A variational approximation scheme for three dimensional elastodynamics with polyconvex energy

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Abstract

We construct a variational approximation scheme for the equations of three dimensional elastodynamics with polyconvex stored energy. The scheme is motivated by some recently discovered geometric identities (Qin [18]) for the null Langangians (the determinant and cofactor matrix), and by an associated embedding of the equations of elastodynamics into an enlarged system which is endowed with a convex entropy. The scheme decreases the energy and its solvability is reduced to the solution of a constrained convex minimization problem. We prove that the approximating process admits regular weak solutions, which in the limit produce a measure-valued solution for polyconvex elastodynamics that satisfies the classical weak form of the geometric identities. This latter property is related to the weak continuity properties of minors of Jacobian matrices, here exploited in a time-dependent setting.

1 Introduction

The equations describing the evolution of a continuous medium with nonlinear elastic response in referential description are

$$\frac{\partial^2 y}{\partial t^2} = \nabla \cdot S(\nabla y), \tag{1.1}$$

where $y : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$ stands for the displacement and S for the Piola-Kirchoff stress tensor. These equations may be expressed in the form of a system of conservation laws

$$\begin{aligned} \frac{\partial}{\partial t} F_{i\alpha} &= \frac{\partial}{\partial x^\alpha} v_i \\ \frac{\partial}{\partial t} v_i &= \frac{\partial}{\partial x^\alpha} S_{i\alpha}(F) \end{aligned} \qquad i, \alpha = 1, 2, 3. \tag{1.2}$$

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where v_i are the components of the velocity $v = \partial_t y$, $F_{i\alpha} = \partial_\alpha y_i$ are the components of the deformation gradient $F = \nabla y$, and we use summation convention over repeated indices.

For a hyperelastic material S is expressed as the gradient of a stored energy function $\sigma : \mathrm{Mat}^{3\times 3} \to [0, \infty)$,

$$S(F) = \frac{\partial \sigma}{\partial F}(F). \tag{1.3}$$

The principle of material frame indifference [23, 1] imposes on σ the requirement that σ remain invariant under rigid rotations, or,

$$\sigma(OF) = \sigma(F) \quad \text{for all orthogonal matrices } O \in O(3). \tag{1.4}$$

In continuum physics, weak solutions of a system of conservation laws are required to satisfy *entropy* inequalities of the form

$$\partial_t \eta + \partial_\alpha q_\alpha \leq 0 \tag{1.5}$$

where η, q are related by a first order partial differential equation. Such inequalities are a manifestation of irreversibility and as such originate from the second law of thermodynamics. For the system of elastodynamics an important entropy pair is

$$\eta = \frac{1}{2}|v|^2 + \sigma(F)\,, \quad q_\alpha = -S_{i\alpha}(F)\, v_i\,, \tag{1.6}$$

in which case inequality (1.5) expresses the dissipation of mechanical energy on shocks.

Convexity of the stored energy function is incompatible with certain physical requirements and is thus ruled out from the list of natural assumptions. It conflicts with the requirement that the energy increase without bound as $\det F \to 0^+$. In addition, convexity of the energy together with the axiom of frame indifference impose restrictions on the induced Cauchy stresses which rule out certain naturally occurring states of stress (*e.g.* Coleman and Noll [5, Sec 8], Ciarlet [4, Sec 4.8]). While there has been substantial progress in handling lack of convexity in elastostatics starting with the work of Ball [2], the analysis is far less developed for elastodynamics: the reader is referred to Dafermos and Hrusa [6] for local existence of smooth solutions, to Klainerman and Sideris [15, 21] for long-time and global existence of strong solutions for small initial data, and to Dafermos [7, 8] for uniqueness of smooth solutions in the class of BV solutions with moderate shocks.

Further complications are presented to the theory of elastodynamics by the existence of (radially symmetric) cavitating solutions, which can actually decrease the mechanical energy [16, 17]. The existence of global weak solutions is a completely open problem, except in one-space dimension: in this case existence was proved using compensated compactness for L^∞ data by DiPerna [12], proving a conjecture of Tartar [22], and later in an L^p setting in [20].

