KINETIC DECOMPOSITION FOR PERIODIC HOMOGENIZATION PROBLEMS

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Abstract. We develop an analytical tool which is adept for detecting shapes of oscillatory functions, is useful in decomposing homogenization problems into limit-problems for kinetic equations, and provides an efficient framework for the validation of multi-scale asymptotic expansions. The main new result concerns a linear hyperbolic homogenization problem which we transform to a hyperbolic limit problem for a kinetic equation. We establish conditions determining an effective equation and counterexamples for the case that such conditions fail. Second, we revisit some already known problems with our approach, applying in particular the kinetic decomposition to the problem of enhanced diffusion; It then leads to a diffusive limit problem for a kinetic equation that in turn yields the known effective equation of enhanced diffusion.

1. Introduction

Homogenization problems appear in various contexts of science and engineering and involve the interaction of two or more oscillatory scales. In this work we focus on the simplest possible mathematical paradigms of periodic homogenization. Our objective is to develop an analytical tool that is capable of understanding the shapes of periodic oscillatory functions when the scales of oscillations are a-priori known (or expected), and use it in order to transform the homogenization problem into a limit problem for a kinetic equation. The calculation of an effective equation becomes then an issue of studying a hyperbolic (or diffusive) limit for the kinetic equation. The procedure is well adapted in identifying the specific characteristics of the underlying homogenization problem and provides an efficient tool for the rigorous justification of multiscale asymptotic expansions.

The main new result of the paper concerns the homogenization of linear transport equation. Namely we provide an effective equation for

$$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t} + a\left(\frac{x}{\varepsilon}\right)\cdot \nabla_x u_\varepsilon = 0, \\ &\operatorname{div} a = 0, \end{aligned} \tag{1.1}$$

for the first time in any case and dimension. To further illustrate the properties and interest of the method, we next apply it to the problem of enhanced diffusion. In this case, the answer has already been known for some time and we of course recover the usual effective equation, in a slighty simplified way.

The main idea is motivated from considerations of kinetic theory. When the statistics of interacting particles is studied it is customary to introduce an empirical measure and to study its statistical properties in the (weak) limit when the number of particles gets large. Likewise, for an oscillating family of functions $\{u_\varepsilon\}$ if we want to study the shape of periodic oscillations at a predetermined scale we may introduce an inner variable that counts the content of oscillation at such scale. For instance, to count oscillations at the scale $\frac{x}{\varepsilon}$ one can introduce

$$f_\varepsilon(x,v) = u_\varepsilon(x)\delta_p(v - \frac{x}{\varepsilon}) \tag{1.2}$$

where δ_p is the periodic delta function, and study the family $\{f_\varepsilon\}$. A-priori bounds for $\{u_\varepsilon\}$ translate to uniform bounds for $\{f_\varepsilon\}$: if for example u_ε is uniformly bounded in L^2, $u_\varepsilon \in_b L^2$, then $f_\varepsilon \in_b L^2(M_p)$ and, along a subsequence,

$$f_\varepsilon \rightharpoonup f \quad \text{weak}\star \text{ in } L^2(M_p)\,, \tag{1.3}$$

where M_p stands for the periodic measures on T^d. In addition, the resulting f is better: $f \in L^2(L^2(\mathbb{T}^d))$.

The above object should be compared to the concept of double-scale limit introduced in the influential works of Nguetseng [19] and Allaire[1] and applied to a variety of homogenization problems [11, 17, 12, 2]. In the double-scale limit one tests the family $\{u_\varepsilon\}$ against oscillating test functions and develops a representation theory for the resulting weak-limits. It turns out, [19], that for a uniformly bounded family $u_\varepsilon \in_b L^2$ and test functions φ periodic in v

$$\int u_\varepsilon(x)\varphi(x, \frac{x}{\varepsilon})dx \to \int\int f(x,v)\varphi(x,v)dxdv \tag{1.4}$$

where $f \in L^2(L^2(\mathbb{T}^d))$. The reader should note that this is precisely the content of (1.2), (1.3), which thus provide an alternative interpretation to the double scale limit. However, what seems to have been missed, perhaps because Nguetseng's analysis [19] proceeds without writing down (1.2) but rather by establishing directly (1.4), is that the measures f_ε satisfy in their own right very interesting equations. This is a consequence of additional properties, like

$$\Big(\nabla_x + \frac{1}{\varepsilon}\nabla_v\Big) f_\varepsilon(x,v) = \nabla_x u_\varepsilon(x)\delta_p(v-\frac{x}{\varepsilon})\,, \tag{1.5}$$

obtained by applying differential operators that annihilate the singular measure. Properties like (1.5), in turn, suggest a procedure for embedding homogenization problems into limit problems for kinetic equations. In the sequel we develop this perspective, using as paradigms the problem of hyperbolic homogenization, and the problem of enhanced diffusion.

The double-scale limit [19] along with the technique of multiscale asymptotic expansions [6] have been quite effective in the development of homogenization theory with considerable progress in several contexts (*e.g.* [18], [1], [5], [11], [12], [17]). Other tools have also been used for the homogenization of linear hyperbolic problems: Among them are of course Young measures, developed by Tartar and used for the homogenization of some particular linear transport equations in two dimensions (see [22] and [23]). Wigner measures (see [13]) may also be mentioned. A closer approach to ours was introduced in [4] and [10]. However, in addition of being differently defined in various directions, the function based on u_ε introduced in [4] and [10] really doubles the variable whereas for our f_ε the variables x and v are still constrained : v is necessarily equal to x/ε mod. $\mathbb{T}^d$. For example the analysis in this paper could not be carried over with simply looking at $g_\varepsilon(x,v) = u_\varepsilon(x+\varepsilon v)$.

As our first example we consider the hyperbolic homogenization problem

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} + a\big(x,\frac{x}{\varepsilon}\big)\cdot\nabla_x u_\varepsilon = 0 \\ u_\varepsilon(0,x) = U^0(x,\frac{x}{\varepsilon})\,, \end{aligned} \tag{1.6}$$

with $a(x,v)$ a divergence free field periodic in v, is transformed to the problem of identifying the hyperbolic limit $\varepsilon \to 0$ of the kinetic initial-value problem

$$\begin{aligned} \frac{\partial f_\varepsilon}{\partial t} + a(x,v)\cdot\nabla_x f_\varepsilon + \frac{1}{\varepsilon}a(x,v)\cdot\nabla_v f_\varepsilon = 0, \\ f_\varepsilon(t=0,x,v) = U^0(x,v)\;\delta_p(v-\frac{x}{\varepsilon}) \end{aligned} \tag{1.7}$$

Homogenization for (1.6) has been studied by Brenier [7] (who solved the ergodic case), and Hou and Xin [17] (2d case) and, in fact, the effective equation is sought - motivated by the double-scale limit - in a class of kinetic equations. E's approach in [11] offers an interesting comparison : It essentially consists in choosing tests functions depending on ε such as to obtain (1.7) integrated over those test functions. However the direct definition of f_ε followed in this paper avoids many pitfalls of [11]. First of all we do not require that the C^∞ functions of the kernel K_x (see (1.8) below) be dense in K_x; This is a delicate property to check (as Fredholm alternative does not hold in general for the operator $a \cdot \nabla_v$) and it is better to be able to proceed without it. Next manipulating Equation (1.7) is easier, making it simpler to avoid algebraic mistakes. Finally our method highlights the difference between the case $a(x,v) = a(v)$ for which we can always give the limit equation and the case where a depends on x which can be ill-posed (at least from the point of view of the double scale limit), see 3.2.

Eq. (1.6) is by no means the only interesting hyperbolic problem for homogenization; we refer to [3], [15], [14] (where a kinetic equation itself is homogenized), and to [2] for an example concerning a Schrödinger equation (the list is of course not exhaustive).

For (1.6), our analysis proceeds by studying the hyperbolic limit for the kinetic equation (1.7). We find that if the kernel of the cell-problem

$$K_x = \Big\{ g \in L^2(\mathbb{R}^d \times \mathbb{T}^d) \quad \Big| \, a(x,v) \cdot \nabla_v g = 0 \ in \ \mathcal{D}' \Big\} \tag{1.8}$$

is *independent* of x, then it is possible to identify the effective equation. Namely, when the vector fielfd $a = a(v)$ is independent of x the effective equation for f reads

$$\begin{aligned} \frac{\partial f}{\partial t} + (Pa) \cdot \nabla_x f = 0 \\ f(t=0,x,v) = PU^0(x,v)\,, \end{aligned} \tag{1.9}$$

where P is the projection operator on the kernel K, and in turn $u = \int_{\mathbb{T}^d} f dv$ (see Theorem 3.1). By contrast, when $a = a(x)$ and K_x depends on x, a counterexample is constructed that shows that the effective equation can not be a pure transport equation (see section 3.2). In section 4, this analysis is extended for homogenization problems where a periodic fine-scale structure is transported by a divergence-free vector field (see equations (4.1) and (4.4)) analogous results to the case of (1.6) are found. Such kinetic equations might turn very useful for devising computational algorithms for the computation of homogenization problems.

A second paradigm is the problem of enhanced diffusion

$$\begin{aligned} \partial_t u_\varepsilon + \frac{1}{\varepsilon} a(x, \frac{x}{\varepsilon}) \cdot \nabla_x u_\varepsilon &= \alpha \triangle_x u_\varepsilon \\ u_\varepsilon(0, x) &= U^0(x, \frac{x}{\varepsilon}) \end{aligned} \tag{1.10}$$

with $a(x, v)$ periodic, divergence-free and with mean $\int_{\mathbb{T}^d} a = 0$. The results formally obtained by multiscale asymptotics have been validated for this problem by McLaughlin, Papanicolaou and Pironneau [18], Avellaneda and Majda [5], and Fannjiang and Papanicolaou [12]. We revisit this problem from the perspective of the kinetic decomposition and transform it to the problem of identifying the $\varepsilon \to 0$ limit

$$\begin{aligned} \frac{\partial f_\varepsilon}{\partial t} + \frac{1}{\varepsilon} a(x, v) \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon^2}\big(a(x, v) \cdot \nabla_v f_\varepsilon - \alpha \triangle_v f_\varepsilon\big) \\ = \alpha \triangle_x f_\varepsilon + \frac{2\alpha}{\varepsilon} \nabla_x \cdot \nabla_v f_\varepsilon \,, \\ f_\varepsilon(t = 0, x, v) = U^0(x, v)\, \delta_p(v - \frac{x}{\varepsilon}). \end{aligned} \tag{1.11}$$

The latter is a limit for the transport-diffusion equation (1.11) in the so-called diffusive scale, and its analysis provides the effective equation (5.7)-(5.9) of enhanced diffusion (see Theorem 5.1). This example indicates the efficiency of this approach in the rigorous validation of multi-scale asymptotic expansions.

Finally, we note that the scales of the drift and of the diffusion in (1.10) may be chosen differently from $1/\varepsilon$ and α, yielding other interesting homogenization problems, see for instance Capdeboscq [8, 9].

The article is organized as follows. Analytical considerations like the proper definition of (1.2), the characterization of the weak limit points of f_ε under various uniform bounds, the differential relations such as (1.5), and the identification of asymptotics for f_ε are developed in section 2 and in appendix I. In section 3, we study the hyperbolic homogenization problem (1.6), derive the effective equation, and produce the counterexample mentioned before. Some material from ergodic theory needed in the derivation is outlined in the appendix II. In section 4, we study the transport via a divergence-free field depending on an oscillating fine-scale, we derive the associated kinetic equation, and discuss the connection of the two formulations via characteristics and the derivation of an effective equation. Finally, in section 5 we study the parabolic homogenization problem (1.10) and derive the enhanced diffusion equation via the kinetic decomposition.

2. Multi-scale decomposition

Let $\{u_\varepsilon(x)\}$ be a family of functions defined on a open set $\Omega \subset \mathbb{R}^d$ that contains periodic oscillations and suppose that the scales of oscillations are either a-priori known (or anticipated). Our goal is to introduce an analytical object that will prompt the anticipated scale(s) of oscillations and quantify the structure of oscillations in the family at the preselected scale(s).

Suppose that periodic oscillations of length ε are anticipated in the family $\{u_\varepsilon\}$. To focus on them we consider a periodic grid with sides of length ε in each coordinate direction. The grid splits the Euclidean space into distinct cubic cells of volume ε^d, and it is arranged so that the centers of the cells occupy the lattice $\varepsilon\mathbb{Z}^d$. Let Ω be placed on that grid, and define a function $\chi_\varepsilon : \Omega \to \varepsilon\mathbb{Z}^d$ that maps the generic $x \in \Omega$ to the center $\chi_\varepsilon(x)$ of the cell containing x. To each point $x \in \Omega$ there is associated a decomposition $(\chi_\varepsilon(x), v)$ where $\chi_\varepsilon(x) \in \varepsilon\mathbb{Z}^d$ stands for the center of the cell that x occupies, and $v \in \mathbb{T}^d$ is the vector difference $x - \chi_\varepsilon(x)$ as measured in units of distance ε, that is $x = \chi_\varepsilon(x) + \varepsilon v$. We introduce the quantity

$$f_\varepsilon(x,v) = u_\varepsilon(x)\delta_p\big(v - \frac{x-\chi_\varepsilon(x)}{\varepsilon}\big)\,, \quad x \in \mathbb{R}^d\,,\ v \in \mathbb{T}^d\,, \tag{2.1}$$

where δ_p stands for a periodization of the usual delta function with period 1 in each coordinate direction, and $\mathbb{T}^d$ stands for the d-dimensional torus, the quotient of $\mathbb{R}^d$ by the subgroup $\mathbb{Z}^d$.

