u(\tau) + s\rho(\xi)u(\xi)$$

If we take the limit $\xi \to s$, then

$$p(\rho(s)) \le \max_{\rho_+ < q < \rho(s+1)} \{3q\bar{u}^2 + p(q)\} + (u_- - u_+) =: \mathcal{A},$$

where $\bar{u} = \max\{|u_-|, |u_+|\}$. $\mathcal{A}$ is independent of μ and ε and, from (H2), $p(\rho) \to \infty$ as $\rho \to \infty$. Hence $\rho(s)$ is bounded by a constant which depends only on $\rho_\pm, u_\pm$ and function p.

Now we consider (2.25). If the solution (ρ, u) is of class C_3 or C_4, we can take $M = \max\{u_-, u_+\}$ for the estimation (2.25). The proof of (2.25) for solutions of class C_1 and C_2 are similar. We prove (2.25) only for the case C_2. Because of the shape of the solution we know that $|u(\xi)| \le \max\{|u_\pm|, |u(s)|\}$. So it is enough to prove that the singular point $s = u(s)$ is bounded by a constant M which is independent of μ and ε. If $s \le 0$, we take $M = \max\{|u_-|, |u_+|\}$. Assume that $s > 0$. Since $(\xi - u)' = 1 - u' > 1$ on (s, ∞) and $s - u(s) = 0$, there is a constant $\alpha \in (s, s+1)$ such that $\alpha - u(\alpha) = 1$. Because of the monotonicity of $(\xi - u)$, we can say $(\xi - u) < 1$ on (s, α) and $(\xi - u) > 1$ on (α, ∞).

$$\rho_+ u(\alpha) = \rho_+ u_+ - \int_\alpha^\infty \rho_+ u' d\zeta \le \rho_+ u_+ - \int_\alpha^\infty \rho u' d\zeta \le \rho_+ u_+ - \int_\alpha^\infty (\zeta - u)\rho u' d\zeta$$
$$\le \rho_+ u_+ - \int_s^\infty (\zeta - u)\rho u' d\zeta = \rho_+ u_+ - \int_s^\infty p(\rho)' d\zeta + \int_s^\infty \varepsilon u'' d\zeta$$
$$= \rho_+ u_+ - p(\rho_+) + p(\rho(s)) \le \rho_+ u_+ - p(\rho_+) + p(\rho_-).$$

(Note that the last inequality is from (H1).) Hence $u(\alpha) \le u_+ - \frac{1}{\rho_+}\{p(\rho_+) - p(\rho_-)\}$. Thus $u(\alpha)$ is bounded by a constant which is independent of ε and μ.

$$\rho_+ u(s) = \rho_+ u(\alpha) - \int_s^\alpha \rho_+ u' d\zeta \le \rho_+ u(\alpha) - \int_s^\alpha \rho u' d\zeta$$
$$= \rho_+ u(\alpha) - \int_s^\alpha (\zeta - u)\rho' d\zeta \le \rho_+ u(\alpha) - \int_s^\alpha \rho' d\zeta$$
$$= \rho_+ u(\alpha) - \rho(\alpha) + \rho(s) \le \rho_+ u(\alpha) - \{\rho_+ - \rho_-\},$$

and $u(s) \le u(\alpha) + \frac{\rho_-}{\rho_+} - 1$. This means $u(s) = s$ is bounded by a constant which is independent of ε and μ. $\square$

The a-priori estimates (2.24) and (2.25), established in Lemma 2.4, are all independent of both of μ and ε. It is well known that if the boundary conditions (ρ_-, u_-) and (ρ_+, u_+) are not close enough, the solution of the Riemann problem may have a vacuum state. Therefore the missing estimate, the lower bound δ of (2.24) for the case C_4, may depend on ε. We have been unable, under the sole Hypotheses (H1) and (H2), to establish the bound: There exists $\delta_\varepsilon > 0$ independent of μ such that

$$0 < \delta_\varepsilon < \rho(\xi) \tag{A}$$

for any solution of (ρ, u) of Class C_4. From here on, in order to simplify the exposition, we admit (A) as an assumption and present the rest of the analysis in Section 2 and 3 under Hypothesis (A). In Section 5 we validate (A) under additional hypotheses on the pressure law $p(\rho)$: for strictly hyperbolic systems with convex laws (cf. Lemma 5.1), or for convex pressure laws under restrictions on $(\rho_\pm, u_\pm)$ that exclude vacuum (cf. Lemma 5.2).

LEMMA 2.5. *Under Hypothesis (A), there exist constants* $0 \le a < 1$ *and* $b \ge 0$, *depending on* $\rho_\pm, u_\pm$ *and* δ *and* M *in* (2.24) *and* (2.25), *such that* (ρ, u) *satisfies* (2.26).

Proof. The proof of $(2.26)_1$ and $(2.26)_2$ are similar and we consider $(2.26)_1$ only. Suppose u is increasing on (s, ∞). Then we can take $b = 0$. Since $u(\xi) < \xi, u'(s) = 0$ and $u(\xi) \to u_+ < \infty$ as $\xi \to \infty$, there is a constant $a \in (0,1)$ such that the line $y = a(\xi - s) + s$ is tangent to the graph of $u(\xi)$ and $u(\xi) \le a(\xi - s) + s$ on (s, ∞). Now we need to show that the slope a is bounded above by a constant which is less than 1 and depends only on $\rho_\pm, u_\pm, \delta$ and M. Let ξ_1 be the tangential point. If the tangential point ξ_1 is far from the fixed point s, for example, $\xi_1 - s > 2(u_+ - s)$, then $a < \frac{1}{2}$. Now we assume $\xi_1 - s \le 2(u_+ - s)$. From (2.11) we get

$$\{p'(\rho(\xi_1)) - (u(\xi_1) - \xi_1)^2\}\rho'(\xi_1) = \varepsilon u''(\xi_1). \tag{2.28}$$

Since $\rho'(\xi_1) > 0$ and $u''(\xi_1) < 0$,

$$\xi_1 - u(\xi_1) > \sqrt{p'(\rho(\xi_1))} \ge \theta,$$

and

$$a = \frac{u(\xi_1) - s}{\xi_1 - s} < \frac{\xi_1 - \theta - s}{\xi_1 - s} \le \frac{\xi_1 - \theta}{\xi_1} \le \frac{2(u_+ - s) - \theta}{2(u_+ - s)}.$$

So we get $a \le \max\{\frac{1}{2}, \frac{2(u_+ - s) - \theta}{2(u_+ - s)}\}$.

Now suppose u is decreasing on (s, ∞). Then we can take $a = 0$. Since $u'(s) = 0$ and $u(\xi) \searrow u_+ > -\infty$ as $\xi \to \infty$, there exists a positive constant b such that $y = -b(\xi - s) + s$ is tangent to the graph of $y = u(\xi)$ and $u(\xi) \ge -b(\xi - s) + s$ on (s, ∞). Now we need to show that the slope $-b$ is bounded below by a constant which depends only on $\rho_\pm, u_\pm, \delta$ and M. Let ξ_1 be the tangential point. Since $|u(\xi_1) - s| < |u_+ - s|$, it is enough to show that $\xi_1 - s > M_0$ for some constant M_0 which depends only on $\rho_\pm, u_\pm, \delta$ and M. In that case $b \le \frac{|u_+ - s|}{M_0}$.

Since $u''(\xi_1) > 0$ and $\rho'(\xi_1) < 0$, (2.28) yields

$$\sqrt{a_0} \le \sqrt{p'(\rho(\xi_1))} \le (\xi_1 - u(\xi_1)).$$

Multiplying by $\rho(\xi_1)$, we get

$$\rho(\xi_1)\theta \le \rho(\xi_1)(\xi_1 - u(\xi_1)) = \int_s^{\xi_1} \rho(\zeta)d\zeta \tag{2.29}$$

$$\delta\theta \le M \int_s^{\xi_1} d\zeta = M(\xi_1 - s). \tag{2.30}$$

Hence $\xi_1 - s \ge \frac{\delta\theta}{M}$. The proof is complete. $\square$

This Corollary is easily derived from Lemma 2.4 and the monotonicity of solutions.

COROLLARY 2.6. *Let $\{(\rho_\varepsilon, u_\varepsilon)\}_{\varepsilon>0}$ be a family of solutions to the boundary-value problem $(\mathcal{P}_\varepsilon)$ corresponding to fixed data $(\rho_\pm, u_\pm)$. Then there exist constants M and δ depending on the data such that*

$$0 < \delta \le \rho_\varepsilon(\xi) \le M \quad ; \quad |u_\varepsilon(\xi)| \le M \tag{2.31}$$

$$TV_{(-\infty,\infty)}\rho_\varepsilon \le M \quad ; \quad TV_{(-\infty,\infty)}u_\varepsilon \le M, \tag{2.32}$$

where δ may depend on ε only if the solution belongs to the category C_4.

3. Existence of Solutions of $(\mathcal{P}_\varepsilon)$**.** To construct solutions of $(\mathcal{P}_\varepsilon)$, we apply the Leray-Schauder degree theory (Rabinowitz [6, Ch V]) to a deformation of maps. Degree theory has been successful to establish connecting trajectories in problems of self-similar viscous limits (Dafermos [1], Slemrod and Tzavaras [8] in parabolic problems and Tzavaras [9], Slemrod [7] in hyperbolic-parabolic problems). In this work we adapt the method in [9] capturing the interplay between hyperbolic and parabolic effects in the system (2.1), but with significant modifications due to the nature of our free-boundary problem.

