SHOCK PROFILES AND SELF-SIMILAR FLUID DYNAMIC LIMITS

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§1. Introduction

Being among the simplest models in the kinetic theory of gases, the Broadwell model has served as a paradigm to understand the phenomenon of relaxation and the transition from a microscopic to a macroscopic description of gases. It consists of the system of semilinear hyperbolic equations

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} &= \frac{1}{\varepsilon}\Big(f_3^2 - f_1 f_2 \Big) \\ \frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} &= \frac{1}{\varepsilon}\Big(f_3^2 - f_1 f_2 \Big) \\ \frac{\partial f_3}{\partial t} &= -\frac{1}{2\varepsilon}\Big(f_3^2 - f_1 f_2 \Big) , \end{aligned} \tag{1.1}$$

and derives from a six-velocity model when specializing to one-dimensional flows, for which the densities of particles moving in directions orthogonal to the flow are all equal. (We refer to Broadwell [B] or to Platkowski and Illner [PI] for the derivation in the kinetic theory context). The function $f = (f_1, f_2, f_3)$ is defined for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ and describes densities of particles: f_1 for particles moving in the positive x-direction, f_2 in the negative x-direction and f_3 in each of the positive or negative y- or z-directions; as a consequence we confine to solutions with $f_j > 0$. The parameter ε stands for the mean free path, a measure of the average distance between successive collisions. The right hand side in (1.1) is called the collision operator and measures the rate of gain (or loss) in densities of particles effected through collisions. It is characterized by the quantity

$$Q(f) = f_3^2 - f_1 f_2 \, . \tag{1.2}$$

The zeroes of $Q(f)$ are the states of equilibrium for the system, $f_3^2 = f_1 f_2$, and are called Maxwellians. Finally, associated with each f are the quantities $\rho_f = f_1 + f_2 + 4 f_3$, $m_f = f_1 - f_2$, measuring the local density and momentum flux in the $x-$direction, respectively.

The limit when the mean free path approaches zero is known as the fluid dynamic limit. For small mean free path the strong interactions of particles allow a macroscopic description of the flow to become meaningful. In the case of the Broadwell model the induced macroscopic "Euler equations" are easy to identify. First rewrite (1.1) in the form

$$\begin{aligned} \frac{\partial}{\partial t}(f_1+f_2+4f_3)+\frac{\partial}{\partial x}(f_1-f_2)=0\,,\\ \frac{\partial}{\partial t}(f_1-f_2)+\frac{\partial}{\partial x}(f_1+f_2)=0\,,\\ \frac{\partial f_3}{\partial t}=-\frac{1}{2\varepsilon}(f_3^2-f_1f_2)\,. \end{aligned}\tag{1.3}$$

It is formally expected that as $\varepsilon\to 0$ the first two equations remain unchanged while the third causes the limit of f to be a local Maxwellian. If we denote by (F_1,F_2,F_3) the limit of f, then $F_3=(F_1F_2)^{1/2}$ and $F=(F_1,F_2)$ satisfies the limiting fluid equations

$$\begin{aligned} \frac{\partial}{\partial t}\Big(F_1+F_2+4(F_1F_2)^{1/2}\Big)+\frac{\partial}{\partial x}(F_1-F_2)=0\,,\\ \frac{\partial}{\partial t}(F_1-F_2)+\frac{\partial}{\partial x}(F_1+F_2)=0\,. \end{aligned}\tag{1.4}$$

The corresponding macroscopic density and momentum of the fluid are given by

$$\rho=F_1+4(F_1F_2)^{1/2}+F_2\,,\qquad m=\rho u=F_1-F_2\,.\tag{1.5}$$

The algebraic system can be easily inverted and leads to an alternative form of the limit "Euler equations", in terms of the macroscopic variables $(\rho\,,\,u)$,

$$\begin{aligned} \frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0\,,\\ \frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho g(u))=0\,, \end{aligned}\tag{1.6}$$

where $g(u):=\frac{1}{3}\Big[2(1+3u^2)^{1/2}-1\Big]$. They form a strictly hyperbolic, genuinely nonlinear system of conservation laws (Caflisch [C]).