We note that the map $x \mapsto (\chi_\varepsilon(x), v)$ is single valued for points that fall into a single cell, but multi-valued for points that fall onto the boundaries between adjacent cells. For the latter points there would be two different decompositions (χ_ε, v) and (χ'_ε, v') associated to the same point $x \in \Omega$. Nevertheless, in that case $x = \chi_\varepsilon + \varepsilon v = \chi'_\varepsilon + \varepsilon v'$ and, due to the use of a periodic delta function,

$$\delta_p\big(v - \frac{x-\chi_\varepsilon}{\varepsilon}\big) = \delta_p\big(v - \frac{x}{\varepsilon}\big) = \delta_p\big(v - \frac{x-\chi'_\varepsilon}{\varepsilon}\big)$$

Hence, both decompositions provide the same outcome in (2.1) with f_ε defined for $x \in \Omega$ and $v \in \mathbb{T}^d$.

The operator $\nabla_x + \frac{1}{\varepsilon}\nabla_v$ annihilates the form $v - \frac{x}{\varepsilon}$ and that - at least formally - yields the formula

$$\big(\nabla_x + \frac{1}{\varepsilon}\nabla_v\big) f_\varepsilon = (\nabla_x u_\varepsilon)\delta_p\big(v - \frac{x}{\varepsilon}\big) \tag{2.2}$$

In the sequel, we provide formal definitions for the decomposition (2.1) and extensions as well as differentiation properties like (2.2) that are helpful in later sections for validating multiscale expansions.

2.1. **Definitions.** We make extensive use of distributions defined on the torus $\mathbb{T}^d$. Such distributions are in one-to-one correspondence with periodic distributions T on $\mathbb{R}^d$ of period 1 in each coordinate direction, that is distributions satisfying for $i = (i_1, ..., i_d) \in \mathbb{Z}^d$ the property $\tau_i T = T$ where τ_i is the shift operator, see [21, p. 229]. The same notation is used for both interpretations of periodic distributions. Let δ_p be the periodic delta function of period 1, defined by its action $< \delta_p, \psi >= \psi(0)$ on continuous periodic test functions $\psi \in C(\mathbb{T}^d)$.

We use the notation $C_p = C(\mathbb{T}^d)$ for the continuous periodic functions, $C_p^\infty = C^\infty(\mathbb{T}^d)$ for periodic test functions and $M_p = M^1(\mathbb{T}^d)$ for the periodic measures, with period 1 in each coordinate direction. Recall that C_p is separable and that bounded sets in $M_p = \big(C_p\big)^*$ are sequentially precompact in the weak-$\star$ topology of M_p.

2.1.1. *The double-scale kinetic decomposition.* Let $\alpha(x)$ be a smooth vector field and $u_\varepsilon \in L^1_{loc}(\Omega)$. We proceed to define the product

$$f_\varepsilon = u_\varepsilon(x)\delta_p\big(v - \frac{\alpha(x)}{\varepsilon}\big)\,. \tag{2.3}$$

Naturally it should act on tensor products $\varphi \otimes \psi$ of test functions via the formula

$$< f_\varepsilon, \varphi \otimes \psi >= \int_{\mathbb{R}^d} u_\varepsilon(x)\varphi(x)\psi\big(\frac{\alpha(x)}{\varepsilon}\big)\,dx\,. \tag{2.4}$$

To define (2.3), we employ the Schwartz kernel theorem [16, Thm 5.2.1]. Consider the linear map

$$\mathcal{K} : C^\infty(\mathbb{T}^d) \to \mathcal{D}'(\Omega) \quad \text{defined by} \quad \mathcal{K}\psi = u_\varepsilon(x)\psi\big(\frac{\alpha(x)}{\varepsilon}\big)$$

If $\psi_n \to 0$ in $C^\infty(\mathbb{T}^d)$ then $\mathcal{K}\psi_n \to 0$ in $\mathcal{D}'(\Omega)$. The kernel theorem implies that there exists a *unique* distribution K such that $< K, \varphi \otimes \psi >= (\mathcal{K}\psi, \varphi)$, that is, K acts on tensor products via (2.4) and is the desired product. It satisfies, for $\theta \in C_c^\infty(\Omega; C^\infty(\mathbb{T}^d))$,

$$< u_\varepsilon \delta_p\big(v - \frac{\alpha(x)}{\varepsilon}\big), \theta >= \int_{\mathbb{R}^d} u_\varepsilon(x)\theta\big(x, \frac{\alpha(x)}{\varepsilon}\big)\,dx\,, \tag{2.5}$$

which can also serve as a direct definition of f_ε. Of course smoothness of $\alpha(x)$ is required for the above definition: at least $\alpha \in C(\Omega; \mathbb{R}^d)$ if f_ε is interpreted as a measure, and more if f_ε is interpreted as a distribution and we need to take derivatives.

We now prove.

Lemma 2.1. *Let* $u_\varepsilon \in W^{1,1}_{loc}(\Omega)$ *and* $\alpha \in C^1(\Omega;\mathbb{T}^d)$. *Then*

$$\big(\nabla_x + \frac{1}{\varepsilon}(\nabla\alpha)^T\nabla_v\big)\left(u_\varepsilon\delta_p\big(v-\frac{\alpha(x)}{\varepsilon}\big)\right) = (\nabla_x u_\varepsilon)\delta_p\big(v-\frac{\alpha(x)}{\varepsilon}\big)$$

Proof. For the k-th coordinate, we have

$$\begin{aligned}
&< \Big(\partial_{x_k} + \frac{1}{\varepsilon}\sum_j \frac{\partial\alpha_j}{\partial x_k}\partial_{v_j}\Big) u_\varepsilon\delta_p\big(v-\frac{\alpha(x)}{\varepsilon}\big),\theta > \\
&\qquad = - < u_\varepsilon\delta_p\big(v-\frac{\alpha(x)}{\varepsilon}\big), \partial_{x_k}\theta + \frac{1}{\varepsilon}\sum_j \partial_{v_j}\Big(\frac{\partial\alpha_j}{\partial x_k}\theta\Big) > \\
&\qquad = -\int_{\mathbb{R}^d} u_\varepsilon(x)\partial_{x_k}\Big(\theta(x,\frac{\alpha(x)}{\varepsilon})\Big)\,dx \\
&\qquad = \int_{\mathbb{R}^d} (\partial_{x_k}u_\varepsilon)(x)\ \theta(x,\frac{\alpha(x)}{\varepsilon})\,dx \\
&\qquad =< (\partial_{x_k}u_\varepsilon)\delta_p\big(v-\frac{\alpha(x)}{\varepsilon}\big),\theta >
\end{aligned}$$

□

When $\alpha(x) = x$ this provides a definition of the product (2.1) and a justification of the formula (2.2).

2.1.2. *A multiscale kinetic decomposition.* We pursue next the construction of decompositions in cases when more than two scales are involved. Suppose that for an oscillating family $\{u_\varepsilon\}$ we wish to focus on oscillations at the scales 1, $\frac{x}{\varepsilon}$ and $\frac{x}{\varepsilon^2}$. We define

$$f_\varepsilon(x,v,w) = u_\varepsilon(x)\delta_p\big(v-\frac{x}{\varepsilon}\big)\delta_p\big(w-\frac{v}{\varepsilon}\big)\,, \qquad x\in\Omega,\ v\in\mathbb{T}^d, w\in\mathbb{T}^d\,, \tag{2.6}$$

or, in terms of the action on test functions, $\theta(x,v,w) \in C^\infty_c(\Omega;C^\infty(\mathbb{T}^d\times\mathbb{T}^d))$ via the formula

$$< f_\varepsilon,\theta >= \int_{\mathbb{R}^d}\theta\big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\big)\,dx\,. \tag{2.7}$$

In a straightforward generalization of Lemma 2.1, f_ε satisfies, for $u_\varepsilon \in W^{1,1}_{loc}(\Omega)$, the differentiation formula

$$\big(\nabla_x + \frac{1}{\varepsilon}\nabla_v + \frac{1}{\varepsilon^2}\nabla_w\big)f_\varepsilon = (\nabla_x u_\varepsilon)(x)\delta_p\big(v-\frac{x}{\varepsilon}\big)\delta_p\big(w-\frac{v}{\varepsilon}\big)\,. \tag{2.8}$$

To motivate the definition (2.6) consider for simplicity the case that $1/\varepsilon$ is an integer. Fix a first grid of size ε and introduce the quantities $\chi_\varepsilon(x)$

and $v = \frac{x-\chi_\varepsilon}{\varepsilon} \in \mathbb{T}^d$ as before. To focus on the scale $\frac{x}{\varepsilon^2}$ we consider a second grid of length ε^2 embedded in the first grid. When $1/\varepsilon$ is an integer, the grids fit perfectly onto one-another. Define the function $\psi_\varepsilon : \mathbb{T}^d \to \varepsilon\mathbb{Z}^d$ that takes the generic point v to the center of the inner cell containing v, and introduce a second inner variable $w = \frac{v-\psi_\varepsilon(v)}{\varepsilon} \in \mathbb{T}^d$ describing the vector distance between v and the center of the inner cell containing v in units of length ε. The process defines a decomposition of the physical space $x \mapsto (\chi_\varepsilon(x), v, \psi_\varepsilon(v), w)$, and allows to define a kinetic function representing three scales by

$$
\begin{aligned}
f_\varepsilon(x,v,w) &= u_\varepsilon(x)\delta_p\big(v - \frac{x-\chi_\varepsilon}{\varepsilon}\big)\delta_p\big(w - \frac{v-\psi_\varepsilon}{\varepsilon}\big) \\
&= u_\varepsilon(x)\delta_p\big(v - \frac{x}{\varepsilon}\big)\delta_p\big(w - \frac{v}{\varepsilon}\big)\,, \qquad x \in \Omega,\ v \in \mathbb{T}^d, w \in \mathbb{T}^d\,,
\end{aligned}
$$

This definition is also valid when $1/\varepsilon$ is not an integer as can be seen by the formula (6.2) in the appendix.

2.2. **Multiscale analysis of uniformly bounded families of functions.** Nguetseng [19] and Allaire [1] introduce the notion of double scale limit, which has been a very effective technical tool in the development of periodic homogenization theory. Their approach does not use the kinetic decomposition (2.1), but the double-scale limit is precisely the weak limit of the measures introduced in (2.1). We review the results of Nguetseng [19] from the perspective of the theory presented here, and produce some further asymptotic analysis of kinetic decompositions for uniformly bounded families of functions. In the sequel, the notation $u_\varepsilon \in_b X$ means that the family $\{u_\varepsilon\}$ belongs in a bounded set of the Banach space X.

2.2.1. *Uniform L^2-bounds.* Suppose first that $\{u_\varepsilon\}$ satisfies $u_\varepsilon \in_b L^2(\Omega)$. We define f_ε by (1.2) and note that

$$
f_\varepsilon \in_b L^2(\Omega; M_p)\,. \tag{2.9}
$$

The Riesz representation theory asserts that there is an isometric isomorphism between the dual of $C_p = C(\mathbb{T}^d)$ and the Banach space of periodic Radon measures $M_p = M^1(\mathbb{T}^d)$ on the torus. Since C_p is separable, bounded sets in M_p are sequentially precompact in the weak-$\star$ topology of M_p. Also, since C_p is separable, so is $L^2(\Omega; C_p)$ and thus bounded sets in $L^2(\Omega; M_p)$ are sequentially precompact in the weak-$\star$ topology of $L^2(\Omega; M_p)$.

As a consequence (2.9) implies that, along a subsequence,

$$f_\varepsilon \rightharpoonup f \qquad \text{weak-}\star \text{ in } L^2(\Omega; M_p) \tag{2.10}$$

with $f \in L^2(\Omega; M_p)$, that is

$$\begin{aligned} < f_\varepsilon, \theta > &= \int u_\varepsilon(x)\theta(x, \frac{x}{\varepsilon})\, dx \\ &\to < f, \theta > = \iint f(x,v)\theta(x,v)\, dx dv \qquad \text{for } \theta \in L^2(\Omega; C_p) \end{aligned} \tag{2.11}$$

Examples. We list some examples that illustrate the properties of this convergence, and refer to Appendix I for their proofs.

1. Note first that

$$\delta_p\big(v - \frac{x}{\varepsilon}\big) \rightharpoonup 1 \qquad \text{weak-}\star \text{ in } L^\infty(\Omega; M_p). \tag{2.12}$$

2. If $u_\varepsilon \to u$ strongly in $L^2(\Omega)$, then

$$u_\varepsilon(x)\delta_p\big(v - \frac{x}{\varepsilon}\big) \rightharpoonup u(x) \qquad \text{weak-}\star \text{ in } L^2(\Omega; M_p). \tag{2.13}$$

3. For $u_\varepsilon = a(\frac{x}{\varepsilon})$. with $a(v)$ a periodic function, we have

$$a(\frac{x}{\varepsilon}) \rightharpoonup \int_{\mathbb{T}^d} a(v) dv \tag{2.14}$$

$$a(\frac{x}{\varepsilon})\delta_p\big(v - \frac{x}{\varepsilon}\big) \rightharpoonup a(v) \tag{2.15}$$

Note that the weak limit (2.14) retains only the information of the average of a while the double scale kinetic limit (2.15) also retains the information of the shape of a.