Let $C^0(\mathbb{R})$ be the Banach space of bounded continuous functions with the C^0-norm, $C^1(\mathbb{R})$ the Banach space of the bounded continuously differentiable functions with bounded derivatives equipped with the C^1-norm, and let

$$X = \{(P,V) \in C^0(\mathbb{R}) \times C^1(\mathbb{R}) : \|(P,V)\|_X < \infty\}$$

be the Banach space equipped with the $C^0 \times C^1$ norm:

$$\|(P,V)\|_X = \sup_{-\infty<\xi<\infty} |P(\xi)| + \sup_{-\infty<\xi<\infty} |V(\xi)| + \sup_{-\infty<\xi<\infty} |V'(\xi)|.$$

For technical reasons we define two sets Y (whose structure is motivated by the a-priori estimates) and Ω where we will apply degree theory.

The set Y consists of all $(P,V) \in X$ which are bounded by

$$0 < \bar{\delta} < P(\xi) < \bar{M}, \qquad \xi \in \mathbb{R} \tag{3.1}$$

$$|V(\xi)| < \bar{M}, \qquad \xi \in \mathbb{R} \tag{3.2}$$

and which satisfy the bounds

$$\bar{A} < 1 - V'(s) < \bar{B}, \tag{3.3}$$

$$\bar{A}|\xi - s| < |\xi - V(\xi)| < \bar{B}|\xi - s|, \qquad \xi \ne s \tag{3.4}$$

for *some* $s \in \mathbb{R}$. Here, $\bar{\delta}, \bar{M}, \bar{A}$ and $\bar{B}$ are fixed constants which satisfy $0 < \bar{\delta} < \delta, 0 < M < \bar{M}$ and $0 < \bar{A} < A < 1 < B < \bar{B}$, where M, δ, A and B are the constants in the estimates (2.24),(2.25) and (2.27). Note that the point s in (3.3) and (3.4) can be different for each $(P,V) \in Y$ and $\bar\delta$ is the only constant which may depend on ε. The geometric meaning of (3.4) is explained by Figure 1. Note that s is a fixed point of V and that such fixed points are bounded by (3.2), i.e., $|s| < \bar M$.

The boundary value problem $(\mathcal{P}_\varepsilon^\mu)$ is transformed to a fixed point problem as follows: For a given $(P,V) \in Y$, consider the linear problem

$$(3.5) \qquad \begin{aligned} (V-\xi)\rho' + Pu' &= 0 \\ (V-\xi)Pu' + p'(P)\rho' &= \varepsilon u'' \end{aligned} \qquad -\infty < \xi < \infty$$

$$(3.6) \qquad \begin{aligned} \rho(\pm\infty) &= \rho_\pm^\mu := \rho_- + \mu(\rho_\pm - \rho_-) \\ u(\pm\infty) &= u_\pm^\mu := u_- + \mu(u_\pm - u_-). \end{aligned}$$

This defines an operator $\mathcal{F} : [0,1] \times Y \to X$ that carries $(\mu,(P,V)) \in [0,1]\times Y$ to the solution of (3.5) and (3.6). From the a-priori estimates of Section 2, any solution (ρ,u) of $(\mathcal{P}_\varepsilon^\mu)$ belongs to Y and is a fixed point of the operator $\mathcal{F}(\mu,\cdot)$.

The Leray-Schauder degree theory will be applied on a bounded open subset Ω of the Banach space X, which contains all possible solutions of $(\mathcal{P}_\varepsilon^\mu)$. The subset $Y \subset X$ defined by (3.1–4) contains all solutions of $(\mathcal{P}_\varepsilon^\mu)$ but is not bounded in the $C^0 \times C^1$ norm. Next, we define the appropriate set Ω.

LEMMA 3.1. *Let $Y \subset X$ be given by (3.1–4). Then, for any $K > 0$, the set*

$$\Omega = \{(P,V) \in Y : |V'(\xi)| < K\}$$

is a bounded open subset of the Banach space X.

Proof. Clearly Ω is bounded. To show that it is open, we fix $(P_o,V_o) \in \Omega$ and find a small positive number ν such that $\|(P,V)-(P_o,V_o)\|_X < \nu$ implies $(P,V) \in \Omega$. The inequalities (3.1) and (3.2) for (P,V) are easy to verify, and we just show (3.3) and (3.4).

Let s_o be the fixed point of V_o (i.e. $V_o(s_o) = s_o$). Since V_o is continuous and bounded, there exist positive constants A_o and B_o such that

$$\bar A|\xi - s_o| < A_o|\xi - s_o| < |\xi - V_o(\xi)| < B_o|\xi - s_o| < \bar B|\xi - s_o|, \qquad \xi \neq s_o;$$

it also follows that $A_o \le 1 - {V_o}'(s_o) \le B_o$. Define

$$\nu_o := \min\{A_o - \bar A, \bar B - B_o\} < 1\,.$$

and note that there exists a positive constant $\kappa < 1$ such that

$$(3.7) \qquad \bar A + \frac{\nu_o}{2} < 1 - {V_o}'(\xi) < \bar B - \frac{\nu_o}{2}, \qquad s_o - \kappa < \xi < s_o + \kappa.$$

Choose

$$\nu := \frac{1}{2}\nu_o\kappa\frac{\bar A}{\bar B} \le \frac{1}{2}\nu_o\kappa \le \frac{1}{2}\nu_o$$

and suppose that $\|(P_o,V_o) - (P,V)\|_X < \nu$. Since V is bounded and continuous, V has a fixed point. Let $V(s) = s$. Then,

$$\nu \ge |V(s) - V_o(s)| = |s - V_o(s)| \ge \bar A|s - s_o|$$

$$|s - s_o| \le \frac{\nu}{\bar A},$$

so that $|s - s_o| \le \kappa\nu_o/2\bar B$ and, in particular, $s \in [s_o - \kappa, s_o + \kappa]$.

Since $|V_o{}'(\xi) - V'(\xi)| < \nu_o/2$, (3.7) implies that

$$\bar{A} < 1 - V'(\xi) < \bar{B} \qquad \text{on } (s_o - \kappa, s_o + \kappa),$$

and (3.3) is satisfied. Since $V(s) = s$, by integrating the above inequility over (s, ξ), we obtain,

$$\bar{A}|\xi - s| < |\xi - V(\xi)| < \bar{B}|\xi - s|, \qquad \xi \in (s_o - \kappa, s_o + \kappa), \quad \xi \neq s,$$

that (3.4) is satisfied on $(s_o - \kappa, s_o + \kappa)$.

On the complementary intervals $s_o + \kappa < \xi$ or $\xi < s_o - \kappa$, we have

$$\begin{aligned} |\xi - V(\xi)| &\le |\xi - V_o(\xi)| + \nu \le B_o|\xi - s_o| + \nu \\ &\le \bar{B}|\xi - s| - (\bar{B} - B_o)|\xi - s| + B_o|s - s_o| + \nu. \end{aligned}$$

From the selections of the parameters

$$\nu \le \kappa\nu_o/2\bar{B}, \quad |s - s_o| \le \kappa\nu_o/2\bar{B}, \quad \nu_o \le \bar{B} - B_o, \quad \bar{B} > 1$$

and since ξ takes values on the interval $|\xi - s| > \kappa(1 - \nu_o/2\bar{B})$, we get

$$\begin{aligned} -(\bar{B} &- B_o)|\xi - s| + B_o|s - s_o| + v \\ &\le -\kappa(\bar{B} - B_o)(1 - \nu_o/2\bar{B}) + \kappa\nu_o B_o/2\bar{B} + \kappa\nu_o/2\bar{B} \\ &\le -\kappa(\bar{B} - B_o) + \kappa\nu_o(1 + \bar{B})/2\bar{B} \\ &\le -\kappa(\bar{B} - B_o)(1 - (1 + \bar{B})/2\bar{B}) \le 0. \end{aligned}$$

So we have the upper bound of (3.4). The other inequality of (3.4) can be shown similarly. So $(P, V) \in \Omega$ and Ω is open. □

3.1 Estimates of the Operator. In this section we check that the map $\mathcal{F}$ is well defined and establish uniform estimates of $(\rho, u) = \mathcal{F}(\mu, (P, V))$ and their derivatives. The derived estimates may depend on $\rho_\pm, u_\pm$ and ε but are independent of the choice of $(\mu, (P, V)) \in [0, 1] \times Y$. Consider a mapping $\mathcal{T}$ which carries $(P, V) \in Y$ to a solution $\mathcal{T}(P, V) := (\rho, u)$ of (3.5) with boundary conditions

$$\rho(\pm\infty) = \rho_\pm - \rho_- \quad ; \quad u(\pm\infty) = u_\pm - u_-. \tag{3.8}$$

It can be easily verified that $(\rho_-, u_-) + \mu\mathcal{T}(P, V)$ is a solution of (3.5) and (3.6), and hence $\mathcal{F}(\mu, (P, V)) = (\rho_-, u_-) + \mu\mathcal{T}(P, V)$.