The justification of the fluid dynamic limit has been the object of several investigations. We refer to Cercignani [Ce] for a survey of the literature on the Boltzmann equation and to Platkowski and Illner [PI] for results on discrete velocity models of kinetic theory. For the Broadwell model, the fluid dynamic limit is understood for smooth solutions of the limit "Euler equations" (Inoue and Nishida [NI], Caflisch and Papanicolaou [CP]). Regarding the case of solutions with shocks, we mention the studies on stability (in time) of traveling wave solutions (Kawashima and Matsumura [KM]) or rarefaction wave solutions (Matsumura [M]) for the Broadwell model, and a recent study by Xin [X], showing that a given piecewise smooth solution with noninteracting shocks of the limit fluid equations can be approximated by solutions of the Broadwell system as $\varepsilon \to 0$, that gives a definitive answer to one direction of the problem. The converse problem, to show that a given family of solutions to the Broadwell system converges globally in time to a fluid-dynamical solution, remains at present open.

Insight in the latter direction is provided by the approach of self-similar fluid dynamic limits, introduced in Slemrod and Tzavaras [ST] and further studied in Tzavaras [T] and Fan [F]. For Riemann, Maxwellian data

$$f(x,0) = \begin{cases} f^+, & x > 0 \\ f^-, & x < 0 \end{cases} \qquad \text{with } f^\pm = (f_1^\pm, f_2^\pm, f_3^\pm) \tag{1.7}$$

$$Q(f^-) = Q(f^+) = 0\,, \quad f_1^\pm\,,\ f_2^\pm\,,\ f_3^\pm\ > 0\,, \tag{M}$$

admissible solutions of the limit fluid equations (1.4) are expected to be self-similar functions of $\xi = x/t$. On the other hand, the Broadwell system does not possess space-time dilational invariance and does not admit self-similar solutions of that type. Motivated by an analogous idea for systems of conservation laws (Dafermos [D]), one considers a modified Broadwell system

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} &= \frac{1}{\varepsilon t}(f_3^2 - f_1 f_2)\,, \\ \frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} &= \frac{1}{\varepsilon t}(f_3^2 - f_1 f_2)\,, \\ \frac{\partial f_3}{\partial t} &= -\frac{1}{2\varepsilon t}(f_3^2 - f_1 f_2)\,, \end{aligned} \tag{1.8}$$

which does preserve the invariance under dilations $(x,t) \to (ax, at)$, $a > 0$. The problem $(1.7-1.8)$ admits self-similar solutions of the form $f = f(x/t)$, that are constructed by solving the singular boundary value problem

$$\begin{aligned} -\xi f_1' + f_1' &= \frac{1}{\varepsilon}(f_3^2 - f_1 f_2) \\ -\xi f_2' - f_2' &= \frac{1}{\varepsilon}(f_3^2 - f_1 f_2)\,, \qquad \text{for } \xi \in [-1,1] \\ -\xi f_3' &= -\frac{1}{2\varepsilon}(f_3^2 - f_1 f_2) \\ f(-1) = f^-\ ,\ f(+1) &= f^+ \end{aligned} \tag{$\mathcal{P}_\varepsilon$}$$

subject to data $f^\pm$ satisfying (M). There are three goals to be attained: (i) To construct solutions f^ε of the problem $(\mathcal{P}_\varepsilon)$ for $\varepsilon > 0$ fixed. (ii) To show that as $\varepsilon \to 0+$ the solutions f^ε converge a.e to a local Maxwellian F solution of the Riemann problem (1.4, 1.7). (iii) To investigate the structure induced on F from emerging through self-similar fluid dynamic limits.

In this article we present a survey of applications of the method to the Broadwell model ([ST], [T] and [F]) and comparisons with studies of shock profiles ([B], [C]). The hope is the approach will be useful in studying relevant questions for other kinetic theory models or for hyperbolic conservation laws with relaxation terms. The presence of relaxation mechanisms is natural in many physical contexts and has been investigated extensively (*e.g.* Liu [L], Chen, Levermore and Liu [CLL]), both in the realm of kinetic theories as well as for models that arise in various branches of continuum physics. One objective is to obtain a quantitative understanding of the regularizing effect induced on shocks by relaxation. In this direction, we mention the comparison of shock profiles with traveling wave solutions of an associated system of viscous conservation laws arising from (1.1) via the Chapman-Enskog expansion [C]. An important difference of the self-similar relaxation investigated here is that it penalizes the whole wave fan simultaneously. Comparisons between the structure entailed by self-similar limits and the Broadwell shock profiles are carried out in the text.