The objective of this article is to provide a variational approximation scheme for the equations of polyconvex elastodynamics, thus establishing a link between the well developed theory of elastostatics and the equations of elastodynamics. For the map $F \mapsto \sigma(F)$ we employ the assumption of polyconvexity, familiar from the work of Morrey and Ball on vectorial minimization problems. This assumption postulates that σ factorizes as

$$\sigma(F) = G(F, \operatorname{cof} F, \det F)\,, \tag{1.7}$$

with G a strictly convex function of F, $\operatorname{cof} F$ and $\det F$ ($\operatorname{cof} F$ is the matrix of the cofactors of F); this encompasses physically realistic models (*e.g.* [4] Sec 4.9, 4.10).

Our analysis is based on the observation of T. Qin [18] that smooth solutions of (1.2) satisfy the *additional conservation laws*

$$\begin{aligned} \frac{\partial}{\partial t}\det F &= \frac{\partial}{\partial x^\alpha}\big((\operatorname{cof} F)_{i\alpha} v_i\big) \\ \frac{\partial}{\partial t}(\operatorname{cof} F)_{k\gamma} &= \frac{\partial}{\partial x^\alpha}(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}v_i). \end{aligned} \tag{1.8}$$

These identities are derivable directly from the first equation in (1.2); one may then append (1.8) to (1.2) and view the resulting system in terms of new variables (v, F, Z, w) with $Z = \operatorname{cof} F$ and $w = \det F$. The resulting system admits, for a (strictly) polyconvex stored energy σ, a (strictly) convex entropy and is thus symmetrizable. We learned about (1.8) from C. Dafermos [8], who attributes them to T. Qin [18] and to unpublished work of P. Lefloch; we thank one of the referees for pointing out the latter work. In [8] the aforementioned enlarged system is used to obtain local existence of classical solutions, and uniqueness of smooth solutions within the class of entropy weak solutions. Furthermore, it is pointed out in [8] that the relations (1.8) may be visualized as the counterparts in a Lagrangian description of trivial conservation laws in the Eulerian description and that, as such, they are geometric constraints that remain valid for weak solutions. Here we recover this property directly (see lemma 5 below), and also show that the relations (1.8) are stable under weak convergence (see remark after lemma 5).

It is worth remarking that there are different possible ways to enlarge the system (1.2), in particular as regards the form of the momentum equation in the new variables. An interesting possibility is suggested by the relationship - pursued in section 2 - between the form of the geometric constraints (1.8) and the notion of null-Lagrangians (Ball, Currie and Olver [3]). It suggests the embedding of the equations of elastodynamics into the enlarged system

$$
\begin{aligned}
&\partial_t v_i = \partial_\alpha \left(\frac{\partial G}{\partial \Xi^A}(F, Z, w) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right) = \partial_\alpha \big(g_{i\alpha}(F, Z, w; F) \big) \\
&\partial_t F_{i\alpha} = \partial_\alpha v_i \\
&\partial_t Z_{k\gamma} = \partial_\alpha (\epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{j\beta} v_i) \\
&\partial_t w = \partial_\alpha \big((\operatorname{cof} F)_{i\alpha} v_i \big).
\end{aligned}
\tag{1.9}
$$

for the augmented set of variables $(v, F, Z, w) \in \mathbb{R}^{22}$, where $\Phi(F) = (F, \operatorname{cof} F, \det F)$ is the vector of null-Lagrangians; the explicit form of $g_{i\alpha}$ is given in (2.15). This enlarged system is obtained in section 2 by appropriate modification of the fluxes and has the important property that it is endowed with a strictly convex entropy for any initial data. Moreover, if the data satisfy that $(F, Z, w) = (F, \operatorname{cof} F, \det F)$ at time $t = 0$, then $(F, Z, w) = (F, \operatorname{cof} F, \det F)$ thereafter (a property that is preserved by the proposed approximation scheme). Since the first equation in (1.2) implies that if F is a gradient initially then it remains a gradient thereafter, the equations of elastodynamics may be regarded as a special constrained evolution of (1.9).