The bounds in (3.1) and Hypothesis (H1) imply the existence of positive constants a_0 and A_0 which satisfy $0 < a_0 < p'(P(\xi)) < A_0 < \infty$ for all $\xi \in \mathbb{R}$ and depend on $\bar{\delta}$ and $\bar{M}$. From (3.5), we get

$$\varepsilon u'' + \frac{\{p'(P) - (V - \xi)^2\}P}{V - \xi} u' = 0, \qquad V(\xi) \neq \xi. \tag{3.9}$$

Since (3.9) has a unique singularity at the fixed point s of V, u' is obtained by

$$u'(\xi) = \begin{cases} c_+ \exp\Big\{ -\dfrac{1}{\varepsilon}\displaystyle\int_{\alpha_+}^{\xi} \dfrac{\{(\zeta - V)^2 - p'(P)\}P}{\zeta - V} d\zeta \Big\} =: c_+ I_+, & s < \xi \\ c_- \exp\Big\{ -\dfrac{1}{\varepsilon}\displaystyle\int_{\alpha_-}^{\xi} \dfrac{\{(\zeta - V)^2 - p'(P)\}P}{\zeta - V} d\zeta \Big\} =: c_- I_-, & \xi < s \end{cases} \tag{3.10}$$

for any $\alpha_- < s < \alpha_+$. In turn, ρ' is obtained by $(3.5)_1$

$$\rho'(\xi) = \frac{P(\xi)}{\xi - V} u'. \tag{3.11}$$

LEMMA 3.2. *Let $(P, V) \in Y$ and $V(s) = s$. Then there exist positive constants $\alpha, \alpha', \beta, \beta'$ and C_ε which depend only on $a_0, A_0, \bar{A}, \bar{B}, \bar{\delta}$ and $\bar{M}$ (C_ε may depend on ε) and satisfy*

$$\frac{1}{C_\varepsilon} |\xi - s|^{\frac{\alpha'}{\varepsilon}} \le I_\pm(\xi) \le C_\varepsilon |\xi - s|^{\frac{\alpha}{\varepsilon}}, \qquad |\xi - s| < 1, \tag{3.12}$$

$$\frac{1}{C_\varepsilon} e^{-\frac{\beta'}{\varepsilon}(\xi - s)^2} \le I_\pm(\xi) \le C_\varepsilon e^{-\frac{\beta}{\varepsilon}(\xi - s)^2}, \qquad |\xi - s| > 1. \tag{3.13}$$

Proof. Let $s < \xi < s + 1$. Then

$$\int_\xi^{s+1} \frac{\{(\zeta - V)^2 - p'(P)\}P}{\zeta - V} d\zeta = \int_\xi^{s+1} (\zeta - V) P d\zeta - \int_\xi^{s+1} \frac{p'(P)P}{\zeta - V} d\zeta$$
$$\le \bar{M}\bar{B} \int_0^1 \zeta d\zeta - \frac{a_0 \bar{\delta}}{\bar{B}} \int_{\xi - s}^1 \frac{1}{\zeta} d\zeta = \mathcal{A} + \alpha \log |\xi - s|,$$

where $\alpha := \frac{a_0 \bar{\delta}}{\bar{B}} > 0$ and $\mathcal{A} := \frac{\bar{M}\bar{B}}{2}$. Also

$$I_+(\xi) = \exp\Big\{ \frac{1}{\varepsilon} \int_\xi^{s+1} \frac{\{(\zeta - V)^2 - p'(P)\}P}{\zeta - V} d\zeta \Big\} \le e^{\frac{\mathcal{A}}{\varepsilon}} e^{\frac{\alpha}{\varepsilon} \log |\xi - s|} = C_\varepsilon |\xi - s|^{\frac{\alpha}{\varepsilon}}.$$

Let $s + 1 < \xi$, then

$$-\int_{s+1}^\xi \frac{\{(\zeta - V)^2 - p'(P)\}P}{\zeta - V} d\zeta = -\int_{s+1}^\xi (\zeta - V) P d\zeta + \int_{s+1}^\xi \frac{p'(P)P}{\zeta - V} d\zeta$$
$$\le -\bar{\delta}\bar{A} \int_1^{\xi - s} \zeta d\zeta + \frac{A_0 \bar{M}}{\bar{A}} \int_1^{\xi - s} \frac{1}{\zeta} d\zeta \le -\beta(\xi - s)^2 + \mathcal{A},$$

where $\beta = \frac{\bar{\delta}\bar{A}}{2} + 1$ and $\mathcal{A}$ is a positive constant which depends on β and $\frac{A_0 \bar{M}}{\bar{A}}$. Also

$$I_+(\xi) = \exp\Big\{ -\frac{1}{\varepsilon} \int_{s+1}^\xi \frac{\{(\zeta - V)^2 - p'(P)\}P}{\zeta - V} d\zeta \Big\} \le e^{\frac{\mathcal{A}}{\varepsilon}} e^{-\frac{\beta}{\varepsilon}(\xi - s)^2} = C_\varepsilon e^{-\frac{\beta}{\varepsilon}(\xi - s)^2}.$$

The rest follows by similar arguments. $\square$

By Lemma 3.2, u' and ρ' are integrable on $(-\infty, \infty)$ and thus (ρ, u) can be calculated by the formulas

$$\rho(\xi) = \begin{cases} (\rho_+ - \rho_-) - c_+ \displaystyle\int_\xi^\infty \frac{P(\zeta) I_+(\zeta)}{\zeta - V(\zeta)} d\zeta, & s < \xi \\ \qquad c_- \displaystyle\int_{-\infty}^\xi \frac{P(\zeta) I_-(\zeta)}{\zeta - V(\zeta)} d\zeta, & \xi < s \end{cases} \tag{3.14}$$

$$u(\xi) = \begin{cases} (u_+ - u_-) - c_+ \displaystyle\int_\xi^\infty I_+(\zeta) d\zeta, & s < \xi \\ \qquad c_- \displaystyle\int_{-\infty}^\xi I_-(\zeta) d\zeta, & \xi < s\,. \end{cases} \tag{3.15}$$

Expressing the continuity of (ρ,u) at $\xi=s$ gives

$$
\begin{aligned}
&c_+\int_s^\infty \frac{P(\zeta)I_+(\zeta)}{\zeta-V(\zeta)}d\zeta + c_-\int_{-\infty}^s \frac{P(\zeta)I_-(\zeta)}{\zeta-V(\zeta)}d\zeta = \rho_+-\rho_- \\
&c_+\int_s^\infty I_+(\zeta)d\zeta \quad + c_-\int_{-\infty}^s I_-(\zeta)d\zeta \quad = u_+-u_-.
\end{aligned}
\tag{3.16}
$$

The determinant Δ of the linear system (3.16) is

$$
\int_s^\infty \frac{P(\zeta)I_+(\zeta)}{\zeta-V(\zeta)}d\zeta\int_{-\infty}^s I_-(\zeta)d\zeta - \int_{-\infty}^s \frac{P(\zeta)I_-(\zeta)}{\zeta-V(\zeta)}d\zeta\int_s^\infty I_+(\zeta)d\zeta > 0.
\tag{3.17}
$$

So there exists a unique solution (c_+,c_-) to (3.16) and the operators $\mathcal{T}$ and $\mathcal{F}$ are well defined.

We now estimate $(\rho,u)=\mathcal{T}(P,V)$, defined by (3.14) and (3.15). Since our objective in this section is to get uniform bounds which are independent of the choice of $(P,V)\in Y$, we consider a generic constant K_ε which may depend on $a_0, A_0, \bar{A}, \bar{B}, \bar{\delta}$, $\bar{M}$ and ε but does not depend on $(P,V)\in Y$.

Considering the lower bounds for $I_\pm$ in Lemma 3.2, the determinant Δ of the linear system (3.16) is bounded from below by a positive constant which depends on $a_0, A_0, \bar{A}, \bar{B}, \bar{\delta}$ and $\bar{M}$. So we get

$$
|c_+|+|c_-|<K_\varepsilon.
\tag{3.18}
$$

Now we estimate ρ, u and their derivatives to see the regularity properties of the operator. Since u', ρ' are given by (3.10) and (3.11) and $I_\pm$ are bounded by (3.12) and (3.13), we have

$$
|u'(\xi)|<K_\varepsilon|\xi-s|^{\frac{\alpha}{\varepsilon}}, \qquad |\xi-s|<1,
\tag{3.19}
$$

$$
|u'(\xi)|<K_\varepsilon e^{-\frac{\beta}{\varepsilon}(\xi-s)^2}, \qquad |\xi-s|>1,
\tag{3.20}
$$

$$
|\rho'(\xi)|=\frac{|P(\xi)|}{|\xi-V(\xi)|}|u'(\xi)|<K_\varepsilon|\xi-s|^{\frac{\alpha}{\varepsilon}-1}, \qquad |\xi-s|<1,
\tag{3.21}
$$

$$
|\rho'(\xi)|=\frac{|P(\xi)|}{|\xi-V(\xi)|}|u'(\xi)|<K_\varepsilon e^{-\frac{\beta}{\varepsilon}(\xi-s)^2}, \qquad |\xi-s|>1.
\tag{3.22}
$$

We also have

$$
u''(\xi)=\frac{1}{\varepsilon}\frac{\{p'(P)-(\xi-V)^2\}P}{\xi-V}c_\pm I_\pm, \qquad \xi\neq s,
\tag{3.23}
$$

and

$$
|u''(\xi)|\le\frac{1}{\varepsilon}\Big(\frac{A_0}{(1-a)|\xi-s|}+(1+b)|\xi-s|\Big)\bar{M}c_\pm I_\pm\le K_\varepsilon|\xi-s|^{\frac{\alpha}{\varepsilon}-1}, \qquad |\xi-s|<1,
\tag{3.24}
$$

$$
|u''(\xi)|\le\frac{1}{\varepsilon}\Big(\frac{A_0}{(1-a)|\xi-s|}+(1+b)|\xi-s|\Big)\bar{M}c_\pm I_\pm\le K_\varepsilon e^{-\frac{\beta}{\varepsilon}(\xi-s)^2}, \qquad |\xi-s|>1.
\tag{3.25}
$$