4. For $u_\varepsilon = a(\frac{x}{\varepsilon})b(\frac{x}{\varepsilon^2})$ with a and b periodic functions.

$$a(\frac{x}{\varepsilon})b(\frac{x}{\varepsilon^2}) \rightharpoonup \int_{\mathbb{T}^d} a(y) dy \int_{\mathbb{T}^d} b(z) dz \tag{2.16}$$

$$a(\frac{x}{\varepsilon})b(\frac{x}{\varepsilon^2})\delta_p\big(v - \frac{x}{\varepsilon}\big) \rightharpoonup a(v) \int_{\mathbb{T}^d} b(z) dz \tag{2.17}$$

$$a(\frac{x}{\varepsilon})b(\frac{x}{\varepsilon^2})\delta_p\big(v - \frac{x}{\varepsilon}\big)\delta_p\big(w - \frac{v}{\varepsilon}\big) \rightharpoonup a(v)\, b(w) \tag{2.18}$$

The proof of (2.18) follows from (6.2) together with a density argument analogous to the one presented in the proof of Lemma 6.3 in Appendix I ; (2.16), (2.17) are consequences of (2.18).

We give a simplified proof of [19, Thm 1] concerning the double-scale limit for sequences that are uniformly bounded in $L^2(\Omega)$.

Proposition 2.2. *Let $u_\varepsilon \in_b L^2(\Omega)$. Then, along a subsequence,*

$$f_\varepsilon \rightharpoonup f \qquad \text{weak-}\star \text{ in } L^2(\Omega\,; M_p)$$

with $f \in L^2(\Omega \times \mathbb{T}^d)$.

Proof. Let $\theta \in C_c^\infty(\Omega; C_p^\infty)$ be a test function. Then

$$< f_\varepsilon, \theta >= \int_\Omega u_\varepsilon(x)\theta(x, \frac{x}{\varepsilon})dx$$

and

$$\begin{aligned} | < f_\varepsilon, \theta > | &\leq \|u_\varepsilon\|_{L^2(\Omega)} \left(\int_\Omega |\theta(x, \frac{x}{\varepsilon})|^2 dx \right)^{1/2} \\ &\leq C \left(\int_\Omega |\theta(x, \frac{x}{\varepsilon})|^2 dx \right)^{1/2} \\ &\overset{(2.12)}{\to} C \left(\int_\Omega \int_{\mathbb{T}^d} |\theta(x,v)|^2 dx dv \right)^{1/2} \end{aligned}$$

Hence, $f_\varepsilon \in_b \left(L^2(\Omega; C_p)\right)^*$, $f_\varepsilon \rightharpoonup f$ weak-$\star$ in $L^2(\Omega; M_p)$ and $f \in L^2(\Omega; M_p)$. Moreover,

$$\frac{| < f, \theta > |}{\|\theta\|_{L^2(\Omega\times\mathbb{T}^d)}} = \lim \frac{| < f_\varepsilon, \theta > |}{\|\theta\|_{L^2(\Omega\times\mathbb{T}^d)}} \leq C$$

and $f \in L^2(\Omega \times \mathbb{T}^d)$. □

2.2.2. *Uniform H^1-bounds.* Next consider the case of families $\{u_\varepsilon\}$ that are uniformly bounded in $H^1(\Omega)$. The first proposition is essentially a rephrasing of [19, Thm 3].

Proposition 2.3. *Let $u_\varepsilon \in_b H^1(\Omega)$. Then, there exist $u \in H^1(\Omega)$, $\pi \in L^2(\Omega; H^1(\mathbb{T}^d))$ such that, along a subsequence,*

$$\begin{aligned} f_\varepsilon = u_\varepsilon \delta_p\big(v - \frac{x}{\varepsilon}\big) &\rightharpoonup u(x) \qquad \text{weak-}\star \text{ in } L^2(\Omega; M_p) \\ (\nabla_x u_\varepsilon - \nabla_x u)\delta_p\big(v - \frac{x}{\varepsilon}\big) &\rightharpoonup \nabla_v \pi(x,v) \qquad \text{weak-}\star \text{ in } L^2(\Omega; M_p) \end{aligned}$$

Proof. Along subsequences (whenever necessary) $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$, $u_\varepsilon \to u$ in $L^2(\Omega)$ and $u_\varepsilon \delta_p\big(v - \frac{x}{\varepsilon}\big) \rightharpoonup u$ weak-$\star$ in $L^2(\Omega; M_p)$. Moreover, Proposition 2.2 implies

$$g_{i,\varepsilon} := \big(\frac{\partial u_\varepsilon}{\partial x_i} - \frac{\partial u}{\partial x_i}\big)\delta_p\big(v - \frac{x}{\varepsilon}\big) \rightharpoonup g_i(x,v)$$

weak-$\star$ in $L^2(\Omega; M_p)$ with $g_i \in L^2(\Omega \times \mathbb{T}^d)$.

For $\varphi \in C_c^\infty(\Omega)$, $\psi_i \in C_p^\infty$, we have

$$-\int_\Omega (u_\varepsilon - u)\left[\varphi(x)\frac{1}{\varepsilon}\frac{\partial\psi_i}{\partial v_i}(\frac{x}{\varepsilon}) + \frac{\partial\varphi}{\partial x_i}(x)\psi_i(\frac{x}{\varepsilon})\right] dxdv$$
$$= \int_\Omega \frac{\partial(u_\varepsilon - u)}{\partial x_i}\varphi(x)\psi_i(\frac{x}{\varepsilon})\, dxdv$$
$$\to \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} g_i(x,v)\varphi(x)\psi_i(v)dxdv$$

We apply the above formula to a test function $\Psi = (\psi_1, ..., \psi_d)$ that satisfies $\mathrm{div}_v\Psi = 0$. Then

$$-\int_{\mathbb{R}^d} (u_\varepsilon - u)\left[\varphi(x)\frac{1}{\varepsilon}\frac{\partial\psi_i}{\partial v_i}(\frac{x}{\varepsilon}) + \frac{\partial\varphi}{\partial x_i}(x)\psi_i(\frac{x}{\varepsilon})\right] dxdv \to 0$$

and we conclude that for a.e. $x \in \Omega$

$$\int_{\mathbb{T}^d}\sum_i g_i(x,v)\psi_i(v)\,dv = 0 \quad \text{for any } \Psi \text{ with } \mathrm{div}_v\Psi = 0. \tag{2.19}$$

A lemma from [19, Lemma 4] then implies there exists $\pi \in L^2(\Omega; H^1(\mathbb{T}^d))$ such that $G = (g_1, ..., g_d) = \nabla_v \pi$. □

The next proposition is novel and establishes the second term in an asymptotic expansion of f_ε when the family $\{u_\varepsilon\}$ is uniformly bounded in $H^1(\Omega)$. With respect to already well-known results (like Prop. 1.14 of [1] for example), it does not only give the double scale limit in the case of a regular sequence u_ε but also establishes rigourously the first order correction in ε.

Proposition 2.4. *Let $u_\varepsilon \in_b H^1(\Omega)$. Then*

$$\begin{aligned}
&\frac{1}{\varepsilon}(\delta_p(v - \frac{x}{\varepsilon}) - 1) \to 0 \quad \textit{in } \mathcal{D}' \\
&g_\varepsilon := u_\varepsilon(x)\frac{1}{\varepsilon}(\delta_p(v - \frac{x}{\varepsilon}) - 1) \in_b H^{-1}(\Omega; L^2(\mathbb{T}^d)) \\
&g_\varepsilon \rightharpoonup g \quad \textit{weak-}\star \textit{ in } H^{-1}(\Omega; L^2(\mathbb{T}^d)) \\
&g \in L^2(\Omega; H^1(\mathbb{T}^d))\,,
\end{aligned} \tag{2.20}$$

and thus f_ε enjoys the asymptotic expansion

$$f_\varepsilon = u_\varepsilon + \varepsilon g + o(\varepsilon) \quad \textit{in } \mathcal{D}'\,.$$

Proof. It is instructive to first give a proof for the case of one dimension. Consider the function $H(v) = v$, $v \in [0,1]$ and let H_p denote its periodic extension of period 1. $H_p \in L^\infty(\mathbb{R})$ satisfies $\partial_v H_p(v) = 1 - \delta_p(v)$ and

$$\frac{1}{\varepsilon}\big(\delta_p(v - \frac{x}{\varepsilon}) - 1\big) = \partial_x H_p(v - \frac{x}{\varepsilon}) \tag{2.21}$$

From standard properties of weak convergence we obtain for $\theta \in C^1_c(\Omega, C_p)$

$$< H_p(v - \frac{x}{\varepsilon}), \theta > \to \int_\Omega \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} H_p(v - w)\theta(x, v) dw dv dx$$

and thus

$$\begin{aligned} < \frac{1}{\varepsilon}\big(\delta_p(v - \frac{x}{\varepsilon}) - 1\big), \theta > &= < \partial_x H_p(v - \frac{x}{\varepsilon}), \theta > \\ &= - < H_p(v - \frac{x}{\varepsilon}), \partial_x \theta > \\ &\rightharpoonup - \int_\Omega \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} H_p(v - w)\partial_x \theta(x, v) dw dv dx \\ &= 0\,. \end{aligned}$$

For functions $u_\varepsilon \in H^1(\Omega)$ we have the identity (note that $u_\varepsilon H_p$ belongs to H^1 and thus the derivatives below make perfect sense)

$$\big(\partial_x + \frac{1}{\varepsilon}\partial_v\big)\big(u_\varepsilon H_p(v - \frac{x}{\varepsilon})\big) = (\partial_x u_\varepsilon) H_p(v - \frac{x}{\varepsilon})$$

which with (2.21) gives the representation

$$u_\varepsilon \frac{1}{\varepsilon}\big(\delta_p(v - \frac{x}{\varepsilon}) - 1\big) = \partial_x \Big(u_\varepsilon H_p(v - \frac{x}{\varepsilon})\Big) - (\partial_x u_\varepsilon) H_p(v - \frac{x}{\varepsilon})\,.$$

Then $(2.20)_2$ and $(2.20)_3$ follow directly. To prove $(2.20)_4$ an additional argument is needed that is shown in the context of the multi-d case.

In the multi-dimensional case we do not have available an exact formula like (2.21) and this difficulty has to be bypassed. For $\theta \in C^1_c(\Omega; C_p)$, let $\operatorname{supp}\theta$ denote the support (in x) of θ, and fix $\varepsilon < \frac{1}{\sqrt{d}} \operatorname{dist}(\operatorname{supp}\theta, \partial\Omega)$. We cover $\operatorname{supp}\theta$ by cubes C_k centered at points $\chi_k \in \varepsilon\mathbb{Z}^d$ of latteral size ε. The number of the cubes covering $\operatorname{supp}\theta$ is of the order $\varepsilon^{-d} O(|\operatorname{supp}\theta|)$, and the covering is arranged so that $\operatorname{supp}\theta \subset \cup_{k=1}^N C_k \subset \Omega$. Observe that by construction $\frac{1}{\varepsilon}\chi_k \in \mathbb{Z}^d$ and compute

$$\begin{aligned} < g_\varepsilon, \theta > &= \int_\Omega u_\varepsilon(x) \frac{1}{\varepsilon}\Big(\theta(x, \frac{x}{\varepsilon}) - \int_{\mathbb{T}^d} \theta(x, v) dv\Big) dx \\ &= \sum_{k \in \mathbb{Z}^d} \int_{C_k} u_\varepsilon(x) \frac{1}{\varepsilon}\Big(\theta(x, \frac{x}{\varepsilon}) - \int_{\mathbb{T}^d} \theta(x, v) dv\Big) dx \\ &= \sum_{k \in \mathbb{Z}^d} \varepsilon^d \int_{\mathbb{T}^d} u_\varepsilon(\chi_k + \varepsilon w) \frac{1}{\varepsilon}\Big(\theta(\chi_k + \varepsilon w, w) - \int_{\mathbb{T}^d} \theta(\chi_k + \varepsilon w, v) dv\Big) dw \\ &= \sum_{k \in \mathbb{Z}^d} \varepsilon^d \int_{\mathbb{T}^d} \frac{1}{\varepsilon}\Big((u_\varepsilon \theta)(\chi_k + \varepsilon w, w) - \int_{\mathbb{T}^d} (u_\varepsilon \theta)(\chi_k + \varepsilon \rho, w) d\rho\Big) dw \end{aligned}$$

We employ the Poincaré inequality

$$\Big| v(z) - \int_{\mathbb{T}^d} v(z') dz' \Big| \le \int_{\mathbb{T}^d} |\nabla v|(z') dz'$$

for $v(z) = (u_\varepsilon\theta)(\chi_k + \varepsilon z, v)$ to obtain

$$\frac{1}{\varepsilon}\Big|(u_\varepsilon\theta)(\chi_k + \varepsilon z, v) - \int_{\mathbb{T}^d}(u_\varepsilon\theta)(\chi_k + \varepsilon\rho, v)d\rho\Big| \\ \leq \int_{\mathbb{T}^d}|\nabla_x(u_\varepsilon\theta)|(\chi_k + \varepsilon\rho, v)d\rho$$

and

$$\begin{aligned} |< g_\varepsilon, \theta >| &\leq \sum_{k\in\mathbb{Z}^d}\varepsilon^d\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}|\nabla_x(u_\varepsilon\theta)|(\chi_k + \varepsilon\rho, w)\,d\rho dw \\ &= \int_\Omega\int_{\mathbb{T}^d}|\nabla_x(u_\varepsilon\theta)|(x, v)\,dxdv \\ &\leq \|u_\varepsilon\|_{H^1(\Omega)}\Big(\|\theta\|_{L^2(\Omega\times\mathbb{T}^d)} + \|\nabla_x\theta\|_{L^2(\Omega\times\mathbb{T}^d)}\Big) \end{aligned}$$

From here we deduce the uniform bound $(2.20)_2$ and that along a subsequence $(2.20)_3$ is valid for some $g \in H^{-1}(\Omega; L^2(\mathbb{T}^d))$.