§2. The limiting fluid equations

First certain properties of the limit fluid equations (1.4) for solutions $F_i > 0$ are presented. These properties are discussed in Caflisch [C]; we give an independent presentation for completeness and to account for extended differences in notation. The characteristic speeds $\lambda_1(F)$, $\lambda_2(F)$ are the solutions of the binomial

$$(F_1 + F_2 + \sqrt{F_1 F_2})\,\lambda_i{}^2 - (F_1 - F_2)\,\lambda_i - \sqrt{F_1 F_2} = 0\,; \tag{2.1}$$

they are real and are given by the expressions

$$\lambda_{1,2}(F) = \frac{1}{2\,(F_1 + F_2 + \sqrt{F_1 F_2})}\left[F_1 - F_2 \pm \sqrt{\left(F_1 + F_2 + 2\sqrt{F_1 F_2}\,\right)^2 - 4F_1 F_2}\;\right], \tag{2.2}$$

where λ_1 corresponds to the minus sign and λ_2 to the plus. One easily checks

$$-1 < \lambda_1(F) < 0 < \lambda_2(F) < 1\,, \qquad \lambda_1(F) < \frac{F_1 - F_2}{F_1 + F_2 + 4\sqrt{F_1 F_2}} = u < \lambda_2(F) \tag{2.3}$$

The corresponding right eigenvectors are given by

$$r_i = \begin{bmatrix} \lambda_i + 1 \\ \lambda_i - 1 \end{bmatrix}, \qquad i = 1,\,2\,.$$

Upon differentiating (2.1) we can express $\frac{\partial\lambda_i}{\partial f_1}$, $\frac{\partial\lambda_i}{\partial f_2}$ in terms of λ_i, and use the resulting relations to compute

$$\nabla\lambda_i \cdot r_i = \frac{(1-{\lambda_i}^2)}{2\sqrt{F_1F_2}} \left(\frac{(F_1+F_2+4\sqrt{F_1F_2})\,\lambda_i - (F_1-F_2)}{(F_1+F_2+\sqrt{F_1F_2})\,2\,\lambda_i - (F_1-F_2)} \right). \tag{2.4}$$

It follows from (2.2), (2.3) and (2.4) that $\nabla\lambda_i \cdot r_i > 0$ for $i = 1, 2$, and the system (1.4) is strictly hyperbolic and genuinely nonlinear.

The shock curves for (1.4) are defined by solving the Rankine-Hugoniot conditions, expressed as the equivalent algebraic system

$$\begin{aligned} -s\,(\,u_1 + 2\,u_3\,) + u_1 &= -s\,(\,w_1 + 2\,w_3\,) + w_1 \\ -s\,(\,u_2 + 2\,u_3\,) - u_2 &= -s\,(\,w_2 + 2\,w_3\,) - w_2 \\ {u_3}^2 - u_1\,u_2 &= {w_3}^2 - w_1\,w_2 = 0 \end{aligned} \tag{2.5}$$

for the states $u = (u_1, u_2, u_3)$ and $w = (w_1, w_2, w_3)$ and the shock speed s. To this end fix one state, say w, with $w_3 = \sqrt{w_1 w_2}$ and consider the increments $[u_i] = u_i - w_i$. Then

$$2\,s\,[u_3] = (1-s)\,[u_1] = -(1+s)\,[u_2] \tag{2.6}$$

$$[u_3]\,([u_3] + 2\,w_3) = [u_1]\,[u_2] + w_2\,[u_1] + w_1\,[u_2] \tag{2.7}$$

Substitution of (2.6) into (2.7) yields

$$[u_3]\left[(1+3\,s^2)\,[u_3] - 2\left[\,(w_1+w_2+w_3)\,s^2 - (w_1-w_2)\,s - w_3\,\right]\right] = 0$$

If $[u_3] = 0$ there is no shock. Thus, using (2.1) and (2.6), we obtain a representation of the shock curve parametrized by the shock speed s in the form

$$\begin{aligned} [u_3] &= \frac{2\,(w_1+w_2+\sqrt{w_1w_2})}{1+3\,s^2}\,\big(s-\lambda_1(w)\big)\,\big(s-\lambda_2(w)\big)\,, \\ [u_1] &= \frac{2s}{1-s}[u_3]\,, \quad [u_2] = -\frac{2s}{s+1}[u_3]\,. \end{aligned} \tag{2.8}$$

Given a triplet $(u, w; s)$ satisfying (2.8) there are two associated shock solutions with speed s of (1.4) one with left state w and right state u and one with the states reversed.