From these estimates we get equicontinuity of ρ, u and u' on any closed set which does not contain the singular point s. (3.19) and (3.20) imply

$$|u'(\xi)| < K_\varepsilon, \qquad -\infty < \xi < \infty, \tag{3.26}$$

and hence u is equicontinuous. From (3.12) and (3.13), we get

$$\int_s^\xi I_+(\zeta)d\zeta < K_\varepsilon(\xi - s)^{\frac{\alpha}{\varepsilon}+1}; \quad \int_s^\xi \frac{PI_+}{|\zeta - V|}d\zeta < K_\varepsilon(\xi - s)^{\frac{\alpha}{\varepsilon}}, \qquad s < \xi < s+1,$$

$$\int_\xi^s I_-(\zeta)d\zeta < K_\varepsilon(s - \xi)^{\frac{\alpha}{\varepsilon}+1}; \quad \int_\xi^s \frac{PI_-}{|\zeta - V|}d\zeta < K_\varepsilon(s - \xi)^{\frac{\alpha}{\varepsilon}}, \qquad s-1 < \xi < s,$$

$$\int_{s+1}^\xi I_+(\zeta)d\zeta < K_\varepsilon e^{-\frac{\beta}{\varepsilon}(\xi-s)^2}; \quad \int_{s+1}^\xi \frac{PI_+}{|\zeta - V|}d\zeta < K_\varepsilon e^{-\frac{\beta}{\varepsilon}(\xi-s)^2}, \qquad s+1 < \xi,$$

$$\int_\xi^{s-1} I_-(\zeta)d\zeta < K_\varepsilon e^{-\frac{\beta}{\varepsilon}(\xi-s)^2}; \quad \int_\xi^{s-1} \frac{PI_-}{|\zeta - V|}d\zeta < K_\varepsilon e^{-\frac{\beta}{\varepsilon}(\xi-s)^2}, \qquad s+1 < \xi.$$

These estimates imply the boundedness of ρ and u

$$|u(\xi)| < K_\varepsilon; \qquad |\rho(\xi)| < K_\varepsilon, \qquad -\infty < \xi < \infty, \tag{3.27}$$

and establish estimates for $\rho(\xi)$ and $u(\xi)$ from (3.14) and (3.15);

$$|u(\xi) - u(s)| < K_\varepsilon|\xi - s|^{\frac{\alpha}{\varepsilon}+1}, \qquad |\xi - s| < 1, \tag{3.28}$$

$$|u(\xi) - u(s)| < K_\varepsilon e^{-\frac{\beta}{\varepsilon}(\xi-s)^2}, \qquad |\xi - s| > 1, \tag{3.29}$$

$$|\rho(\xi) - \rho(s)| < K_\varepsilon|\xi - s|^{\frac{\alpha}{\varepsilon}}, \qquad |\xi - s| < 1, \tag{3.30}$$

$$|\rho(\xi) - \rho(s)| < K_\varepsilon e^{-\frac{\beta}{\varepsilon}(\xi-s)^2}, \qquad |\xi - s| > 1. \tag{3.31}$$

The estimates (3.26) and (3.27) imply that the image $\mathcal{T}(Y)$ of Y under the mapping $\mathcal{T}$ is bounded under the $C^0 \times C^1$ norm.

3.2 Existence. If (ρ, u) is a solution of $(\mathcal{P}_\varepsilon^\mu)$, then (ρ, u) is a fixed point under $\mathcal{F}(\mu, \cdot)$. So (ρ, u) is a image under the mapping $\mathcal{F}(\mu, \cdot)$ and the previous estimate (3.26) can be considered as an a-priori estimate of solutions of $(\mathcal{P}_\varepsilon^\mu)$. Now we fix K of Lemma 3.1 with $\bar{K} := K_\varepsilon + 1$ and consider

$$\Omega = \{(P, V) \in Y : |V'(\xi)| < \bar{K}\}. \tag{3.32}$$

LEMMA 3.3. *The mapping $\mathcal{T} : \bar{\Omega} \to X$ is a compact operator.*

Proof. First, we show that $\mathcal{T}(\bar{\Omega})$ is precompact in X. Let (ρ_n, u_n) be a sequence in $\mathcal{T}(\bar{\Omega})$. Since u_n' is uniformly bounded by (3.26), u_n is equicontinuous. The equicontinuity of ρ_n follows from (3.21), (3.22) and (3.30) and the one of u_n' from (3.19),(3.24) and (3.25).

For example we consider ρ_n. Let $\eta > 0$ be given. From (3.30) there exists $\delta_1 > 0$ such that $|\rho_n(\xi_1) - \rho_n(\xi_2)| < \eta$ for all $\xi_1, \xi_2 \in I = [-\delta_1, \delta_1]$. From (3.21) and (3.22) ρ_n' is uniformly bounded on $I^c = (-\infty, -\delta_1) \cup (\delta_1, \infty)$ and there exists $\delta_2 > 0$ such that $|\rho_n(\xi_1) - \rho_n(\xi_2)| < \eta$ for all $\xi_1, \xi_2 \in I^c$ with $|\xi_1 - \xi_2| < \delta_2$. If we take $\delta = \min(\delta_1, \delta_2)$, we see that $|\rho_n(\xi_1) - \rho_n(\xi_2)| < 2\eta$ for $|\xi_1 - \xi_2| < \delta$. So ρ_n is equicontinuous.

From (2.25) and (2.26) ρ_n, u_n, u_n' are also uniformly bounded and the Ascoli-Arzela theorem implies the existence of a subsequence, rename it ρ_n, u_n, which converges uniformly on every compact set. By taking a subsequence again, we can assume that the singular points s_n converge to s.

Let ρ, u and u_1 be the limit of ρ_n, u_n and ${u_n}'$. We can easily verify that $u' = u_1$ and $(\rho, u) \in X$. Since the singular points s_n are bounded by $\bar{M}$, we can choose $L > 0$ which satisfies

$$\begin{aligned} &|{u_n}'(\xi)| < \eta, && L < |\xi| \\ |u_n(\xi)| < \eta;\quad &|\rho_n(\xi)| < \eta, && \xi < -L \\ |u_n(\xi) - (u_+ - u_-)| < \eta;\quad &|\rho_n(\xi) - (\rho_+ - \rho_-)| < \eta, && L < \xi \end{aligned}$$

from (3.20), (3.31) and (3.32). The limit (ρ, u) also satisfies these estimates. From these estimates together with the fact that $\rho_n, u_n, {u_n}'$ converge uniformly on $[-L, L]$, we can take N such that

$$\|(\rho_n, u_n) - (\rho, u)\|_X = \sup_{-\infty<\xi<\infty} |\rho_n - \rho| + \sup_{-\infty<\xi<\infty} |u_n - u| + \sup_{-\infty<\xi<\infty} |{u_n}' - u'| < 6\eta,$$

whenever $n > N$. So $(\rho_n, u_n) \to (\rho, u)$ in X, and $\mathcal{T}(\bar{\Omega})$ is precompact.

Now we show that the mapping $\mathcal{T} : \bar{\Omega} \to X$ is continuous. Let $(P_n, V_n) \in \bar{\Omega}$ and $(P_n, V_n) \to (P, V)$ in X. Let $(\rho_n, u_n) = \mathcal{T}(P_n, V_n)$ and $(\rho, u) = \mathcal{T}(P, V)$. The sequence $\{(\rho_n, u_n)\}$ has at least one limit point (ρ^o, u^o). Let (ρ_{n_k}, u_{n_k}) be a subsequence converges to (ρ^o, u^o). Then,

$$\rho_{n_k}(\xi) = \begin{cases} (\rho_+ - \rho_-) - c_+^{n_k} \displaystyle\int_\xi^\infty \frac{P_{n_k}(\zeta) I_+^{n_k}(\zeta)}{\zeta - V_{n_k}(\zeta)} d\zeta, & s_{n_k} < \xi \\ \quad c_-^{n_k} \displaystyle\int_{-\infty}^\xi \frac{P_{n_k}(\zeta) I_-^{n_k}(\zeta)}{\zeta - V_{n_k}(\zeta)} d\zeta, & \xi < s_{n_k} \end{cases} \tag{3.33}$$

$$u_{n_k}(\xi) = \begin{cases} (u_+ - u_-) - c_+^{n_k} \displaystyle\int_\xi^\infty I_+^{n_k}(\zeta) d\zeta, & s_{n_k} < \xi \\ \quad c_-^{n_k} \displaystyle\int_{-\infty}^\xi I_-^{n_k}(\zeta) d\zeta, & \xi < s_{n_k}\ , \end{cases} \tag{3.34}$$

where $c_+^{n_k}$ and $c_-^{n_k}$ are solutions of (3.16) with $I_\pm^{n_k}$ given by

$$I_\pm^{n_k}(\xi) = \exp\Big\{ -\frac{1}{\varepsilon} \int_{\pm 1}^\xi \frac{\{(\zeta - V_{n_k})^2 - p'(P_{n_k})\} P_{n_k}}{\zeta - V_{n_k}} d\zeta \Big\}.$$

Because of the bounds in (3.12) and (3.13) we can take the limit as $k \to \infty$ inside the integrals in (3.33) and (3.34). Since $(\rho_{n_k}, u_{n_k}) \to (\rho^o, u^o)$ in X, we get

$$(\rho^o, u^o) = \mathcal{T}(P, V) = (\rho, u).$$

This says that $(\rho_n, u_n) = \mathcal{T}(P_n, V_n)$ has exactly one limit point (ρ, u), i.e. $(\rho_n, u_n) \to (\rho, u)$ in X. $\square$

We conclude with a theorem guaranteeing existence for solutions of $(\mathcal{P}_\varepsilon)$ under extra assumption (A). In Section 5 we justify (A) under additional assumptions on the pressure function $p(\rho)$ or on the Riemann data $(\rho_\pm, u_\pm)$.