In addition, we have

$$\begin{aligned} \nabla_v g_\varepsilon &= \varepsilon\big(\nabla_x + \frac{1}{\varepsilon}\nabla_v\big)g_\varepsilon - \varepsilon\nabla_x g_\varepsilon \\ &= (\nabla_x u_\varepsilon)\Big(\delta_p(v - \frac{x}{\varepsilon})\Big) - \varepsilon\nabla_x g_\varepsilon \end{aligned}$$

Therefore,

$$< \nabla_v g_\varepsilon, \theta >= \int_\Omega \nabla_x u_\varepsilon(x)\Big(\theta(x, \frac{x}{\varepsilon}) - \int_{\mathbb{T}^d}\theta(x, v)dv\Big)dx + \varepsilon < g_\varepsilon, \nabla_x\theta >$$

and passing to the limit $\varepsilon \to 0$

$$\begin{aligned} |< \nabla_v g, \theta >| &= \lim|< \nabla_v g_\varepsilon, \theta >| \\ &\leq \overline{\lim}\Big[\int_\Omega|\nabla_x u_\varepsilon(x)\theta(x, \frac{x}{\varepsilon})|dx + \int_\Omega\int_{\mathbb{T}^d}|\nabla_x u_\varepsilon||\theta|dvdx\Big] \\ &\leq C\|\theta\|_{L^2(\Omega\times\mathbb{T}^d)} \end{aligned}$$

This inequality holds for any $\theta \in C_c^1(\Omega \times \Pi^d)$ so by density it means that $\nabla_v g$ belongs to the dual of L^2 or

$$\nabla_v g \in L^2(\Omega \times \mathbb{T}^d)$$

Moreover, we have $\int_{\mathbb{T}^d} g = 0$ and thus using the Poincaré inequality we obtain$(2.20)_4$.

To see the first property, consider a test function $\theta = \varphi \otimes \psi$ which is a tensor product of $\varphi \in C_c^\infty(\Omega)$ and $\psi \in C^\infty(\mathbb{T}^d)$. Then

$$
\begin{aligned}
&< \frac{1}{\varepsilon}\big(\delta_p(v - \frac{x}{\varepsilon}) - 1\big), \varphi \otimes \psi > \\
&= \sum_{k \in \mathbb{Z}^d} \varepsilon^d \int_{\mathbb{T}^d} \frac{1}{\varepsilon} \varphi(\chi_k + \varepsilon w) \Big(\psi(w) - \int_{\mathbb{T}^d} \psi\Big) dw \\
&= \sum_{k \in \mathbb{Z}^d} \varepsilon^d \int_{\mathbb{T}^d} \frac{\varphi(\chi_k + \varepsilon w) - \varphi(\chi_k) - \nabla\varphi(\chi_k) \cdot \varepsilon w}{\varepsilon} \big(\psi(w) - \int_{\mathbb{T}^d} \psi\big) dw \\
&\quad + \sum_{k \in \mathbb{Z}^d} \varepsilon^d \nabla\varphi(\chi_k) \cdot \int_{\mathbb{T}^d} w \Big(\psi(w) - \int_{\mathbb{T}^d} \psi\Big) dw \\
&= O(\varepsilon) + \int_\Omega \nabla\varphi(x) dx \cdot \int_{\mathbb{T}^d} w \Big(\psi(w) - \int_{\mathbb{T}^d} \psi\Big) dw \\
&\to 0
\end{aligned}
$$

as φ is of compact support. Since

$$
\frac{1}{\varepsilon}\big(\delta_p(v - \frac{x}{\varepsilon}) - 1\big) \in_b H^{-1}(\Omega; M_p) = \big(H_0^1(\Omega; C_p)\big)^*
$$

and finite sums of tensor products $\sum_j \varphi_j \otimes \psi_j$ are dense in $H_0^1(\Omega; C_p)$ we obtain $(2.20)_1$. $\square$

3. Homogenization of hyperbolic equations

In this section we consider certain homogenization problems for transport equations. First we develop an example where the effective equation can be calculated with the help of the double scale kinetic decomposition. Then we provide a counter-example where the double scale limit is not the right object to treat the effective equation.

3.1. **Effective equation.** Consider the transport equation

$$
\begin{aligned}
&\frac{\partial u_\varepsilon}{\partial t} + a\big(\frac{x}{\varepsilon}\big) \cdot \nabla_x u_\varepsilon = 0 \\
&u_\varepsilon(0, x) = U^0(x, \frac{x}{\varepsilon})
\end{aligned}
\tag{3.1}
$$

We assume that $a(v)$ is a C^1 vector field, periodic with period 1, and satisfying $\operatorname{div} a = 0$, and that the initial data $U^0 \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ is 1-periodic in v and satisfy the uniform bounds

$$
\int_{\mathbb{R}^d} |U^0(x, \frac{x}{\varepsilon})|^2 dx \le C \,. \tag{h$_d$}
$$

Under this hypothesis standard energy estimates for (3.1) imply the uniform bound on solutions

$$\int_{\mathbb{R}^d} |u_\varepsilon(t,x)|^2 dx \leq \int_{\mathbb{R}^d} |U^0(x, \frac{x}{\varepsilon})|^2 dx \leq C \quad \forall t > 0\,. \tag{3.2}$$

We introduce

$$f_\varepsilon(t,x,v) = u_\varepsilon(t,x)\delta_p\big(v - \frac{x}{\varepsilon}\big) \quad t \in \mathbb{R}_+,\, x \in \mathbb{R}^d,\, v \in \mathbb{T}^d\,, \tag{3.3}$$

and use Lemma 2.1 to check that f_ε satisfies

$$\frac{\partial f_\varepsilon}{\partial t} + a(v)\cdot\nabla_x f_\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_v f_\varepsilon = 0, \quad \text{in } \mathcal{D}' \tag{3.4}$$

$$f_\varepsilon(t=0,x,v) = U^0(x,v)\ \delta_p(v - \frac{x}{\varepsilon}) \tag{3.5}$$

with periodic boundary conditions ($v \in \mathbb{T}^d$). The uniform bound (3.2) implies

$$f_\varepsilon \in_b L^\infty\big([0,\ \infty),\ L^2(\mathbb{R}^d, M_p)\big)\,, \tag{3.6}$$

and thus by Proposition 2.2, along a subsequence if necessary,

$$f_\varepsilon \rightharpoonup f \quad \text{weak-}\star \text{ in } L^\infty\big([0,\ \infty),\ L^2(\mathbb{R}^d, M_p)\big) \tag{3.7}$$

with f enjoying the improved regularity

$$f \in L^\infty([0,\ \infty),\ L^2(\mathbb{R}^d \times \mathbb{T}^d))\,. \tag{3.8}$$

Our objective is to calculate the effective limit of (3.1) by computing the hydrodynamic limit problem for the kinetic equation (3.4)-(3.5). Note that if f satisfies a well-posed problem then this provides a complete determination of the weak limit of u_ε since

$$u_\varepsilon = \int_{\mathbb{T}^d} f_\varepsilon dv \rightharpoonup \int_{\mathbb{T}^d} f dv = u$$

We introduce

$$K = \Big\{ g \in L^2(\mathbb{T}^d) \quad \Big|\ a(v)\cdot\nabla_v g = 0 \ in\ \mathcal{D}' \Big\}. \tag{3.9}$$

and remark that K is the space of solutions of the cell-problem obtained by the method of multiscale asymptotic expansion [6] for the homogenization problem (3.1) (see [17], [11]). Let P denote the L^2-projection operator on the kernel K. We prove

Theorem 3.1. *Let a be a C^1 periodic vector field satisfying $\mathrm{div} a = 0$. Under hypothesis* (h_d) *the effective limit of problem* (3.1) *is obtained as* $u = \int_{\mathbb{T}^d} f dv$

where $f \in L^\infty([0,\ \infty)\,,\ L^2(\mathbb{R}^d \times \mathbb{T}^d))$, $f(t,x,\cdot) \in K$ *for a.e.* (t,x), *and* f *is the unique solution of the kinetic problem*

$$\begin{aligned} &\frac{\partial f}{\partial t} + (Pa)\cdot\nabla_x f = 0 \\ &f(t=0,x,v) = PU^0(x,v)\,, \end{aligned} \tag{3.10}$$

where P *is the projection operator on the kernel* K.

Proof. Let f_ε and f be as in (3.6), (3.7) and (3.8). The proof is split in three steps:

Step 1 : The limit f *belongs to* K. The kernel K is defined in (3.9). We may consider elements of K as functions of t, x and v instead of only v, as t and x play the role of parameter in the definition of K. Thus we have

$$K_x = \Big\{ g \in L^2(\mathbb{R}^d \times \mathbb{T}^d) \quad \Big|\ a(v)\cdot\nabla_v g = 0 \ \mathit{in}\ \mathcal{D}' \Big\}, \tag{3.11}$$

and

$$K_{t,x} = \Big\{ g \in L^\infty([0,\ \infty),\ L^2(\mathbb{R}^d \times \mathbb{T}^d)) \quad \Big|\ a(v)\cdot\nabla_v g = 0 \ \mathit{in}\ \mathcal{D}' \Big\}. \tag{3.12}$$

We may also define all the

$$K^p = \Big\{ g \in L^p(\mathbb{T}^d)) \quad \Big|\ a(v)\cdot\nabla_v g = 0 \ \mathit{in}\ \mathcal{D}' \Big\},$$

and their extensions K^p_x and $K^p_{t,x}$.

The convergence (3.7) states that for ϕ in $L^1([0,\ \infty),\ L^2(\mathbb{R}^d,\ C_p))$ we have

$$\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{T}^d} \phi(t,x,v)\,df_\varepsilon \longrightarrow \int_0^\infty \int_{\mathbb{R}^d\times\mathbb{T}^d} \phi(t,x,v)\,df.$$

Take $\phi \in C_c^\infty([0,\ \infty)\times\mathbb{R}^d\times\mathbb{T}^d)$ and compute

$$\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{T}^d} a(v)\cdot\nabla_v\phi\,df_\varepsilon = -\varepsilon\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{T}^d} (\partial_t\phi + a(v)\cdot\nabla_x\phi)\,df_\varepsilon.$$

Passing to the limit, we conclude that

$$\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{T}^d} a(v)\cdot\nabla_v\phi\,df = 0.$$

On the other hand $f \in L^\infty([0,\ \infty), L^2(\mathbb{R}^d\times\mathbb{T}^d))$, so $f \in K_{t,x}$.

Step 2 : The limit equation. Consider a function $\phi \in K_x$. We wish to mollify ϕ and use it as a test function in the weak form of (3.4). Since K_x depends only parametrically in x, we may select ϕ to be compactly supported in x.

Take $H(x) \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} H(x)\,dx = 1$, and $\bar H(v) \in C_c^\infty((0,\ 1)^d)$ with $\int_{\mathbb{T}^d} \bar H(v)\,dv = 1$. For any ϕ define

$$\phi_n = \int_{\mathbb{R}^d} n^d\, H\left(n(x-y)\right)\ \phi(y,v)\,dy,$$
$$\phi_{n,m} = \int_{\mathbb{T}^d} \bar H_m(v-\eta)\ \phi_n(x,\eta)\,d\eta,$$

with $\bar H_m(v) = m^d \sum_{k\in\mathbb{Z}^d} \bar H(m(v+k))$, periodic and well defined for all m as $\bar H(mv)$ is compactly supported in $(0,\ 1/m)^d$.

Then for any $\phi \in K_x$ we have

$$\int_{\mathbb{R}^d\times\mathbb{T}^d} \phi(y,\eta) a(\eta)\cdot\nabla \bar H_m(v-\eta) H_n(x-y)\,dy d\eta = 0\,.$$

Thus, for a Lipschitz continuous,

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{T}^d} a(v)\cdot\nabla_v \phi_{n,m}\,df_\varepsilon \\
&\qquad = \int_{\mathbb{R}^d\times\mathbb{T}^d}\int_{\mathbb{T}^d} (a(v)-a(\eta))\cdot\nabla \bar H_m\left(v-\eta\right)\phi_n(x,\eta)\,d\eta\,df_\varepsilon \\
&\qquad = \int_{\mathbb{R}^d\times\mathbb{T}^d}\int_{\mathbb{T}^d}\int_0^1 \big(\zeta\cdot\nabla a(v-(1-t)\zeta)\big) \\
&\qquad\qquad\qquad \cdot \sum_{k\in\mathbb{Z}^d} m^{d+1}\nabla \bar H\left(m(\zeta+k)\right)\phi_n(x,v-\zeta)\,dt\,d\zeta\,df_\varepsilon.
\end{aligned}$$

Notice that $m^{d+1}\zeta\otimes\nabla\bar H(m\zeta)$ converges in the sense of distributions toward $C\,(Id)\,\delta$ with C a numerical constant. Moreover thanks to (3.6), and to the fact that $\phi_n \in L^2(\mathbb{T}^d,\ C_c(\mathbb{R}^d))$ and $\nabla a \in C(\mathbb{T}^d)$, we may pass to the limit in m in the previous equality and find

$$\lim_{m\to\infty}\int_{\mathbb{R}^d\times\mathbb{T}^d} a(v)\cdot\nabla_v\phi_{n,m}\,df_\varepsilon = C\int_{\mathbb{R}^d\times\mathbb{T}^d} \operatorname{div} a(v)\,\phi_n(x,v)\,df_\varepsilon = 0,$$

as a is divergence free. Multiplying (3.4) by $\phi_{n,m}$ and taking first the limit $m\to\infty$ and then the limit $\varepsilon\to 0$, we find that for any $\phi\in K_x$ compactly supported in x we have

$$\partial_t \int_{\mathbb{R}^d\times\mathbb{T}^d} \phi_n f\,dx\,dv - \int_{\mathbb{R}^d\times\mathbb{T}^d} a(v)\cdot\nabla_x\phi_n\,f\,dx\,dv = 0.$$

This relation can now easily be extended by approximation to any $\phi\in K_x$.

