MULTISCALE ANALYSIS FOR INTERACTING PARTICLES: RELAXATION SYSTEMS AND SCALAR CONSERVATION LAWS

by

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ABSTRACT. We investigate the derivation of semilinear relaxation systems and scalar conservation laws from a class of stochastic interacting particle systems. These systems are Markov jump processes set on a lattice, they satisfy detailed mass balance (but not detailed balance of momentum) and are equipped with multiple scalings. Using a combination of correlation function methods with compactness and convergence properties of semidiscrete relaxation schemes we prove that, at a mesoscopic scale, the interacting particle system gives rise to a semilinear hyperbolic system of relaxation type, while at a macroscopic scale it yields a scalar conservation law. Rates of convergence are obtained in both scalings.

1. Introduction

We investigate a class of interacting particle systems (IPS) set on a one dimensional lattice. The IPS consists of two kinds of particles: the first kind is of mass ℓ and moves to the left with velocity -1, while the second kind is of mass 1 (or more generally of mass m) and moves to the right with velocity $+1$. The IPS evolves under a combination of two stochastic mechanisms: First, streaming of particles following their velocity at exponentially distributed times. Second, an exchange mechanism where a fixed number of ℓ particles moving with velocity $+1$ merge to create a single particle moving with velocity -1 (or more generally m particles moving with velocity -1); vice versa, one particle (or more generally m particles) moving with velocity -1 breaks into ℓ particles moving with velocity $+1$. Both processes occur with configuration dependent rates (see Sec. 2). The exchange mechanism satisfies detailed mass balance, but violates detailed balance of linear momentum. The objective of this article is to study the behavior of the particle system in two scaling limits. We will prove: (i) At a mesoscopic scale, after suitable averaging and space rescaling, the IPS converges to

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a relaxataion system equipped with a mass balance law. (ii) There exists a continuum of macroscopic scalings, of both lattice size and time, so that the averaged particle system relaxes to the entropy solution of a scalar conservation law.

The particle systems studied here are generalizations of the IPS modeling the Carleman system in De Masi and Presutti[(1)]. The streaming mechanisms are the same in both cases. By contrast the "collision" mechanisms differ: While in the Carleman case, two particles with velocity $+1$ are converted into two particles with velocity -1, for the present IPS the velocity exchange process is not symmetric. As a result, while the Carleman model yields at a macroscopic limit a trivial conservation law, for the particle system considered here - when $\ell \neq m$ - the macroscopic limit is described by a quasilinear hyperbolic equation.

The mesoscopic limit of the IPS is described by relaxation systems of the type

$$
\begin{aligned}
\partial_t w + U_0 \cdot \nabla w &= \frac{1}{\varepsilon} \sum_{i=1}^{N} G_i(w, z_i) \\
\partial_t z_i + U_i \cdot \nabla z_i &= -\frac{1}{\varepsilon} G_i(w, z_i)\,, \qquad i = 1, ..., N,
\end{aligned}
\tag{1.1}
$$

studied by Katsoulakis and Tzavaras[(2)]. The system (1.1) describes the dynamics of the state vector (w, Z), $Z = (z_1, \ldots, z_N)$, where U_0, U_1,$\ldots$, $U_N \in \mathbb{R}^N$ are convective velocities, and $G_i(w, z_i)$ are smooth functions that are decreasing in w and increasing in z_i, and $\varepsilon > 0$ is a relaxation parameter. It is assumed that each equation $G_i(w, z_i) = 0$ has a unique solution $z_i = g_i(w)$, with $g_i(w)$ a smooth strictly increasing function. The curve

$$
w \mapsto (w, g_1(w), \ldots, g_N(w))\,, \quad w \in \mathbb{R}\,,
$$

constitutes the manifold of local equilibria (or Maxwellian states). Solutions of (1.1) satisfy the conservation law (corresponding to mass balance)

$$
\partial_t (w + \sum_{i=1}^{N} z_i) + \operatorname{div}(U_0 w + \sum_{i=1}^{N} U_i z_i) = 0\,.
$$

As $\varepsilon \to 0$, the local equilibria are enforced, and the limiting dynamics is described by weak solutions of

$$
\partial_t (w + \sum_{i=1}^{N} g_i(w)) + \operatorname{div}(U_0 w + \sum_{i=1}^{N} U_i g_i(w)) = 0\,.
\tag{1.2}
$$

It is shown in Ref. 2 that solutions of (1.1) are precompact in L^1 and that the moment $u = w + \sum_{i=1}^{N} z_i$ converges as $\varepsilon \to 0$ to an entropy solution of (1.2).

The analogy to the kinetic theory of dilute gases is transparent. In an ideal gas the molecules are caricatured as a set of hard spheres, evolving under Newtonian dynamics and undergoing elastic collisions. A reduced - mesoscopic - description of this IPS is given by the Boltzmann equation. The Boltzmann equation still does not describe macroscopic quantities but rather the evolution of the probability distribution of particles in the phase space. When the mean free path goes to zero, the solution of the Boltzmann equation relaxes to a Maxwellian distribution and the process yields a macroscopic description via fluid equations. In principle, one would hope to derive rigorously the fluid equations directly from particle dynamics, using the Boltzmann equation as a mesoscopic-intermediate step. This goal seems quite ambitious as only parts of this program are at present rigorous, while the remaining obstacles are difficult to overcome.

Here, we pursue this general strategy in the much simpler context of a theory equipped with only one detailed balance law, the balance of mass. We also restrict to one-space dimension. The proposed IPS can be naturally interepreted as a microscopic model for a convection-reaction process. Building on Refs 1 and 2 we will show : (a) At mesoscopic scales the IPS approximates the one-dimensional version of the relaxation model (1.1) (*cf.* Theorem 2.1). (b) At macroscopic scales it approximates the entropy solution of a scalar conservation law; this is accomplished for a continuum of space/time scales given as powers of the lattice size (Theorem 2.2). Error estimates are obtained for the convergence of the stochastic IPS at both the mesoscopic and macroscopic limits.

In contrast to the Newtonian dynamics of rarefied gases which is deterministic, the particle systems considered have stochastic dynamics. A quite surprising fact concerning the derivation of discrete velocity systems from microscopic considerations was discovered by Uchyiama[(3)]: There exist models of deterministic particle dynamics with finitely many velocities that do not give rise to discrete velocity kinetic equations. On the other hand the Carleman and Broadwell systems as well as (1.1) can be rigorously derived from particle systems with stochastic dynamics (see Refs. 4 and 5).

Convergence of stochastic IPS to various discrete velocity models is established in Caprino-De Masi-Presutti-Pulvirenti[(4),(5)], Caprino-Pulvirenti[(6)], Rezakhanlou-Tarver[(7)]. The convergence from mesoscopic to macroscopic equations, in the hyperbolic context, falls in the realm of relaxation limits. We refer to Liu[(8)], Chen-Levermore-Liu[(9)], Natalini[(10),(11)], Katsoulakis-Tzavaras[(2)] and references in these works for studies of relaxation approximations, and to Jin-Xin[(12)], Tveito-Winther[(13)], Katsoulakis-Kossioris-Makridakis[(14)] for studies of numerical relaxation schemes. Concerning the passage from stochastic IPS to macroscopic equations, there exist two approaches: Investigations that establish directly, without use of mesoscopic equations, the convergence for the exclusion and zero-range processes to scalar conservation laws, Rezakhanlou[(15)]. Investigations where the study of the mesoscopic-scale equations is the main ingredient in the derivation of the macroscopic equations, like the study in Perthame-Pulvirenti[(16)] of a class of IPS related to scalar conservation laws, as well as the derivation of geometric evolutions, such as motion by mean curvature, from Ising systems (*e.g.* Bonaventura[(17)], Katsoulakis-Souganidis[(18)]). The stochastic IPS in Ref. 16 is patterned after the kinetic formulation of (1.2) of Lions-Perthame-Tadmor[19]. The IPS considered here are patterned after Ref. 4-5 and the relaxation system in Ref. 2, and have a small number of species. We point out that the techniques presented here may be relevant in the study of lattice Boltzmann models (cellular automata) with a single conserved quantity.

From a technical standpoint, our analysis relies on interweaving two tools. First, the correlation function method introduced by Lanford[20], for the short time derivation of the Boltzmann equation from Newtonian dynamics, and developed for stochastic IPS by Caprino-DeMasi-Presutti-Pulvirenti[4]. Second, on compactness and convergence properties for semidiscrete approximations of (1.1). Semi-discrete relaxation schemes are derived as approximations of stochastic IPS by means of the correlation function method, and their study is a key ingredient of the rigorous derivation of (1.2) from IPS.

The contents are organized as follows. In Section 2 we introduce the interacting particle system associated to (1.1), and present the main results on convergence of the IPS and rates of convergence. In Sections 3 and 4 we prove Theorems 2.1 and 2.2. In Section 5 we prove compactness-convergence properties and rates of convergence, in both mesoscopic and macroscopic scales, for semidiscrete approximations of (1.1). In the Appendix we present two key approximation lemmata that use the technique of doubling of variables of the Kruzhkov[21] and Kuznetzov[22] theory. The first is due to Bouchut and Perthame[23] and concerns the conservation law (1.2). The second shows that the Kruzhkov theory may be extended to the class of contractive relaxation systems (1.1). These results are the basis for obtaining rates of convergence for the IPS. Part of the results presented here were announced in Ref. 24.

2. Interacting Particle Systems with relaxation

2a. The one-dimensional model. We introduce a stochastic Interacting Particle System (IPS) set on a one-dimensional lattice which, in a mesoscopic scaling, will converge to the relaxation system (1.1), while, in a macroscopic scaling of lattice size and time, will approximate the entropy solution of a conservation law. The macroscopic system is set in the interval $[0,1]$ with periodic boundary conditions. In the usual fashion, weak solutions u of the scalar conservation law

$$\begin{cases} \partial_t u + F(u)_x = 0, & x \in [0,1], t > 0, \\ u(x,0) = u_0(x)\,, & u(0,t) = u(1,t)\,, \end{cases} \tag{2.1}$$

are required to satisfy the Kruzhkov entropy conditions

$$\partial_t |u-k| + \partial_x \big[(F(u) - F(k))\operatorname{sgn}(u-k) \big] \le 0, \quad \text{in } \mathcal{D}', \text{ for all } k \in \mathbb{R}\,.$$

The IPS is set on the discrete torus of the lattice $\mathbb{Z}$, denoted by

$$\mathbb{Z}_\nu = \{0, 1, ..., M\}\,,$$

where $\nu = M^{-1}$ is the inverse of the positive integer M. There are two types of particles on the lattice $\mathbb{Z}_\nu$: particles of mass ℓ moving with velocity -1 and particles of mass 1 moving with velocity $+1$. Let $\eta(i,\sigma)$ be the number of particles at the lattice site $i \in \mathbb{Z}_\nu$ with velocity $\sigma \in \{-1,1\}$. We define the configuration

$$\eta = \{\eta(i,\sigma)\,,(i,\sigma) \in \mathbb{Z}_\nu \times \{-1,1\}\}\,,$$

and the set of all possible configurations $X = \mathbb{N}^{\mathbb{Z}_\nu \times \{-1,1\}}$, called configuration space. The updating is done through two mechanisms, "streaming" and "interaction" of particles:

I. A particle at i with velocity σ, denoted by (i,σ), jumps to $(i+\sigma,\sigma)$ keeping its velocity and moving one lattice unit in the direction σ, at a rate $V\gamma^{-1}$ with $V,\gamma > 0$ and $\gamma << 1$.

II. A number of ℓ particles at the site i with velocity $\sigma = +1$ interact to produce a single particle at the same site i with opposite velocity $\sigma = -1$. Conversely, a single particle at the site i with velocity $\sigma = -1$ breaks up into ℓ particles at the site i with velocity $\sigma = +1$. The two processes occur at rates $c(\eta,1,i)$ and $c(\eta,-1,i)$ respectively, satisfying the conditions:

(i) $c(\eta,1,i) = D_\ell(\eta(i,1)) := \eta(i,1)[\eta(i,1)-1]...[\eta(i,1)-\ell+1]\,,$

(ii) $c(\eta,-1,i) = \eta(i,-1)$.

Note that if μ is a product Poisson measure on X, with varying densities over the lattice, such that

$$E_\mu\eta(i,1) = w(i) \geq 0, \quad E_\mu\eta(i,-1) = z(i) \geq 0\,, \quad i \in \mathbb{Z}_\nu\,,$$

then

$$E_\mu c(\eta,-1,i) = z(i), \quad E_\mu c(\eta,1,i) = w(i)^\ell\,.$$

The streaming and interaction mechanisms, endowed with *periodic conditions*, give rise to a jump Markov process,

$$\eta_\cdot = \left\{\eta_t(i,\sigma) \in \mathbb{N} : t \geq 0, (i,\sigma) \in \mathbb{Z}_\nu \times \{-1,1\}\right\},$$

set on the configuration space X. The process is completely described by its generator

$$L^\gamma = \gamma^{-1}L_s + L_c\,,$$

where

$$\begin{aligned}
L_s f(\eta) =& V\sum_{(i,\sigma)} \eta(i,\sigma)[f(\eta+\delta_{(i+\sigma,\sigma)} - \delta_{(i,\sigma)}) - f(\eta)],\\
L_c f(n) =& \sum_i c(\eta,1,i)[f(\eta+\delta_{(i,-1)} - \ell\delta_{(i,1)}) - f(\eta)]+\\
& c(\eta,-1,i)[f(\eta+\ell\delta_{(i,1)} - \delta_{(i,-1)}) - f(\eta)],
\end{aligned}$$

$f : X \mapsto \mathbb{R}$ cylindrical, and $\delta_{(i,\sigma)}(j,\sigma') = 1$ if $(j,\sigma') = (i,\sigma)$ and equal to 0 otherwise. We refer to Refs. 25 and 26 for the construction of stochastic processes, associated with interacting particle systems.

Remark 2.1 (a) In the above model the particles do not undergo true collisions but rather an interaction mechanism according to the rates $c(\eta, \pm 1, i)$. The interaction mechanism obeys *detailed balance of mass.*
(b) The analysis concerning the mesoscopic limit of the IPS applies to certain models involving real collisions, such as the Ruijgrok-Wu model[27], for which the collision rates are

$$c(\eta,1,i) = \alpha_1 \eta(i,1) + \alpha_3 \eta(i,1)\eta(i,-1), \;\; c(\eta,-1,i) = \alpha_2 \eta(i,-1)\,, \quad i \in \mathbb{Z}_\nu\,,$$

with a_1, a_2, a_3 positive constants. In this case $E_\mu c(\eta,-1,i) = \alpha_2 z(i)$ and $E_\mu c(\eta,1,i) = \alpha_1 w(i) + \alpha_3 w(i) z(i)$.
(c) We may also consider a particle system consisting of particles of mass ℓ traveling with velocity -1 and particles of mass m traveling with velocity $+1$ and where ℓ particles of velocity $+1$ are converted into m particles of velocity -1 and vice versa; then

$$c(\eta,-1,i) = D_m(\eta(i,-1)) = \eta(i,1)[\eta(i,1)-1]...[\eta(i,1)-m+1]\,,$$

while $c(\eta,1,i)$ satisfies Condition (i). Our analysis extends to such systems in a straightforward manner.
(d) The above particle systems generalize the IPS modeling the Carleman system in De Masi-Presutti[1]. There, the streaming mechanism is given also by the generator L_s, while the definition of L_c is modified so that 2 particles of velocity 1 are converted into 2 particles of velocity -1. Due to this symmetry, the Carleman system yields a trivial conservation law in the hydrodynamic limit. It turns out for the particle systems considered here that, at least when $\ell \neq m$, the hydrodynamic limit is described by the hyperbolic conservation law (2.1).

Heuristically, one can think of the continuous process $\eta_.$ through its Markov chain approximation, which, for time step $\Delta t << 1$, is constructed as follows (for simplicity we describe the interaction mechanism only). Let η_0 be a configuration of the initial measure on X. A site $(i,\sigma) \in \mathbb{Z}_\nu \times \{-1,1\}$ is selected at random and a particle (or ℓ particles, depending on the velocity) at i with velocity σ is converted to ℓ particles (or one particle) with velocity $-\sigma$ with probability $c(\eta_0,\sigma,i)\Delta t$, while η_0 remains unchanged with probability $1 - c(\eta_0,\sigma,i)\Delta t$. Thus we construct the configuration $\eta_{\Delta t}$; we then pick randomly a new site $(i',\sigma') \in \mathbb{Z}_\nu \times \{-1,1\}$ and complete the previous procedure to construct $\eta_{2\Delta t}$, and so on.