THEOREM 3.4. *Suppose a pressure function $p(\rho)$ satisfies (H1), (H2). If solutions (ρ,u) of $(\mathcal{P}_\varepsilon^\mu)$ satisfy the a-priori lower bound (A), then the boundary value problem $(\mathcal{P}_\varepsilon)$ has a solution $(\rho(\xi),u(\xi))$ for any $\varepsilon>0$.*

Proof. We define the map $\mathcal{F}:[0,1]\times\bar{\Omega}\to X$ by $\mathcal{F}(\mu,P,V)=(\rho_-,u_-)+\mu\mathcal{T}(P,V)$. If (ρ,u) is a solution of $(\rho,u)=(\rho_-,u_-)+\mu\mathcal{T}(\rho,u)$ in Ω, then (ρ,u) is a solution of $(\mathcal{P}_\varepsilon^\mu)$ with $(\rho,u)\in\Omega$. We apply the Leray-Schauder degree theory (Rabinowitz [6, Ch V]) to solve

$$(\rho,u)-\mu\mathcal{T}(\rho,u)=(\rho_-,u_-)\,,\quad \mu\in[0,1]\,. \tag{3.35}$$

We have already shown that $\mathcal{T}:\bar{\Omega}\to X$ is compact. The map $\mu\mathcal{T}:[0,1]\times\bar{\Omega}\to X$ is also compact, thus the Leray-Schauder degree of $I-\mu\mathcal{T}$ is well defined. For any solution (ρ,u) of (3.35), $\frac{1}{\mu}\{(\rho,u)-(\rho_-,u_-)\}\in\mathcal{T}(\bar{\Omega})$. So u satisfies (3.26). Hence by Lemma 2.4 and 2.5 and the definition of Ω, any solution (ρ,u) of (3.35) lies in the interior of Ω. Therefore

$$d(I-\mu\mathcal{T},\Omega,(\rho_-,u_-))=d(I,\Omega,(\rho_-,u_-))=1\,,\quad \mu\in[0,1]$$

(3.35) admits at least one solution for each $\mu\in[0,1]$. □

4. The Structure of the Solution of the Riemann Problem. In this section we consider a sequence $(\rho_\varepsilon,u_\varepsilon)$ of solutions of $(\mathcal{P}_\varepsilon)$ obeying the estimates (2.24), (2.25) and (2.26). Since the bounds M are independent of ε, $TV_{(-\infty,\infty)}u_\varepsilon<2M$ and $TV_{(-\infty,\infty)}\rho_\varepsilon<2M$. On account of the Helly's theorem, there exists a subsequence, which we call $(\rho_\varepsilon,u_\varepsilon)$ again, such that $(\rho_\varepsilon,u_\varepsilon)$ converges pointwise to a function (ρ,u) of bounded variation as $\varepsilon\to0$. By taking further subsequences, if necessary, we assume that $(\rho_\varepsilon,u_\varepsilon)$ belongs to one of the four categories in Section 2.3. Since the singular points s_ε of $(\rho_\varepsilon,u_\varepsilon)$ are uniformly bounded, we may also assume that $s_\varepsilon\to s$ as $\varepsilon\to0$, for some s. The limit ρ and u inherit the monotonicity properties of ρ_ε and u_ε, but the monotonicities are no longer strict.

4.1 Solution of the Riemann Problem. We construct solutions of $(\mathcal{P})$ as limits of solutions $(\rho_\varepsilon,u_\varepsilon)$ of $(\mathcal{P}_\varepsilon)$, and study the structure of the emerging limit. First we consider the structure under the condition of $\rho>0$. In that case, we can assume that the lower bound δ for ρ_ε is independent of ε and the constants in (2.24)–(2.26) are all independent of ε.

THEOREM 4.1. *Let $(\rho_\varepsilon,u_\varepsilon),\varepsilon>0$ be the solution of $(\mathcal{P}_\varepsilon)$. Then there exists a subsequence $(\rho_{\varepsilon_n},u_{\varepsilon_n}),\varepsilon_n\to0$ such that the sequence of singular points $s_{\varepsilon_n}\to s$ and $(\rho_{\varepsilon_n},u_{\varepsilon_n})$ converges pointwise to a weak solution (ρ,u) of $(\mathcal{P})$. Furthermore, if $\rho>0$, then there exist constants $\beta_-<\alpha_-<s<\alpha_+<\beta_+$ such that*

$$(\rho(\xi),u(\xi))=\begin{cases}(\rho_-,u_-), & \xi<\beta_-,\\ (\rho(s),u(s)), & \alpha_-<\xi<\alpha_+,\\ (\rho_+,u_+), & \beta_+<\xi.\end{cases} \tag{4.1}$$

Proof. We omit the sub-index of ε_n and simply use ε for the subsequence. The first part of the theorem has been established already and we consider the other parts. Integrate $(2.1)_2$ on (s_ε,ξ) to get

$$\varepsilon {u_\varepsilon}'=\rho_\varepsilon {u_\varepsilon}^2+p(\rho_\varepsilon)-\xi\rho_\varepsilon u_\varepsilon+\int_{s_\varepsilon}^{\xi}\rho_\varepsilon u_\varepsilon d\zeta-p(\rho_\varepsilon(s_\varepsilon)). \tag{4.2}$$

Since $\rho_\varepsilon, u_\varepsilon$ and $p(\rho_\varepsilon)$ are bounded by a constant M,

$$|u_\varepsilon{}'(\xi)| \le \frac{M}{\varepsilon}(|\xi|+1), \qquad -\infty < \xi < \infty. \tag{4.3}$$

If we multiply u_ε to $(2.1)_2$ and integrate it on any bounded interval (a_0, b_0), then integration by parts gives

$$\varepsilon \int_{a_0}^{b_0} (u_\varepsilon{}')^2 d\zeta \le M(|a_0|+|b_0|+1) + \varepsilon M(u_\varepsilon{}'(b_0) - u_\varepsilon{}'(a_0)). \tag{4.4}$$

From (4.3), we have $\varepsilon(u_\varepsilon{}'(b_0) - u_\varepsilon{}'(a_0)) \le M(|a_0|+|b_0|+1)$. So

$$\varepsilon \int_{a_0}^{b_0} (u_\varepsilon{}')^2 d\zeta \le M(|a_0|+|b_0|+1). \tag{4.5}$$

Let φ be a test function with supp$\varphi \subset [a_0, b_0]$. Then,

$$\begin{aligned} \Big|\int_{-\infty}^{\infty} \varepsilon u_\varepsilon{}''\varphi d\zeta\Big| = \Big|\varepsilon \int_{-\infty}^{\infty} u_\varepsilon{}'\varphi' d\zeta\Big| &\le \varepsilon\Big(\int_{a_0}^{b_0}(u_\varepsilon{}')^2 d\zeta\Big)^{\frac12}\Big(\int_{a_0}^{b_0}(\varphi')^2 d\zeta\Big)^{\frac12} \\ &\le \varepsilon^{\frac12} M(|a_0|+|b_0|+1)\Big(\int_{a_0}^{b_0}(\varphi')^2 d\zeta\Big)^{\frac12}. \end{aligned} \tag{4.6}$$

Hence

$$\lim_{\varepsilon\to0} \varepsilon \int_{-\infty}^{\infty} u_\varepsilon{}''\varphi d\zeta = 0, \tag{4.7}$$

and

$$\begin{aligned} &\int_{-\infty}^{\infty} \rho u(\zeta\varphi)' d\zeta - \int_{-\infty}^{\infty}(\rho u^2 + p(\rho))\varphi' d\zeta \\ &\quad = \lim_{\varepsilon\to0}\int_{-\infty}^{\infty} \rho_\varepsilon u_\varepsilon(\zeta\varphi)' d\zeta - \lim_{\varepsilon\to0}\int_{-\infty}^{\infty}(\rho_\varepsilon u_\varepsilon{}^2 + p(\rho_\varepsilon))\varphi' d\zeta \\ &\quad = \lim_{\varepsilon\to0} \varepsilon\int_{-\infty}^{\infty} u_\varepsilon{}''\varphi d\zeta = 0. \end{aligned}$$

That is (ρ, u) is a weak solution of $(\mathcal{P})$.

Now we consider the structure of the limit solution under the condition $\rho > 0$. In that case there exist constants $\delta > 0$ and $M > 0$ such that

$$\delta < \rho(\xi) < M, \qquad \xi \in \mathbb{R} \tag{4.8}$$

for all small $\varepsilon > 0$. From (H1) there exist constants $a_0 > 0$ and $A_0 > 0$ such that

$$a_0 \le p'(\rho) \le A_0, \qquad \xi \in \mathbb{R}. \tag{4.9}$$

So the constants $\beta_- < \alpha_- < s < \alpha_+ < \beta_+$ and α, β in Lemma 2.3 are independent of ε. Taking the limit $\varepsilon \to 0$ in Lemma 2.3, we obtain (4.1). □

If (ρ, u) is a solution of $(\mathcal{P})$, then it can be easily verified that $(\rho(x,t), u(x,t)) = (\rho(\frac{x}{t}), u(\frac{x}{t}))$ is a weak solution of the Riemann problem (1.1) and (1.2). So the solution of the Riemann problem has been established through the vanishing viscosity process. Theorem 4.1 provides information on the structure of solutions that do not have a vacuum state. We conclude this section with establishing the Rankine-Hugoniot jump conditions. Since (ρ, u) is of bounded variation, it has right and left limits at each $\xi \in \mathbb{R}$. Let $\mathcal{S}$ be the points of jump discontinuity of either ρ or u.