Let us denote $\bar a$ the orthogonal projection on K of a. The new function $\bar a$ belongs to $L^\infty(\mathbb{T}^d)$ as the projection operator P is continuous on every $L^p(\mathbb{T}^d)$ for all $1\le p\le\infty$, but does not necessarily have any further regularity, Lipschitz for instance (see the appendix where we recall the basic properties of P). Now as $f \in K_{t,x}$ and $\nabla_x \phi_n \in K_x$, then $\nabla_x \phi_n \, f \in K^1_{t,x}$ and consequently

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{T}^d} \phi_n f\,dx\,dv - \int_{\mathbb{R}^d\times\mathbb{T}^d} \bar{a}(v)\cdot\nabla_x\phi_n\,f\,dx\,dv = 0.$$

On the other hand, the projection operator P may be trivially extended on $K_{t,x}$ from K as t and x are only parameters and of course it commutes with derivatives in t or x. Now, for any $\phi \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$

$$\int_{\mathbb{R}^d\times\mathbb{T}^d} (\phi - P\phi)\,f\,dx\,dv = 0, \int_{\mathbb{R}^d\times\mathbb{T}^d} \bar{a}\cdot\nabla(\phi_n - P\phi_n)\,f\,dx\,dv = 0.$$

Finally for any $\phi \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$, we have that

$$\partial_t \int_{\mathbb{R}^d\times\mathbb{T}^d} \phi_n f\,dx\,dv - \int_{\mathbb{R}^d\times\mathbb{T}^d} \bar{a}(v)\cdot\nabla_x\phi_n\,f\,dx\,dv = 0.$$

This implies that $f \in K_{t,x}$ is a solution in the sense of distribution to

$$\partial_t f + \bar{a}(v)\cdot\nabla_x f = 0. \tag{3.13}$$

Step 3 : Conclusion. Let us begin with the identification of the initial value $f(t=0)$ which has a sense since $\partial_t f \in L^\infty([0,\ \infty),\ H^{-1}(\mathbb{R}^d,\ L^2(\mathbb{T}^d)))$ because of (3.13) and as $f \in L^\infty([0,\ \infty),\ L^2(\mathbb{R}^d \times \mathbb{T}^d)))$. For every $\phi \in K_x$, as

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{T}^d} \phi_n df_\varepsilon = \int_{\mathbb{R}^d\times\mathbb{T}^d} a(v)\cdot\nabla_x\phi_n\,df_\varepsilon, \tag{3.14}$$

then $\int_{\mathbb{R}^d\times\mathbb{T}^d} \phi_n df_\varepsilon(t,.,.)$ has a limit as $t \to 0$ and this limit is, thanks to (3.5)

$$\int_{\mathbb{R}^d} \phi_n(x, x/\varepsilon)\,U^0(x, x/\varepsilon)\,dx.$$

Moreover because of (3.14), we may pass to the limit in ε and deduce that

$$\int_{\mathbb{R}^d\times\mathbb{T}^d} \phi_n f(t,x,v)\,dxdv \underset{t\to 0}{\longrightarrow} \int_{\mathbb{R}^d\times\mathbb{T}^d} \phi_n f(0,x,v) = \int_{\mathbb{R}^d\times\mathbb{T}^d} \phi_n U^0(x,v).$$

On the other hand we of course have for any $\phi \in L^2$ as $f \in K_{t,x}$

$$0 = \int_{\mathbb{R}^d\times\mathbb{T}^d} (\phi_n - P\phi_n) f(t,x,v)\,dxdv \underset{t\to 0}{\longrightarrow} \int_{\mathbb{R}^d\times\mathbb{T}^d} (\phi_n - P\phi_n) f(0,x,v).$$

Combining the last two equalities we get that

$$f(t=0,x,v) = P\,U^0(x,v). \tag{3.15}$$

Finally, we notice that Eq. (3.13) combined with (3.15) has a unique solution in the space of distribution, through standard arguments of kinetic theory and as, even though $\bar{a}$ is only bounded, it does not depend on x.

Therefore any extracted subsequence of f_ε has only one possible limit and the whole sequence f_ε converges toward the solution of (3.13) with (3.15). $\square$

EXAMPLES. We calculate the equation for the double scale limit f and the associated effective equation for certain examples, always within the framework of (3.1).

1. First consider the case that $a(v)$ is ergodic. Then

$$K = \{g \in L^2(\mathbb{T}^d) : g = const.\}$$
$$P_K g = \int_{\mathbb{T}^d} g dv =: \overline{g}$$

The equation for f becomes

$$\partial_t f + \overline{a} \cdot \nabla_x f = 0$$

and of course $u = \int_{\mathbb{T}^d} f dv$.

2. Consider next the homogenization problem

$$\partial_t u_\varepsilon + b(\frac{x_2}{\varepsilon})\partial_{x_1} u_\varepsilon = 0$$
$$u_\varepsilon(0, x_1, x_2) = U^0(x_1, x_2, \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})$$

where $u_\varepsilon = u_\varepsilon(t, x)$, $x = (x_1, x_2) \in \mathbb{R}^2$, and the vector field $a(x_1, x_2) = (b(x_2), 0)$ corresponds to a shear flow with $b(x_2) \neq 0$ for *a.e.* x_2. We compute

$$K = \{g \in L^2(\mathbb{T}^2) : b(v_2)\partial_{v_1} g = 0\} = \{g = \psi(v_2) \bigm| \forall \psi \in L^2(\mathbb{T}^1)\}$$
$$P_K g = \int_0^1 g(v_1, v_2) dv_1.$$

Since $f \in K$ we conclude that $f = f(t, x_1, x_2, v_2)$ and satisfies the problem

$$\partial_t f + b(v_2)\partial_{x_1} f = 0$$
$$f(0, x_1, x_2, v_2) = P_K U^0 = \int_0^1 U^0(x_1, x_2, v_1, v_2) dv_1$$

The weak limit $u = \int_{\mathbb{T}^2} f$ satisfies the integrated equation.

3. It is possible to give a more general framework for the situation of the previous example. Suppose that the divergence free vector field a is such that the following description of K is true: There exist functions $\xi_1, \dots \xi_N$, $N \leq d$ from $\mathbb{T}^d$ to $\mathbb{R}$. These functions are local coordinates in the sense that they may be completed by $\xi_{N+1}, \dots \xi_d$ and that the change of coordinates v to $(\xi_1(v), \dots, \xi_d(v))$ is a C^1 diffeomorphism from $\mathbb{T}^d$ to some domain $O \subset \mathbb{R}^d$. And finally

$$K = \{\psi(\xi_1(v), \dots, \xi_N(v)) \mid \forall \psi \in L^2(O)\}.$$

For instance in dimension $d = 2$, as div $a = 0$, there is always $\xi : \mathbb{R}^d \to \mathbb{R}$ such that $a = \nabla^\perp \xi$. Now if in addition ξ is a periodic regular function with $\nabla\xi(v) \neq 0$ for all v, which is a non trivial assumption, then K is exactly the set of functions $\psi(\xi)$.

In that case, we may define $g(t, x, \xi) = f(t, x, V(\xi_1, \ldots, \xi_d))$ with V the inverse change of variables. Then g does not depend on $\xi_{N+1}, \ldots, \xi_d$ and it simply satisfies

$$\partial_t g + b(\xi_1, \ldots, \xi_N) \cdot \nabla_x g = 0,$$

with $b(\xi) = \bar{a}(V(\xi)) = \int a(V(\xi)) d\xi_{N+1} \ldots d\xi_d$.

4. Notice now that the kernel K, endowed with the usual L^2 scalar product, is a Hilbert space and so that the kernel K admits an orthonormal basis $\{\psi_k(v)\}$, possibly countable (but this is unclear, there is no reason why K as a Hilbert space should be separable). Since $f \in K$ it will be given in a Fourier expansion

$$f = \sum_{k=1}^{\infty} m_k(t, x)\psi_k(v) \quad \text{where } m_k =< f, \psi_k >.$$

Moreover, we see that

$$< P_K(a)f, \psi_k >=< P_K(af), \psi_k >=< af, \psi_k >$$

and one computes that the set of moments m_k satisfies the initial value problem

$$\partial_t m_k + \sum_{j=1}^{d} \Big(\sum_{n=1}^{\infty} < a_j\psi_n, \psi_k > \frac{\partial m_n}{\partial x_j} \Big) = 0$$
$$m_k(0, x) =< P_K U^0(x, \cdot), \psi_k >= \int_{\mathbb{T}^d} U^0(x, v)\overline{\psi_k(v)} dv$$

As the wave speed a is real, $< a_j\psi_n, \psi_k >=< \psi_n, a_j\psi_k >$, and the system of moments is an infinite symmetric hyperbolic system.

3.2. **The multiscale case: A counter example.** A natural extension of the previous analysis is to deal with transport coefficients depending on more than one scale. Consider for example the equation

$$\partial_t u_\varepsilon + a_\varepsilon \cdot \nabla_x u_\varepsilon = 0, \tag{3.16}$$

with $a_\varepsilon = a(x, x/\varepsilon)$ and $a(x, v)$ a Lipschitz function, or even with $a_\varepsilon = a(x, x/\varepsilon, x/\varepsilon^2)$ (or with as many scales as one cares to introduce). Assume again that $\text{div}_v a(x, v) = 0$ and $\text{div}_w a(x, v, w) = 0$.

Is it possible to derive an equation for the double scale limit (or for the triple scale limit when $a(x, x/\varepsilon, x/\varepsilon^2)$) in the case of (3.16)?

In fact, it is relatively easy to show that the previous approach does not work! Everything goes as before in the beginning; upon defining

$$f_\varepsilon(t,x,v,w) = u_\varepsilon(t,x)\,\delta_p(v - x/\varepsilon)\,\delta_p(w - v/\varepsilon),$$

as in paragraph 2.1.2, one simply obtains the generalized kinetic equation

$$\partial_t f_\varepsilon + \nabla_x \cdot (a(x,v,w) f_\varepsilon) + \frac{1}{\varepsilon} a \cdot \nabla_v f_\varepsilon + \frac{1}{\varepsilon^2} a \cdot \nabla_w f_\varepsilon = 0.$$

However it is not always possible to derive a well posed problem for the hydrodynamic limit, even in the simple setting where a depends only on x, v and the equation for $f_\varepsilon(t,x,v)$ is

$$\partial_t f_\varepsilon + \nabla_x \cdot (a(x,v) f_\varepsilon) + \frac{1}{\varepsilon} a(x,v) \cdot \nabla_v f_\varepsilon = 0.$$

Indeed the only information that we have is that any limit f belongs to the kernel which now depends on v and x

$$K = \{f \in L^2(\mathbb{R}^d \times \mathbb{T}^d) \mid a(x,v) \cdot \nabla_v f = 0\}.$$

On the other hand, when projecting the equation on K, it is not possible to handle the term with the x derivative as projection on K and differentiation in x no longer commute. This is associated to the possibility that the dimensionality of K may vary with x.

In addition, at the level of the double scale limit, this is not a mere technical problem, rather the double scale limit is in general not unique and depends on the choice of the extracted subsequence in ε.

This can be simply seen for the problem

$$\begin{aligned} &\partial_t u_\varepsilon + a(x) \cdot \nabla_x u_\varepsilon = 0, \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}^2, \\ &u_\varepsilon(t = 0, x) = U^0(x, x/\varepsilon). \end{aligned} \tag{3.17}$$

The oscillations are due only to the initial data as the transport coefficient no longer depends on ε, and u the weak limit of u_ε satisfies the same equation. Take now

$$a_1(x) = 1, \quad a_2(x) = x_1\, \mathbb{1}_{0 \le x_1 \le 1} + 1\, \mathbb{1}_{x_1 > 1},$$

so that a is Lipschitz and divergence free, and select the initial data

$$U^0(x,v) = K(x)\, L(v_2),$$

with K and L two C^∞ functions, L periodic on $\mathbb{R}$ of period 1 and with zero average, and K compactly supported with support in x_1 in $[-1,\ -1/2]$.

As the average of $U^0(x,v)$ in v vanishes for all x, the weak limit u of u_ε is uniformly 0.

For $1 \leq t \leq 3/2$, the support in x_1 of the solution u_ε is entirely in the interval $[0,\ 1]$. Therefore any double scale limit f should satisfy

$$\partial_{v_1} f + x_1 \partial_{v_2} f = 0.$$

It is easy to check that the only L^2 solutions to this last equations are the functions which depend only on x and not on v. Therefore for $1 \leq t \leq 3/2$ the double scale limit is equal to u, i.e. uniformly vanishes.

Let us finally compute the double scale limit for $t > 2$ and check that it does not vanish. For that introduce the characteristics $X(t,x)$

$$\partial_t X(t,x) = a(X(t,x)), \quad X(0,x) = x.$$

We need the characteristics only for those x which belong to the support of U^0, that is for $-1 < x_1 < -1/2$.