Next, we present the main results concerning the mesoscopic and macroscopic behavior of the particle system defined by $\eta_.$. First we motivate the meso- and macroscopic scales through a formal discussion. Assume that the initial distribution μ_0 of the process $\eta_.$ is a product Poisson measure on X, i.e. for any $B = \{\eta \in X : \eta(i,\sigma) = k_{i,\sigma}\,,\, (i,\sigma) \in \mathbb{Z}_\nu \times \{-1,1\}\,\}$,

$$\mu_0(B) = \prod_{(i,\sigma)\in\mathbb{Z}_\nu \times\{-1,1\}} \nu_{\rho_0}\big(\{\eta(i,\sigma) = k_{i,\sigma}\}\big)\,,$$

where ν_{ρ_0} is a Poisson measure with varying density on $\mathbb{Z}_\nu \times \{-1,1\}$, $\rho_0 = \rho_0(i,\sigma)$, $(i,\sigma) \in \mathbb{Z}_\nu \times \{-1,1\}$:

$$\nu_{\rho_0}\big(\{\eta(i,\sigma)=k\}\big) = e^{-\rho_0(i,\sigma)}\frac{\rho_0(i,\sigma)^k}{k!}\,.$$

Then, Doob's Theorem[25] implies that, under solely the influence of the streaming mechanism L_s, the distribution μ_t at any time t is still a product Poisson measure with possibly different densities. For the particle system generated by $L^\gamma = \gamma^{-1}L_s + L_c$, we expect that, since the streaming mechanism L_s is dominant for $\gamma << 1$, the distribution μ_t of η_t will also be a product Poisson measure in the asymptotic limit $\gamma \to 0$. This observation suggests the "propagation of chaos" property, i.e. that the occupation numbers $\eta_t(i,\sigma)$ become independent - for different (i,σ)'s and all times - as $\gamma \to 0$, if they are independent at $t=0$. Furthermore, it suggests that, given an initial product Poisson measure μ_0, then for all positive times the process $\eta_.$ is at an approximate *local equilibrium* described by a product Poisson measure. The limit product Poisson measure on X is characterized by the expected occupation numbers $E_{\mu_0}\eta_t(i,\sigma)$ at each point (i,σ). These satisfy

$$\begin{aligned}\frac{d}{dt}E_{\mu_0}\eta_t(i,1) + \frac{V}{\gamma}[E_{\mu_0}\eta_t(i,1) - E_{\mu_0}\eta_t(i-1,1)] &= \ell E_{\mu_0}[c(\eta_t,-1,i) - c(\eta_t,1,i)]\,,\\ \frac{d}{dt}E_{\mu_0}\eta_t(i,-1) + \frac{V}{\gamma}[E_{\mu_0}\eta_t(i,-1) - E_{\mu_0}\eta_t(i+1,-1)] &= E_{\mu_0}[c(\eta_t,1,i) - c(\eta_t,-1,i)]\,.\end{aligned} \tag{2.2}$$

Equation (2.2) is a semi-discrete approximation of a semilinear hyperbolic system provided the following conditions are met: (a) γ is interpreted as a space discretization Δx of $[0,1]$, and (b) the term $E_{\mu_0}[c(\eta_t,-\sigma,i) - c(\eta_t,-\sigma,i)]$ can be written as a function of the expected occupation numbers $E_{\mu_0}\eta_t(i,\sigma)$. Condition (a) can be easily met; condition (b) is only expected in the limit $\gamma \to 0$, due to the assumptions on the interaction rates $c(\eta,\sigma,i)$ and the heuristic discussion on the anticipated asymptotic behavior of the distribution of η_t (propagation of chaos property).

Henceforth, the process $\eta_.$ is set on the periodic lattice

$$\mathbb{Z}_\nu = \mathbb{Z}_\gamma = \{0,1,...,\gamma^{-1}\}\,,$$

with γ^{-1} an integer. Then $\{\gamma i\}_{i\in\mathbb{Z}_\gamma}$ is a discretization of the interval $[0,1]$ with step size $\Delta x = \gamma$. We consider the quantities

$$E_{\mu_0}\eta_t(i,1)\,,\quad E_{\mu_0}\ell\eta_t(i,-1)\,,$$

describing the expected values of the masses of the particle system. From (2.2) and the "propagation of chaos" property, their limiting values, as $\gamma \to 0$, $\rho^+(i,t)$ and $\rho^-(i,t)$ are expected to satisfy the relaxation system

$$\left\{\begin{aligned}\rho_t^+ + V\rho_x^+ &= G(\rho^+,\rho^-)\\ \rho_t^- - V\rho_x^- &= -G(\rho^+,\rho^-)\end{aligned}\right. \quad,\quad (x,t)\in[0,1]\times(0,T)\,, \tag{2.3}$$

where

$$G(\rho^+,\rho^-) = \rho^- - g(\rho^+)\,, \quad g(\rho^+) = \ell(\rho^+)^\ell\,.$$

subject to periodic conditions in $[0,1]$. We emphasize that (2.3) is derived from the expected values of the mass occupation numbers $E_{\mu_0}\eta_t(i,1)$, $E_{\mu_0}\ell\eta_t(i,-1)$ and, for convenience, we define a new process $\hat{\eta}_.$ as follows:

$$\hat{\eta}_t(i,1) = \eta_t(i,1)\,, \quad \hat{\eta}_t(i,-1) = \ell\eta_t(i,-1)\,.$$

The derivation of (2.3) from the process $\hat{\eta}_.$ is justified in Theorem 2.1: if the initial data of (2.3) are the limits of the average occupation numbers associated with the initial measure μ_0, then the average (mass) occupation numbers $\big(E_{\mu_0}\hat{\eta}_t(i,1), E_{\mu_0}\hat{\eta}_t(i,-1)\big)$ converge to the solution (ρ^+,ρ^-), as $\gamma \to 0$.

Returning to Remark 2.1(c), we note that the rates $c(\eta,\sigma,i)$ can be selected so that $G(\rho^+,\rho^-) = m\ell[(\rho^-/\ell)^m - (\rho^+/m)^\ell]$, where m and ℓ are positive integers; the analysis below applies to all such models.

After the space/time hyperbolic rescaling

$$(x,t) \mapsto (x/\varepsilon, t/\varepsilon)\,, \tag{2.4}$$

the system (2.3) - defined on $[0,1/\varepsilon]\times(0,T/\varepsilon)$ - becomes a relaxation system,

$$\begin{cases} \rho_t^+ + V\rho_x^+ = \frac{1}{\varepsilon}G(\rho^+,\rho^-) \\ \rho_t^- - V\rho_x^- = -\frac{1}{\varepsilon}G(\rho^+,\rho^-) \end{cases} , \quad (x,t)\in[0,1]\times(0,T)\,, \tag{2.5}$$

equipped with the conservation law

$$\frac{\partial}{\partial t}(\rho^+ + \rho^-) + V\frac{\partial}{\partial x}(\rho^+ - \rho^-) = 0\,, \tag{2.6}$$

and admitting the local equilibria

$$\mathcal{C} = \{(\rho^+,\rho^-)\in\mathbb{R}^2 : G(\rho^+,\rho^-) = \rho^- - g(\rho^+) = 0\}\,.$$

For G decreasing in ρ^+ and increasing in ρ^-, the system (2.5) is an L^1 contraction and, as $\varepsilon\to 0$, the "mass density" $u^\varepsilon = \rho^+ + \rho^-$ converges to the entropy solution $u = \rho^+ + g(\rho^+)$ of the conservation law

$$\partial_t(\rho^+ + g(\rho^+)) + V\partial_x(\rho^+ - g(\rho^+)) = 0\,,\ \ (x,t)\in[0,1]\times(0,T)\,, \tag{2.7}$$

with periodic boundary conditions. (See Ref. 2 for the initial value problem; the proof for periodic boundary conditions follows without serious modifications.)

Next, we ask if there are space/time scalings so that the IPS $\eta_\cdot$ yields in the limit entropy solutions of (2.7). It is proved in Theorem 2.2 that there is a positive number r^* and a continuum of hyperbolic scalings,

$$\mathbb{Z}_\gamma \times (0,T) \;\mapsto\; \mathbb{Z}_{\gamma\varepsilon} \times (0, T/\varepsilon)\,,$$

where $\gamma^{-1}\varepsilon^{-1}$ is a positive integer and $\varepsilon = \gamma^r$ for any $r < r^*$, such that, when the process $\eta_\cdot$ is observed in this space/time window, the averaged total particle number $E_{\mu_0}\eta_t(i,1) + \eta_t(i,-1)$ converges to the entropy solution of (2.7).

Before stating our main results we introduce some notation. For $n \in \mathbb{N}$, define the sets

$$\mathcal{M}^n_\gamma = \big\{\underline{\zeta} = (\zeta_1, \dots, \zeta_n) : \zeta_1 \neq \cdots \neq \zeta_n \text{ and } \zeta_k = (i_k, \sigma_k) \in \mathbb{Z}_\gamma \times \{-1,1\}\big\}$$

and

$$\mathcal{M}^n_{\gamma\varepsilon} = \big\{\underline{\zeta} = (\zeta_1, \dots, \zeta_n) : \zeta_1 \neq \cdots \neq \zeta_n \text{ and } \zeta_k = (i_k, \sigma_k) \in \mathbb{Z}_{\gamma\varepsilon} \times \{-1,1\}\big\}\,.$$

Also recall that

$$\hat\eta_t(i,1) = \eta_t(i,1)\,, \quad \hat\eta_t(i,-1) = \ell\eta_t(i,-1)\,.$$

A. Mesoscopic limit. We consider the process $\eta_\cdot$ defined on the periodic lattice $\mathbb{Z}_\gamma$. Let μ_0^γ be a slowly varying Poisson product measure on X such that

$$E_{\mu_0^\gamma}\hat\eta(i,1) = \rho_0^+(\gamma i) \;\text{ and }\; E_{\mu_0^\gamma}\hat\eta(i,-1) = \rho_0^-(\gamma i)\,, \;\; i \in \mathbb{Z}_\gamma\,, \tag{2.8}$$

where $\rho_0^\pm$ are periodic in $[0,1]$.

Theorem 2.1 *Let $\big(\rho^+(x,t), \rho^-(x,t)\big)$ be the solution of (2.3) emanating from periodic initial data $\rho_0^+, \rho_0^- \in BV \cap L^\infty([0,1])$. Then, for $T>0$ there exists $\hat r$ determined by (4.17) such that for any $\gamma > 0$*

$$\sup_{t\in[0,T]} \gamma^n \sum_{\underline{\zeta}\in\mathcal{M}^n_\gamma} \big|E_{\mu_0^\gamma} \prod_{k=1}^n \hat\eta_t(i_k,\sigma_k)a - \prod_{k=1}^n \rho^{\sigma_k}(\gamma i_k, t)\big| = O(\gamma^{\hat r})\,,$$

where $O(\cdot)$ depends on n and the L^∞ and BV norms of the initial data.

B. Macroscopic limits. Here we define the process $\eta_\cdot$ on the periodic lattice $\mathbb{Z}_{\gamma\varepsilon}$ and assume that:

(i) μ_0^γ is a slowly varying Poisson product measure on X such that

$$E_{\mu_0^\gamma}\hat\eta(i,1) = \rho_0^+(\gamma\epsilon i)\,, \quad E_{\mu_0^\gamma}\hat\eta(i,-1) = \rho_0^-(\gamma\epsilon i)\,, \;\; i \in \mathbb{Z}_{\gamma\varepsilon}\,.$$

(ii) $\rho_0^\pm \in BV \cap L^\infty([0,1])$.

(iii)$\|u_{0\varepsilon} - u_0\|_{L^1(\mathbb{R})} = o(1)$, where $u_{0\varepsilon} = \rho_0^+ + \rho_0^-$, and $\epsilon = \epsilon(\gamma) = \gamma^r$.

Theorem 2.2 *Let u be the solution of (2.7) corresponding to the relaxation limit of (2.5). There is a constant $r^* > 0$ determined by (4.14) such that for $r < r^*$, $\varepsilon = \gamma^r$, any fixed n and $T > 0$, we have*

$$\lim_{\gamma\to 0} \sup_{t\in[0,T/\epsilon]} (\gamma\varepsilon)^n \sum_{i_1\neq\cdots\neq i_n} |E_{\mu_0^\gamma} \prod_{k=1}^n [\hat\eta_t(i_k,1)+\hat\eta_t(i_k,-1)] - \prod_{k=1}^n u(\gamma\epsilon i_k,\epsilon t)| = 0\,.$$

In addition, if $\|G(\rho_0^+,\rho_0^-)\|_{L^1([0,1])} = O(\epsilon)$, we have the error estimate

$$\sup_{t\in[0,T/\epsilon]} (\gamma\varepsilon)^n \sum_{i_1\neq\cdots\neq i_n} |E_{\mu_0^\gamma} \prod_{k=1}^n [\hat\eta_t(i_k,1)+\hat\eta_t(i_k,-1)] - \prod_{k=1}^n u(\gamma\epsilon i_k,\epsilon t)| = O(\sqrt{\varepsilon})\,,$$

where $O(\cdot)$ depends on n and the L^∞ and BV norms of the initial data.

Remark 2.2 Similar statements to Theorems 2.1 and 2.2 can be proved for the general correlation functions of the process $\eta_.$: for any n points (not necessarily distinct), $\zeta_1,\ldots,\zeta_n$, where $\zeta_k = (i_k,\sigma_k) \in \mathbb{Z}_\nu \times \{-1,1\}$, $k = 1,\ldots,n$, we define the n-correlation function

$$f_t(\xi;\mu_0) := E_{\mu_0} D(\xi,\eta_t)\,,$$

where $\xi : \mathbb{Z}_\nu \times \{-1,1\} \mapsto \mathbb{N}$, $\xi(\zeta) = \sum_{k=1}^n \delta_{\zeta_k}(\zeta)$, ($\delta_{\zeta_k}(\zeta) = 1$ if $\zeta = \zeta_k$, and 0 otherwise) and $|\xi| = \sum_\zeta \xi(\zeta)$. Also,

$$D(\xi,\eta) = \prod_{\zeta\in\mathbb{Z}_\nu\times\{-1,1\}} D_{\xi(\zeta)}(\eta(\zeta))\,,$$

where

$$D_k(n) = \begin{cases} 1\,, & k=0\,,\\ 0\,, & k>n\,,\\ n(n-1)\cdots(n-k+1)\,, & k\le n\,. \end{cases}$$

If $\zeta_1 \neq \cdots \neq \zeta_n$, then $E_{\mu_0} D(\xi,\eta_t) = E_{\mu_0} \prod_{k=1}^n \eta_t(i_k,\sigma_k)$, as in the statements of Theorems 2.1 and 2.2. On the other hand, if $\zeta_1 = \cdots = \zeta_n$, then

$$E_{\mu_0} D(\xi,\eta_t) = E_{\mu_0} \eta(i_1,\sigma_1)[\eta(i_1,\sigma_1)-1]\cdots[\eta(i_1,\sigma_1)-n+1]\,.$$

Without significant changes –we only need to modify the definition of the v-functions in Section 3–we may show a more general version of Theorem 2.1 for the correlation functions:

$$\sup_{t\in[0,T]} \gamma^n \sum_{|\xi|=n} |E_{\mu_0^\gamma} D(\xi,\hat\eta_t) - \prod_{\zeta=(i,\sigma)} \rho^\sigma(\gamma i,t)^{\xi(\zeta)}| = O(\gamma^{\hat r})\,. \tag{2.9}$$

An analogous statement, corresponding to Theorem 2.2, also holds. As pointed out in the formal discussion earlier, Theorems 2.1 and 2.2 and more specifically their versions for general correlation functions indicate that the process $\eta_.$ remains in both space/time windows in approximate local equilibrium, *i.e.* as $\gamma \to 0$ its distribution is a product Poisson measure with slowly varying densities, given by the solution of (2.3) in the case of the mesoscopic limit and by the entropy solution of (2.7) in the macroscopic case.

We conclude with remarks on the technical aspects of the proofs. In view of the relationship between (2.3), (2.5) and (2.7), to prove Theorem 2.2 it suffices to demonstrate that, for a long time interval $[0, T\varepsilon^{-1}]$,

$$E_{\mu_0^\gamma} \hat{\eta}_t(i, \pm 1) \approx \rho^{\pm}(\gamma i, t)\,, \tag{2.10}$$

where (ρ^+, ρ^-) solves (2.3) and $\varepsilon = \gamma^{r^*}$, for some $r^* > 0$ to be determined. (For Theorem 2.1 the same asymptotic estimate must be established in the shorter interval $[0, T]$.) An important tool for handling such questions is the *correlation function* method in Refs. 20, 4 and 5. There are the following problems to resolve. While the IPS and the systems in Refs 4 and 5 conserve the total number of particles for $\gamma > 0$, it is conceivable that as $\gamma \to 0$ the number of particles at a given site tends to infinity. In fact, the correlation function method yields (2.10) for short times, but, as there is no available global a priori bound for the correlation functions (as it happens in birth-death IPS), it is conceivable that they blow up in finite time. This suggests to discretize in time, in order to obtain the required asymptotic approximations in $[0, T]$ or $[0, T\varepsilon^{-1}]$. A second problem, associated with the macroscopic limit of Theorem 2.2 (and not present in the mesoscopic limits of Refs. 4 and 5 or in Theorem 2.1), is that the time scale $t \mapsto t/\varepsilon$ is long and the errors in the approximation (2.10) can add up as $\varepsilon \to 0$. Both are overcome by controlling the error at each time step, using the L^1-contraction property and the L^∞ stability of (2.3) (see Lemmata 4.1-2).

Finally the IPS, being a jump process, is close only to a semidiscrete approximation of (2.3). The aforementioned strategy will work provided that semidiscrete schemes converge to the entropy solution of (2.7). The convergence, stability and error estimates for semidiscrete schemes are addressed in Section 5. The convergence rates, in Theorems 2.1 and 2.2, follow from a combination of error estimates for semidiscrete schemes and the short time analysis for the correlation functions, presented in Section 3.

2b. The multidimensional model. We present a generalization of the interacting particle model in several space dimensions, which at mesoscopic scales is expected to give rise to the relaxation system (1.1), while at macroscopic scales is expected to converge to the conservation law (1.2). The particle system is set on the discrete torus of the lattice $\mathbb{Z}^N$,

$$\mathbb{Z}^N_\nu = \mathbb{Z}^N \mod (M)\,,$$

where $\nu = M^{-1}$ is the inverse of the positive integer M. At each lattice site $q \in \mathbb{Z}^N_\nu$ there are $N+1$ species of particles each moving with their velocity $\sigma \in U = \{e_0, e_1, ..., e_N\}$. For velocities we take $e_1, ..., e_N$ to be the usual basis of $\mathbb{R}^N$ and $e_0 = -\sum_{i=1}^N \omega_i e_i$, where ω_i are positive constants such that $q + e_0 \in \mathbb{Z}^N_\nu$ for all $q \in \mathbb{Z}^N_\nu$. We denote by $\eta(q, \sigma)$ the number of particles at the site $q \in \mathbb{Z}_\nu^N$ with velocity $\sigma \in U$ and define the configuration space $X = \mathbb{N}^{\mathbb{Z}_\nu^N \times U}$. The IPS is set on the periodic lattice $\mathbb{Z}_\nu^N$, the updating is done through mechanisms of "streaming" and "interaction" of particles, and it gives rise to a jump Markov process,

$$\eta_\cdot = \left\{ \eta_t(q,\sigma) \in \mathbb{N} : t \geq 0, (q,\sigma) \in \mathbb{Z}_\nu^N \times U \right\},$$

on the configuration space X.