We work with periodic solutions on the torus $\mathbb{T}^3$ and develop a variational approximation method inspired by the method of time discretization (Kinderlehrer and Pedregal [13], Demoulini [9, 10]). The scheme is implicit-explicit (see (3.1)) and makes efficient use of the null-Lagrangians and the interpretation of $\operatorname{cof} F$ and $\det F$ as constraints propagated by the evolution. It leads to the following constrained variational problem: given $v^0(x), F^0(x), Z^0(x), w^0(x)$,

$$
\min \int_{\mathbb{T}^3} \left(\frac{1}{2}(v - v^0)^2 + G(F, Z, w) \right) \, dx
\tag{1.10}
$$

over the affine subspace

$$
\mathcal{C} := \Big\{ (v, F, Z, w) : \mathrm{T}^3 \to \mathbb{R}^{22} \ \text{ subject to the constraints} \tag{1.11}
$$
$$
\begin{aligned}
&\frac{1}{h}(F_{i\alpha} - F^0_{i\alpha}) = \partial_\alpha v_i \,, \\
&\frac{1}{h}(Z_{k\gamma} - Z^0_{k\gamma}) = \partial_\alpha (\epsilon_{ijk}\epsilon_{\alpha\beta\gamma} F^0_{j\beta} v_i) \,, \\
&\frac{1}{h}(w - w^0)\, dx = \partial_\alpha ((\operatorname{cof} F^0)_{i\alpha} v_i) \qquad \Big\} .
\end{aligned}
$$

The minimization problem is solvable in L^p spaces and for each $h > 0$ provides a map S_h, the solution operator of (3.5)-(3.8),

$$
(v, F, Z, w) = S_h(v^0, F^0, \operatorname{cof} F^0, \det F^0)
$$

which is well defined and decreases the energy (see lemmas 1 and 2).

The iterates of the map S_h define approximate solutions (V^h, F^h, Z^h, w^h) which have a weak limit point (v, F, Z, w). The approximate solutions generate a measure-valued solution of the equations of polyconvex elastodynamics. More precisely, the weak limit (v, F, Z, w) satisfies a measure-valued version of the momentum equation, but $Z = \operatorname{cof} F$ and $w = \det F$ are weakly continuous and they satisfy the classical weak form of equations (1.8) (see the main theorem). The preservation of geometric constraints under weak convergence is an important feature of our analysis and represents a dynamic version of the known weak continuity properties of determinants.

The fact that we obtain only a measure-valued solution is a shortcoming owing to poor understanding of compactness properties for multi-dimensional conservation laws. Nevertheless, it is worth noticing that the approximating scheme (3.1) has regular weak solutions that decrease the energy. Also in cases with better compactness properties, such as the equations of nonlinear viscoelasticity, the method of time-discretization produces classical weak solutions [10].

Another shortcoming is that we do not require $\det F > 0$ or that the map y be injective, and strictly speaking $y(t, x)$ may not be interpreted as an elastic motion. Both of these deficiencies can be overcome in the one-dimensional case: using the method of compensated compactness [22], the one dimensional analogue of the present approximation scheme yields regular weak solutions that dissipate all convex entropies [11].

2 Null-Lagrangians and the symmetrization of three-dimensional elastodynamics

Consider the system of three-dimensional elastodynamics (1.1) in the strictly polyconvex case, *i.e.* when S is the gradient of a stored energy function $\sigma : \mathrm{Mat}^{3\times 3} \to [0,\infty)$ which factorizes as a strictly convex function of the minors of F:

$$\sigma(F) = G \circ \Phi(F), \tag{2.1}$$

with $G : \mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R} \to \mathbb{R}$ strictly convex and

$$\Phi(F) = (F, \operatorname{cof} F, \det F). \tag{2.2}$$

Here the cofactor matrix $\operatorname{cof} F$ and the determinant $\det F$ are given by

$$\begin{aligned} (\operatorname{cof} F)_{i\alpha} &= \frac{1}{2}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}F_{k\gamma}, \\ \det F &= \frac{1}{6}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{i\alpha}F_{j\beta}F_{k\gamma} = \frac{1}{3}(\operatorname{cof} F)_{i\alpha}F_{i\alpha}. \end{aligned} \tag{2.3}$$

First, we review the results on symmetrizing (1.2) with polyconvex energy [18, 8], emphasizing the connection with null-Lagrangians. Restrict, for the present, to the case of smooth maps $y : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$. Recall that all the components of $\Phi^A(F)$ in (2.2), for $A = 1, \dots, 19$, are null Lagrangians (see [3]), *i.e.* :

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla y)\right) \equiv 0. \tag{2.4}$$

The kinematic relations

$$F_{i\alpha} = \frac{\partial y_i}{\partial x^\alpha} \quad \text{and} \quad v_i = \frac{\partial y_i}{\partial t} \tag{2.5}$$

imply that

$$\frac{\partial F_{i\alpha}}{\partial t} = \frac{\partial v_i}{\partial x^\alpha}. \tag{2.6}$$

Therefore if we set

$$\Xi(t,x) = \Phi(F(t,x)) \tag{2.7}$$

then (2.4) implies

$$\frac{\partial \Xi^A}{\partial t} = \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\frac{\partial v_i}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)v_i\right). \tag{2.8}$$

A justification of this calculation for less regular $y(t,x)$ is given below in lemma 5.