LEMMA 4.2. *The solution (ρ, u) satisfies the Rankine-Hugoniot jump condition at all $\xi \in \mathcal{S}$:*

$$\xi[\rho(\xi+) - \rho(\xi-)] = \rho(\xi+)u(\xi+) - \rho(\xi-)u(\xi-) \tag{4.10}$$

$$\xi[\rho(\xi+)u(\xi+) - \rho(\xi-)u(\xi-)] = \rho(\xi+)u^2(\xi+) + p(\rho(\xi+)) - \rho(\xi-)u^2(\xi-) - p(\rho(\xi-)). \tag{4.11}$$

Proof. We integrate (2.5) over $(\xi-\delta, \xi+\delta)$, along $(\rho_\varepsilon, u_\varepsilon)$, and take the limit as $\varepsilon \to 0$, to get

$$(\xi+\delta-u(\xi+\delta))\rho(\xi+\delta) - (\xi-\delta-u(\xi-\delta))\rho(\xi-\delta) = \int_{\xi-\delta}^{\xi+\delta} \rho(\zeta)d\zeta. \tag{4.12}$$

The limit of (4.12) as $\delta \to 0$ gives

$$\xi[\rho(\xi+) - \rho(\xi-)] = \rho(\xi+)u(\xi+) - \rho(\xi-)u(\xi-).$$

If we integrate of (2.6) over $(\xi-\delta, \xi+\delta)$, along $(\rho_\varepsilon, u_\varepsilon)$, and perform the same process, then we get the other equality. □

4.2 Structure of the Limit Solutions. In this section we study the structure of the limit (ρ, u) without the strict positiveness of the density limit, i.e., $\rho \geq 0$, but with the convexity hypothesis

$$p''(\rho) \geq 0 \quad \text{for} \quad \rho > 0. \tag{H3}$$

Hypothesis (H3) includes both cases of genuine nonlinearity and linear degeneracy. Lemma 4.3 takes two important properties of solutions to $(\mathcal{P}_\varepsilon^\mu)$ under Hypothesis (H3). From (2.11) we obtain

$$(p'(\rho_\varepsilon) - (u_\varepsilon - \xi)^2)\rho_\varepsilon{}' = \varepsilon u_\varepsilon{}''. \tag{4.13}$$

LEMMA 4.3. *Let $(\rho_\varepsilon, u_\varepsilon)$ be a solution of $(\mathcal{P}_\varepsilon^\mu)$ and s_ε be the singular point. Then (i) If u_ε is increasing on (s_ε, ∞), then $u_\varepsilon{}' \leq 1$ on (s_ε, ∞). (ii) If u_ε is increasing on $(-\infty, s_\varepsilon)$, then $u_\varepsilon{}' \leq 1$ on $(-\infty, s_\varepsilon)$. (iii) If u_ε is decreasing on (s_ε, ∞) ,then there exists exactly one $\xi \in (s_\varepsilon, \infty)$ such that $u_\varepsilon{}''(\xi) = 0$. (iv) If u_ε is decreasing on $(-\infty, s_\varepsilon)$, then there exists exactly one $\xi \in (-\infty, s_\varepsilon)$ such that $u_\varepsilon{}''(\xi) = 0$.*

Proof. Let u_ε be increasing on (s_ε, ∞) and suppose there exist $\xi \in (s_\varepsilon, \infty)$ such that $u_\varepsilon{}'(\xi) > 1$. Since $u_\varepsilon{}'(s_\varepsilon) = 0$ and $u_\varepsilon{}'(\xi) \to 0$ as $\xi \to \infty$, there exist $\xi_1, \xi_2 \in (s_\varepsilon, \infty)$ such that $u_\varepsilon{}'(\xi_1) = u_\varepsilon{}'(\xi_2) = 1$, $u_\varepsilon{}''(\xi_1) > 0$, $u_\varepsilon{}''(\xi_2) < 0$ and $u_\varepsilon{}' > 1$ on (ξ_1, ξ_2). In that case, $(\xi - u_\varepsilon)^2$ is decreasing on (ξ_1, ξ_2) and

$$\begin{aligned} p'(\rho_\varepsilon(\xi_1)) - (u_\varepsilon(\xi_1) - \xi_1)^2 &= \varepsilon \frac{u_\varepsilon{}''(\xi_1)}{\rho'(\xi_1)} > 0, \\ p'(\rho_\varepsilon(\xi_2)) - (u_\varepsilon(\xi_2) - \xi_2)^2 &= \varepsilon \frac{u_\varepsilon{}''(\xi_2)}{\rho'(\xi_2)} < 0. \end{aligned} \tag{4.14}$$

Hence

$$p'(\rho_\varepsilon(\xi_2)) < (u_\varepsilon(\xi_2) - \xi_2)^2 < (u_\varepsilon(\xi_1) - \xi_1)^2 < p'(\rho_\varepsilon(\xi_1)). \tag{4.15}$$

From (H3) p' is increasing for $\rho > 0$. So $\rho_\varepsilon(\xi_2) < \rho_\varepsilon(\xi_1)$ which contradicts the fact that ρ_ε is strictly increasing on (s_ε, ∞) when u_ε is strictly increasing on (s_ε, ∞). A similar argument gives (ii).

Let u_ε be decreasing on (s_ε, ∞). Since u_ε is decreasing on (s_ε, ∞), ρ_ε is decreasing on (s_ε, ∞), too. Since $u_\varepsilon{}'(\xi) \to 0$ as $\xi \to \infty$ and $u_\varepsilon{}'(s_\varepsilon) = 0$, there exists $\xi_1 \in (s_\varepsilon, \infty)$

such that ${u_\varepsilon}''(\xi_1) = 0$. Now suppose that there is $\xi_2 > \xi_1$ such that ${u_\varepsilon}''(\xi_2) = 0$, too. If we put ξ_1 and ξ_2 into (4.13), we get

$$\begin{aligned} p'(\rho_\varepsilon(\xi_1)) - (u_\varepsilon(\xi_1) - \xi_1)^2 = 0, \\ p'(\rho_\varepsilon(\xi_2)) - (u_\varepsilon(\xi_2) - \xi_2)^2 = 0. \end{aligned} \tag{4.16}$$

So we have

$$p'(\rho_\varepsilon(\xi_1)) = (u_\varepsilon(\xi_1) - \xi_1)^2 < (u_\varepsilon(\xi_2) - \xi_2)^2 = p'(\rho_\varepsilon(\xi_2)). \tag{4.17}$$

Since p' is increasing, $\rho_\varepsilon(\xi_1) \le \rho_\varepsilon(\xi_2)$. But ρ_ε is an decreasing function. So there exists exactly one point $\xi \in (s_\varepsilon, \infty)$ such that ${u_\varepsilon}''(\xi) = 0$. Property (iv) is proved similarly. $\square$

Properties (i) and (ii) in Lemma 4.3 provide the structure of rarefaction waves and (iii) and (iv) provide the structure of shock waves. Since (2.18) implies the convergence of the double index sequence $u_{\varepsilon_1}(s_{\varepsilon_2})$ for viscous solutions u_ε which belong to the category of C_1, C_2, C_3, we have $u(s) = s$. For the case of C_4, the convergence is from (i) and (ii) of Lemma 4.3. So we get $u(s) = s$. In the following two lemmas we study the continuity of the limit solution.

Lemma 4.4. *Let a solution (ρ, u) of $(\mathcal{P})$ be a limit of viscous solutions $(\rho_\varepsilon, u_\varepsilon)$ of $(\mathcal{P}_\varepsilon)$ and s be the limit of singular point s_ε. Then $u(s) = s$ and ρ and u are continuous at $\xi = s$.*

Proof. First, we suppose $\rho(s) > 0$. Then, from Theorem 4.1, $\rho(\xi)$ and $u(\xi)$ are constant on a neighborhood of $\xi = s$. So ρ and u are continuous at $\xi = s$.

Now suppose $\rho(s) = 0$. Since $\rho_\pm$ are positive, this is possible only when $(\rho_\varepsilon, u_\varepsilon)$ belongs to Category C_4 in Section 2.3. So u_ε are increasing on $(-\infty, \infty)$ and $|{u_\varepsilon}'(\xi)| \le 1$. We already know that $\{u_\varepsilon\}$ is uniformly bounded. The Ascoli-Arzela theorem implies the limit u is continuous on $(-\infty, \infty)$. From the Rankine-Hugoniot jump condition at $\xi = s$ we have $p(\rho(s+)) = p(\rho(s-))$. Then (H1) implies $\rho(s+) = \rho(s-)$. $\square$

Lemma 4.4 implies that the limit of viscous solutions is continuous at a singular point $s = u(s)$ in both cases of shock or rarefaction waves. Now we consider the continuity of the rarefaction waves on $\mathbb{R}$.

Lemma 4.5. *Let a solution (ρ, u) of (P) be the limit of viscous solutions of $(\mathcal{P}_\varepsilon)$ and s be a singular point, i.e. $u(s) = s$. If $u_\varepsilon(\xi)$ are increasing on (s, ∞), then the limit u and ρ are continuous on (s, ∞). If $u_\varepsilon(\xi)$ are increasing on $(-\infty, s)$, then the limit u and ρ are continuous on $(-\infty, s)$.*

Proof. From Lemma 2.4, $|u_\varepsilon(\xi)|$ is uniformly bounded by a constant M which is independent of ε. We also know from Lemma 4.3 that $|{u_\varepsilon}'(\xi)| < 1$. So $\{u_\varepsilon\}$ is uniformly bounded and equicontinuous. So, from the Ascoli-Arzela Theorem, u is continuous. From $(2.11)_1$

$$|{\rho_\varepsilon}'| = \frac{|\rho_\varepsilon {u_\varepsilon}'|}{|u_\varepsilon - \xi|} < \frac{\rho_\varepsilon}{|u_\varepsilon - \xi|}. \tag{4.18}$$

Suppose $\rho(s) > 0$. Then, ρ is constant on an open set $(s, s+\delta)$ for some δ which is independent of ε. The constant a in (2.26) is independent of ε. So $|{\rho_\varepsilon}'|$ is bounded above uniformly on the open interval $(s + c, \infty)$ for any $c > 0$. The Ascoli-Arzela theorem implies that $|\rho|$ is continuous on $(s+c, \infty)$ for any $c > 0$, that is ρ is continuous on (s, ∞).

Now we consider the case of $\rho(s) = 0$. This case is divided into two cases.