I. Let $V_0, V_1, \ldots, V_N > 0$, $\gamma > 0$. In the streaming mechanism, a particle (q, e_i), located at the site $q \in \mathbb{Z}_\nu^N$ with velocity $\sigma = e_i\,, i = 0, ..., N$, jumps to $(q + e_i, e_i)$ keeping the same velocity, at a rate $V_i \gamma^{-1}$ with $\gamma << 1$.

II. The particles moving along perpendicular directions or in the same direction do not collide with each other. The particles at (q, e_0) collide with particles at $(q, e_i)\,, i = 1, ..., N$ and exchange velocities: e_0 is converted to e_i with rate $c(\eta, e_0, e_i, q)$ and e_i is converted to e_0 with rate $c(\eta, e_i, e_0, q)$, for all $i = 1, ..., N$ and $q \in \mathbb{Z}_\nu^N$. The rates satisfy the conditions:

(i) $c(\eta, \sigma, \sigma', q)$ is a polynomial in $\eta(q, \sigma)$,

(ii) $c(\eta, \sigma, \sigma', q) \geq 0$ for all $\eta \in X$ with equality holding if $\eta(q, \sigma) = 0$, and

(iii) if μ is a product Poisson measure on X such that $E_\mu \eta(q, e_0) = w(q) \geq 0$, $E_\mu \eta(q, e_i) = z_i(q) \geq 0\,, q \in \mathbb{Z}_\nu^N$, then

$$E_\mu c(\eta, e_i, e_0, q) = z_i(q)\,, \quad E_\mu c(\eta, e_0, e_i, q) = g_i(w(q))\,,$$

for given increasing polynomials $g_i\,, i = 1, ..., N$, with $g_i(0) = 0$.

The resulting process is described by its generator $L^\gamma = \gamma^{-1} L_s + L_c\,,$ where

$$L_s f(\eta) = \sum_{i=0}^N \sum_{q \in \mathbb{Z}_\nu^N} V_i \eta(q, e_i)[f(\eta + \delta_{(q+e_i, e_i)} - \delta_{(q, e_i)}) - f(\eta)],$$

$$L_c f(\eta) = \sum_{i=1}^N \sum_{q \in \mathbb{Z}_\nu^N} \Big\{ c(\eta, e_i, e_0, q)[f(\eta + \delta_{(q,e_0)} - \delta_{(q,e_i)}) - f(\eta)] + c(\eta, e_0, e_i, q)[f(\eta + \delta_{(q,e_i)} - \delta_{(q,e_0)}) - f(\eta)] \Big\},$$

$f : X \mapsto \mathbb{R}$ cylindrical, and $\delta_{(q,\sigma)}(q', \sigma') = 1$ if $(q', \sigma') = (q, \sigma)$, and equal to 0 otherwise.

Following (2.2) we formally see that the average particle numbers

$$E_{\mu_0} \eta_t(q, e_0) = w^\gamma(\gamma q, t)\,, \quad E_{\mu_0} \eta_t(q, e_i) = z_i^\gamma(\gamma q, t)\,, \quad i = 1, ..., N\,,$$

converge to a solution $(w, z_1, ..., z_N)$ of (1.1), where $\varepsilon = 1$, $G_i(w, z_i) = z_i - g_i(w)$, $U_i = V_i e_i\,, i = 1, ..., N$, and $U_0 = -V_0(\omega_1, \ldots, \omega_N) = V_0 e_0$. Note that $|U_i| = V_i$, $i = 1, ..., N$, and $|U_0| = V_0 |e_0|$. At macroscopic scales we expect to obtain the conservation law (1.2), as in Theorem 2.2. In fact, the analysis in Section 5 holds for more than one space dimensions, so we could conclude the proof with the techniques of this paper, provided one shows that the Lemmata in Section 3 hold for the multidimensional model for certain classes of rates.

Observe that the particle system introduced in Section 2a fits into the above form, and one could easily state special multi-dimensional models that satisfy detailed balance of mass.

3. Preliminary Lemmata

In this section we present some background material necessary for the sequel. The proofs are straightforward modifications of results in Refs 4 and 5 for the Carleman and Broadwell systems. We also refer to the monograph by De Masi and Presutti[1].

We first comment on the convergence of the particle system of Section 2a to the semilinear hyperbolic system (2.3). The proof for short times follows along the aforementioned results for the Carleman and Broadwell systems; the basic technique goes back to Lanford[20], who derived the Boltzmann equation from the hard sphere dynamics for short times (see also Spohn[28]). We briefly sketch the steps of the proof and we refer to Ref. 1 for more details and references to related works.

We consider the n-correlation functions defined in Remark 2.2, $f_t(\xi;\mu_0) := E_{\mu_0} D(\xi,\eta_t)$. Instead of studying directly the hierarchy of equations for $f_t(\xi;\mu_0)$, we define a suitable dual process $\xi_\cdot$ with generator $\gamma^{-1}L_s^* + L_1$, where L_s^* is the formal adjoint of L_s, and the operator L_1 will be defined below. Then we consider the direct product of the $\xi_\cdot$ and $\eta_\cdot$ processes and denote by $\hat{E}$ the corresponding expectation. We have,

$$\frac{d}{ds}\hat{E}D(\xi_{t-s},\eta_s) = \hat{E}L_c D(\xi_{t-s},\eta_s) - \hat{E}L_1 D(\xi_{t-s},\eta_s)\,.$$

The operator L_1 is selected so that in the previous relation the quadratic and higher order terms cancel, and thus the right hand side depends only on combinations of terms of the type $\hat{E}\xi_{t-s}(i,\sigma)D(\bar{\xi}_{t-s},\eta_s)$, where $\bar{\xi}_{t-s}$ denotes perturbations of ξ_{t-s}. As a consequence of this fact, we have that $\hat{E}D(\xi_0,\eta_t) = f_t(\xi_0;\mu_0)$ is bounded for small enough t's. The convergence for short times follows by direct comparison of $f_t(\xi;\mu_0)$ and $\prod_{\zeta=(i,\sigma)} \rho^\sigma(\gamma i,t)^{\xi(\zeta)}$ (see Remark 2.2), and use of the Central Limit Theorem for the independent process corresponding to L_s. Again, we refer to Ref. 1 for the details.

We introduce the upwind discretization of the relaxation system (2.3):

$$\begin{aligned}
&\frac{\partial}{\partial t}\rho(j,\ 1,t) + \frac{V}{\gamma}\big(\rho(j,\ 1,t) - \rho(j-1,\ 1,t)\big) = G\big(\rho(j,\ 1,t),\rho(j,-1,t)\big)\\
&\frac{\partial}{\partial t}\rho(j,-1,t) - \frac{V}{\gamma}\big(\rho(j+1,-1,t) - \rho(j,-1,t)\big) = -G\big(\rho(j,\ 1,t),\rho(j,-1,t)\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad (j,t)\in\mathbb{Z}_\gamma\times(0,\infty)\,,
\end{aligned} \tag{3.1}$$

with periodic boundary conditions at $j=0$, $j=\gamma^{-1}$. Note the relation between (3.1) and the equation (2.2) for the average occupation numbers. We denote by $\rho(j,\pm 1,t\mid f)$ the solution of (3.1) when emphasizing the dependence on the initial data $\rho(j,\pm1,0) = f^\pm(j)$, $j\in\mathbb{Z}_\gamma$. Solutions of (3.1) are defined for all $t>0$ and are periodic with period γ^{-1}, see Sec. 5. In a similar way, we define (3.1) on the periodic lattice $\mathbb{Z}_{\gamma\varepsilon}$.

Also, given n distinct points $\zeta_k = (i_k,\sigma_k)\in\mathbb{Z}_\gamma\times\{-1,1\}$, we define $\underline{\zeta} = (\zeta_1,\ldots,\zeta_n)$ and the v^n*-functions,*

$$v_t^n(\underline{\zeta} \mid \eta) = E_\eta\Big[\prod_{k=1}^{n}\Big(\hat{\eta}_t(i_k,\sigma_k) - \rho(i_k,\sigma_k,t \mid \hat{\eta})\Big)\Big]\,, \tag{3.2}$$

where E_η denotes the expectation conditioned on $\eta_0 = \eta$. A more general definition of v-functions is needed if we want to prove (2.9) (see Refs 4 and 5), however the technical results below remain essentially unaltered.

Lemma 3.1. *For each* $\gamma > 0$, *let* $\eta \in X$ *be such that*

$$\eta(i,\sigma) < \gamma^{-\xi}\,, \quad \textit{for all} \quad (i,\sigma) \in \mathbb{Z}_\gamma \times \{-1,1\}\,.$$

Then, for $\xi < \beta < \frac{1}{4}$ *and all* $n \in \mathbb{N}$, *there is a positive constant* c_n *depending only on* n *such that for all* $\underline{\zeta}$ *and* $0 < t \le \gamma^\beta$,

$$|v_t^n(\underline{\zeta} \mid \eta)| < c_n \gamma^{-\xi n}\Big(\frac{\gamma}{t}\Big)^{n/4}\,.$$

Proof. For the proof we refer to Ref. 1 (Ch. 5), where the same property is proved for the Carleman system. See also Refs. 4 and 5, and references in De Masi-Presutti[1], for similar estimates on correlation functions for stochastic cellular automata, the Broadwell model and Glauber-Kawasaki dynamics. □

Now we define the set of "good" realizations of the process $\eta_\cdot$, i.e. the ones that are close to the discretized equations (3.1) in a suitable metric defined below. First, let $P_t^\gamma((i,\sigma),(j,\sigma))$ or $P_t^\gamma(i,j;\sigma)$ for simplicity, be the Green's function associated with the discrete part of (3.1), i.e., the probability a particle at the site $i \in \mathbb{Z}_\gamma$ with velocity σ lands after time t to the site $j \in \mathbb{Z}_\gamma$ keeping the same velocity. We define the seminorm

$$\|f\|_\gamma = \sup_i \Big|\sum_{(j,\sigma')} P_{\gamma^{1/4}}(i,j;\sigma') f(\eta(j,\sigma'))\Big| \tag{3.3}$$

where $f \in L^\infty(X)$.

The choice $t = \gamma^{1/4}$ in the definition of the seminorm is not the only one possible. Notice that by such a choice, the main contribution of $P_{\gamma^{1/4}}(i,j;\sigma)$ is on the sites j in an interval centered at $i - \sigma\gamma^{-1}\gamma^{1/4}$ with length approximately $\gamma^{-3/8} = \big(\gamma^{-1}\gamma^{1/4}\big)^{1/2}$. Thus the seminorm is an average over a mesoscopic interval of size $\gamma^{-3/8}$ (since $3/8 < 1$), and is sufficient to suppress the local fluctuations of η_t in a rough analogy to a law of large numbers.

Let $\{t_k\}_{k=1}^m$ be a partition of $[0,T]$ with mesh $t_{k+1} - t_k = h$ and define $H_{m,h}$ as the set of all configurations $\eta_\cdot$ satisfying the following properties:

$$\begin{cases} \text{(i)} & \hat{\eta}^{(k)} < \gamma^{-\xi} \\[2ex] \text{(ii)} & \|\hat{\eta}^{(0)}(\cdot) - \rho_0(\cdot)\|_\gamma < \gamma^\theta \\[2ex] \text{(iii)} & \|\hat{\eta}^{(k)}(\cdot) - \rho(\cdot,h \mid \hat{\eta}^{(k-1)})\|_\gamma \le \gamma^\theta \quad k = 1,\dots,m, \end{cases} \tag{3.4}$$

where $\hat{\eta}_{t_k} = \hat{\eta}^{(k)}$ and the parameters θ, h, ξ will be determined later. Then, Chebyshev's inequality, Lemma 3.1 and the decay estimate $P_t^\gamma(i,j;\sigma) = O(\varepsilon^{1/2}t^{-1/2})$ on the Green's function give the following (see Ref. 1 for a similar proof):

Lemma 3.2. *Let the initial measure of the process be a product Poisson measure μ_0^γ satisfying (2.8) and let*

$$\xi < \beta < \theta < \frac{1}{4}, \tag{3.5}$$

For any $\kappa > 0$, there exists a positive constant c_κ depending on κ, ξ and θ such that for any $\gamma > 0$ and mesh $t_{k+1} - t_k = h = \gamma^\beta$ we have

$$P_{\mu_0^\gamma}(H_{m,h}) > 1 - c_\kappa \gamma^\kappa .$$

4. Proofs of Theorems 2.1 and 2.2

We first present the proof of Theorem 2.2; the proof of Theorem 2.1 follows along the same lines but it is simpler, since it is not necessary to work on long time intervals of size ε^{-1}.

Consider the process $\hat{\eta}_\cdot$ and the semi-discrete scheme (3.1) on the periodic lattice $\mathbb{Z}_{\gamma\varepsilon}$ (note that Lemmas 3.1-2 remain true if $\mathbb{Z}_\gamma$ is replaced by $\mathbb{Z}_{\gamma\varepsilon}$. For simplicity of notation we write from now on η instead of $\hat{\eta}$ and we will return to the original notations at the end of the section. Recall that $\hat{\eta}$ is just a rescaling of η (see Sec. 2). In the proof of Theorem 2.1 (respectively Theorem 2.2), we denote by $\rho(i,\sigma,t)$ be the solution of (3.1) defined on $\mathbb{Z}_\gamma$ (resp. $\mathbb{Z}_{\gamma\varepsilon}$) and emanating from data $\rho_0^\pm(\gamma i)$ (resp. $\rho_0^\pm(\varepsilon\gamma i)$). We use the notation $\rho(i,\sigma,t|f)$ for solutions of (3.1) emanating from data $f^\pm$. Finally, $\rho^\sigma(x,t)$ with $\sigma = \pm 1$ will stand for solutions of the relaxation system (2.3) (or (2.5)) defined on $[0,1]$.

Lemma 4.1. *Let $H_{m,h}$ be the set of realizations defined in Section 3 with corresponding partition mesh,*

$$t_{k+1} - t_k = h = \gamma^\beta, \quad 1 \le k \le m, \tag{4.1}$$

and $t_0 = 0$ and $t_m = \tau$. If the parameter ξ satisfies, in addition to (3.5),

$$\xi \le \frac{1}{4\ell}, \tag{4.2}$$

where ℓ is the positive integer arising in the definition of the generator L_c, then for any $\tau \in [0, T\varepsilon^{-1}]$,

$$\begin{aligned}\gamma\varepsilon \sum_{i\in\mathbb{Z}_{\gamma\varepsilon}} |\rho(i,\sigma,h \mid \eta^{(k)}) - \rho(i,\sigma,t_{k+1})| \le{}& \gamma\varepsilon \sum_{i\in\mathbb{Z}_{\gamma\varepsilon}} |\rho(i,\sigma,h \mid \eta^{(k-1)}) - \rho(i,\sigma,t_k)| \\ &+ O(\gamma^\theta + \gamma^{1/4-\xi\ell}).\end{aligned}$$

Proof. First observe that, for any $(i,\sigma)\in\mathbb{Z}_{\gamma\varepsilon}\times\{-1,1\}$,

$$\rho(i,\sigma,h\mid\eta^{(k)})-\rho(i,\sigma,t_{k+1})=\rho(i,\sigma,h-\gamma^{1/4}\mid\rho(\cdot,\gamma^{1/4}\mid\eta^{(k)}))-\rho(i,\sigma,t_{k+1})\,. \tag{4.3}$$

Using the variation of parameters formula, we obtain

$$\rho(i,\sigma,\gamma^{1/4}\mid\eta^{(k)})=T^{\gamma}_{\gamma^{1/4}}\eta^{(k)}+\int_0^{\gamma^{1/4}}T^{\gamma}_{\gamma^{1/4}-s}G\Big(\rho(\cdot,1,s\mid\eta^{(k)}),\rho(\cdot,-1,s\mid\eta^{(k)})\Big)\,ds\,,$$

where T^{γ}_t the semigroup associated with the kernel $P^{\gamma}_t(i,j;\sigma)$.