§3. Shock profiles for the Broadwell system

The regularizing effect induced on shocks by relaxation can be understood by studying traveling wave solutions for (1.1) (*c.f.* Broadwell [B], Caflisch [C]). If we introduce the ansatz

$$f_j^\varepsilon = v_j(\frac{x-st}{\varepsilon})\,; \quad \tau = \frac{x-st}{\varepsilon}\,, \quad v_j(\pm\infty) = f_j^\pm \tag{3.1}$$

and look for a traveling wave connecting two end states f_- and f_+, then v_j satisfy the system of differential equations

$$(-s+1)\frac{dv_1}{d\tau} = Q(v)\,, \quad -(s+1)\frac{dv_2}{d\tau} = Q(v)\,, \quad -s\frac{dv_3}{d\tau} = -\frac{1}{2}Q(v)\,. \tag{3.2}$$

After some simple manipulations, it turns out that $v = (v_1, v_2, v_3)$ must satisfy the set of equations

$$2s\,(v_3(\tau) - f_3^-) = (1-s)\,(v_1(\tau) - f_1^-) = -(1+s)\,(v_2(\tau) - f_2^-)\,, \tag{3.3}$$

$$\frac{dv_3}{d\tau} = \frac{1}{2s}(v_3^2 - v_1 v_2)\,, \tag{3.4}$$

that $v(\pm\infty)$ must be Maxwellian states satisfying the Rankine-Hugoniot conditions (2.5), and thus the jumps $[f_j] = f_j^+ - f_j^-$ of the data are connected with the shock speed s through relations (2.8). Equation (3.4) may be written in the form

$$\frac{dv_3}{d\tau} = \frac{1+3s^2}{2s(1-s^2)}(v_3 - f_3^-)(v_3 - f_3^+)\,, \tag{3.5}$$

the solution of which is given by the explicit formula

$$v_3(\tau) = \frac{f_3^+ + f_3^-}{2} + \frac{f_3^+ - f_3^-}{2}\,\tanh\left(\frac{1+3s^2}{4s(s^2-1)}\,(f_3^+ - f_3^-)\,(\tau - \tau_0)\right) \tag{3.6}$$

where the constant τ_0 determines the shift of the shock profile. Then $v_1(\tau)$ and $v_2(\tau)$ are obtained from (3.3).

The question arises which shock solutions of (1.4) have associated shock profiles. From (3.6), a triplet $(f^+, f^-; s)$ satisfying (2.8) admits a shock profile with f^- as left state and f^+ as right state if and only if $s(s^2-1)(f_3^+ - f_3^-) > 0$. It follows, upon using (2.8) to express $[f_3]$ (with $w = f_3^-$ and with $w = f_3^+$) and (2.3), that the latter condition is equivalent to

$$\begin{aligned} &\lambda_1(f^+) < s < \lambda_1(f^-) \qquad \text{for } s < 0\,, \\ &\lambda_2(f^+) < s < \lambda_2(f^-) \qquad \text{for } s > 0\,. \end{aligned} \tag{3.7}$$

These are the Lax shock conditions for (1.4).

§4. Self-similar fluid dynamic limits

We turn now to our main objective the study of the system $(\mathcal{P}_\varepsilon)$ and its limits $\varepsilon \to 0$. Regarding the question of existence of solutions we have

Theorem 1 (Slemrod and Tzavaras [ST], Fan [F]). *For any $f^{\pm}$ satisfying (M) and $\varepsilon > 0$ the boundary-value problem $(\mathcal{P}_\varepsilon)$ has a solution f^ε continuously differentiable in $(-1,0) \cup (0,1)$ and Hölder continuous with exponent $0 < \alpha_\varepsilon \leq 1$ at the singular points $\xi = -1, 0, +1$. The regularity improves as ε decreases and for $\varepsilon < \varepsilon_0$ the function f^ε is Lipshitz continuous on $[-1,1]$.*