As $a_1 = 1$, we simply have

$$X_1(t,x) = x_1 + t.$$

As to X_2, as long as $X_1 < 0$ or $t < t_0 = -x_1$ (remember $x_1 \in [-1,\ -1/2]$) it is equal to x_2. For $t_0 < t < t_1 = 1 - x_1$ (corresponding to X_1 in $[0,\ 1]$), we have

$$\partial_t X_2 = X_1 = x_1 + t.$$

As a consequence

$$X_2(t_1) = x_2 + x_1\,(t_1 - t_0) + \frac{t_1^2}{2} - \frac{t_0^2}{2} = x_2 + x_1 + \frac{(1-x_1)^2}{2} - \frac{x_1^2}{2} = x_2 + 1/2.$$

After $t > t_1$, $\partial_t X_2 = 1$ and so

$$X_2(t) = x_2 + 1/2 + t - t_1 = x_2 + 1/2 + t - 1 + x_1 = x_2 + X_1 - 1/2.$$

With this, the solution u_ε is given for $t > 2$ by

$$\begin{aligned} u_\varepsilon(t,x) &= u_\varepsilon(0, x_1 - t, x_2 - x_1 + 1/2) \\ &= K(x_1 - t, x_2 - x_1 + 1/2)\, L(x_2/\varepsilon - x_1/\varepsilon + 1/2\varepsilon). \end{aligned}$$

For every $\alpha \in [0,\ 1]$, choose a subsequence ε_n such that $1/2\varepsilon_n$ converges to α modulo 1. Then the double scale limit associated to this subsequence is the function

$$f(t,x,v) = K(x_1 - t, x_2 - x_1 + 1/2)\, L(v_2 - v_1 + \alpha).$$

Instead of one unique limit, we obtain a whole family which clearly indicates the ill-posedness of the problem at the level of the double scale limit.

4. Transport of oscillating fine-scale

An interesting question that can be studied using the techniques developped in section 3 is the problem of transport of an oscillatory fine-scale structure under a divergence-free vector field. Consider the homogenization problem

$$\begin{aligned} &\partial_t u_\varepsilon + \nabla_x \cdot \left(A\Big(t, x, \frac{\varphi(t,x)}{\varepsilon}\Big) u_\varepsilon \right) = 0 \\ &u_\varepsilon(0,x) = U^0(x, \frac{x}{\varepsilon}) \end{aligned} \tag{4.1}$$

where $\varphi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a C^2 map describing the fine scale of oscillations that satisfies for some $c > 0$

$$\begin{aligned} &\varphi(\cdot, t) \text{ is surjective and invertible for } t \text{ fixed} \\ &\det \nabla(\varphi^{-1}(\cdot, t)) \geq c > 0 \\ &\varphi(\cdot, 0) = id \end{aligned} \tag{h$_{so}$}$$

and $a_\varepsilon = A(t, x, \frac{\varphi(t,x)}{\varepsilon})$ is a divergence free field.

The latter is guaranteed provided $A(t, x, v)$ is a C^1 vector field 1-periodic in v such that

$$\begin{aligned} &\nabla_x \cdot A(t, x, v) = 0 \\ &\mathrm{tr}\left(\nabla_v A \nabla_x \varphi\right) = \sum_{i,j=1}^d \frac{\partial A_i}{\partial v_j} \frac{\partial \varphi_j}{\partial x_i} = 0 \end{aligned} \tag{h$_{tvf}$}$$

The initial data $U^0 \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ are 1-periodic in v and satisfy the uniform bounds

$$\int_{\mathbb{R}^d} |U^0(x, \frac{x}{\varepsilon})|^2 dx \leq C\,. \tag{h$_d$}$$

Then standard energy estimates imply that solutions of (4.1) satisfy the uniform bound

$$\int_{\mathbb{R}^d} |u_\varepsilon(t,x)|^2 dx \leq \int_{\mathbb{R}^d} |U^0(x, \frac{x}{\varepsilon})|^2 dx \leq C\,. \tag{4.2}$$

Our objective is to calculate an effective equation for the weak limits of $\{u_\varepsilon\}$. The counter example of section 3.2 indicates that we can not expect to do that in full generality. A more precise statement of what will be achieved is that we will identify conditions on the vector field A and the structure function φ under which an effective equation is calculated.

4.1. **Reformulation via a kinetic problem.** We introduce the "kinetic function"

$$f_\varepsilon = u_\varepsilon(t,x)\delta_p\big(v - \frac{\varphi(t,x)}{\varepsilon}\big)\,, \tag{4.3}$$

which is well defined (see section 2.1.1) as a measure. Due to the identities in lemma 2.1 of section 2.1.1, it is possible to transform the homogenization problem (4.1) into a hyperbolic limit for a kinetic initial value problem:

Lemma 4.1. *If u_ε a weak solution of* (4.1) *then f_ε in* (4.3) *verifies in $\mathcal{D}'$ the kinetic problem*

$$\begin{aligned} &\partial_t f_\varepsilon + \nabla_x\cdot(Af_\varepsilon) + \nabla_v(\frac{1}{\varepsilon}Bf_\varepsilon) = 0\,,\\ &\qquad\qquad f_\varepsilon(0,x,v) = U^0(x,v)\delta_p(v - \frac{x}{\varepsilon})\,, \end{aligned} \tag{4.4}$$

where the vector field $B(t,x,v)$, defined by

$$B_i = \big(\partial_t + A\cdot\nabla_x\big)\varphi_i\,,\quad i = 1,...,d\,, \tag{4.5}$$

is 1*-periodic in v and divergence-free, $\nabla_v\cdot B = 0$.*

Proof. Let u_ε be a weak solution of (4.1). By (h_{tvf}), B defined in (4.5) is 1-periodic in v and divergence free. We consider a test function $\theta \in C^1_c\big([0,\infty)\times\mathbb{R}^d; C^1(\mathbb{T}^d)\big)$ and compute

$$\begin{aligned} &\int_0^\infty\int_{\mathbb{R}^d}\int_{\mathbb{T}^d}\big[\partial_t\theta + \nabla_x(A\cdot\theta) + \frac{1}{\varepsilon}\nabla_v(B\cdot\theta)\big]df_\varepsilon\\ &\qquad = \int_0^\infty\int_{\mathbb{R}^d} u_\varepsilon(x,t)\Big(\partial_t\theta + A\cdot\nabla_x\theta + \frac{1}{\varepsilon}B\cdot\nabla_v\theta\Big)(t,x,\frac{\varphi}{\varepsilon})dxdt\\ &\qquad = \int_0^\infty\int_{\mathbb{R}^d} u_\varepsilon\left[\partial_t\big(\theta(t,x,\frac{\varphi}{\varepsilon})\big) + A(t,x,\frac{\varphi}{\varepsilon})\cdot\nabla_x\big(\theta(t,x,\frac{\varphi}{\varepsilon})\big)\right]dxdt\\ &\qquad = -\int_{\mathbb{R}^d} U^0(x,\frac{x}{\varepsilon})\theta(0,x,\frac{x}{\varepsilon})dx\\ &\qquad = -\int_{\mathbb{R}^d}\int_{\mathbb{T}^d}\theta(0,x,v)df_\varepsilon(0,x,v)\,; \end{aligned}$$

that is, f_ε is a weak solution of (4.4). □

An alternative, albeit formal, derivation of (4.4) may be obtained by studying characteristics. The characteristic curve of (4.1) emanating from the point y is defined by

$$\begin{cases} \frac{dx}{dt} = A(x,t,\frac{\varphi}{\varepsilon})\\ x(0,y) = y \end{cases}$$

and is denoted by $x = X(t;y)$. Along such curves we have

$$\frac{d}{dt}\Big(\frac{\varphi}{\varepsilon}\Big) = \frac{1}{\varepsilon}\Big(\varphi_t + A(x,t,\frac{\varphi}{\varepsilon})\cdot\nabla_x\varphi\Big)$$

The two equations together can be embedded into the system of ordinary differential equations

$$\begin{cases} \frac{dx}{dt} = A(t,x,v) \\ \frac{dv}{dt} = \frac{1}{\varepsilon}\Big(\varphi_t + A(x,t,v)\cdot\nabla_x\varphi\Big) \end{cases} \tag{4.6}$$

in the following sense: If $(Y(t;y,u), U(t;y,u))$ is the solution of (4.6) emanating from the point (y,u) then

$$X(t;y) = Y(t;y,\varphi(y,0))$$
$$\frac{\varphi(X(t;y),t)}{\varepsilon} = U(t;y,\varphi(y,0))$$

Note that $(4.4)_1$ is precisely the Liouville equation associated to the characteristic system (4.6).

4.2. **Conditions leading to an effective equation.** Our next goal is to derive an effective equation for the hydrodynamic limit of (4.4). We first show that under hypothesis (h_{so}) the definition (4.3) still induces good properties for the weak limit points of $\{f_\varepsilon\}$.

Lemma 4.2. *Under hypotheses* (h_{so}) *and* (4.2),

$$\delta_p\big(v - \frac{\varphi(x,t))}{\varepsilon}\big) \rightharpoonup 1 \qquad \text{in } \mathcal{D}' \tag{4.7}$$

$$f_\varepsilon \in_b L^\infty\big((0,\infty); L^2(\mathbb{R}^d; M_p)\big) \tag{4.8}$$

and, along a subsequence (if necessary),

$$f_\varepsilon \rightharpoonup f \qquad \text{weak-}\star \text{ in } L^\infty\big((0,\infty); L^2(\mathbb{R}^d; M_p)\big) \tag{4.9}$$

$$\text{with } f \in L^\infty\big((0,\infty); L^2(\mathbb{R}^d \times \mathbb{T}^d)\big) \tag{4.10}$$

Proof. For $\theta \in C_c^\infty\big((0,\infty)\times\mathbb{R}^d; C^\infty(\mathbb{T}^d)\big)$ we have

$$\begin{aligned} < \delta_p\big(v - \frac{\varphi(x,t))}{\varepsilon}\big), \theta > &= \int_0^\infty \int_{\mathbb{R}^d} \theta(x,t,\frac{\varphi(x,t))}{\varepsilon})dxdt \\ &= \int_0^\infty \int_{\mathbb{R}^d} \theta(\varphi^{-1}(y,t),t,\frac{y}{\varepsilon})|\det\nabla_y(\varphi^{-1})|dydt \\ &\to \int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} \theta(t,x,v)dvdxdt \end{aligned}$$

Note next that for (x,t) fixed

$$\|f_\varepsilon(x,t,\cdot)\|_{M_p} = |u_\varepsilon(x,t)|$$

and thus (4.8) and (4.9) follow from (4.2). Finally,

$$
\begin{aligned}
| < f_\varepsilon, \theta > | &\le \|u_\varepsilon\|_{L^\infty(L^2)} \int_0^\infty \left(\int_{\mathbb{R}^d} \big|\theta(x,t,\frac{\varphi(x,t))}{\varepsilon})\big|^2 dx \right)^{\frac{1}{2}} dt \\
&\le C \int_0^\infty \left(\int_{\mathbb{R}^d} \big|\theta(x,t,\frac{\varphi(x,t))}{\varepsilon})\big|^2 dx \right)^{\frac{1}{2}} dt \\
&\to C \int_0^\infty \left(\int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \big|\theta(x,t,v)\big|^2 dx dv \right)^{\frac{1}{2}} dt
\end{aligned}
$$

and (4.10) follows. □

Remark 4.3. A hypothesis of the type of (h_{so}) is essential for the validity of (4.7) and accordingly for (4.10). For instance, in the extreme case that φ is a constant map, $\varphi(x,t) \equiv c$, it is possible by choosing appropriate sequences $\varepsilon_n \to 0$ to achieve any weak limit

$$
\delta_p\big(v - \frac{c}{\varepsilon_n}\big) \rightharpoonup \delta_p(v - v_o) \quad \text{with any } 0 < v_o < 1.
$$

The regularity of f is then no better than the regularity of $\{f_\varepsilon\}$ and (4.10) is of course violated.

We conclude by providing a formal derivation of an effective equation. Consider the $\varepsilon \to 0$ limit of (4.4)-(4.5) and recall that, by (4.2) and lemma 4.2, we have $f_\varepsilon \rightharpoonup f$ as in (4.9) and (4.10). Define the set

$$
K_{t,x} = \left\{ g \in L^\infty((0,\infty); L^2(\mathbb{R}^d \times \mathbb{T}^d)) \;\Big|\; B(t,x,v)\cdot \nabla_v g = 0 \right\}.
$$

The set $K = N(\mathcal{B})$ is the null space of the operator $\mathcal{B} := B \cdot \nabla_v$ and in general it will depend on (t,x). We will derive the effective equation under the hypothesis

$$
\text{(H)} \qquad N(\mathcal{B}) \;\text{ is independent of } (x,t)
$$

Then we have the decomposition

$$
L^2 = N(\mathcal{B}) \oplus \overline{R(\mathcal{B}^T)} = K \oplus K^\perp
$$

and the spaces remain the same for any point (x,t). Let P denote the L^2-projection on the set K. Any $\theta \in L^2(\mathbb{T}^d)$ can be decomposed as

$$
\theta = P\theta + (I - P)\theta =: \psi + \phi
$$

Moreover the differentiation operators ∂_t and ∇_x commute with the projector P.