Recall that $G(w,z)=z-g(w)=z-\ell w^{\ell}$; since $\eta_{.}\in H_{m,h}$, and due to (4.2) and Theorem 5.1 (ii), we obtain that

$$\int_0^{\gamma^{1/4}}T^{\gamma}_{\gamma^{1/4}-s}G\Big(\rho(\cdot,1,s\mid\eta^{(k)}),\rho(\cdot,-1,s\mid\eta^{(k)})\Big)\,ds=O(\gamma^{1/4-\xi\ell})\,, \tag{4.4}$$

where $O(\cdot)$ depends only on ℓ. Therefore by (3.4)(iii),

$$\begin{aligned}\rho(i,\sigma,\gamma^{1/4}\mid\eta^{(k)})&=T^{\gamma}_{\gamma^{1/4}}\rho(\cdot,\sigma,h\mid\eta^{(k-1)})+O(\gamma^{\theta}+\gamma^{1/4-\xi\ell})\\&=\rho(i,\sigma,\gamma^{1/4}\mid\rho(\cdot,\sigma,h\mid\eta^{(k-1)}))+O(\gamma^{\theta}+\gamma^{1/4-\xi\ell})\,.\end{aligned}\tag{4.5}$$

By Theorem 5.1(i) we have, after summing (4.3) over all $i\in\mathbb{Z}_{\gamma\varepsilon}$, multiplying with the lattice size $\gamma\varepsilon$, and using (4.5),

$$\begin{aligned}\gamma\varepsilon\sum_{i\in\mathbb{Z}_{\gamma\varepsilon}}|\rho(i,\sigma,h\mid\eta^{(k)})-\rho(i,\sigma,t_{k+1})|&\le\gamma\varepsilon\sum_{i\in\mathbb{Z}_{\gamma\varepsilon}}|\rho(i,\sigma,\gamma^{1/4}\mid\eta^{(k)})-\rho(i,\sigma,t_k+\gamma^{1/4})|\\&=\gamma\varepsilon\sum_{i\in\mathbb{Z}_{\gamma\varepsilon}}|\rho(i,\sigma,\gamma^{1/4}\mid\rho(\cdot,\sigma,h\mid\eta^{(k-1)}))-\rho(i,\sigma,t_k+\gamma^{1/4})|+O(\gamma^{\theta}+\gamma^{1/4-\xi\ell})\\&\le\gamma\varepsilon\sum_{i\in\mathbb{Z}_{\gamma\varepsilon}}|\rho(i,\sigma,h\mid\eta^{(k-1)})-\rho(i,\sigma,t_k)|+O(\gamma^{\theta}+\gamma^{1/4-\xi\ell})\,.\end{aligned}$$

Lemma 4.2. *Under the assumptions of Lemmata 3.2 and 4.1 we have that for $\tau=mh\in[0,T\varepsilon^{-1}]$, and $\eta_{.}\in H_{m,h}$:*

(i)
$$\gamma\varepsilon\sum_{i\in\mathbb{Z}_{\gamma\varepsilon}}|\rho(i,\sigma,h\mid\eta^{(m-1)})-\rho(i,\sigma,\tau)|\le T\varepsilon^{-1}h^{-1}O(\gamma^{\theta}+\gamma^{1/4-\xi\ell})\,.$$

(ii) *There is a positive constant c such that*

$$\begin{aligned}0\le\rho(i,1,h\mid\eta^{(m-1)})&\le c+L\\0\le\rho(i,-1,h\mid\eta^{(m-1)})&\le g(c+L)\end{aligned}$$

where $g(w)=\ell w^{\ell}$ and $L=T\varepsilon^{-1}h^{-1}O(\gamma^{\theta}+\gamma^{1/4-\xi\ell})$.

Proof. (i) Let $\tau > 0$ be fixed, $\tau \in [0, T\epsilon^{-1}]$. We divide $[0,\tau]$ in m subintervals as in the statement of Lemma 4.1. Note also that working as in Lemma 4.1, (3.4) implies that

$$\rho(i,\sigma,\gamma^{1/4} \mid \eta_0) = T^{\gamma}_{\gamma^{1/4}}\rho_0 + O(\gamma^{\theta} + \gamma^{1/4-\xi\ell}) = \rho(i,\sigma,\gamma^{1/4}) + O(\gamma^{\theta} + \gamma^{1/4-\xi\ell})\,; \tag{4.6}$$

therefore, by Theorem 5.1 (i),

$$\gamma\varepsilon \sum_{i\in\mathbb{Z}_{\gamma\varepsilon}} |\rho(i,\sigma,h \mid \eta_0) - \rho(i,\sigma,h)| = O(\gamma^{\theta} + \gamma^{1/4-\xi\ell})\,.$$

By iteration and using Lemma 4.1, we obtain that

$$\gamma\varepsilon \sum_{i\in\mathbb{Z}_{\gamma\varepsilon}} |\rho(i,\sigma,h \mid \eta^{(m-1)}) - \rho(i,\sigma,t_m)| \le mO(\gamma^{\theta} + \gamma^{1/4-\xi\ell})\,.$$

Since $t_m = hm = \tau \le T\varepsilon^{-1}$, we conclude

$$\gamma\varepsilon \sum_{i\in\mathbb{Z}_{\gamma\varepsilon}} |\rho(i,\sigma,h \mid \eta^{(m-1)}) - \rho(i,\sigma,\tau)| \le T\epsilon^{-1}h^{-1}O(\gamma^{\theta} + \gamma^{1/4-\xi\ell})\,.$$

(ii) The left hand inequalities follow from the positivity of $\eta.$ and (5.7) in Theorem 5.1. Select c so that $g'(w) \ge 1$ for $w \ge c$, and $\rho_0^+ \le c$, $\rho_0^- \le g(c)$ are fulfilled. Then

$$g(w) + a \le g(w+a) \quad \text{for } w \ge c \text{ and } a > 0.$$

We set $l_0 = O(\gamma^{\theta} + \gamma^{1/4-\xi\ell})$, as in (4.5) and (4.6), and proceed to prove recursively:

$$\begin{aligned} \rho(i,1,h \mid \eta^{(m-1)}) &\le c + ml_0 \\ \rho(i,-1,h \mid \eta^{(m-1)}) &\le g(c + ml_0) \end{aligned}$$

Equation (4.6) and Theorem 5.1 (ii) imply the result for $m = 1$. Similarly, (4.5) and Theorem 5.1 (ii) imply the result for any m. $\square$

We now turn towards the proof of Theorem 2.2:

Proof of Theorem 2.2. Let $\tau > 0$ be fixed, $\tau \in [0, T\epsilon^{-1}]$. We divide $[0,\tau]$ in m subintervals as in the statement of Lemmata 4.1 and 4.2. We set

$$\epsilon = \epsilon(\gamma) = \gamma^{r} \quad \text{and} \quad h = h(\gamma) = \gamma^{\beta}\,, \tag{4.7}$$

where r, β will be chosen in the course of the proof.

As in the definition of the v^n-functions in (3.2), we pick n distinct points $i_k \in \mathbb{Z}_{\gamma\varepsilon}, k = 1, ..., n$. Then,

$$\begin{aligned} &E_{\mu_0^{\gamma}}\Big[\prod_{k=1}^{n}[\eta_\tau(i_k,1) + \eta_\tau(i_k,-1)] - \prod_{k=1}^{n}[\rho(i_k,1,\tau) + \rho(i_k,-1,\tau)]\Big] \\ &\quad = (I) + (II) + (III) + (IV)\,, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
(I) =& E_{\mu_0^\gamma}(1-\chi_{H_{m,h}}) \prod_{k=1}^{n} (\eta_\tau(i_k,1)+\eta_\tau(i_k,-1)) \\
(II) =& E_{\mu_0^\gamma}\chi_{H_{m,h}} \Big[\prod_{k=1}^{n} \Big(\eta_\tau(i_k,1)+\eta_\tau(i_k,-1)\Big) \\
&- \prod_{k=1}^{n} \Big(\rho(i_k,1,h \mid \eta^{(m-1)})+\rho(i_k,-1,h \mid \eta^{(m-1)})\Big)\Big] \\
(III) =& E_{\mu_0^\gamma}\chi_{H_{m,h}} \Big[\prod_{k=1}^{n} \Big(\rho(i_k,1,h \mid \eta^{(m-1)})+\rho(i_k,-1,h \mid \eta^{(m-1)})\Big) \\
&- \prod_{k=1}^{n} \Big(\rho(i_k,1,\tau)+\rho(i_k,-1,\tau)\Big)\Big] \\
(IV) =& -E_{\mu_0^\gamma}(1-\chi_{H_{m,h}})\Big[\prod_{k=1}^{n} (\rho(i_k,1,\tau)+\rho(i_k,-1,\tau))\Big] .
\end{aligned}$$

Since $\rho(\cdot,\pm 1,\tau)$ is bounded in L^∞ (cf. Theorem 5.1(ii)), Lemma 3.2 yields that

$$\text{(IV)} = O(\gamma^\kappa) , \tag{4.9}$$

where κ can be chosen arbitrarily large and $O(\cdot)$ depends on the L^∞ bounds of $\rho(\cdot,\pm 1,\tau)$ and the constant c_κ in Lemma 3.2.

Furthermore, we can rewrite the term (II),

$$\begin{aligned}
(II) = E_{\mu_0^\gamma}\chi_{H_{m,h}} E_{\eta^{(m-1)}} \Big[& \prod_{k=1}^{n} (\eta_\tau(i_k,1)+\eta_\tau(i_k,-1)) - \\
& \prod_{k=1}^{n} (\rho(i_k,1,h \mid \eta^{(m-1)})+\rho(i_k,-1,h \mid \eta^{(m-1)}))\Big] .
\end{aligned}$$

We now substitute $\eta_\tau(i_k,\pm 1) = w_\tau(i_k,\pm 1)+\rho(i_k,\pm 1,h \mid \eta^{(m-1)})$; recall that for any $l = 1,...,2n$, the correlation functions are defined as $v_h^l(\underline{\zeta} \mid \eta) = E_\eta \prod_{k=1}^{l} w_\tau(i_k,\sigma_k)$, where $\underline{\zeta} = \big((i_1,\sigma_1),...,(i_l,\sigma_l)\big)$. Using Lemma 3.1 we get, after some rearrangements, that the term (II) is dominated by a linear combination of v^l-functions, $l = 1,...,2n$ and finally obtain:

$$\text{(II)} = O(\gamma^{1/4-\xi} h^{-1/4}) , \tag{4.10}$$

where $O(\cdot)$ depends on the constants c_l, $l = 1,...,2n$ in Lemma 3.1 and the L^∞ bounds of $\rho(\cdot,\pm 1,\tau)$ and $\rho(\cdot,\pm 1,h \mid \eta^{(m-1)})$; notice that the latter are also bounded by Lemma 4.2(ii) and the choice of ε and h in (4.13), guaranteeing that $T\epsilon^{-1}h^{-1}O(\gamma^\theta+\gamma^{1/4-\xi\ell})$ vanishes as $\gamma \to 0$.

We now turn to the term (III). For suitable positive constants $l_k, k = 1, ..., n$ depending on L^∞ bounds of $\rho(\cdot, \pm 1, \tau)$ and $\rho(\cdot, \pm 1, h \mid \eta^{(m-1)})$, we have,

$$\text{(III)} \le E_{\mu_0^\gamma} \chi_{H_{m,h}} \sum_{k=1}^{n} l_k [\rho(i_k, \sigma_k, h \mid \eta^{(m-1)}) - \rho(i_k, \sigma_k, \tau)] \,,$$

therefore by Lemma 4.2,

$$\begin{aligned} (\gamma\varepsilon)^n \sum_{i_1 \ne \cdots \ne i_n} |\text{(III)}| \le & C\gamma\varepsilon \sum_{i \in \mathbb{Z}_{\gamma\varepsilon}} |\rho(i, \sigma_i, h \mid \eta^{(m-1)}) - \rho(i, \sigma_i, \tau)| \\ = & T\varepsilon^{-1} h^{-1} O(\gamma^\theta + \gamma^{1/4 - \xi\ell}) \,, \end{aligned} \tag{4.11}$$

where $C = n \max_k l_k$.

We conclude with the term (I). By Lemma 3.2 and the Cauchy-Schwarz inequality we have that for any $\kappa \in \mathbb{N}$,

$$\begin{aligned} |E_{\mu_0^\gamma} (1 - \chi_{H_{m,h}}) \prod_{k=1}^{n} (\eta_\tau(i_k, 1) + \eta_\tau(i_k, -1))|^2 & \le c_\kappa \gamma^\kappa E_{\mu_0^\gamma} \big(\prod_{k=1}^{n} \eta_\tau(i_k, 1) + \eta_\tau(i_k, -1) \big)^2 \\ & \le c_\kappa \gamma^\kappa E_{\mu_0^\gamma} \big(\sum_{k=1}^{n} \eta_\tau(i_k, 1) + \eta_\tau(i_k, -1) \big)^{2n} \\ & \le c_\kappa \gamma^\kappa E_{\mu_0^\gamma} \big(\sum_{i \in \mathbb{Z}_{\gamma\varepsilon}} \eta_\tau(i, 1) + \eta_\tau(i, -1) \big)^{2n} . \end{aligned}$$

Since the total number of particles in the system is conserved, then the last term in the previous relation is bounded by

$$c_\kappa \gamma^\kappa E_{\mu_0^\gamma} \big(\sum_{i \in \mathbb{Z}_{\gamma\varepsilon}} \eta_0(i, 1) + \eta_0(i, -1) \big)^{2n} .$$

Now notice that $\sum_{i \in \mathbb{Z}_{\gamma\varepsilon}} \eta_0(i, 1) + \eta_0(i, -1)$ is Poisson with mean $\sum_{i \in \mathbb{Z}_{\gamma\varepsilon}} \rho_0^+(i) + \rho_0^-(i)$, which in turn implies that its $2n$-th moment is bounded by a linear combination of the $2l$ powers of the mean, $l = 1, ..., n$. In view of the boundedness of the initial data of (3.1) in L^1, we have

$$E_{\mu_0^\gamma} \big(\sum_{i \in \mathbb{Z}_{\gamma\varepsilon}} \eta_0(i, 1) + \eta_0(i, -1) \big)^{2n} \le C\gamma^{-2n} \varepsilon^{-2n} \,,$$

for a constant C depending on the L^1 norms of $\rho_0^\pm$. Thus,

$$\text{(I)} = O(\gamma^{\kappa/2 - n} \varepsilon^{-n}) \,, \tag{4.12}$$

where $O(\cdot)$ depends on c_κ and the L^1 norm of the initial data. Note that the right hand side vanishes for κ large enough and ε polynomial in γ.

Henceforth, we return to the notation of Section 2 – recall that throughout Section 4 we used η instead of $\hat{\eta}$. Putting together (4.9)–(4.12), we obtain

$$\begin{aligned}(4.13)\qquad &(\gamma\varepsilon)^n \sum_{i_1\neq\cdots\neq i_n} \Big| E_{\mu_0^\gamma} \prod_{k=1}^n \big(\hat{\eta}_\tau(i_k,1)+\hat{\eta}_\tau(i_k,-1)\big) - \prod_{k=1}^n \big(\rho(i_k,1,\tau)+\rho(i_k,-1,\tau)\big)\Big| \\ &\qquad = O\big(\gamma^{\kappa/2-n}\varepsilon^{-n} + \gamma^{1/4-\xi}h^{-1/4} + (\gamma^\theta + \gamma^{1/4-\xi\ell})\varepsilon^{-1}h^{-1} + \gamma^\kappa\big)\,,\end{aligned}$$

where κ can be chosen arbitrarily large. Set $\varepsilon=\gamma^r$, $h=\gamma^\beta$. We now pick r, β, ξ such that (4.1), (4.2) and the conditions of Lemma 3.2 are satisfied, and the exponents in (4.13) are positive, i.e.

$$(4.14)\qquad \begin{aligned} &\xi < \beta < \theta < \frac{1}{4} \\ &\xi < \frac{1}{4\ell} \\ &\beta < 1-4\xi \\ &r+\beta < \min\big(\theta\,, \frac{1}{4}-\xi\ell\big)\,. \end{aligned}$$

The pair $\big(\rho(i,1,t\varepsilon^{-1}),\rho(i,-1,t\varepsilon^{-1})\big)$, $i\in\mathbb{Z}_{\gamma\varepsilon}$, $t\in[0,T]$, solves (5.1_a) with γ replaced by $\gamma\varepsilon$. By Theorem 5.7, we then have

$$(4.15)\qquad \sup_{[0,T\epsilon^{-1}]} \gamma\varepsilon \sum_{i\in\mathbb{Z}_{\gamma\epsilon}} |(\rho(i,1,\tau)+\rho(i,-1,\tau)) - u(\gamma\epsilon\, i,\tau\epsilon)| = O(\sqrt{\varepsilon+\gamma\varepsilon})\,.$$

We conclude by combining (4.13) and (4.15). □

Finally, we turn towards the proof of Theorem 2.1:

Proof of Theorem 2.1. The proof proceeds along the lines of Theorem 2.2, when we set $\varepsilon=1$: we first obtain (4.13), with $\varepsilon=1$,

$$\begin{aligned}(4.16)\qquad &\gamma^n \sum_{\underline{\zeta}\in\mathcal{M}_\gamma^n} |E_{\mu_0^\gamma} \prod_{k=1}^n \hat{\eta}_t(i_k,\sigma_k) - \prod_{k=1}^n \rho(i_k,\sigma_k,t)| = \\ &\qquad = O\big(\gamma^{\kappa/2-n} + \gamma^{1/4-\xi}h^{-1/4} + (\gamma^\theta+\gamma^{1/4-\xi\ell})h^{-1} + \gamma^\kappa\big)\,.\end{aligned}$$

Again we set $h=\gamma^\beta$ and pick β, ξ such that (4.1), (4.2) and the conditions of Lemma 3.2 are satisfied and the exponents in (4.16) are positive:

$$(4.17)\qquad \begin{aligned} &\xi < \beta < \theta < \frac{1}{4} \\ &\xi < \frac{1}{4\ell} \\ &\beta < \min\big(1-4\xi\,,\theta\,, \frac{1}{4}-\xi\ell\big)\,. \end{aligned}$$

By Theorem 5.6 (a), we have

$$(4.18)\qquad \sup_{[0,T]} \gamma \sum_{i\in\mathbb{Z}_\gamma} |(\rho(i,\pm1,\tau) - \rho^\pm(\gamma\, i,\tau)| = O(\sqrt{\gamma})\,,$$

where (ρ^+,ρ^-) solve (2.3). We conclude by combining (4.16) and (4.18). □

5. Semi-discrete relaxation schemes

In this section, we consider an upwind space-discretization of the relaxation system (2.5). To simplify the forrest of indices and superscripts, we replace (ρ^+, ρ^-) with (w, z). The discretization then reads:

$$
\begin{aligned}
\frac{\partial}{\partial t} w_j + \frac{V}{\gamma}[w_j - w_{j-1}] &= \frac{1}{\epsilon} G(w_j, z_j) \\
\frac{\partial}{\partial t} z_j - \frac{V}{\gamma}[z_{j+1} - z_j] &= -\frac{1}{\epsilon} G(w_j, z_j) \,,
\end{aligned}
\tag{5.1$_a$}
$$

with initial data

$$w_j(0) = w_{j0} \,, \quad z_j(0) = z_{j0} \,,$$

where $(j, t) \in \mathbb{Z} \times (0, \infty)$ and $\gamma > 0$ is the (uniform) grid size. Solutions of (5.1_a) satisfy the conservation law

$$\frac{\partial}{\partial t}(w_j + z_j) + V \frac{w_j - w_{j-1}}{\gamma} - V \frac{z_{j+1} - z_j}{\gamma} = 0 \,. \tag{5.1$_b$}$$

We assume that $G \in C^1(\mathbb{R}^2; \mathbb{R})$ satisfies the properties: $G(0, 0) = 0$,

(A) $G(\cdot, z)$ is strictly decreasing, $G(w, \cdot)$ is strictly increasing.