The functions f^ε are extended to the whole real line by setting $f^\varepsilon = f^-$ on $(-\infty, -1)$ and $f^\varepsilon = f^+$ on $(1, \infty)$. Regarding the self-similar fluid-dynamic limit we have

Theorem 2 (Slemrod and Tzavaras [ST]). *Let $\{f^\varepsilon\}_{\varepsilon > 0}$ be a family of extended solutions of $(\mathcal{P}_\varepsilon)$ corresponding to data $f^{\pm}$ subject to (M). Then:*

(i) *There are positive constants m_j, M_j and K_j, $j = 1,2,3$, depending on the boundary data $f^{\pm}$ but independent of ε such that*

$$0 < m_j \leq f_j^\varepsilon(\xi) \leq M_j\,, \qquad \xi \in [-1,1] \tag{4.1}$$

$$TV_{[-1,1]}\, f_j^\varepsilon \leq K_j \tag{4.2}$$

(ii) *There exists a subsequence $\{f^{\varepsilon_n}\}$ with $\varepsilon_n \to 0$ and a positive, bounded function F of bounded variation such that $f^{\varepsilon_n} \to F$ pointwise on the reals. The function F satisfies*

$$F_3 = \sqrt{F_1\, F_2} \qquad \textit{for a.e. } \xi \in [-1,1] \tag{4.3}$$

and the balance of mass and momentum equations

$$\begin{aligned} -\xi \frac{d}{d\xi}(F_1 + F_2 + 4(F_1 F_2)^{1/2}) + \frac{d}{d\xi}(F_1 - F_2) = 0\,, \\ -\xi \frac{d}{d\xi}(F_1 - F_2) + \frac{d}{d\xi}(F_1 + F_2) = 0\,, \end{aligned} \tag{4.4}$$

in the sense of measures.

Remarkably the problem of existence is more difficult than performing the $\varepsilon \to 0$ limit. There are two approaches to tackle the existence question. In [ST] a Fredholm alternative type of theory for singular boundary value problems is developed. The theory has a wide range of potential applicability, but at the final stage one must show that a certain linear but non-autonomous boundary-value problem has no eigensolutions. As there are no general techniques for such questions this is a technical obstacle. For the Carleman model this method works for any Maxwellian data, but for the Broadwell restrictions on $f^{\pm}$, of technical nature, had to be imposed. These were removed in [F], where the existence question is addressed by first desingularizing the system and then taking a dynamical systems point of view for the resulting boundary-value problem. Fan [F] fixes $f^{\pm}$ and uses a shooting method to construct solutions on $(-1,0)$ and $(0,1)$ separately, and then shows that the traces of the constructed solutions at $\xi = 0$ intersect transversally.

The uniform variation estimates are based on the following observation. The collision operator $Q(f^\varepsilon)$ satisfies the differential equation

$$\frac{dQ(f^\varepsilon)}{d\xi} = \frac{1}{\varepsilon}\Big(\frac{f_1^\varepsilon}{\xi+1} - \frac{f_2^\varepsilon}{1-\xi} + \frac{f_3^\varepsilon}{\xi}\Big)\,Q(f^\varepsilon) = \frac{1}{\varepsilon}\,c^\varepsilon\,Q(f^\varepsilon)\,. \tag{4.5}$$

The uniqueness theorem for differential equations implies that on each of the intervals $(-1,0)$ and $(0,1)$ either $Q(f^\varepsilon)$ is identically zero or it never vanishes. $(\mathcal{P}_\varepsilon)$ in turn implies that the f_j^ε are monotone or constant on the respective intervals, and (4.1), (4.2) follow by a case analysis and a use of the balance of mass equation for the L^∞ bounds. One also obtains the uniform estimate

$$\int_{-1}^{1} \Big|\frac{Q(f^\varepsilon)}{\varepsilon}\Big|\,d\tau \le C \tag{4.6}$$

Helly's theorem implies there exists a subsequence $\{f^{\varepsilon_n}\}$ and a function F of bounded variation so that $f^{\varepsilon_n} \to F$ pointwise on $\mathbb{R}$. Use of (4.1) and (4.6) shows that the components $F_j > 0$ and F is a local Maxwellian. Passing to the limit $\varepsilon \to 0$ in $(\mathcal{P}_\varepsilon)$, we deduce F satisfies (4.4) in the sense of measures.