Thus (1.1) can be embedded into the system of conservation laws:

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial G}{\partial \Xi^A}(\Xi)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) \tag{2.9}$$

$$\frac{\partial \Xi^A}{\partial t} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)v_i\right). \tag{2.10}$$

Here $\Xi = (F,Z,w)$ takes values in $\mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R}$ and is treated as a new dependent variable. (Since the components of F constitute the first nine components of Ξ, equation (2.6) is included as the first part of (2.10).) Smooth evolutions of the system (2.9)-(2.10) preserve the constraints $\Xi^A = \Phi^A(F)$, and thus it can be regarded as an enlargement of the original system (at least for such solutions). An advantage of this formulation is that the enlarged system admits a strictly convex entropy:

$$\eta(v,F,Z,w) = \frac{1}{2}|v|^2 + G(F,Z,w). \tag{2.11}$$

Using the formulas on derivatives of determinants and cofactor matrices,

$$\frac{\partial \det F}{\partial F_{i\alpha}} = (\operatorname{cof} F)_{i\alpha} \tag{2.12}$$

$$\frac{\partial (\operatorname{cof} F)_{i\alpha}}{\partial F_{j\beta}} = \epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{k\gamma}\,, \tag{2.13}$$

the enlarged system may be written in the explicit form

$$\begin{aligned}
&\partial_t v_i = \partial_\alpha\left(\frac{\partial G}{\partial \Xi^A}(F,Z,w)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) = \partial_\alpha\big(g_{i\alpha}(F,Z,w;F)\big)\\
&\partial_t F_{i\alpha} = \partial_\alpha v_i\\
&\partial_t Z_{k\gamma} = \partial_\alpha(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}v_i)\\
&\partial_t w = \partial_\alpha\big((\operatorname{cof} F)_{i\alpha}v_i\big)
\end{aligned} \tag{2.14}$$

where $g_{i\alpha}$ are given by the explicit form

$$\begin{aligned}
g_{i\alpha}(F,Z,w;F^0) &= \frac{\partial G}{\partial F_{i\alpha}} + \frac{\partial G}{\partial Z_{k\gamma}}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F^0_{j\beta} + (\operatorname{cof} F^0)_{i\alpha}\frac{\partial G}{\partial w}\\
&= DG(F,Z,w)\circ D\Phi(F^0)\,.
\end{aligned} \tag{2.15}$$

We note that if $(F,Z,w) = \Phi(F)$ initially then it remains so thereafter, and recall the well known property of (1.2): if F is a gradient initially then it remains a gradient thereafter. In this sense the equations of elasticity (1.1) may be regarded as a special evolution of (2.14).

3 The variational approximation scheme

In this section we introduce a variational approximation scheme for the equations of elastodynamics. The general approach is inspired by the method of time discretization [13, 9], but with some modifications: firstly, time-discretization is applied to the *enlarged system* (2.9) - (2.10) and secondly an implicit-explicit scheme is now employed (as in [10]). The scheme makes efficient use of the null-Lagrangians and of the interpretation of $\operatorname{cof} F$ and $\det F$ as constraints propagated by the evolution.

Successive iterates are constructed by discretizing (2.9)-(2.10) as follows: given the $(J-1)^{th}$ iterates (v^{J-1}, Ξ^{J-1}) the J^{th} iterates are constructed by solving

$$
\begin{aligned}
\frac{v_i^J - v_i^{J-1}}{h} &= \frac{\partial}{\partial x^\alpha}\left(\frac{\partial G}{\partial \Xi^A}(\Xi^J)\,\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{J-1})\right) \\
\frac{(\Xi^J - \Xi^{J-1})^A}{h} &= \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{J-1})v_i^J\right).
\end{aligned} \tag{3.1}
$$

We also define a discretization of the energy by

$$
\mathcal{E}^J = \int_\Omega \Big(\frac{1}{2}|v^J|^2 + G(\Xi^J)\Big)\,dx. \tag{3.2}
$$