(B) The zero level set of the graph of G is a C^1 curve $(w, g(w))$, with g strictly increasing and $g(\pm\infty) = \pm\infty$.

(C) Given $\mathcal{R}^{a,b}$, there is $c = c(a, b) > 0$ such that
$$G(w, z)\big(g^{-1}(z) - w\big) \geq c[z - g(w)]^2 \,, \text{ for } w, z \in \mathcal{R}^{a,b},$$

where $\mathcal{R}^{a,b} := [a, b] \times [g(a), g(b)]$. At times (C) is replaced by the hypothesis:

(C') Given $\mathcal{R}^{a,b}$, there is $c = c(a, b) > 0$ such that
$$|G(w, z)| \geq c|z - g(w)| \text{ and } G_z(w, z) - G_w(w, z) \geq c \,, \text{ for } w, z \in \mathcal{R}^{a,b}.$$

Both (C) and (C') follow from the assumption $G_z(w, z) \geq r$ for some $r > 0$. All hypotheses are satisfied for $G(w, z) = z - g(w)$ with g strictly increasing.

In Sections 5a and 5b, we develop an existence theory for the semi-discrete scheme (5.1). In Sections 5c, 5d and 5e, we investigate the convergence to the relaxation system (2.5) as ε is kept fixed and $\gamma \to 0$, and the convergence to the scalar conservation law (2.7) as both $\gamma \to 0$ and $\varepsilon \to 0$. The theory proceeds along the lines of Ref. 2; the point is that upwind space discretization preserves the estimate structure of the relaxation system (2.5).

5a. Preliminaries. In the sequel, π_ν denotes the space of bounded sequences $f = \{f_j\}_{j\in\mathbb{Z}}$ that are periodic with integer period $M = \nu^{-1}$, that is $f_{j+M} = f_j$ for $j \in \mathbb{Z}$. Also, l^p will stand for the space of p-summable sequences $f = \{f_j\}_{j\in\mathbb{Z}}$ if $1 \le p < \infty$, and l^∞ for the space of bounded sequences with the usual norms; recall that $l^1 \subset l^p \subset l^\infty$. We pursue two parallel existence theories: (i) for initial data w_0, z_0 in l^1, and (ii) for initial data w_0, z_0 in π_ν that are periodic sequences with period M.

If we set $w(t) = \{w_j(t)\}_{j\in\mathbb{Z}}$, $z(t) = \{z_j(t)\}_{j\in\mathbb{Z}}$, then solutions of (5.1) may be visualized as functions $w, z : [0,T] \to \mathbb{R}^{\mathbb{Z}}$. (The dependence of w and z on γ and ε will be suppressed.) For local existence, (5.1) is viewed as an infinite system of ordinary differential equations indexed by $j \in \mathbb{Z}$. One can check that the right hand side (consisting of the convective and the reacting terms) is a locally Lipshitz map from l^∞ to l^∞, and that the standard existence theory for ordinary differential equations can be extended to this case, and yields:
(a) If w_0, $z_0 \in l^\infty$ then there exists a unique solution w, $z \in C([0,T]; l^\infty)$ of (5.1) for some T small.
(b) If $T^\star$ is the maximal time of existence then $\limsup_{t\to T^*} \|w(t)\|_\infty + \|z(t)\|_\infty = \infty$.
(c) If w_0, $z_0 \in \pi_\nu$ then the solution $w(t)$, $z(t) \in \pi_\nu$ for each t.
The time regularity of this solution depends on the smoothness of G. If G is Lipshitz then w, $z \in C^1([0,T]; l^\infty)$, if G is more regular then the time-regularity of w, z also improves. Finally, if there is an a-priori bound for $(w(t), z(t))$ in l^∞ then (b) implies that the maximal time of existence $T^\star = \infty$.

Given a function $v : \mathbb{Z} \mapsto \mathbb{R}$, we define for $j \in \mathbb{Z}$ the backward and forward differencing operators

$$\delta_b v_j = \frac{v_j - v_{j-1}}{\gamma} \quad \text{and} \quad \delta_f v_j = \frac{v_{j+1} - v_j}{\gamma}\,. \tag{5.2}$$

The operators δ_b and δ_f satisfy the relations

$$\delta_b |v_j| \le (\operatorname{sgn} v_j)\delta_b v_j\,, \quad -\delta_f |v_j| \le -(\operatorname{sgn} v_j)\delta_f v_j\,, \tag{5.3}$$

and, for any convex, twice differentiable function $\Psi(v)$,

$$\delta_b \Psi(v_j) \le \Psi'(v_j)\delta_b v_j \quad \text{and} \quad -\delta_f \Psi(v_j) \le -\Psi'(v_j)\delta_f v_j\,. \tag{5.4}$$

Property (5.3) can be established by direct computation, while (5.4) follows from an application of the Taylor theorem

$$\Psi(b) = \Psi(a) + \Psi'(a)(b-a) + \Big[\int_0^1 \int_0^t \Psi''(sb + (1-s)a)\, ds dt \Big] (b-a)^2$$

to the convex function Ψ. In fact, if Ψ is strictly convex, with $\Psi''(v) \ge c > 0$, then we have the stronger variants of (5.4)

$$\begin{aligned} \delta_b \Psi(v_j) + \frac{\gamma c}{2}(\delta_b v_j)^2 &\le \Psi'(v_j)\delta_b v_j \\ -\delta_f \Psi(v_j) + \frac{\gamma c}{2}(\delta_f v_j)^2 &\le -\Psi'(v_j)\delta_f v_j\,. \end{aligned} \tag{5.5}$$

5b. Global existence theory and estimates. This section carries the existence theory for the system (5.1).

Theorem 5.1. *Let G satisfy Hypotheses $(A - B)$ and $w_0, z_0 \in l^\infty$. There exists a unique globally defined solution (w, z) of (5.1) such that*
(a) if $w_0, z_0 \in l^1$ then $(w, z) \in C([0,T]; l^1)$, for any $T > 0$,
(b) if $w_0, z_0 \in \pi_\nu$ then $(w, z) \in C([0,T]; l^\infty)$, $T > 0$, and $w(t)$, $z(t)$ take values in π_ν.
In either case (a) or (b) the following hold:
(i) If (w, z), $(\bar{w}, \bar{z})$ are two solutions of (5.1) then, for $t > 0$,

$$\sum_{j \in J} |w_j(t) - \bar{w}_j(t)| + |z_j(t) - \bar{z}_j(t)| \leq \sum_{j \in J} |w_{j0} - \bar{w}_{j0}| + |z_{j0} - \bar{z}_{j0}| \,. \tag{5.6}$$

(ii) If $(w_{j0}, z_{j0}) \in [a, b] \times [g(a), g(b)]$ for $j \in \mathbb{Z}$, then

$$(w_j(t), z_j(t)) \in \mathcal{R}^{a,b} := [a, b] \times [g(a), g(b)] \qquad \text{for } j \in \mathbb{Z}\,, \ \ t > 0\,, \tag{5.7}$$

i.e., the region $\mathcal{R}^{a,b}$ is positively invariant.
(iii) For $t > 0$, we have the Total Variation Diminishing (TVD) property,

$$\sum_{j \in J} |w_j(t) - w_{j-1}(t)| + |z_j(t) - z_{j-1}(t)| \leq \sum_{j \in J} |w_{j0} - w_{(j-1)0}| + |z_{j0} - z_{(j-1)0}| \,. \tag{5.8}$$

The sums in (5.6) and (5.8) are over $J = \mathbb{Z}$ for the case (a) of l^1 initial data, and over one period, $J = \mathbb{Z}_\nu$, for the case (b) of periodic initial data.

Proof. Let (w, z) be the solution of (5.1) defined on a maximal interval of existence $[0, T^\star)$, and let $T < T^\star$. Since $w, z \in C([0,T]; l^\infty)$, it follows that for some $r > 0$, $|w_j(t)| \leq r$ and $|z_j(t)| \leq r$ for $j \in \mathbb{Z}$ and $t \in [0, T]$.

First we consider l^1 initial data and prove (a) and (i-iii). We begin by showing that $w(t)$, $z(t) \in l^1$ for $t \in [0, T]$. In the ball B_r, we have $|G(w, z)| \leq C(|w| + |z|)$. Then, a rough estimation on (5.1) gives

$$\frac{\partial}{\partial t} \sum_{j=-n}^{n} \big(|w_j| + |z_j|\big) \leq \frac{2V}{\gamma} \sum_{j=-n-1}^{n+1} \big(|w_j| + |z_j|\big) + \frac{2C}{\varepsilon} \sum_{j=-n}^{n} \big(|w_j| + |z_j|\big)\,,$$

and the function

$$f_n(t) = \sum_{j=-n}^{n} |w_j(t)| + |z_j(t)|$$

satisfies the recursive relation

$$\begin{aligned} f_n(t) &\leq a + b \int_0^t f_{n+1}(s)\, ds \\ f_n(t) &\leq M_n \end{aligned} \tag{5.9}$$

with $b = \left(\frac{2V}{\gamma} + \frac{2C}{\varepsilon}\right)$, $M_n = 4nr$, and $a = \|w_0\|_1 + \|z_0\|_1$. As the following Lemma shows, (5.9) yields an exponential bound on $f_n(t)$:

Lemma 5.2. *Suppose the functions $f_n(t) \geq 0$, $n = 1, 2, \ldots$, satisfy, for some positive constants a, b, T and M_n, the recursive relations (5.9). If $\frac{(bT)^n M_{n+k}}{n!} \to 0$ as $n \to \infty$ for each $k = 1, 2, \ldots$, then*

$$f_n(t) \leq a\, e^{bt}\,, \quad \text{for } n = 1, 2, \ldots \text{ and } 0 \leq t \leq T. \tag{5.10}$$

Proof of Lemma 5.2. Iterating (5.9) we obtain, after k iterations,

$$f_n(t) \leq a\,(1 + bt + \cdots + \frac{b^k t^k}{k!}) + \frac{b^{k+1} t^{k+1}}{(k+1)!} M_{n+k+1}\,.$$

Passing to the limit $k \to \infty$, yields (5.10). $\square$

For the case at hand, Lemma 5.2, in conjunction with the bound $M_n \leq 2nr$, implies

$$f_n(t) = \sum_{j=-n}^{n} |w_j(t)| + |z_j(t)| \leq \Big(\|w_0\|_1 + \|z_0\|_1 \Big)\, e^{bt}\,,$$

$w(t)$, $z(t) \in l^1$ for $0 \leq t \leq T$, and $w_{\pm n}(t) \to 0$, $z_{\pm n}(t) \to 0$ as $n \to \infty$.

Let now (w, z) and $(\bar{w}, \bar{z})$ be two solutions of (5.1). They satisfy the equations

$$\begin{aligned} \frac{\partial}{\partial t}(w_j - \bar{w}_j) + V\delta_b(w_j - \bar{w}_j) &= \frac{1}{\varepsilon}[G(w_j, z_j) - G(\bar{w}_j, \bar{z}_j)] \\ \frac{\partial}{\partial t}(z_j - \bar{z}_j) - V\delta_f(z_j - \bar{z}_j) &= -\frac{1}{\varepsilon}[G(w_j, z_j) - G(\bar{w}_j, \bar{z}_j)] \end{aligned}$$

and the conservation law

$$\frac{\partial}{\partial t}(w_j - \bar{w}_j) + (z_j - \bar{z}_j) + V\delta_b(w_j - \bar{w}_j) - V\delta_f(z_j - \bar{z}_j) = 0\,.$$

We multiply the former equations by $\operatorname{sgn}(w_j - \bar{w}_j)$, $\operatorname{sgn}(z_j - \bar{z}_j)$ respectively, and use (5.3) and Hypothesis (A) to obtain:

$$\begin{aligned} &\frac{\partial}{\partial t}(|w_j - \bar{w}_j| + |z_j - \bar{z}_j|) + V\delta_b|w_j - \bar{w}_j| - V\delta_f|z_j - \bar{z}_j| \\ &\quad \leq \frac{1}{\varepsilon}[\operatorname{sgn}(w_j - \bar{w}_j) - \operatorname{sgn}(z_j - \bar{z}_j)[G(w_j, z_j) - G(\bar{w}_j, \bar{z}_j)] \leq 0\,. \end{aligned} \tag{5.11}$$

In turn, this yields

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(\sum_{j=-n}^{n} |w_j - \bar{w}_j| + |z_j - \bar{z}_j| \Big) \\ &\quad + \frac{V}{\gamma}\Big[|w_n - \bar{w}_n| - |w_{-n-1} - \bar{w}_{-n-1}| + |z_{-n} - \bar{z}_{-n}| - |z_{n+1} - \bar{z}_{n+1}| \Big] \leq 0\,. \end{aligned}$$

Letting $n \to \infty$, we deduce that (w, z), $(\bar{w}, \bar{z})$ satisfy

$$\sum_{j \in \mathbb{Z}} |w_j(t) - \bar{w}_j(t)| + |z_j(t) - \bar{z}_j(t)| \leq \sum_{j \in \mathbb{Z}} |w_{j0} - \bar{w}_{j0}| + |z_{j0} - \bar{z}_{j0}| \,, \tag{5.12}$$

and (5.5) follows for $w_0, z_0 \in l^1$.

To see (ii), note that the conservation law gives

$$\sum_{j \in \mathbb{Z}} (w_j(t) - \bar{w}_j(t)) + (z_j(t) - \bar{z}_j(t)) = \sum_{j \in \mathbb{Z}} (w_{j0} - \bar{w}_{j0}) + (z_{j0} - \bar{z}_{j0}) \,, \tag{5.13}$$

which, together with (5.12), implies

$$\sum_{j \in \mathbb{Z}} (w_j(t) - \bar{w}_j(t))^+ + (z_j(t) - \bar{z}_j(t))^+ \leq \sum_{j \in \mathbb{Z}} (w_{j0} - \bar{w}_{j0})^+ + (z_{j0} - \bar{z}_{j0})^+ \,. \tag{5.14}$$

Using the the stationary solutions $\bar{w}_j(t) = \kappa$, $\bar{z}_j(t) = g(\kappa)$, $\kappa = a,\, b$, as comparison functions in (5.14), leads to (ii). The invariant regions (5.7), in conjunction with Hypothesis (B), imply that solutions of (5.1) emanating from l^∞ data stay uniformly in j in an invariant set, and thus solutions of (5.1) exist globally in time. The Total Variation Diminishing property (iii) is an immediate consequence of (i).

If w_0, z_0 are in π_ν, the solutions $w(t)$ and $z(t)$ take values in π_ν for $t > 0$. When summing (5.11) over a set of indices $\mathbb{Z}_\nu$ with length one period M, the contributions at the endpoints cancel and we obtain

$$\sum_{j \in \mathbb{Z}_\nu} |w_j(t) - \bar{w}_j(t)| + |z_j(t) - \bar{z}_j(t)| \leq \sum_{j \in \mathbb{Z}_\nu} |w_j(0) - \bar{w}_j(0)| + |z_j(0) - \bar{z}_j(0)|$$

The rest of the proof proceeds along similar lines. □

The semidiscrete scheme (5.1) satisfies two variants of discrete, Kruzhkov-type inequalities. Relation (5.16) is a discretized version of the Kruzhkov *entropy conditions* for the conservation law (2.7). Relation (5.15) is a semidiscrete version of the entropy inequalities for the relaxation system (2.5); its importance is seen in Theorem A.2 of the Appendix.

Corollary 5.3. *Under the Hypotheses (A-B), solutions of (5.1) satisfy:*
(i) For $\kappa,\, \lambda \in \mathbb{R}$, $j \in \mathbb{Z}$ *and* $t > 0$,

$$\begin{aligned} &\partial_t\big(|w_j - \kappa| + |z_j - \lambda|\big) + V\delta_b|w_j - \kappa| - V\delta_f|z_j - \lambda|) \\ &\qquad \leq \frac{1}{\varepsilon} G(w_j, z_j)\left(\mathrm{sgn}(w_j - \kappa) - \mathrm{sgn}(z_j - \lambda)\right) \end{aligned} \tag{5.15}$$

(ii) For $\kappa \in \mathbb{R}$, $j \in \mathbb{Z}$ *and* $t > 0$

$$\partial_t\big(|w_j - \kappa| + |z_j - g(\kappa)|\big) + V\delta_b|w_j - \kappa| - V\delta_f|z_j - g(\kappa)|) \leq 0 \,. \tag{5.16}$$

Proof. For κ, λ fixed, (5.1_a) and (5.2) give

$$\partial_t |w_j - \kappa| + V\delta_b |w_j - \kappa| \le \frac{1}{\varepsilon} G(w_j, z_j)\operatorname{sgn}(w_j - \kappa)\,,$$
$$\partial_t |z_j - \lambda| - V\delta_f |z_j - \lambda| \le -\frac{1}{\varepsilon} G(w_j, z_j)\operatorname{sgn}(z_j - \lambda)\,,$$

whence (5.15) follows. Relation (5.16) is a consequence of (5.11), when $(\bar w, \bar z)$ is selected an equilibrium solution $(\kappa, g(\kappa))$. □

5c. Distance from equilibrium. For the remainder of the section, we present the statements for data w_0, $z_0 \in \pi_\nu$. Analogous statements hold for l^1 data with minor modifications.