For a function of bounded variation the right and left limits $F(\xi-)$, $F(\xi+)$ exist at any ξ and its domain can be decomposed into two disjoint subsets : $\mathcal{C}$ the points of continuity of F, and $\mathcal{S}$ the points of jump discontinuity of F, with $\mathcal{S}$ at most countable. The components F_j inherit the monotonicity properties of f_j^ε. On account of (4.6)

$$\begin{aligned} F_3(\xi) &= \big(F_1(\xi)\,F_2(\xi)\big)^{1/2}\,, \qquad \xi \in \mathcal{C}\,,\\ F_3(\xi\pm) &= \big(F_1(\xi\pm)\,F_2(\xi\pm)\big)^{1/2}\,, \qquad \xi \in \mathcal{S}\,. \end{aligned} \tag{4.7}$$

(4.4) implies that at any point $\xi \in \mathcal{S}$ the Rankine-Hugoniot conditions are satisfied

$$\begin{aligned} -\xi\,\big(\,[F_1 + F_2 + 4\sqrt{F_1F_2}\,]\,\big|_{\xi-}^{\xi+}\big) + [F_1 - F_2]\,\big|_{\xi-}^{\xi+} &= 0\\ -\xi\,\big(\,[F_1 - F_2]\,\big|_{\xi-}^{\xi+}\big) + [F_1 + F_2]\,\big|_{\xi-}^{\xi+} &= 0 \end{aligned} \tag{4.8}$$

By construction $F = f_-$ on $(-\infty,-1]$ and $F = f_+$ on $[1,\infty)$. The function $\Big(F_1(\frac{x}{t}), F_2(\frac{x}{t})\Big)$ is easily seen to be a weak solution of the Riemann problem (1.4), (1.7).

§5. Structure induced by fluid dynamic limits

In the previous section we outlined the construction of a solution to the Riemann problem (1.4), (1.7) via self-similar fluid dynamic limits. In this section we discuss the structure of the resulting limit, emphasizing the behavior at points of discontinuity. We outline the main ingredients of the technique and refer to Tzavaras [T] for details of the presentation.

First, the appropriate framework for passing to the $\varepsilon \to 0$ limit in $(\mathcal{P}_\varepsilon)$ is that of measures. Consider the functions

$$\Phi^\varepsilon(\xi) = \int_{-\infty}^{\xi} \frac{Q(f^\varepsilon(\tau))}{\varepsilon}\, d\tau \tag{5.1}$$

and note that Φ^ε takes constant values outside $[-1, 1]$. The sequence $\{\Phi^\varepsilon\}$ is of uniformly bounded variation on $\mathbb{R}$. Helly's selection principle implies there exists a subsequence, denoted again by $\{\Phi^\varepsilon\}$, and a function of bounded variation Φ such that $\Phi^\varepsilon \to \Phi$ pointwise on $\mathbb{R}$. In turn, Helly's convergence theorem implies

$$< \nu^\varepsilon, \varphi >:= \int \varphi \frac{Q(f^\varepsilon)}{\varepsilon}\, d\xi = \int \varphi\, d\Phi^\varepsilon \to \int \varphi\, d\Phi =:< \nu, \varphi > \tag{5.2}$$

for any $\varphi \in C_c(\mathbb{R})$, continuous function with compact support. By the Riesz representation theorem ν^ε, ν may be viewed as finite (signed) Borel-Stieltjes measures, both supported on $[-1, 1]$, ν^ε is generated by Φ^ε and ν is generated by $\bar{\Phi}$, the right continuous version of Φ defined by $\bar{\Phi}(x) = \Phi(x+)$. Equation (5.2) states that $\nu^\varepsilon \rightharpoonup \nu$ in the weak-$\star$ topology of measures. It allows to pass to the limit $\varepsilon \to 0$ in $(\mathcal{P}_\varepsilon)$ and obtain, for any test function $\varphi \in C_c^1(\mathbb{R})$,

$$\begin{aligned} \int F_1 \left((\xi - 1)\, \varphi \right)' d\xi &=< \nu, \varphi > \\ \int F_2 \left((\xi + 1)\, \varphi \right)' d\xi &=< \nu, \varphi > \\ \int F_3 \left(\xi\, \varphi \right)' d\xi &= -\frac{1}{2} < \nu, \varphi > \end{aligned} \tag{5.3}$$

with $F_3 = \sqrt{F_1 F_2}$. It is suggested by (5.3) and can be justified by an analysis near the singular points that $\operatorname{supp} \nu$ is precisely the set of points where f is not a constant state. Equation (5.3) implies the weak form (4.4), but carries additional information if further properties of ν are known.