Case 1 : Suppose $\rho(\xi) > 0$ on (s, ∞). It is enough to prove that ρ is continuous on $(s+\delta, \infty)$ for any $\delta > 0$. Suppose $u(s+\delta) = s+\delta$. Then, since $u_\varepsilon{}'(\xi) \leq 1$ for all ε, $u(\xi) = \xi$ on $[s, s+\delta]$. From $(2.11)_1$, we have $\rho = 0$ on $[s, s+\delta]$. So $u(s+\delta) < s+\delta$. Since $|u(\xi) - \xi|$ is increasing, we have $|u(\xi) - \xi| > s + \delta - u(s+\delta)$ on $(s+\delta, \infty)$. So, from (4.18), $\rho_\varepsilon{}'$ is uniformly bounded on $(s+\delta, \infty)$. Hence ρ is continuous on$(s+\delta, \infty)$ by the Ascoli-Arzela theorem.
Case 2 : Suppose $\rho(\xi) = 0$ on $(s, s+\tau]$ and $\rho(\xi) > 0$ on $(s+\tau, \infty)$ for some $\tau > 0$. The proof of the continuity on $(s+\tau, \infty)$ is the same as the Case 1. We just prove the continuity at $s+\tau$. The Rankine-Hugoniot jump condition should be satisfied at $s+\tau$. If we write the condition with $\rho((s+\tau)-) = 0$ we get

$$\begin{aligned} (s+\tau)\rho((s+\tau)+) &= \rho((s+\tau)+)u((s+\tau)+) \\ (s+\tau)\rho((s+\tau)+)u((s+\tau)+) &= \rho((s+\tau)+)u((s+\tau)+)^2 + p(\rho((s+\tau)+)). \end{aligned} \tag{4.19}$$

So we get $u((s+\tau)+) = s+\tau$ and $p(\rho((s+\tau)+)) = 0$. So $\rho((s+\tau)+) = 0$ and ρ is continuous. □

The previous lemmas provide regularity properties for the limit solution (ρ, u). Let $\mathcal{S}$ be the set of points of discontinuity of (ρ, u) and $\mathcal{C}$ be the set of points of continuity. If u is increasing, (ρ, u) is continuous from Lemma 4.5. If u is decreasing, we can easily verify that there exists at most one point of discontinuity in $(-\infty, s)$ and (s, ∞) from Lemma 4.3 (iii),(iv).

Now we consider the relationship between the characteristic speeds of the problem $(\mathcal{P})$ and the weak derivative of the limit solution (ρ, u). Let

$$m = \rho u, \quad U = \begin{pmatrix} \rho \\ m \end{pmatrix}, \quad F(U) = \begin{pmatrix} m \\ \frac{m^2}{\rho} + p(\rho) \end{pmatrix}. \tag{4.20}$$

We know that the eigenvalues $\lambda_\pm$ of ∇F are given by

$$\lambda_\pm(\rho, u) = u \pm \sqrt{p'(\rho)}. \tag{4.21}$$

We use the notation $\lambda_\pm(\xi) = \lambda_\pm(\rho(\xi), u(\xi))$ and $\lambda_\pm^\varepsilon(\xi) = \lambda_\pm(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$.

Let $d\mu = (d\mu_1, d\mu_2) = \frac{dU}{d\xi}$ be the vector valued measure which corresponds to the weak derivative of U, i.e. corresponding to the linear functional:

$$\phi \to -\int \phi'(\xi) U(\xi) d\xi, \qquad \phi \in C_c^1(\mathbb{R}). \tag{4.22}$$

We apply the Volpert product([11]) of $\nabla F(U)$ and $d\mu$ to the equation $(\mathcal{P})$ to get

$$\big(\hat{\nabla F}(U) - \xi I\big) d\mu = 0 \tag{4.23}$$

in the sense of measures, where the averaged superposition $\hat{\nabla F}(U)$ of U by ∇F is given by

$$\hat{\nabla F}(U)(\xi) = \int_0^1 \nabla F(U(\xi-) + s(U(\xi+) - U(\xi-)))ds.$$

Let $\xi \in \mathcal{C} \cap \operatorname{supp}\mu$. Since there is at most one point of discontinuity, $\hat{\nabla F}(U) = \nabla F(U)$ in a neighborhood of ξ. Suppose the determinant of $(\nabla F(U) - \xi I)$ is not zero, for example $\det(\nabla F(U) - \xi I) > 0$. Then there exists a neighborhood N of ξ such that $\det(\nabla F(U) - \zeta I) > \delta > 0$ for all $\zeta \in N$. But from (4.23), the measures $\det(\nabla F(U) - \zeta I) d\mu_{1,2} = 0$ on N, which contradicts the fact that $\xi \in \operatorname{supp}\mu$. So ξ is an eigenvalue of $\nabla F(U(\xi))$. We summarize these facts in a lemma :

LEMMA 4.6. *Let a solution (ρ, u) of $(\mathcal{P})$ be a limit of viscosity solutions $(\rho_\varepsilon, u_\varepsilon)$ of $(\mathcal{P}_\varepsilon)$ with a singular point s. Let $d\mu$ be the measure of* (4.22). *Then we have: (i) If u is increasing, then (ρ, u) is continuous. If u is decreasing, then there exists at most one point of discontinuity in $(-\infty, s)$ and (s, ∞). (ii) If (ρ, u) is continuous at $\xi \in supp\mu$, $\xi = \lambda_+(\xi)$ on (s, ∞) and $\xi = \lambda_-(\xi)$ on $(-\infty, s)$.*

We conclude the section with a theorem which provides the structure of the limit solution (ρ, u) of the viscosity solutions $(\rho_\varepsilon, u_\varepsilon)$ which obey the a-priori estimates (2.24)–(2.26).

THEOREM 4.7. *Let (ρ, u) be a solution of the Riemann problem $(\mathcal{P})$ through the method of self-similar zero-viscosity limits and s be the limit of singular points. (i) If u is increasing on (s, ∞), then $\lambda_+(\xi)$ is continuous on (s, ∞) and*

$$\lambda_+(\xi) = \begin{cases} \lambda_+(s), & s < \xi < \lambda_+(s), \\ \xi, & \lambda_+(s) < \xi < \lambda_+(\rho_+, u_+), \\ \lambda_+(\rho_+, u_+), & \lambda_+(\rho_+, u_+) < \xi. \end{cases} \tag{4.24}$$

(ii) If u is increasing on $(-\infty, s)$, then $\lambda_-(\xi)$ is continuous on $(-\infty, s)$ and

$$\lambda_-(\xi) = \begin{cases} \lambda_-(s), & \lambda_-(s) < \xi < s, \\ \xi, & \lambda_-(\rho_-, u_-) < \xi < \lambda_-(s), \\ \lambda_-(\rho_-, u_-), & \xi < \lambda_-(\rho_-, u_-). \end{cases} \tag{4.25}$$

(iii) If u is decreasing on (s, ∞), then $\lambda_+(\xi)$ has an unique discontinuity on (s, ∞) and

$$\lambda_+(\xi) = \begin{cases} \lambda_+(s), & s < \xi < \dfrac{\rho_+ u_+}{(\rho_+ - \rho(s))}, \\ \lambda_+(\rho_+, u_+), & \dfrac{\rho_+ u_+}{(\rho_+ - \rho(s))} < \xi. \end{cases} \tag{4.26}$$

(iv) If u is decreasing on $(-\infty, s)$, then u has an unique discontinuity on $(-\infty, s)$ and

$$\lambda_-(\xi) = \begin{cases} \lambda_-(s), & \dfrac{\rho_- u_-}{(\rho_- - \rho(s))} < \xi < s, \\ \lambda_-(\rho_-, u_-), & \xi < \dfrac{\rho_- u_-}{(\rho_- - \rho(s))}. \end{cases} \tag{4.27}$$

Proof. We always consider the eigenvalue λ_- on the interval $(-\infty, s)$ and λ_+ on the other side (s, ∞). If u is increasing, then u and ρ are continuous. So they should be constant out of suppμ and $\lambda_\pm$ are continuous and increasing, too. With these facts we can easily check that suppμ is a connected sub-intervals of $(-\infty, s]$ or $[s, \infty)$. If not, $\lambda_\pm$ is discontinuous. So the structure of $\lambda_\pm$ should follow (i) and (ii).

If u is decreasing, then $\lambda_\pm$ are also decreasing and there exists at most one point of discontinuity on $(-\infty, s)$ and (s, ∞). Since $\lambda_\pm = \xi$ on suppμ and $\lambda_\pm$ are decreasing, suppμ should be the point of discontinuity and $\lambda_\pm$ be constant before and after the discontinuity. We can find the point of the discontinuity from the Rankine-Hugoniot jump condition, and $\lambda_\pm$ should follow (iii) and (iv). □

In summary, the limit of viscosity solutions has an intermediate state which is connected to the boundary states by rarefaction waves ((i) and (ii)) or shocks ((iii) and (iv)).

COROLLARY 4.8. *If the system* (1.1) *is strictly hyperbolic, i.e. there exists* $c > 0$ *such that*

$$p'(\rho) \geq c^2 > 0, \qquad \rho > 0, \tag{4.28}$$

then the emerging limit does not have a vacuum state.