There are two mechanisms for controlling the distance of a solution (w, z) from the line of equilibria $(\kappa, g(\kappa))$. The first is based on an "entropy" estimate in conjunction with Hypothesis (c): Consider the strictly convex functions $\Phi(w) = \frac{1}{2}w^2$ and $\Psi(z) = \int_0^z g^{-1}(\xi)d\xi$. We multiply the first equation in (5.1_a) by w_j, the second by $\Psi'(z_j)$, and use (5.4) to arrive at

$$\frac{\partial}{\partial_t}\Big(\frac{1}{2}w_j^2 + \Psi(z_j)\Big) + V\delta_b \frac{1}{2}w_j^2 - V\delta_f \Psi(z_j) + \frac{1}{\varepsilon}\big(g^{-1}(z_j) - w_j\big)\, G(w_j, z_j) \le 0\,.$$

In turn, the periodicity of $(w(t), z(t))$ and Hypothesis (C) imply

$$\begin{aligned}\sum_{j\in\mathbb{Z}_\nu} \gamma\Big(\frac{1}{2}w_j^2(t) + \Psi(z_j(t))\Big) + \frac{c}{\varepsilon}\int_0^t \gamma \sum_{j\in\mathbb{Z}_\nu} \big(z_j - g(w_j)\big)^2\, ds \\ \le \sum_{j\in\mathbb{Z}_\nu} \gamma\Big(\frac{1}{2}w_{j0}^2 + \Psi(z_{j0})\Big)\,.\end{aligned} \tag{5.17}$$

The second way for controlling the distance from equilibrium is based on a Lyapunov function for the associated system of ordinary differential equations.

Lemma 5.4. *Under Hypotheses (A), (B) and (C′), solutions of (5.1) satisfy*

$$\begin{aligned}\frac{1}{\varepsilon}\sum_{j\in\mathbb{Z}_\nu} \gamma |G(w_j(t), z_j(t))| \le & \frac{1}{\varepsilon} e^{-\frac{c}{\varepsilon}t} \sum_{j\in\mathbb{Z}_\nu} \gamma G(w_{j0}, z_{j0}) \\ & + \frac{C}{c}\sum_{j\in\mathbb{Z}_\nu} |w_{j0} - w_{(j-1)0}| + |z_{(j-1)0} - z_{j0}|\end{aligned} \tag{5.18}$$

where c is the constant in Hypothesis (C′), and C depends on the invariant domain $\mathcal{R}^{a,b}$ and V.

Proof. The function $G_j = G(w_j, z_j)$ satisfies the equation

$$\partial_t G_j + \frac{1}{\varepsilon}\big(G_z(w_j, z_j) - G_w(w_j, z_j)\big) G_j = -V G_w(w_j, z_j)\delta_b w_j + V G_z(w_j, z_j)\delta_f z_j$$

Using (C'), (5.7) and the TVD property (5.8) of the scheme, we see that

$$\begin{aligned}\frac{d}{dt}\sum_{j\in\mathbb{Z}_\nu}\gamma|G_j| + \frac{c}{\varepsilon}\sum_{j\in\mathbb{Z}_\nu}\gamma|G_j| &\le C\sum_{j\in\mathbb{Z}_\nu}\gamma|\delta_b w_j| + \gamma|\delta_f z_j| \\ &\le C\sum_{j\in\mathbb{Z}_\nu}|w_{j0} - w_{(j-1)0}| + |z_{(j-1)0} - z_{j0}|\end{aligned}$$

where C depends on the invariant region $\mathcal{R}^{a,b}$ and V. Then (5.18) follows from integrating the inequality. $\square$

5d. Compactness and convergence properties of semi-discrete schemes. We now consider the relation between solutions of the semidiscrete scheme (5.1) and 1-periodic solutions of the relaxation system (2.5) or the conservation law (2.7). As we are interested in constructing 1-periodic solutions, the number of grid points per interval I, of length one period, is $M = \gamma^{-1}$ with M an integer. The grid size γ is taken uniform and $\mathbb{Z}_\nu = \mathbb{Z}_\gamma$.

We assume that the data $\{(w_{j0}, z_{j0})\}_{j\in\mathbb{Z}}$ of the semi-discrete scheme are in π_γ and satisfy

$$(\mathrm{h}_1)\qquad \sum_{j\in\mathbb{Z}_\gamma}\gamma|w_{j0}| + \gamma|z_{j0}| \le C\,, \qquad (w_{j0},\, z_{j0}) \in \mathcal{R}^{a,b} \ \text{ for } j\in\mathbb{Z}\,,$$

$$(\mathrm{h}_2)\qquad \sum_{j\in\mathbb{Z}_\gamma}|w_{(j+1)0} - w_{j0}| + |z_{(j+1)0} - z_{j0}| \le K$$

for some constants C, K and a, b independent of γ and ε. To see the relevance of these assumptions, note that if $w_0^\varepsilon(x)$, $z_0^\varepsilon(x)$ are stable in $L^1\cap L^\infty(I)$ and 1-periodic, then w_0^ε, z_0^ε can be approximated in $L^1(I)$ by the piecewise constant approximants

$$w_0^{\gamma,\varepsilon} = \sum_{j\in\mathbb{Z}} w_{j0}\,\chi_{I_j}(x)\,, \quad z_0^{\gamma,\varepsilon} = \sum_{j\in\mathbb{Z}} z_{j0}\,\chi_{I_j}(x)\,, \tag{5.19}$$

where $I_j = [j\gamma, (j+1)\gamma)$, χ_{I_j} is the characteristic function of the interval I_j, and the interpolants (w_{j0}, z_{j0}) satisfy (h_1). If $w_0^\varepsilon(x)$, $z_0^\varepsilon(x)$ are also stable in $BV(I)$ then the interpolants satisfy (h_1) and (h_2).

Let $(w_j(t), z_j(t))$, $j\in\mathbb{Z}$, be the corresponding solution of (5.1) and define the *approximate* solution $(w^{\gamma,\varepsilon}, z^{\gamma,\varepsilon})$ of (2.5), for $(x,t)\in\mathbb{R}\times[0,\infty)$, by piecewise constant approximation

$$w^{\gamma,\varepsilon}(x,t) := \sum_{j\in\mathbb{Z}} w_j(t)\,\chi_{I_j}(x)\,, \qquad z^{\gamma,\varepsilon}(x,t) := \sum_{j\in\mathbb{Z}} z_j(t)\,\chi_{I_j}(x)\,. \tag{5.20}$$

Then $(w^{\gamma,\varepsilon}, z^{\gamma,\varepsilon})$ is 1-periodic in x, uniformly bounded

$$(w^{\gamma,\varepsilon}, z^{\gamma,\varepsilon}) \in \mathcal{R}^{a,b} \quad \text{for } x\in\mathbb{R},\, t>0, \tag{5.21}$$

and its total variation measure is

$$\begin{aligned}\int_I |D_x w^{\gamma,\varepsilon}|(x,t) &= \sum_{j\in\mathbb{Z}_\gamma}|w_{j+1}(t) - w_j(t)| \\ \int_I |D_x z^{\gamma,\varepsilon}|(x,t) &= \sum_{j\in\mathbb{Z}_\gamma}|z_{j+1}(t) - z_j(t)|\,.\end{aligned} \tag{5.22}$$

Theorem 5.5. *Let $\{w_{j0}, z_{j0}\}_{j\in\mathbb{Z}}$ satisfy $(h_1 - h_2)$.*
(a) Assume that G satisfies (A-B). Then $\{w^{\gamma,\varepsilon}\}$, $\{z^{\gamma,\varepsilon}\}$ are precompact in $L^1(I\times[0,T])$, when ε is fixed and $\gamma\to 0$. Along a subsequence $(w^{\gamma_n,\varepsilon}, z^{\gamma_n,\varepsilon}) \to (w^\varepsilon, z^\varepsilon)$ in $L^1(I\times[0,T])$, where $(w^\varepsilon, z^\varepsilon)$ is a 1-periodic solution of the relaxation system (2.5).
(b) Assume that G satisfies (A-B) and (C) and let $u^{\gamma,\varepsilon} = w^{\gamma,\varepsilon} + z^{\gamma,\varepsilon}$. Then, the family $\{u^{\gamma,\varepsilon}\}$ is precompact in $L^1(I\times[0,T])$ as $\varepsilon\to 0$ and $\gamma\to 0$. Along a subsequence $u^{\gamma_n,\varepsilon_n} \to u = w + g(w)$ in $L^1(I\times[0,T])$, where w is a 1-periodic entropy solution of the conservation law (2.7).

Proof. First we show (a). It follows from (5.8), (5.22) and (h_2) that, for $h\in\mathbb{R}$,

$$\int_I |w^{\gamma,\varepsilon}(x+h,t) - w^{\gamma,\varepsilon}(x,t)|\,dx \le |h| \int_I |D_x w^{\gamma,\varepsilon}|(x,t) \le K\,|h|\,.$$

For $k>0$, (5.20) and (5.1) imply

$$\begin{aligned}
\int_I |w^{\gamma,\varepsilon}(x,t+k) - w^{\gamma,\varepsilon}(x,t)|\,dx &= \sum_{j\in\mathbb{Z}_\gamma} \gamma |w_j(t+k) - w_j(t)| \\
&\le \sum_{j\in\mathbb{Z}_\gamma} \int_t^{t+k} \big[V|w_j(s) - w_{j-1}(s)| + \frac{1}{\varepsilon}\gamma |G(w_j(s), z_j(s))| \big] ds \\
&\le Vk \sum_{j\in\mathbb{Z}_\gamma} |w_{j0} - w_{(j-1)0}| + \frac{C}{\varepsilon}\,k \sum_{j\in\mathbb{Z}_\gamma} \gamma = (K + \frac{C}{\varepsilon})\,k\,.
\end{aligned}$$

Similar statements hold for the functions $z^{\gamma,\varepsilon}$. We conclude that $\{w^{\gamma,\varepsilon}\}$ and $\{z^{\gamma,\varepsilon}\}$ are precompact in L^1 as $\gamma\to 0$ and ε is kept fixed.

Consider now a subsequence $\gamma_n \to 0$ such that $w^{\gamma_n,\varepsilon}\to w^\varepsilon$, $z^{\gamma_n,\varepsilon}\to z^\varepsilon$ in $L^1(I\times[0,T])$ and a.e., and $w^{\gamma_n,\varepsilon}(\cdot,0)\to w^\varepsilon(\cdot,0)$, $z^{\gamma_n,\varepsilon}(\cdot,0)\to z^\varepsilon(\cdot,0)$ in $L^1(I)$. Using (5.1_a), we see that $(w^\varepsilon, z^\varepsilon)$ is a 1-periodic function in x that satisfies (2.5) in $\mathcal{D}'$ and emanates from the data $(w^\varepsilon(\cdot,0), z^\varepsilon(\cdot,0))$.

We turn to (b). Observe that $u^{\gamma,\varepsilon} = w^{\gamma,\varepsilon} + z^{\gamma,\varepsilon}$ satisfies

$$\int_I |u^{\gamma,\varepsilon}(x+h,t) - u^{\gamma,\varepsilon}(x,t)|\,dx \le K\,|h|$$

for $h\in\mathbb{R}$. For $k>0$ a time increment using the equation (5.1_b) we deduce

$$\begin{aligned}
\int_I |u^{\gamma,\varepsilon}(x,t+k) - u^{\gamma,\varepsilon}(x,t)|\,dx &= \gamma \sum_{j\in\mathbb{Z}_\gamma} |(w_j+z_j)(t+k) - (w_j+z_j)(t)| \\
&\le V \int_t^{t+k} \sum_{j\in\mathbb{Z}_\gamma} |w_j(s) - w_{j-1}(s)| + |z_{j+1}(s) - z_j(s)| ds \le VKk\,.
\end{aligned}$$

Hence, $\{u^{\gamma,\varepsilon}\}$ is precompact in L^1 as $\varepsilon\to 0$ and $\gamma\to 0$.

Next, (5.20), (C), (5.17) and (h_1) together imply

$$\int_0^T \int_I |z^{\gamma,\varepsilon} - g(w^{\gamma,\varepsilon})|^2 dxdt = \int_0^T \sum_{j\in\mathbb{Z}_\gamma} \gamma (z_j - g(w_j))^2 ds \le C\varepsilon\,.$$

Let $\varphi \in C_c^\infty(\mathbb{R}\times[0,\infty))$ be a test function. A straightforward calculation, using (5.1) and (5.20), shows that

$$\iint_{\mathbb{R}\times[0,\infty)} \varphi_t(w^{\gamma,\varepsilon} + z^{\gamma,\varepsilon}) - V\frac{\varphi(x,t)-\varphi(x+\gamma,t)}{\gamma} w^{\gamma,\varepsilon} + V\frac{\varphi(x-\gamma,t)-\varphi(x,t)}{\gamma} z^{\gamma,\varepsilon}\,dxdt$$
$$+ \int_{\mathbb{R}} \varphi(x,0)u^{\gamma,\varepsilon}(x,0)\,dx = 0\,.$$

Consider a subsequence γ_n, $\varepsilon_n \to 0$, such that $u^{\gamma_n,\varepsilon_n} \to u$ in $L^1(I\times[0,T])$ and a.e., $u^{\gamma_n,\varepsilon_n}(\cdot,0) = w^{\gamma_n,\varepsilon_n}(\cdot,0) + z^{\gamma_n,\varepsilon_n}(\cdot,0) \to u(\cdot,0)$ in $L^1(I)$ and $e^{\gamma_n,\varepsilon_n} = z^{\gamma_n,\varepsilon_n} - g(w^{\gamma_n,\varepsilon_n}) \to 0$ a.e. Since g is strictly increasing, there is w such that $w + g(w) = u$, $w^{\gamma_n,\varepsilon_n} \to w$ a.e., and $z^{\gamma_n,\varepsilon_n} = g(w^{\gamma_n,\varepsilon_n}) + e^{\gamma_n,\varepsilon_n} \to g(w)$ a.e. Passing to the limit γ_n, $\varepsilon_n \to 0$, we conclude that $u = w + g(w)$ is a 1-periodic weak solution of (2.7) emanating from the initial data $u(\cdot,0)$. A similar argument, starting from (5.16), shows that w satisfies, for $\kappa\in\mathbb{R}$,

$$\partial_t(|w-\kappa| + |g(w)-g(\kappa)|) + V\partial_x(|w-\kappa| - |g(w)-g(\kappa)|) \le 0$$

in $\mathcal{D}'$. □

5e. Error estimates. In this section we use the results of the Appendix, to obtain error estimates for the limit processes: (i) ε fixed and $\gamma\to 0$, and (ii) $\varepsilon\to 0$ and $\gamma\to 0$. We employ the notation: $(w^\varepsilon, z^\varepsilon)$ for solutions of the relaxation system (2.5) associated with the data $(w_0^\varepsilon, z_0^\varepsilon)$; u or w, $u = w + g(w)$, for the solution of the conservation law (2.7) associated with the initial data $u_0 = w_0 + g(w_0)$; finally, $(w^{\gamma,\varepsilon}, z^{\gamma,\varepsilon})$ for the *approximate* solution, defined in (5.20), associated with the piecewise constant data $(w_0^{\gamma,\varepsilon}, z_0^{\gamma,\varepsilon})$ in (5.19).

Theorem 5.6. *Let G satisfy (A-B) and suppose that $(w_0^\varepsilon, z_0^\varepsilon)\in BV(I)$.*
(a) If $(w_0^{\gamma,\varepsilon}, z_0^{\gamma,\varepsilon})$ satisfy $(h_1 - h_2)$, then for any $T>0$ there is a constant $\hat C$, depending on T, ε, the bounds in Hypotheses (h_1-h_2) and the BV-norm of $(w_0^\varepsilon, z_0^\varepsilon)$, such that

$$\|w^{\gamma,\varepsilon}(\cdot,t) - w^\varepsilon(\cdot,t)\|_{L^1(I)} + \|z^{\gamma,\varepsilon}(\cdot,t) - z^\varepsilon(\cdot,t)\|_{L^1(I)} \le \|w_0^{\gamma,\varepsilon} - w_0^\varepsilon\|_{L^1(I)} + \|z_0^{\gamma,\varepsilon} - z_0^\varepsilon\|_{L^1(I)} + \hat C\gamma^{1/2}\,. \tag{5.23$_a$}$$

for $0\le t\le T$, as $\gamma\to 0$.
(b) If $(w_0^{\gamma,\varepsilon}, z_0^{\gamma,\varepsilon})$ satisfy (h_1), then for any $T>0$ there is $\hat C$, depending on T, ε, the bound in Hypothesis (h_1) and the BV-norm of $(w_0^\varepsilon, z_0^\varepsilon)$, such that

$$\|w^{\gamma,\varepsilon}(\cdot,t) - w^\varepsilon(\cdot,t)\|_{L^1(I)} + \|z^{\gamma,\varepsilon}(\cdot,t) - z^\varepsilon(\cdot,t)\|_{L^1(I)} \le \|w_0^{\gamma,\varepsilon} - w_0^\varepsilon\|_{L^1} + \|z_0^{\gamma,\varepsilon} - z_0^\varepsilon\|_{L^1} + \hat C\gamma^{1/3}\,. \tag{5.23$_b$}$$

for $0 \le t \le T$, *as* $\gamma \to 0$.