The second ingredient is a representation formula for $Q(f^\varepsilon)/\varepsilon$. Using (4.5) and $(\mathcal{P}_\varepsilon)$ it is shown in [T] that

$$\frac{Q(f^\varepsilon)}{\varepsilon} = \begin{cases} (f_1^\varepsilon(0) - f_1^-)\, \mu_-^\varepsilon & \text{on } (-1, 0) \\ (f_2^\varepsilon(0) - f_2^+)\, \mu_+^\varepsilon & \text{on } (0, 1) \end{cases} \tag{5.4}$$

where μ_-^ε, μ_+^ε are given by

$$\mu_-^\varepsilon(\xi) = \frac{e^{\frac{1}{\varepsilon}\,g^\varepsilon(\xi)}}{\int_{-1}^{0} \frac{1}{1-\zeta}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta} = \frac{\exp\{\frac{1}{\varepsilon}\int_{\alpha_-}^{\xi} c^\varepsilon(s)\, ds\}}{\int_{-1}^{0} \frac{1}{1-\zeta}\, \exp\{\frac{1}{\varepsilon}\int_{\alpha_-}^{\zeta} c^\varepsilon(s)\, ds\}\, d\zeta} \quad \xi \in (-1,0)\,, \tag{5.5}$$

$$\mu_+^\varepsilon(\xi) = \frac{e^{\frac{1}{\varepsilon}\,g^\varepsilon(\xi)}}{\int_{0}^{1} \frac{1}{\zeta+1}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta} = \frac{\exp\{\frac{1}{\varepsilon}\int_{\alpha_+}^{\xi} c^\varepsilon(s)\, ds\}}{\int_{0}^{1} \frac{1}{\zeta+1}\, \exp\{\frac{1}{\varepsilon}\int_{\alpha_+}^{\zeta} c^\varepsilon(s)\, ds\}\, d\zeta} \quad \xi \in (0,1)\,, \tag{5.6}$$

and α_-, α_+ are any fixed points with $-1 < \alpha_- < 0 < \alpha_+ < 1$. Note that because of their form (along subsequences) $\mu_+^\varepsilon \rightharpoonup \mu_+$ and $\mu_-^\varepsilon \rightharpoonup \mu_-$ weak-$\star$ in measures. We deduce from (5.2) and (5.4) that ν arises as a limit of probability measures and $\operatorname{supp}\nu \subset \operatorname{supp}\mu_- \cup \operatorname{supp}\mu_+$.

Remarkably, the function c^ε in (5.5 – 5.6) and (4.5) is connected to the wave speeds of the hyperbolic system (1.4), as it can be expressed in the form

$$c^\varepsilon = -\frac{1}{(1-\xi^2)\,\xi}\,\left(f_1^\varepsilon + f_2^\varepsilon + f_3^\varepsilon\right)\left(\xi - \lambda_1(f^\varepsilon)\right)\left(\xi - \lambda_2(f^\varepsilon)\right), \tag{5.7}$$

where $\lambda_{1,2}(f)$ are given by

$$\lambda_{1,2}(f) = \frac{1}{2\,(f_1+f_2+f_3)}\left[f_1 - f_2 \pm \sqrt{(f_1+f_2+2f_3)^2 - 4f_1f_2}\,\right]. \tag{5.8}$$

A comparison of (5.8) and (2.2) shows that $\lambda_{1,2}(f)$ coincide with the wave speeds $\lambda_{1,2}(F)$ along Maxwellian states. Hence, along the convergent sequence $f^\varepsilon \to F$,

$$c^\varepsilon \to c\; = -\frac{1}{(1-\xi^2)\,\xi}\,\left(F_1+F_2+F_3\right)\left(\xi - \lambda_1(F)\right)\left(\xi-\lambda_2(F)\right) \tag{5.9}$$

pointwise on $(-1,0)\cup(0,1)$.