For $\psi \in N(\mathcal{B})$, using $\nabla_x \cdot A = \nabla_v \cdot B = 0$, we derive from (4.4) that

$$
\partial_t < f, \psi > + \nabla_x < Af, \psi > = 0 \tag{4.11}
$$

where the brackets denote the usual inner product in $L^2(\mathbb{T}^d)$. One easily sees that $f(t,x,\cdot) \in K$ for a.e. (t,x). Given $\theta \in L^2(\mathbb{T}^d)$ let $\psi = P\theta$. Then

$$< f, \theta >=< f, \psi >$$

and

$$< A_i f, \psi >=< P(A_i f), \psi >=< P(A_i f), \theta >$$

Since $f \in K$ we have $P(A_i f) = P(A_i) f$ and we conclude that (4.11) can be expressed in the form

$$\partial_t < f, \theta > + \nabla_x \cdot < P(A) f, \theta >= 0 \,, \quad \theta \in L^2(\mathbb{T}^d) \,.$$

The effective equation thus takes the form

$$\partial_t f + \nabla_x \cdot (PA) f = 0 \,. \tag{4.12}$$

The above derivation of (4.12) is formal and is based on hypothesis (H). This assumption is quite restrictive especially when viewed together with (h_{tvf}) that has to be satisfied simultaneously. We view this equation as a theoretical framework of when an effective equation can be computed. To derive it rigorously one needs an analysis as in Theorem 3.1 and we will not pursue the details here. The counter example in section 3.2 indicates that the hypothesis (H) is essential.

We list two examples that can be viewed under the above framework. First, the homogenization problem (3.1) is a special case of (4.1) with the obvious identifications. A second example is given by the problem

$$\begin{aligned} \partial_t u_\varepsilon + a(x) \cdot \nabla_x u_\varepsilon = 0 \\ u_\varepsilon(x,0) = U(x, \frac{x}{\varepsilon}) \end{aligned} \tag{4.13}$$

where a is a divergence free field, $\nabla_x \cdot a = 0$. Define $\varphi(t,x)$ to be the backward characteristic emanating from the point x. Then $\varphi = (\varphi_1, ..., \varphi_n)$ satisfies

$$\partial_t \varphi_i + a(x) \cdot \nabla_x \varphi_i = 0$$

$$\varphi_i(0,x) = x_i$$

The problem (4.13) fits under the framework of (4.1) under the selections

$$A(t,x,v) = a(x) \,. \quad B_i(t,x,v) = \big(\partial_t + a(x) \cdot \nabla_x\big)\varphi_i = 0 \,.$$

The kinetic equation for $f_\varepsilon = u_\varepsilon \delta_p\big(v - \frac{\varphi}{\varepsilon}\big)$ becomes

$$\partial_t f_\varepsilon + a(x) \cdot \nabla_x f_\varepsilon = 0$$

$$f_\varepsilon(0,x,v) = U(x,v)\delta_p\big(v - \frac{x}{\varepsilon}\big)$$

while the limiting f satisfies the same transport equation with initial condition $f(0,x,v) = U(x,v)$. Hence, it is computed explicitly by

$$f(t,x,v) = U(\varphi(t,x),v)\,.$$

5. Enhanced diffusion

In this section we study the enhanced diffusion problem

$$\begin{aligned} \partial_t u_\varepsilon + \frac{1}{\varepsilon} a(x,\frac{x}{\varepsilon}) \cdot \nabla_x u_\varepsilon &= \alpha \triangle_x u_\varepsilon \\ u_\varepsilon(0,x) &= U^0(x,\frac{x}{\varepsilon}) \end{aligned} \tag{5.1}$$

where $a(x,v)$ is a Lipschitz vector field periodic (with period 1) in v that satisfies

$$\nabla_x \cdot a = \nabla_v \cdot a = 0\,, \quad \int_{\mathbb{T}^d} a(x,v)dv = 0\,, \tag{h$_{vf}$}$$

$\alpha > 0$ is constant and $U^0 \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$. We use this as an example to develop the methodology of section 2. For previous work and a commentary on the significance of this problem we refer to Avellaneda-Majda [5], Fannjiang-Papanicolaou [12] and references therein. It is assumed that the initial data oscillates at the scale ε and satisfy the uniform bound

$$\int_{\mathbb{R}^d} |U^0(x,\frac{x}{\varepsilon})|^2 dx \le C\,. \tag{h$_d$}$$

Standard energy estimates then imply the uniform bounds

$$\int_{\mathbb{R}^d} |u_\varepsilon(t,x)|^2 dx + \alpha \int_0^t \int_{\mathbb{R}^d} |\nabla u_\varepsilon|^2 dx dt \le \int_{\mathbb{R}^d} |U^0(x,\frac{x}{\varepsilon})|^2 dx \le C \tag{5.2}$$

for solutions of (5.1).

We introduce the kinetic decomposition

$$f_\varepsilon(t,x,v) = u_\varepsilon(t,x)\delta_p\big(v - \frac{x}{\varepsilon}\big) \quad t \in \mathbb{R}_+,\, x \in \mathbb{R}^d,\, v \in \mathbb{T}^d\,, \tag{5.3}$$

and use Lemma 2.1 to obtain that f_ε satisfies the transport-diffusion equation

$$\begin{aligned} \frac{\partial f_\varepsilon}{\partial t} + \frac{1}{\varepsilon} a(x,v) \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon^2}\big(a(x,v) \cdot \nabla_v f_\varepsilon - \alpha \triangle_v f_\varepsilon\big) & \\ = \alpha \triangle_x f_\varepsilon + \frac{2\alpha}{\varepsilon} \nabla_x \cdot \nabla_v f_\varepsilon\,, & \qquad \text{in } \mathcal{D}' \\ f_\varepsilon(t=0,x,v) = U^0(x,v)\ \delta_p(v - \frac{x}{\varepsilon}) & \end{aligned} \tag{5.4}$$

with periodic boundary conditions on the torus in the v variable. Our objective is to analyze the $\varepsilon \to 0$ limit of this problem and through this process to calculate the effective equation satisfied by the weak limit of u_ε. We note that this is a hydrodynamic limit problem in the diffusive scaling for the kinetic equation (5.4).

We prove.

Theorem 5.1. *Under hypothesis* (h_{vf}) *and* (h_d) *we have the following asymptotic behavior for* f_ε *as* $\varepsilon \to 0$*:*

$$f_\varepsilon(t,x,v) \rightharpoonup u(t,x) \quad \textit{weak-}\star \textit{ in } L^\infty([0,\ \infty),\ L^2(\mathbb{R}^d, M_p) \tag{5.5}$$

$$f_\varepsilon(t,x,v) = u_\varepsilon(t,x) + \varepsilon g(t,x,v) + o(\varepsilon)\,, \quad \textit{in } \mathcal{D}' \tag{5.6}$$

where u *of class* (5.2) *and* $g \in L^2((0,\infty)\times\Omega; H^1(\mathbb{T}^d))$ *satisfy respectively*

$$\partial_t u - \alpha \triangle_x u + \nabla_x \cdot \int_{\mathbb{T}^d} a(x,v) g(t,x,v) = 0 \tag{5.7}$$

$$\alpha \triangle_v g - \nabla_v \cdot a g = a \cdot \nabla_x u \tag{5.8}$$

The weak limit u *satisfies the effective diffusion equation*

$$\partial_t u = \alpha \sum_{i,j} \partial_{x_i} \Big(\Big(\delta_{ij} + \int_{\mathbb{T}^d} \nabla_v \chi_i \cdot \nabla_v \chi_j dv \Big) \partial_{x_j} u \Big)$$

$$u(0,x) = \int_{\mathbb{T}^d} U^0(x,v)\, dv$$

where χ_k*,* $k = 1,...,d$*, is the solution of the cell problem*

$$\alpha \triangle_v \chi_k - \nabla_v \cdot a \chi_k = a \cdot e_k \tag{5.9}$$

Proof. Let f_ε be defined as in (5.3). Then f_ε satisfies the problem (5.4) and $u_\varepsilon = \int_{\mathbb{T}^d} f_\varepsilon$. The proof is split in three steps:

Step 1 : Characterization of the weak limit. From (5.2) and Lemma 2.1 we obtain uniform bounds for f_ε:

$$\begin{aligned} &f_\varepsilon \in_b L^\infty([0,\ \infty)\,;\ L^2(\mathbb{R}^d, M_p)\,, \\ &\big(\nabla_x + \frac{1}{\varepsilon}\nabla_v\big) f_\varepsilon \in_b L^2((0,\infty)\times\mathbb{R}^d\,;\ M_p) \end{aligned} \tag{5.10}$$

Using (a slight variant of) Proposition 2.2 we see that, along a subsequence if necessary, f_ε satisfies

$$\begin{aligned} &f_\varepsilon \rightharpoonup f \quad \text{weak-}\star \text{ in } L^\infty([0,\ \infty);\ L^2(\mathbb{R}^d, M_p) \\ &f \in L^\infty([0,\ \infty)\,;\ L^2(\mathbb{R}^d \times \mathbb{T}^d))\,, \end{aligned} \tag{5.11}$$

that is

$$\int \theta(t,x,v) d f_\varepsilon(t,x,v) \to \int \theta(t,x,v) f(t,x,v) dt dx dv$$

for $\theta \in L^1((0,\infty); L^2(\mathbb{R}^d, C_p))$.

Passing to the limit $\varepsilon \to 0$ in (5.4) and using (5.10) we see that f satisfies

$$\alpha \triangle_v f - \nabla_v \cdot a f = 0 \,, \quad \text{in } \mathcal{D}',$$

and for any test function θ

$$< \nabla_v f, \theta >= \lim_{\varepsilon \to 0} < \nabla_v f_\varepsilon, \theta >= 0 \,.$$

Hence, $\nabla_v f = 0$ in $\mathcal{D}'$ and

$$f(t,x,v) = \int_{\mathbb{T}^d} f dv =: u(t,x) \tag{5.12}$$

Step 2 : Asymptotics of f_ε. Define next

$$g_\varepsilon(t,x,v) = \frac{1}{\varepsilon}\big(f_\varepsilon(t,x,v) - u_\varepsilon(t,x)\big) \,, \quad t \in \mathbb{R}_+ \,,\, x \in \mathbb{R}^d \,,\, v \in \mathbb{T}^d \,, \tag{5.13}$$

where $u_\varepsilon = \int_{\mathbb{T}^d} f_\varepsilon$. We proceed along the lines of Proposition 2.4 replacing the bounds of that proposition by the bound (5.2) and accounting for the extra dependence in time. After minor modifications in the proof we obtain for any $T > 0$

$$\begin{aligned} &g_\varepsilon \in_b L^2\big((0,T)\,;\, H^{-1}(\Omega, M_p)\big) \\ &g_\varepsilon \rightharpoonup g \quad \text{weak-}\star \text{ in } L^2\big((0,T)\,;\, H^{-1}(\Omega, M_p)\big) \\ &g \in L^2((0,\infty) \times \Omega\,;\, H^1(\mathbb{T}^d))\,, \quad \int_{\mathbb{T}^d} g = 0 \,. \end{aligned} \tag{5.14}$$

Accordingly, f_ε enjoys the asymptotic expansion

$$f_\varepsilon = u_\varepsilon + \varepsilon g + o(\varepsilon) \quad \text{in } \mathcal{D}' \,.$$

On the other hand, on account of (5.1), (5.4) and (h_{vf}), it follows that u_ε and g_ε satisfy

$$\partial_t u_\varepsilon - \alpha \triangle_x u_\varepsilon + \nabla_x \cdot \int_{\mathbb{T}^d} a(x,v) \frac{f_\varepsilon - u_\varepsilon}{\varepsilon} = 0 \tag{5.15}$$

and

$$\alpha \triangle_v g_\varepsilon - \nabla_v \cdot a g_\varepsilon = \varepsilon(\partial_t f_\varepsilon - \alpha \triangle_x f_\varepsilon) - 2\alpha \nabla_x \cdot \nabla_v f_\varepsilon + \nabla_x \cdot a f_\varepsilon \tag{5.16}$$

Using (5.11), (5.12) and (5.14), we pass to the limit $\varepsilon \to 0$ and deduce that u, g satisfy (5.7) and (5.8) respectively.

Step 3 : Characterization of the limit problem. Due to its regularity the solution g of (5.8) is unique and can be expressed in the form

$$g = \nabla_x u(t,x) \cdot \chi(x,v)$$

where $\chi = (\chi_1, ..., \chi_d)$ is the solution of the cell problem

$$\alpha \triangle_v \chi_k - \nabla_v \cdot a \chi_k = a \cdot e_k = a_k \,. \tag{5.17}$$

A direct computation shows that solutions of (5.9) satisfy the property

$$\frac{1}{2}\int_{\mathbb{T}^d}(a_i\chi_k + a_k\chi_i)dv = -\alpha\int_{\mathbb{T}^d}\nabla_v\chi_i\cdot\nabla_v\chi_k dv$$

and (5.7) may be written in the equivalent forms

$$\begin{aligned}\partial_t u &= \sum_{i,j}\partial_{x_i}\left(\Big(\alpha\delta_{ij} - \frac{1}{2}\int_{\mathbb{T}^d}(a_i\chi_j + a_j\chi_i)\,dv\Big)\partial_{x_j}u\right)\\ &= \alpha\sum_{i,j}\partial_{x_i}\left(\Big(\delta_{ij} + \int_{\mathbb{T}^d}\nabla_v\chi_i\cdot\nabla_v\chi_j dv\Big)\partial_{x_j}u\right)\end{aligned}$$

The latter is a diffusion equation with positive definite diffusion matrix

$$D_{ij} = \delta_{ij} + \int_{\mathbb{T}^d}\nabla_v\chi_i\cdot\nabla_v\chi_j dv$$

$$\sum_{ij} D_{ij}\nu_i\nu_j = |\nu|^2 + \int_{\mathbb{T}^d}|\nabla_v(\chi\cdot\nu)|^2 dv\,, \quad \nu\in\mathbb{R}^d,$$

determined through the solution of (5.9). □

6. Appendix I

We prove a lemma that illuminates the nature of multiscale decompositions and is used in producing examples.