We note that in part (b) of the previous Theorem, the hypothesis $w_0^\varepsilon, z_0^\varepsilon \in BV(I)$ refers to the *approximated* solution of the relaxation system (2.5), while the assumption (h_1) is on the data of the semidiscrete approximating scheme (5.1). As noted at the end of the proof of Theorem 5.6 (b), the hypothesis $w_0^\varepsilon, z_0^\varepsilon \in BV(I)$ can be replaced by a weaker hypothesis on the L^1-modulus of continuity of $w_0^\varepsilon, z_0^\varepsilon$ at the expense of obtaining a slower rate of convergence than $\gamma^{1/3}$. In this case we may also obtain a weaker version of Theorem 2.1.

The next theorem provides error estimates for the convergence $\gamma \to 0$, $\varepsilon \to 0$ to the conservation law. We assume, without loss of generality in the BV framework, that g is globally Lipschitz. The data (w_{j0}, z_{j0}) are assumed to satisfy

$$\sum_{j \in \mathbb{Z}_\gamma} \gamma |G(w_{j0}, z_{j0})| \le C\varepsilon \tag{h$_3$}$$

with C independent of γ and ε. Then we have:

Theorem 5.7. *Let G satisfy $(A-B)$ and (C'), and suppose that $u_0 \in BV(I)$ and $(w_0^{\gamma,\varepsilon}, z_0^{\gamma,\varepsilon})$ satisfy $(h_1 - h_3)$. Then for any $T > 0$ there is a constant $\hat{C}$, depending on T, the Lipshitz norm of $g(w)$, the bounds $(h_1 - h_3)$, and the BV-norm of u_0, such that*

$$\|u^{\gamma,\varepsilon}(\cdot,t) - u(\cdot,t)\|_{L^1(I)} \le \|u_0^{\gamma,\varepsilon} - u_0\|_{L^1(I)} + \hat{C}(\varepsilon + \gamma)^{1/2}, \tag{5.24}$$

for $0 \le t \le T$, *as* $\gamma \to 0$ *and* $\varepsilon \to 0$.

Proof of Theorem 5.6. We first show that $(w^{\gamma,\varepsilon}, z^{\gamma,\varepsilon})$ is an approximate solution of (2.5), in the sense that $(A.11)$ in the Appendix is satisfied. Fix κ and $\lambda \in \mathbb{R}$ and $T > 0$. Using (5.20) we compute for $(x,t) \in \mathbb{R} \times [0,T]$

$$|w^{\gamma,\varepsilon} - \kappa| = \sum_{j \in \mathbb{Z}} |w_j - \kappa|\, \chi_{I_j}, \quad |z^{\gamma,\varepsilon} - \lambda| = \sum_{j \in \mathbb{Z}} |z_j - \lambda|\, \chi_{I_j}.$$

Then (5.15) implies that $(w^{\gamma,\varepsilon}, z^{\gamma,\varepsilon})$ satisfies, in $\mathcal{D}'$,

$$\begin{aligned} S &:= \partial_t\big(|w^{\gamma,\varepsilon} - \kappa| + |z^{\gamma,\varepsilon} - \lambda|\big) + V\partial_x\big(|w^{\gamma,\varepsilon} - \kappa| - |z^{\gamma,\varepsilon} - \lambda|\big) \\ &\qquad - \frac{1}{\varepsilon} G(w^{\gamma,\varepsilon}, z^{\gamma,\varepsilon})\big(\mathrm{sgn}(w^{\gamma,\varepsilon} - \kappa) - \mathrm{sgn}(z^{\gamma,\varepsilon} - \lambda)\big) \\ &= \sum_{j \in \mathbb{Z}} \Big(\partial_t(|w_j - \kappa| + |z_j - \lambda|) - G(w_j, z_j)(\mathrm{sgn}(w_j - \kappa) - \mathrm{sgn}(z_j - \lambda))\Big)\, \chi_{I_j} \\ &\qquad + V \sum_{j \in \mathbb{Z}} \Big(\gamma\delta_b|w_j - \kappa| - \gamma\delta_b|z_j - \lambda|\Big)\delta(x - j\gamma) \\ &\le V f_w - V f_z, \end{aligned} \tag{5.25}$$

where f_w and f_z are given by

$$
\begin{aligned}
f_w &= \sum_{j\in\mathbb{Z}} \delta_b |w_j - \kappa| \Big(\gamma \delta(x - j\gamma) - \chi_{I_j} \Big) \\
f_z &= \sum_{j\in\mathbb{Z}} \delta_b |z_j - \lambda| \Big(\gamma \delta(x - j\gamma) - \chi_{I_j} \Big)
\end{aligned}
\tag{5.26}
$$

and $\delta(x - j\gamma)$ stands for the delta function at the point $j\gamma$.

We proceed to estimate the error terms f_w and f_z. First, we operate under the framework of both Hypotheses $(h_1 - h_2)$. Let $\varphi(x,t)$ be a test function and let $a_j = \delta_b |w_j - \kappa|$. Then

$$
\begin{aligned}
(f_w, \varphi) &= \sum_{j\in\mathbb{Z}} a_j \Big(\gamma \varphi(j\gamma, t) - \int_{I_j} \varphi(y,t) dy \Big) \\
&= - \sum_{j\in\mathbb{Z}} a_j \int_{I_j} \int_{j\gamma}^{y} \frac{\partial \varphi}{\partial x}(z,t)\, dz dy = -(F_w, \frac{\partial \varphi}{\partial x}) \,,
\end{aligned}
$$

i.e. $f_w = \partial_x F_w$ in $\mathcal{D}'$. Furthermore, F_w can be viewed as a Radon measure on $I \times [0,T]$, defined by

$$
(F_w, \psi) = \sum_{j\in\mathbb{Z}_\gamma} a_j \int_{I_j} \int_{j\gamma}^{y} \psi(z,t)\, dz dy \,, \quad \text{for } \psi \in C^0(I \times [0,T]),
$$

and F_w is estimated in $\mathcal{M}(I \times [0,T])$ by

$$
|(F_w, \psi)| \le \frac{\gamma}{2} \Big(\sum_{j\in\mathbb{Z}_\gamma} |w_j(t) - w_{j-1}(t)| \Big) \|\psi\|_{C^0} \,.
$$

Under Hypotheses $(h_1 - h_2)$, (5.8) implies that

$$
f_w = \partial_x F_w \,, \quad \text{with } \|F_w\|_{\mathcal{M}_{x,t}} \le K\gamma \,,
\tag{5.27}
$$

and, in a similar fashion,

$$
f_z = \partial_x F_z \,, \quad \text{with } \|F_z\|_{\mathcal{M}_{x,t}} \le K\gamma \,,
\tag{5.28}
$$

where the constant K in (5.27) and (5.28) is as in (h_2).

In summary, $(w^{\gamma,\varepsilon}, z^{\gamma,\varepsilon})$ satisfies

$$
S \le V \partial_x F_w - V \partial_x F_z \,.
$$

where F_w and F_z are bounded in measures as in (5.27-5.28). For the selections $R = m$ any integer, $\delta = \Delta > 0$, $\nu = 0$, $M = V$, and $B_t = B(0, m + M(T-t) + \delta)$ we apply Theorem A.2.

We set m_0 to be the smallest integer larger than $MT + \delta$, and take account of the fact that the moduli of continuity E^t and E^x of the solution $(w^\varepsilon, z^\varepsilon)$ of (2.5) are estimated by

$$E^t \leq (m + m_0) C_1 \delta\,, \quad E^x \leq (m+m_0) C_2 \delta\,, \tag{5.29}$$

where C_1, C_2 depend on the BV-norm of $(w_0^\varepsilon, z_0^\varepsilon)$, and C_1 also depends on ε. Also, that

$$E^H \leq (m+m_0) C_3 \frac{\gamma}{\delta}\,,$$

where C_3 depends on T and K in (5.27-5.28). Then Theorem A.2 implies that, for $\delta > 0$ and m any integer,

$$\begin{aligned}
&\int_I |w^{\gamma,\varepsilon}(x,T) - w^\varepsilon(x,T)| + |z^{\gamma,\varepsilon}(x,T) - z^\varepsilon(x,T)| dx \leq \\
&\quad (1 + \frac{m_0}{m}) \int_I |w^{\gamma,\varepsilon}(x,0) - w^\varepsilon(x,0)| + |z^{\gamma,\varepsilon}(x,0) - z^\varepsilon(x,0)| dx + C(1+\frac{m_0}{m})(\delta + \frac{\gamma}{\delta})\,.
\end{aligned}$$

The optimum estimate follows from the selection $\delta = \gamma^{1/2}$ and $m \to \infty$ and leads to (5.23_a).

We indicate a second way for treating the errors f_w and f_z that requires the weaker Hypothesis (h_1), but still requires knowledge on the L^1-modulus of continuity for $(w^\varepsilon, z^\varepsilon)$. Let φ be a test function and set $\delta_b|w_j - \kappa| = \frac{b_j - b_{j-1}}{\gamma}$, where $b_j = |w_j - \kappa| - |\kappa|$. Then (5.26) implies

$$\begin{aligned}
(f_w, \varphi) &= -\sum_{j\in\mathbb{Z}} \frac{b_j - b_{j-1}}{\gamma} \int_{I_j} \int_{j\gamma}^{y} \frac{\partial \varphi}{\partial x}(z,t)\, dz dy \\
&= -\frac{1}{\gamma} \sum_{j\in\mathbb{Z}} b_j \Big(\int_{I_j} \int_{j\gamma}^{y} \frac{\partial \varphi}{\partial x}(z,t)\, dz dy - \int_{I_{j+1}} \int_{(j+1)\gamma}^{y} \frac{\partial \varphi}{\partial x}(z,t)\, dz dy \Big) \\
&= \frac{1}{\gamma} \sum_{j\in\mathbb{Z}} b_j \int_{I_j} \int_{j\gamma}^{y} \int_z^{z+\gamma} \frac{\partial^2 \varphi}{\partial x^2}(w,t)\, dw dz dy = (\mathcal{F}_w, \frac{\partial^2 \varphi}{\partial x^2})\,,
\end{aligned}$$

i.e. $f_w = \partial_x^2 \mathcal{F}_w$ in $\mathcal{D}'$. The error term $\mathcal{F}_w$ is viewed as a Radon measure on $I \times [0,T]$, defined by

$$(\mathcal{F}_w, \psi) = \frac{1}{\gamma} \sum_{j\in\mathbb{Z}} b_j \int_{I_j} \int_{j\gamma}^{y} \int_z^{z+\gamma} \psi(w,t)\, dw dz dy \quad \text{for } \psi \in C^0(I \times [0,T]),$$

and estimated in $\mathcal{M}(I \times [0,T])$ by

$$|(\mathcal{F}_w, \psi)| \leq \frac{\gamma^2}{2} \Big(\sum_{j \in \mathbb{Z}_\gamma} |b_j| \Big) \|\psi\|_{C^0} \leq \frac{\gamma}{2} \Big(\sum_{j\in\mathbb{Z}_\gamma} \gamma |w_j(t)| \Big) \|\psi\|_{C^0}\,.$$

Under Hypothesis (h_1), (5.6) implies

$$f_w = \partial_x^2 \mathcal{F}_w\,, \qquad \text{with } \|\mathcal{F}_w\|_{\mathcal{M}_{x,t}} \leq C\gamma\,, \tag{5.30}$$

and, in a similar fashion,

$$f_z = \partial_x^2 \mathcal{F}_z \,, \qquad \text{with } \|\mathcal{F}_z\|_{\mathcal{M}_{x,t}} \le C\gamma \,. \tag{5.31}$$

Now $(w^{\gamma,\varepsilon}, z^{\gamma,\varepsilon})$ is viewed as an approximate solution, based on the inequality

$$S \le V\partial_x^2 \mathcal{F}_w - V\partial_x^2 \mathcal{F}_z \,. \tag{5.32}$$

Applying Theorem A.2 on the domain $B_t = B(0, m + M(T-t) + \delta)$, with m any integer and $\delta > 0$, and using (5.29) and the implication of (A.10)

$$E^L \le (m + m_0)C_4 \frac{\gamma}{\delta^2} \,,$$

we obtain

$$\begin{aligned} &\int_I |w^{\gamma,\varepsilon}(x,T) - w^\varepsilon(x,T)| + |z^{\gamma,\varepsilon}(x,T) - z^\varepsilon(x,T)| dx \le \\ &\quad (1 + \frac{m_0}{m}) \int_I |w^{\gamma,\varepsilon}(x,0) - w^\varepsilon(x,0)| + |z^{\gamma,\varepsilon}(x,0) - z^\varepsilon(x,0)| dx + (1 + \frac{m_0}{m}) C(\delta + \frac{\gamma}{\delta^2}) \,. \end{aligned}$$

The optimum estimate corresponds to $\delta = \gamma^{1/3}$ and $m \to \infty$ and yields (5.23_b). The hypothesis $w_0^\varepsilon, z_0^\varepsilon \in BV(I)$ can be replaced in part (b) by a weaker hypothesis on the L^1-modulus of continuity of $w_0^\varepsilon, z_0^\varepsilon$ at the expense of obtaining a slower rate of convergence than $\gamma^{1/3}$. □

Proof of Theorem 5.7. Let $u^{\gamma,\varepsilon} = w^{\gamma,\varepsilon} + z^{\gamma,\varepsilon}$. We show that $w^{\gamma,\varepsilon}$ satisfies the approximate entropy inequalities (A.8). Given $k \in \mathbb{R}$ define $\kappa \in \mathbb{R}$ such that $k = \kappa + g(\kappa)$. Since g is increasing we see that

$$\begin{aligned} |w^{\gamma,\varepsilon} - \kappa| + |g(w^{\gamma,\varepsilon}) - g(\kappa)| &= |w^{\gamma,\varepsilon} - \kappa| + |z^{\gamma,\varepsilon} - g(\kappa)| - J^{\gamma,\varepsilon} \,, \\ V|w^{\gamma,\varepsilon} - \kappa| - V|g(w^{\gamma,\varepsilon}) - g(\kappa)| &= V|w^{\gamma,\varepsilon} - \kappa| - V|z^{\gamma,\varepsilon} - g(\kappa)| + VJ^{\gamma,\varepsilon} \,. \end{aligned} \tag{5.33}$$

where

$$\begin{aligned} J^{\gamma,\varepsilon}(x,t) &= |z^{\gamma,\varepsilon} - g(\kappa)| - |g(w^{\gamma,\varepsilon}) - g(\kappa)| \,, \\ |J^{\gamma,\varepsilon}(x,t)| &\le |g(w^{\gamma,\varepsilon}) - z^{\gamma,\varepsilon}| \,. \end{aligned} \tag{5.34}$$

Next, (5.16), (5.25) with $\lambda = g(\kappa)$, and (5.33) imply

$$\begin{aligned} L &:= \partial_t\big(|w^{\gamma,\varepsilon} - \kappa| + |g(w^{\gamma,\varepsilon}) - g(\kappa)|\big) + V\partial_x\big(|w^{\gamma,\varepsilon} - \kappa| - |g(w^{\gamma,\varepsilon}) - g(\kappa)|\big) \\ &\le V\partial_x F_w - V\partial_x F_z - \partial_t J^{\gamma,\varepsilon} + V\partial_x J^{\gamma,\varepsilon} \,, \end{aligned} \tag{5.35}$$

where F_w and F_z are estimated as in (5.27-5.28) and $J^{\gamma,\varepsilon}$ is given in (5.34). On account of Lemma 5.4 and Hypotheses (C′) and (h_2), (h_3), we have

$$
\begin{aligned}
\sup_{t\in[0,T]} \int_I |J^{\gamma,\varepsilon}| dx &\le \sup_{t\in[0,T]} \sum_{j\in\mathbb{Z}_\gamma} \frac{\gamma}{c} |G(w_j(t), z_j(t))| \\
&\le C \sum_{j\in\mathbb{Z}_\gamma} \gamma |G(w_{j0}, z_{j0})| + C\varepsilon \sum_{j\in\mathbb{Z}_\gamma} |w_{j0} - w_{(j-1)0}| + |z_{(j+1)0} - z_{j0}| \le C\varepsilon\,.
\end{aligned}
\tag{5.36}
$$

We now apply Theorem A.1 on the domain $B_t = B(0, m + M(T-t) + \delta)$ with m integer and $\delta > 0$. For $u_0 \in BV(I)$ the moduli of continuity E^t and E^x of u are estimated by

$$
E^t \le (m + m_0) C\delta\,, \quad E^x \le (m + m_0) C\delta
$$

with C independent of ε. Then Theorem A.1, in conjunction with (5.36) and (A.10), implies that for any $\delta > 0$ and m integer

$$
\begin{aligned}
\int_I |u^{\gamma,\varepsilon}(x,T) - u(x,T)| dx &\le (1 + \frac{m_0}{m}) \| u_0^{\gamma,\varepsilon}(x,0) - u(x,0)\|_{L^1(I)} \\
&\quad + C(1 + \frac{m_0}{m})(\delta + \varepsilon + \frac{\gamma + \varepsilon}{\delta})\,.
\end{aligned}
$$

The optimum bound is obtained for $\delta = (\gamma + \varepsilon)^{1/2}$ and $m \to \infty$ and leads to (5.24). □

Appendix: Kruzkhov estimates for relaxation systems

In Ref. 2, a class of $N+1$ semilinear hyperbolic systems is studied, set in $\mathbb{R}^N \times [0,\infty)$, describing the dynamics of the state vector (w, Z), with $Z = (z_1, \ldots, z_N)$:

$$
\begin{aligned}
\partial_t w + U_0 \cdot \nabla w &= \frac{1}{\varepsilon} \sum_{i=1}^N G_i(w, z_i) \\
\partial_t z_i + U_i \cdot \nabla z_i &= -\frac{1}{\varepsilon} G_i(w, z_i)\,, \qquad i = 1, \ldots, N\,.
\end{aligned}
\tag{A.1}
$$

Here $U_0, U_1, \ldots, U_N \in \mathbb{R}^N$ are convective velocity vectors, $G_i : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are smooth functions, strictly decreasing in the w variable and increasing in the z_i variable, and $\varepsilon > 0$ is a relaxation parameter. It is assumed that the equations $G_i(w,z) = 0$ have a unique solution $z_i = g_i(w)$ with $g_i(w)$ strictly increasing and (globally) Lipschitz. The curve

$$
w \mapsto (w, g_1(w), \ldots, g_N(w))\,, \quad w \in \mathbb{R}\,,
$$

constitutes the manifold of local equilibria (or Maxwellian states).