Use of the above ingredients, together with an analysis of the behavior near the singular points, gives ([T], Lemma 4.3, Proposition 4.4)

Proposition 3 . *There are constants $\lambda_{1\pm}$, $\lambda_{2\pm}$ with $-1 < \lambda_{1-} < \lambda_{1+} < 0 < \lambda_{2-} < \lambda_{2+} < 1$ such that* $\operatorname{supp}\nu \subset \left(\operatorname{supp}\mu_- \cup \operatorname{supp}\mu_+\right) \subset [\lambda_{1-},\lambda_{1+}]\cup[\lambda_{2-},\lambda_{2+}]$. *Moreover*

(i) *If $\xi \in \operatorname{supp}\mu_+$ then $\int_{\alpha_+}^{\zeta} c(s)ds \le \int_{\alpha_+}^{\xi} c(s)ds$ for any $\zeta \in (0,1)$*

(ii) *If $\xi \in \operatorname{supp}\mu_-$ then $\int_{\alpha_-}^{\zeta} c(s)ds \le \int_{\alpha_-}^{\xi} c(s)ds$ for any $\zeta \in (-1,0)$.*

The maximization properties (i) and (ii) capture the effect induced on shocks by self-similar fluid dynamic limits. To illustrate, fix $\xi < 0$, $\xi \in \mathcal{S}$. Then $\xi \in \operatorname{supp}\mu_-$, the function g achieves its global maximum in $(-1,0)$ at ξ and thus $c(\xi+) \le 0$, $c(\xi-) \ge 0$. Using (5.9) and (2.3) we deduce

$\lambda_1(F(\xi+)) \leq \xi \leq \lambda_1(F(\xi-))$, a weak form of the Lax shock conditions. In fact the complete behavior of F can de characterized and the final result is stated in the first part of Theorem 4.

The last topic is to discuss the relation between self-similar limits and shock profiles. Let $\xi \in \mathcal{S}$ a point of discontinuity for F, and note that $F(\xi-) \neq F(\xi+)$ are Maxwellian states satisfying (4.8). Given a sequence of points $\{\xi_\varepsilon\}$ with $\xi_\varepsilon \to \xi$, estimates $(4.1-4.2)$ imply the functions $\{v_j^\varepsilon\}$ defined by $v_j^\varepsilon(\zeta) = f_j^\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta)$ are of uniformly bounded variation in the new variable $-\infty < \zeta < \infty$. This accounts for a shift of the shock in the original solution and the introduction of the stretched variable ζ. Helly's theorem and a diagonal argument in turn imply the existence of a subsequence and a function $v = (v_1, v_2, v_3)$ such that

$$f_j^\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) \to v_j(\zeta) \qquad \text{pointwise for } \; -\infty < \zeta < \infty\,. \tag{5.10}$$

Then v_j are solutions of the traveling wave equations (3.2) (*c.f.* second part of Theorem 4).

Theorem 4 (Tzavaras [T]). *The limiting function F constructed in Theorem 3 has the behavior: F stays constant on each connected component of* $\mathbb{R} - \operatorname{supp}\nu$. *Also there exist disjoint closed sets I_{λ_k}, with each I_{λ_k} associated with the k-th characteristic speed, such that* $\operatorname{supp}\nu = I_{\lambda_1} \cup I_{\lambda_2}$ *and*

(i) *either I_{λ_k} is empty, or*

(ii) *I_{λ_k} contains a single point in $\mathcal{S}$, in which case F is a shock wave on I_{λ_k} satisfying* (4.8) *and the Lax shock conditions, or*

(iii) *I_{λ_k} is a full interval of points in $\mathcal{C}$, in which case F is a k-rarefaction wave on I_{λ_k}.*

Given a point of discontinuity $\xi \in \mathcal{S}$, there exists a choice of the sequence $\{\xi_\varepsilon\}$ such that $v(\zeta)$ in (5.10) *is a (smooth) solution of* (3.2) *that satisfies* $\lim_{\zeta\to-\infty} v(\zeta) = F(\xi-)$, $\lim_{\zeta\to\infty} v(\zeta) = F(\xi+)$.

We remark that the proof is of purely analytical nature. Geometric properties such as the genuine nonlinearity of (1.4) and the form of the shock curves (2.8) are only used in the last step to exclude contact discontinuities and simplify the emerging structure of F.

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