Proof. Theorem 4.7 implies that $\lambda_+(\xi)$ is constant on the interval $(s, \lambda_+(s)) \neq \phi$ and $(\lambda_+(\rho_+, u_+), \infty)$. Since $\sqrt{p'(\rho)}$ is increasing on (s, ∞), $u(\xi)$ is also constant on those intervals. From $(1.6)_1$ ρ is also constant. Now we consider the interval $(\lambda_+(s), \lambda_+(\rho_+, u_+))$. From $(1.6)_2$ we get $c^2\rho' \leq (\xi - u)\rho$, and hence there exists a constant C such that

$$\rho' \leq C\rho.$$

So we have

$$\rho(\xi) \leq \rho(\lambda_+(s))e^{C(\xi - \lambda_+(s))}.$$

Suppose the solution has a vacuum state, i.e. $\rho(s) = 0$. Then, since ρ is constant on $(s, \lambda_+(s))$, $\rho(\lambda_+(s)) = 0$, and hence ρ is zero on $(\lambda_+(s), \lambda_+(\rho_+, u_+))$. So $\rho(\infty) = 0$ which contradicts the boundary condition $\rho(\infty) = \rho_+ > 0$. □

5. Convex pressure laws. In Lemma 2.4 the a-priori estimates (2.24), (2.25) are established except for the lower bound for ρ of the case C_4. In this section we complete the a-priori estimates in two cases under the convex pressure laws (H3). First, we consider the case of strictly hyperbolic systems.

The equation $(2.22)_1$ can be written as a first order linear equation for ρ :

$$\rho' + \frac{u'}{u - \xi}\rho = 0, \qquad \xi \neq s, \tag{5.1}$$

where s is the singular point. Then the solution is given by

$$\rho(\xi) = \begin{cases} \rho_+^\mu e^{-\int_\xi^\infty \frac{u'}{\zeta - u} d\zeta}, & s < \xi \\ \rho_-^\mu e^{-\int_{-\infty}^\xi \frac{u'}{u - \zeta} d\zeta}, & \xi < s, \end{cases} \tag{5.2}$$

where $\rho_-^\mu = \rho_-$ and $\rho_+^\mu = \rho_- + \mu(\rho_+ - \rho_-)$ are the boundary conditions of (2.23).

LEMMA 5.1. *Let a solution* (ρ, u) *of* $(\mathcal{P}_\varepsilon^\mu)$ *belong to the class* C_4. *If the system* (1.1) *is strictly hyperbolic, i.e.*

$$p'(\rho) \geq c^2 > 0, \qquad \rho > 0, \tag{5.3}$$

then there exists a constant $\delta > 0$ *which satisfies* (2.24) *and is independent of* ε *and* μ.

Proof. Since u is increasing on $\mathbb{R}$, $u' \leq 1$ by Lemma 4.3 and $\xi - u(\xi)$ is also increasing. So there exists $\xi_1 > s$ such that $0 < \xi - u(\xi) \leq \frac{c}{2}$ on (s, ξ_1) and $\frac{c}{2} \leq \xi - u(\xi)$ on (ξ_1, ∞). Then

$$\rho(\xi_1) = \rho_+^\mu e^{-\int_{\xi_1}^\infty \frac{u'}{\zeta - u} d\zeta} \geq \rho_+^\mu e^{-\frac{2}{c}(u_+ - u_-)}. \tag{5.4}$$

Integrating (2.28) on (s, ξ_1) we get

$$\varepsilon \int_s^{\xi_1} u''(\zeta) d\zeta = \varepsilon u'(\xi_1) \leq \varepsilon \tag{5.5}$$

and

$$\int_s^{\xi_1} (p'(\rho) - (\zeta - u)^2)\rho' d\zeta \geq \frac{3c^2}{4} \int_s^{\xi_1} \rho' d\zeta \geq \frac{3c^2}{4}(\rho(\xi_1) - \rho(s)). \tag{5.6}$$

From the above estimations ρ is bounded below by

$$\rho(s) \geq \rho_+^\mu e^{-\frac{2}{c}(u_+ - u_-)} - \frac{4\varepsilon}{3c^2} \geq \min\{\rho_-, \rho_+\} e^{-\frac{2}{c}(u_+ - u_-)} - \frac{4\varepsilon}{3c^2}. \tag{5.7}$$

The positive lower bound for ρ is obtained for small ε. □

We return to general convex pressure laws (H3) and consider the function

$$g(\rho) = \frac{p(\rho)}{\rho}, \qquad \rho > 0. \tag{5.8}$$

Either the function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is invertible or the system is strictly hyperbolic. Consider the case when g has an inverse g^{-1}.

LEMMA 5.2. *Let a solution (ρ, u) of $(\mathcal{P}_\varepsilon^\mu)$ belong to the class C_4. If the boundary conditions $(\rho_\pm, u_\pm)$ satisfy*

$$u_+ - u_- < \max_{m>0}(m \ln(\frac{\rho_-^\mu}{g^{-1}(m^2)})) + \max_{m>0}(m \ln(\frac{\rho_+^\mu}{g^{-1}(m^2)})), \tag{5.9}$$

then there exists a constant $\delta > 0$ which satisfies (2.24) *and is independent of ε and μ.*

Proof. Let s be the singular point of the solution (ρ, u). Since u is increasing in $\mathbb{R}$, we have $u_- < u(s) < u_+$. If (5.9) holds,

$$u_+ - u(s) < \max_{m>0}(m \ln(\frac{\rho_+^\mu}{g^{-1}(m^2)})) \tag{5.10}$$

or

$$u(s) - u_- < \max_{m>0}(m \ln(\frac{\rho_-^\mu}{g^{-1}(m^2)})). \tag{5.11}$$

We assume that (5.10) holds. Then there exists $m > 0$ such that $u_+ - u(s) < m \ln(\frac{\rho_+^\mu}{g^{-1}(m^2)})$ or equivalently

$$g(\rho_+^\mu e^{-\frac{(u_+ - u(s))}{m}}) - m^2 > 0. \tag{5.12}$$

Since $\xi - u(\xi)$ is increasing, there exists $\xi_1 > s$ such that $0 < \xi - u(\xi) \leq m$ on (s, ξ_1) and $m \leq \xi - u(\xi)$ on (ξ_1, ∞). Then

$$\rho(\xi_1) = \rho_+^\mu e^{-\int_{\xi_1}^\infty \frac{u'}{\zeta - u} d\zeta} \geq \rho_+^\mu e^{-\frac{(u_+ - u(s))}{m}}. \tag{5.13}$$

We can easily check that $g(\rho)$ is increasing for $\rho > 0$ and $g(\rho(\xi_1)) - m^2 > 0$. Integrating (2.28) on (s, ξ_1) we get

$$\varepsilon \int_s^{\xi_1} u''(\zeta) d\zeta = \varepsilon u'(\xi_1) \leq \varepsilon \tag{5.14}$$

and

$$\int_s^{\xi_1} (p'(\rho) - (\zeta - u)^2)\rho' d\xi \geq (\frac{p(\rho(\xi_1)) - p(\rho(s))}{\rho(\xi_1) - \rho(s)} - m^2)(\rho(\xi_1) - \rho(s)). \tag{5.15}$$

The convexity Hypothesis (H3) implies

$$\frac{p(\rho(\xi_1)) - p(\rho(s))}{\rho(\xi_1) - \rho(s)} - m^2 > \frac{p(\rho(\xi_1))}{\rho(\xi_1)} - m^2 = g(\rho(\xi_1)) - m^2 > 0. \tag{5.16}$$

So the density ρ is bounded below by

$$\rho(s) \geq \min\{\rho_+, \rho_-\} e^{-\frac{1}{m}(u_+ - u(s))} - \frac{\varepsilon}{g(\xi_1) - m^2} > 0 \tag{5.17}$$

for a sufficiently small $\varepsilon > 0$. The situation is similar if (5.11) holds. □

One can check that, if

$$u_+ - u_- < \max_{m>0}(m \ln(\frac{\rho_-}{g^{-1}(m^2)})) + \max_{m>0}(m \ln(\frac{\rho_+}{g^{-1}(m^2)})) \tag{5.18}$$

holds and $\rho_- \leq \rho_+$, then (5.9) holds. If $\rho_- > \rho_+$ and instead of using the continuation of the boundary data (2.23) one uses

$$\begin{aligned} \rho(\pm\infty) = \rho_\pm^\mu &:= \rho_+ + \mu(\rho_\pm - \rho_+) \\ u(\pm\infty) = u_\pm^\mu &:= u_+ + \mu(u_\pm - u_+) \end{aligned} \tag{5.19}$$

then again (5.9) holds. Thus (5.18) provides a sufficient condition which prevents vacuum from appearing. It is known that admissible solutions of (1.1) and (1.2) do not have a vacuum state if and only if

$$u_+ - u_- < \int_0^{\rho_-} \frac{\sqrt{p'(\rho)}}{\rho} d\rho + \int_0^{\rho_+} \frac{\sqrt{p'(\rho)}}{\rho} d\rho. \tag{5.20}$$

While (5.18) is a sufficient condition to avoid vacuum, simple numerical computations show that it is not a necessary. In the case of γ-laws, $p(\rho) = \rho^\gamma$ for $\gamma > 1$, the condition (5.18) corresponds to

$$u_+ - u_- < \ln\left(\frac{2}{\gamma - 1}\right)\left[\left(\frac{2}{\rho_-(\gamma - 1)}\right)^{\frac{1-\gamma}{2}} + \left(\frac{2}{\rho_+(\gamma - 1)}\right)^{\frac{1-\gamma}{2}}\right]. \tag{5.21}$$

Now we summarize the previous lemmas and the results of Section 3 in the theorem :

THEOREM 5.3. *Suppose $p(\rho)$ satisfies (H1), (H2) and (H3). If the system (1.1) is strictly hyperbolic or the initial data $(\rho_\pm, u_\pm)$ satisfy (5.18), then the boundary value problem $(\mathcal{P})$ has a solution (ρ, u) which is a $\varepsilon \to 0$ of solutions of $(\mathcal{P}_\varepsilon)$. The function (ρ, u) has the structure stated in Theorem 4.7 and does not contain vacuum. $(\rho(x/t), u(x/t))$ is a solution of Riemann problem (1.1),(1.2).*

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