Solutions of (A.1) satisfy the conservation law

$$
\partial_t (w + \sum_{i=1}^N z_i) + \operatorname{div}(U_0 w + \sum_{i=1}^N U_i z_i) = 0\,.
\tag{A.2}
$$

As $\varepsilon \to 0$, the local equilibria are enforced, and the limiting dynamics is described by weak solutions of

$$(A.3) \qquad \partial_t(w + \sum_{i=1}^{N} g_i(w)) + \mathrm{div}(U_0 w + \sum_{i=1}^{N} U_i g_i(w)) = 0\,.$$

It is shown in Ref. 2 that solutions of (A.1) emanating from BV-stable data are compact in the strong L^1-topology, and the moment $u^\varepsilon = w + \sum_{i=1}^N z_i$ converges, in the zero-relaxation limit $\varepsilon \to 0$, to an entropy solution of (A.3). In addition any scalar multidimensional conservation law

$$(A.4) \qquad u_t + \sum_{i=1}^{N} \partial_{x_i} F_i(u) = 0\,, \quad x \in \mathbb{R}^n, t > 0,$$

can be written in the form (A.3) and thus be recovered as an asymptotic limit of a relaxation system (A.1), provided its characteristic speeds satisfy, relative to the convective velocities of (A.1), a multidimensional analog of the subcharacteristic condition.

It is a well known fact that the Kruzhkov[21] entropy conditions

$$(\mathrm{A.5}) \qquad \partial_t |u-k| + \sum_{i=1}^{n} \partial_{x_i} \big[(F_i(u) - F_i(k))\mathrm{sgn}(u-k) \big] \leq 0\,, \qquad \text{in } \mathcal{D}', \text{ for all } k \in \mathbb{R}.$$

guarantee uniqueness of weak solutions of (A.4). The system (A.1) is equipped with a family of entropies,

$$(\mathrm{A.6}) \qquad \begin{aligned} \partial_t \big(|w-\kappa| + \sum_i |z_i - \lambda_i|\big) + \mathrm{div}\Big(U_0|w-\kappa| + \sum_i U_i|z_i - \lambda_i|\Big) \leq \\ \frac{1}{\varepsilon}\sum_i G(w, z_i)\big[\mathrm{sgn}(w-k) - \mathrm{sgn}(z_i - \lambda_i)\big]\,, \end{aligned}$$

for $\kappa, \lambda_i \in \mathbb{R}\,, i = 1, ..., N$. If λ_i are selected $\lambda_i = g_i(\kappa)$, then the right hand side in (A.6) vanishes, and in the $\varepsilon \to 0$ limit we obtain the entropy conditions (A.5) for $k = \kappa + g_i(\kappa)$, (see Ref. 2).

The following theorems use the technique of doubling of variables of Kruzhkov[21] and Kuznetzov[22], in the form formulated by Bouchut and Perthame[23], to provide error estimates. Theorem A.1 provides estimates between approximate solutions of (A.1) and entropy solutions of the conservation law (A.4), and is an adaptation to (A.3) of Ref. 23 (Thm 2.1). Theorem A.2 deals with estimates between approximate solutions of (A.1) and weak solutions of (A.1) for $\varepsilon > 0$ fixed, and it indicates that the entropy conditions (A.6) allow to extend the Kruzhkov theory to the class of contractive relaxation systems (A.1).

Theorem A.1. *Let $w, \bar{w} \in L^\infty_{loc}([0,\infty), L^1_{loc}(\mathbb{R}^n))$ be right continuous with values in $L^1_{loc}(\mathbb{R}^n)$. Assume that w satisfies, for $\kappa \in \mathbb{R}$, the entropy conditions*

$$\partial_t\big(|w-\kappa| + \sum_i |g_i(w) - g_i(\kappa)|\big) + \operatorname{div}\Big(U_0|w-\kappa| + \sum_i U_i|g_i(w)-g_i(\kappa)|\Big) \leq 0\,, \tag{A.7}$$

and $\bar{w}$ satisfies, for $\kappa \in \mathbb{R}$,

$$\begin{aligned}\partial_t\big(|\bar{w}-\kappa| + \sum_i |g_i(\bar{w}) - g_i(\kappa)|\big) &+ \operatorname{div}\Big(U_0|\bar{w}-\kappa| + \sum_i U_i|g_i(\bar{w})-g_i(\kappa)|\Big) \\ &\leq K_\kappa + \partial_t J_\kappa + \operatorname{div} H_\kappa + \sum_{i,j=1}^N \partial_{x_i x_j} L_\kappa^{(ij)}\,,\end{aligned} \tag{A.8}$$

in $\mathcal{D}'$, where K_κ, J_κ, H^i_κ, $L^{(ij)}_\kappa$ are local Radon measures that satisfy, for some nonnegative κ-independent Radon measures α_K, α_J, α^i_H and $\alpha^{(ij)}_L$, the bounds

$$\begin{aligned}|K_\kappa(x,t)| &\leq \alpha_K(x,t)\,, \qquad |H^i_\kappa(x,t)| \leq \alpha^i_H(x,t)\,, \quad i = 1,2,\ldots N\,,\\ |J_\kappa(x,t)| &\leq \alpha_J(x,t)\,, \qquad |L^{(ij)}_\kappa(x,t)| \leq \alpha^{(ij)}_L(x,t)\,, \;\; i,j = 1,2,\ldots N\,,\end{aligned}$$

in the sense of measures. Moreover, we assume $\alpha_J \in L^\infty_{loc}([0,\infty), L^1_{loc}(\mathbb{R}^n))$. Then, for any $T \geq 0$, $x_0 \in \mathbb{R}^N$, $R > 0$, $\delta > 0$, $\Delta > 0$, $\nu \geq 0$, and letting

$$M = Lip\Big(U_0 w + \sum_i U_i g_i(w)\Big)\,, \quad B_t = B\big(x_0, R + M(T-t) + \Delta + \nu\big)\,,$$

we have

$$\begin{aligned}\int_{|x-x_0|<R} &|w(x,T) - \bar{w}(x,T)| + \sum_i |g_i\big(w(x,T)\big) - g_i\big(\bar{w}(x,T)\big)|dx \\ &\leq \int_{B_0} |w(x,0) - \bar{w}(x,0)| + \sum_i |g_i\big(w(x,0)\big) - g_i\big(\bar{w}(x,0)\big)|dx \\ &\quad + C(E^t + E^x + E^K + E^J + E^H + E^L)\,,\end{aligned} \tag{A.9}$$

where C is a uniform constant and

$$\begin{aligned}E^t &= \max_{t=0,T} \sup_{0<s-t<\delta} \int_{B_t} |w(x,s) - w(x,t)|dx\,,\\ E^x &= \max_{t=0,T} \sup_{|h|<\Delta} \int_{B_t} |w(x+h,t) - w(x,t)|dx\,,\\ E^K &= \iint_{0<t\leq T, x\in B_t} a_K(x,t)\,, \quad E^H = \frac{1}{\Delta}\sum_{i=1}^N \iint_{0<t\leq T, x\in B_t} \alpha^i_H(x,t)\,,\\ E^L &= \frac{1}{\Delta^2} \sum_{1\leq i,j\leq N} \iint_{0\leq t\leq T, x\in B_t} \alpha^{ij}_L(x,t)\,,\\ E^J &= (1 + \frac{T}{\delta} + \frac{MT}{\Delta+\nu}) \sup_{(0,2T)} \int_{B_t} \alpha_J(x,t)dx\,.\end{aligned} \tag{A.10}$$

Proof. The theorem is an adaptation from Theorem 2.1 in Ref. 23. To see that, let $u = w + \sum_i g_i(w)$, and for $k \in \mathbb{R}$ define $\kappa \in \mathbb{R}$ such that $k = \kappa + \sum_i g_i(\kappa)$. Since g_i are strictly increasing, $\mathrm{sgn}(u-k) = \mathrm{sgn}(w-\kappa)$ and

$$|u-k| = |w-\kappa| + \sum_i |g_i(w) - g_i(\kappa)|\,,$$
$$(F^j(u) - F^j(k))\mathrm{sgn}(u-k) = U_0^j|w-\kappa| + \sum_i U_i^j|g_i(w)-g_i(\kappa)|\,,$$

where $U_i = (U_i^1, ..., U_i^N)$, $i = 0, 1, ..., N$. Thus, the entropy conditions (A.7), (A.8) are written in the form (A.5) and Theorem 2.1 in Ref. 23 provides the result. □

We now turn to Kruzkhov estimates for (A.1) *away* from the local equilibrium $\varepsilon \approx 0$:

Theorem A.2. *Let $(w, Z), (\bar{w}, \bar{Z}) \in L^\infty_{loc}([0,\infty), L^1_{loc}(\mathbb{R}^n))$ be right continuous with values in $L^1_{loc}(\mathbb{R}^n)$. Assume that (w, Z) satisfies for $\kappa \in \mathbb{R}$, $\Lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ the entropy conditions (A.6) and $(\bar{w}, \bar{Z})$ satisfies, for $\kappa \in \mathbb{R}, \Lambda \in \mathbb{R}^N$,*

$$\begin{aligned}
&\partial_t\big(|\bar{w}-\kappa| + \sum_i |\bar{z}_i - \lambda_i|\big) + \mathrm{div}\Big(U_0|\bar{w}-\kappa| + \sum_i U_i|\bar{z}_i - \lambda_i|\Big)\\
&\quad\le \frac{1}{\varepsilon}\sum_i G(\bar{w}, \bar{z}_i)\big[\mathrm{sgn}(\bar{w}-k) - \mathrm{sgn}(\bar{z}_i - \lambda_i)\big]\\
&\quad+ K_{\kappa,\Lambda} + \partial_t J_{\kappa,\Lambda} + \mathrm{div} H_{\kappa,\Lambda} + \sum_{i,j=1}^N \partial_{x_i x_j} L^{(ij)}_{\kappa,\Lambda}\,,
\end{aligned}\tag{A.11}$$

in $\mathcal{D}'$, where $K_{\kappa,\Lambda}$, $J_{\kappa,\Lambda}$, $H^i_{\kappa,\Lambda}$, $L^{(ij)}_{\kappa,\Lambda}$ are local Radon measures that satisfy, for some nonnegative (κ,Λ)-independent Radon measures α_K, α_J, α^i_H and $\alpha^{(ij)}_L$, the inequalities

$$\begin{aligned}
&|K_{\kappa,\Lambda}(x,t)| \le \alpha_K(x,t)\,, \qquad |H^i_{\kappa,\Lambda}(x,t)| \le \alpha^i_H(x,t)\,, \quad i = 1,2,...N\,,\\
&|J_{\kappa,\Lambda}(x,t)| \le \alpha_J(x,t)\,, \qquad |L^{(ij)}_{\kappa,\Lambda}(x,t)| \ \le \alpha^{(ij)}_L(x,t)\,, \ \ i,j = 1,2,...N\,,
\end{aligned}$$

in the sense of measures. Moreover, we assume $\alpha_J \in L^\infty_{loc}([0,\infty), L^1_{loc}(\mathbb{R}^n))$. Then, for any $T \ge 0$, $x_0 \in \mathbb{R}^N$, $R > 0$, $\delta > 0$, $\Delta > 0$, $\nu \ge 0$, and for $M = \max_{0\le i\le N}|U_i|$, $B_t = B(x_0, R + M(T-t) + \Delta + \nu)$, we have

$$\begin{aligned}
&\int_{|x-x_0|<R} |w(x,T) - \bar{w}(x,T)| + \sum_i |z_i(x,T) - \bar{z}_i(x,T)|dx\\
&\qquad\le \int_{B_0} |w(x,0) - \bar{w}(x,0)| + \sum_i |z_i(x,0) - \bar{z}_i(x,0)|dx\\
&\qquad+ C(E^t + E^x + E^K + E^J + E^H + E^L)\,,
\end{aligned}\tag{A.12}$$

where C is a uniform constant,

$$E^t = \max_{t=0,T} \sup_{0<s-t<\delta} \int_{B_t} |w(x,s) - w(x,t)| + \sum_i |z_i(x,s) - z_i(x,t)| dx \,,$$

$$E^x = \max_{t=0,T} \sup_{|h|<\Delta} \int_{B_t} |w(x+h,t) - w(x,t)| + \sum_i |z_i(x+h,t) - z_i(x,t)| dx \,,$$

and E^K, E^H, E^L and E^J satisfy (A.10 c-e).

Proof. In contrast to Theorem A.1, the terms $\frac{1}{\varepsilon} G_i(w, z_i)$ cannot be treated as errors because ε is not necessarily small. We first perform a doubling of variables in (A.6) and (A.11). Given two functions $\Phi \in C_c^\infty((0,\infty)\times\mathbb{R}^N)$ and $\zeta \in C_c^\infty((-\infty,0)\times\mathbb{R}^N)$, with $\Phi \geq 0$, $\zeta \geq 0$, we set

$$\phi(x,t,y,s) = \Phi(x,t)\zeta(x-y,t-s) \geq 0 \,. \tag{A.13}$$

Consider the weak form of (A.11) for the test function $\Psi = \phi(\cdot,\cdot,y,s)$ with $(y,s) \in \mathbb{R}^N \times (0,\infty)$ fixed and $\kappa = w(y,s)$, $\lambda_i = z_i(y,s)$. Similarly, consider the entropy inequality (A.6) for the test function $\Psi = \phi(x,t,\cdot,\cdot)$, for $(x,t) \in \mathbb{R}^N \times (0,\infty)$ fixed and $\kappa = \bar{w}(x,t)$, $\lambda_i = \bar{z}_i(x,t)$. We add the two relations, integrate with respect to all variables and use the relations $\partial_t\zeta = -\partial_s\zeta$ and $\nabla_x\zeta = -\nabla_y\zeta$ to obtain

$$\begin{aligned}
-\iiiint \zeta(t-s,x-y)\Big[&|w(y,s)-\bar{w}(x,t)|\big(\Phi_t + U_0\nabla_x\Phi\big)(x,t) \\
&+ \sum_i |z_i(y,s)-\bar{z}_i(x,t)|\big(\Phi_t + U_i\nabla_x\Phi\big)(x,t)\Big] dsdtdydx \\
&\leq \frac{1}{\varepsilon}\iiiint \phi(x,t,y,s)\sum_i \Big(G_i\big(w(y,s),z_i(y,s)\big) - G_i\big(\bar{w}(x,t),\bar{z}_i(x,t)\big)\Big) \\
&\quad\times\Big[\operatorname{sgn}\big(w(y,s)-\bar{w}(x,t)\big) - \operatorname{sgn}\big(z_i(y,s)-\bar{z}_i(x,t)\big)\Big] dsdtdydx + R^\alpha \,,
\end{aligned} \tag{A.14}$$

where

$$\begin{aligned}
R^\alpha = \iiiint &\alpha_K(t,x)\phi(t,x,s,y) + \alpha_J(t,x)|\partial_t\phi(t,x,s,y)| \\
&+ \sum_j \alpha_H^j(t,x)|\partial_{x_j}\phi(t,x,s,y)| + \sum_{1\leq i,\, j\leq N} \alpha_L^{ij}(t,x)|\partial^2_{x_i x_j}\phi(t,x,s,y)| dsdtdydx \,.
\end{aligned}$$

The monotonicity assumptions on G_i imply

$$\begin{aligned}
\sum_i &\Big(G_i\big(w(y,s),z_i(y,s)\big) - G_i\big(\bar{w}(x,t),\bar{z}_i(x,t)\big)\Big) \\
&\times\Big[\operatorname{sgn}\big(w(y,s)-\bar{w}(x,t)\big) - \operatorname{sgn}\big(z_i(y,s)-\bar{z}_i(x,t)\big)\Big] \leq 0 \,,
\end{aligned}$$

and thus, for any $\varepsilon > 0$,

$$\begin{aligned}
-\iiiint \zeta(t-s,x-y)\Big[&\Big(|w(y,s)-\bar{w}(x,t)| + \sum_i |z_i(y,s)-\bar{z}_i(x,t)|\Big)\Phi_t(x,t) \\
&+ \Big(U_0|w(y,s)-\bar{w}(x,t)| + \sum_i U_i|z_i(y,s)-\bar{z}_i(x,t)|\Big)\cdot\nabla_x\Phi(x,t)\Big] dsdtdydx \leq R^\alpha \,.
\end{aligned} \tag{A.15}$$

Although (A.15) involves differences of $N+1$ functions, because of the Lipshitz continuity of the flux relative to the conserved quantity,

$$\Big|U_0|w-\bar{w}|+\sum_i U_i|z_i-\bar{z}_i|\Big| \leq \max_{i=0,\ldots,N}|U_i|\Big(|w-\bar{w}|+\sum_i|z_i-\bar{z}_i|\Big),$$

we may conclude the proof by selecting suitable Φ and ζ and reproducing the steps in the proof of Theorem 2.1 in Ref. 23. □

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