Lemma 6.1. *Let Ω be an open subset of $\mathbb{R}^d$, $\theta\in C_c(\Omega)$, $\varphi\in C(\mathbb{T}^d)$, $\psi\in C(\mathbb{T}^d)$, and suppose that $\delta=\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$. Then, as $\varepsilon\to 0$,*

$$\int_\Omega \theta(x)\varphi(\frac{x}{\varepsilon})dx \to \int_{\mathbb{T}^d}\varphi(z)dz\int_\Omega\theta(x)dx \tag{6.1}$$

$$\int_\Omega \theta(x)\varphi(\frac{x}{\varepsilon})\psi(\frac{x}{\varepsilon\delta})dx \to \int_{\mathbb{T}^d}\psi(y)dy\int_{\mathbb{T}^d}\varphi(z)dz\int_\Omega\theta(x)dx \tag{6.2}$$

Proof. We refer to [6] for the proof of (6.1). To show (6.2) observe that

$$\begin{aligned}&\int_\Omega \theta(x)\varphi(\frac{x}{\varepsilon})\psi(\frac{x}{\varepsilon\delta})dx - \int_{\mathbb{T}^d}\psi(y)dy\int_{\mathbb{T}^d}\varphi(z)dz\int_\Omega\theta(x)dx\\ &\quad= \int_\Omega \theta(x)\varphi(\frac{x}{\varepsilon})\psi(\frac{x}{\varepsilon\delta})dx - \int_{\mathbb{T}^d}\psi(y)dy\int_\Omega\theta(x)\varphi(\frac{x}{\varepsilon})dx\\ &\quad\quad+\int_{\mathbb{T}^d}\psi(y)dy\left(\int_\Omega\theta(x)\varphi(\frac{x}{\varepsilon})dx - \int_{\mathbb{T}^d}\varphi(z)dz\int_\Omega\theta(x)dx\right)\\ &\quad=: I_1 + I_2\end{aligned}$$

and that $I_2\to 0$ as $\varepsilon\to 0$, by (6.1).

Consider a covering of $\operatorname{supp}\theta$ by cubes $\bar{C}_k$ centered at points $\chi_k\in\varepsilon\delta\mathbb{Z}^d$ of latteral size $\varepsilon\delta$. We can arrange the cubes so that $\operatorname{supp}\theta\subset\cup_{k=1}^{\bar{N}}\bar{C}_k\subset\Omega$

and their number $\bar N$ satisfies $\bar N(\varepsilon\delta)^d = O(|\operatorname{supp}\theta|)$. We have

$$\int_\Omega \theta(x)\varphi(\frac{x}{\varepsilon})\psi(\frac{x}{\varepsilon\delta})dx = \sum_{k=1}^{\bar N}\int_{\bar C_k}\theta(x)\varphi(\frac{x}{\varepsilon})\psi(\frac{x}{\varepsilon\delta})dx$$
$$= \sum_{k=1}^{\bar N}(\varepsilon\delta)^d\int_{\mathbb{T}^d}\theta(\bar\chi_k+\varepsilon\delta z)\varphi(\frac{1}{\varepsilon}\bar\chi_k+\delta z)\psi(z)dz$$

and

$$I_1 = \sum_{k=1}^{\bar N}\int_{\bar C_k}\theta(x)\varphi(\frac{x}{\varepsilon})\psi(\frac{x}{\varepsilon\delta})dx - \int_{\mathbb{T}^d}\psi(z)dz\ \sum_{k=1}^{\bar N}\int_{\bar C_k}\theta(x)\varphi(\frac{x}{\varepsilon})dx$$
$$= \int_{\mathbb{T}^d}\psi(z)\sum_{k=1}^{\bar N}(\varepsilon\delta)^d\int_{\mathbb{T}^d}\Big(\theta(\bar\chi_k+\varepsilon\delta z)\varphi(\frac{1}{\varepsilon}\bar\chi_k+\delta z)$$
$$- \theta(\bar\chi_k+\varepsilon\delta y)\varphi(\frac{1}{\varepsilon}\bar\chi_k+\delta y)\Big)\,dy\,dz$$

Since $\bar N = O(\frac{1}{(\varepsilon\delta)^d})$ and $\lim_{\varepsilon\to 0}\delta = 0$, we deduce $I_2 \to 0$ as $\varepsilon \to 0$ and (6.2). □

Remark 6.2. Both equations can be extended for test functions $\theta \in C(\bar\Omega)$ provided that $\Omega \subset \mathbb{R}^d$ is a bounded open set and its boundary $\partial\Omega$ is of finite $d-1$ Hausdorff dimension.

The derivation of (2.12), (2.15) and (2.18) is based on Lemma 6.1 together with density arguments of the type described below for the case of (2.12). The proof of the remaining relations follow similar lines and are omitted.

Lemma 6.3. *We have*

$$\delta_p\big(v - \frac{x}{\varepsilon}\big) \rightharpoonup 1 \qquad \textit{weak-}\star \textit{ in } L^\infty(\Omega, M_p)$$

Proof. We need to show that for $\theta \in L^1(\Omega, C_p)$ we have

$$\int_\Omega \theta(x,\frac{x}{\varepsilon})dx \to \int_\Omega\int_{\mathbb{T}^d}\theta(x,v)dxdv$$

Equation (6.1) justifies that for $\theta = \chi(x)\otimes\varphi(v)$ a tensor product with $\chi \in C_c(\Omega)$ and $\varphi \in C_p$ and by a density argument also for $\chi \in L^1(\Omega)$, $\varphi \in C_p$.

To complete the proof we need to show that finite sums of tensor products $\sum_j \chi_j \otimes \varphi_j$ are dense in $L^1(\Omega, C_p)$. Fix $\theta \in L^1(\Omega, C_p)$ and consider a decomposition of the torus $\mathbb{T}^d$ into squares of size $1/n$. Take a partition of unity $\varphi_i \in C_p$, $i = 1, ..., n^d$, with each φ_i supported in a square of size $2/n$ and $\sum_i \varphi_i = 1$. Let v_i be the center of each square and define

$$\theta_n(x,v) = \sum_i \theta(x,v_i)\varphi_i(v)$$

Clearly, θ_n is a sum of tensor products. Now define

$$\sup_{v\in\mathbb{T}^d} |\theta(x,v) - \theta_n(x,v)| \le \sup_{v\in\mathbb{T}^d, |h|<\frac{2}{n}} |\theta(x,v) - \theta(x,v+h)| =: g_n(x)$$

and thus

$$\|\theta - \theta_n\|_{L^1(C_p)} \le \int_\Omega |g_n(x)| dx$$

Note that $g_n(x) \to 0$ for a.e. $x \in \Omega$ and that $|g_n(x)| \le 2\sup_{v\in\mathbb{T}^d} |\theta(x,v)|$. The latter is an L^1 function by the very definition of θ, and the dominated convergence theorem implies $\int_\Omega |g_n| dx \to 0$. □

7. Appendix II: Some basic results of ergodic theory

The purpose of this appendix is to recall some well known properties of the classical ergodic theory for the projection on the kernel

$$K = \{f \in L^2(\mathbb{T}^d) \mid a(v)\cdot\nabla_v f(v) = 0\},$$

where the last equality is of course in the sense of distributions.

Let us define the characteristics associated with a which are the solutions on $\mathbb{T}^d$ of the following differential equation

$$\partial_t T(t,v) = a(T(t,v)), \quad T(0,v) = v.$$

Then assuming that

$$a \in W^{1,\infty}(\mathbb{T}^d), \quad \nabla_v \cdot a = 0, \tag{7.1}$$

the characteristics are well defined and for a fixed t, the transform $v \to T(t,v)$ is a mesure preserving homeomorphism of $\mathbb{T}^d$. We then have the well-known theorem (see Sinai [20] for more details)

Theorem 7.1. *For every $f \in L^p(\mathbb{T}^d)$ with $1 \le p < \infty$, there exists a unique function in $L^p(\mathbb{T}^d)$, denoted by Pf, such that*

$$\frac{1}{t}\int_0^t f(T(s,v))\, ds \longrightarrow Pf(v), \text{ as } t\to\infty, \text{ strongly in } L^p(\mathbb{T}^d).$$

Moreover Pf satisfies in the sense of distribution

$$a(v)\cdot\nabla_v Pf(v) = 0,$$

and if $f \in L^2(\mathbb{T}^d)$, then Pf is exactly the orthogonal projection of f on K.

This immediately implies the

Corollary 7.2. *The orthogonal projection on K may be extended as an operator on $L^p(\mathbb{T}^d)$ for every $1 \leq p < \infty$. In addition if $f \in L^2 \cap L^p(\mathbb{T}^d)$ with $1 \leq p \leq \infty$ ($p = \infty$ allowed), then $P_K f$ also belongs to $L^2 \cap L^p(\mathbb{T}^d)$.*

Proof of Theorem 7.1. This proof exactly corresponds to the one in [20] in the particular case which we consider.

Notice that if, for $f \in L^p(\mathbb{T}^d)$, $\frac{1}{t}\int_0^t f(T(s,v))\,ds$ converges to Pf then trivially

$$Pf(T(t,v)) = Pf(v) \quad \forall\, t.$$

Therefore we automatically have in the sense of distribution that

$$a(v) \cdot \nabla_v P f = 0.$$

Take now f in L^p and assume first that there exists $g \in L^p$ with $a \cdot \nabla_v g = 0$ and $h \in W^{1,p}(\mathbb{T}^d)$ such that

$$f = g + a \cdot \nabla_v h.$$

This is not true in general for all f (Fredholm's alternative does not hold for $a \cdot \nabla$ in general). Then notice that in the sense of distribution

$$\partial_t \left(g(T(t,v))\right) = a(T(t,v)) \cdot \nabla_v g(T(t,v)) = 0,$$

and so

$$g(T(t,v)) = g(T(0,v)) = g(v).$$

On the other hand

$$a(T(t,v)) \cdot \nabla_v h(T(t,v)) = \partial_t \left(h(T(t,v))\right),$$

and therefore

$$\frac{1}{t}\int_0^t f(T(s,v))\,ds = g(v) + \frac{h(T(t,v)) - h(v)}{t}.$$

Consequently in this case $\frac{1}{t}\int_0^t f(T(s,v))\,ds$ converges to g which is unique as a consequence. This proves the theorem on the set

$$L_p = \{g + a(v) \cdot \nabla_v h(v) \mid h \in W^{1,\infty}(\mathbb{T}^d),\ g \in L^p(\mathbb{T}^d) \text{ with } a \cdot \nabla_v g = 0\}.$$

Let us first prove that L_2 defined as the particular case of L_p with $p = 2$ or

$$L_2 = \{g + a(v) \cdot \nabla_v h(v) \mid h \in W^{1,\infty}(\mathbb{T}^d),\ g \in L^2(\mathbb{T}^d) \text{ with } a \cdot \nabla_v g = 0\},$$

is dense in $L^2(\mathbb{T}^d)$. If L_2 is not dense, then there exists $f \in L^2 \setminus \{0\}$ orthogonal to L_2. This implies that for all $h \in W^{1,\infty}(\mathbb{T}^d)$

$$\int_{\mathbb{T}^d} f(v)\, a(v) \cdot \nabla_v h(v)\, dv = 0,$$

or in other words $f \in K$. But $K \subset L_2$ and f should consequently be orthogonal to K, which is impossible. Notice that Pf necessarily is the orthogonal projection on K as $a \cdot \nabla h$ belongs to $K^\perp$.

Now for any $f \in L^p$. If $1 \leq p < 2$, take $g_n + a \cdot \nabla_v h_n = f_n \in L_2$ converging toward f in L^p (first take $\hat{f}_n \in L^2$ and then select f_n by diagonal extraction). We have that

$$\begin{aligned} \| \frac{1}{t} \int_0^t f(T(s,v))\, ds - \frac{1}{t'} \int_0^{t'} f(T(s,v))\, ds \|_{L^p} \\ \leq 2\|f - f_n\|_{L^p} + \left(\frac{1}{t} + \frac{1}{t'} \right) \|h_n\|_{L^p}. \end{aligned}$$

So the sequence $\frac{1}{t}\int_0^t f(T(s,v))\, ds$ is of Cauchy in L^p and hence converges to a unique limit Pf.

Finally if $f \in L^\infty(\mathbb{T}^d)$, then $f \in L^2(\mathbb{T}^d)$ and $\frac{1}{t}\int_0^t f(T(s,v))\, ds$ converges to Pf in L^2. As the first quantity is uniformly bounded in L^∞, $Pf \in L^\infty$ and the convergence holds in every L^p, $p < \infty$. By interpolation, one eventually obtains the desired result for $f \in L^p(\mathbb{T}^d)$. □

Note that from the proof, if Fredholm's alternative is true for $a \cdot \nabla$ then one obtains a rate of convergence for any $f \in L^2$

$$\left\| \frac{1}{t} \int_0^t f(T(s,v))\, ds - Pf \right\|_{L^2} = O(t^{-1}). \tag{7.2}$$

Many other particular properties then automatically hold for the kernel K (for instance $C^\infty \cap K$ dense in K). However it is clear from this rate that this is not possible for many a. Even simple constant field like $a = (1, \sqrt{2})$ in $2d$ cannot imply (7.2) for any $f \in L^2$...

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