Assumptions

To avoid inessential difficulties we will work with periodic boundary conditions, *i.e.* the spatial domain Ω is taken to be the three dimensional torus $\mathbb{T}^3$. The indices $i, j, \dots$ generally run over $1, \dots 3$ while $A, B, \dots$ run over $1, \dots, 19$. Also, we use the notation $L^p = L^p(\mathbb{T}^3)$ and $L^\infty(L^p) = L^\infty((0,T); L^p(\mathbb{T}^3))$. Write $Q_T = [0,T]\times\mathbb{T}^3$ and $Q_\infty = [0,\infty)\times\mathbb{T}^3$ and $\overset{\circ}{Q}_\infty = (0,\infty)\times\mathbb{T}^3$. Finally, we work under the following convexity and growth assumptions on G:

(H1) $G \in C^2(\mathrm{Mat}^{3\times3}\times\mathrm{Mat}^{3\times3}\times\mathbb{R}; [0,\infty))$ is a strictly convex function i.e. $\exists \gamma > 0$ such that $D^2 G \geq \gamma > 0$.

(H2) $G(F, Z, w) \geq c_1|F|^p + c_2|Z|^q + c_3|w|^r - c_4$ where $p \in (4,\infty)$ and $q, r \in [2,\infty)$ are fixed.

(H3) $G(F, Z, w) \leq c\big(|F|^p + |Z|^q + |w|^r + 1\big)$ with p, q, r as in (H2).

(H4) $|\partial_F G|^{\frac{p}{p-1}} + |\partial_Z G|^{\frac{p}{p-2}} + |\partial_w G|^{\frac{p}{p-3}} \leq C\big(|F|^p + |Z|^q + |w|^r + 1\big)$ with the exponents p, q, r as in (H2).

The last condition ensures certain integrability properties for the functions $g_{i\alpha}$ defined in (3.9) (see lemmas 1 and 2) and guarantees their Young measure representation. An example of a function satisfying (H1)-(H4) is $\tilde{G}(F,Z,w) = \alpha|F|^6 + |F|^2 + \beta|Z|^3 + |Z|^2 + w^2$ for α, β non-negative.

The main theorem has two parts, concerning, respectively, the solvability and the convergence of the discretization scheme (3.1). In the following it is assumed that we have Cauchy data $(y(0), \partial_t y(0)) \in W^{1,p} \times L^2$ for (1.1) with the property that if $F(0) = \nabla y(0)$ then $(F(0), \operatorname{cof} F(0), \det F(0)) \in L^p \times L^q \times L^r$; this triple, together with $v^0 = \partial_t y(0)$, shall be taken as the zeroth iterate (v^0, F^0, Z^0, w^0) for the process.

Main theorem *The discretization* (3.1) *can be solved, for all $h > 0$, by a constrained minimization method, and has the property that the energy* (3.2) *is decreasing in J. As $h \to 0$ the approximations generate a measure-valued solution to* (2.9)-(2.10) *for which the momentum equation* (2.9) *is satisfied in a* measure-valued *sense, but the constraint equation* (2.10) *is satisfied in the* classical weak *sense. To be precise, there exists*

$$(v, \Xi) = (v, F, Z, w) \in L^\infty(L^2) \oplus L^\infty(L^p) \oplus L^\infty(L^q) \oplus L^\infty(L^r)$$

and a Young measure $(\boldsymbol{\nu}_{x,t})_{x,t \in Q_\infty}$ such that for $i = 1, \dots 3$

$$-\int \phi(0,x) v_i(0,x)\, dx + \int v_i \partial_t \phi\, dx dt = \int \langle \boldsymbol{\nu}, g_{i\alpha} \rangle\, \partial_\alpha \phi\, dx dt \tag{3.3}$$

and for $A = 1, \dots 19$

$$-\int \phi(0,x) \Xi^A(0,x) dx + \int \Xi^A \partial_t \phi dx dt = \int \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \right) \partial_\alpha \phi\, dx dt \tag{3.4}$$

for all smooth ϕ, compactly supported in time. Furthermore, there exists a map y, with space and time derivatives F, v respectively, such that (1.1) *is satisfied in the measure-valued sense.*

The theorem is proved in several steps. First, we show that the scheme is well defined and that it decreases the total energy. The discretization (3.1) is written in an explicit form: the J^{th} iterates are given by

$$(v^J, \Xi^J) = (v^J, F^J, Z^J, w^J) = (S_h)^J (v^0, F^0, Z^0, w^0)$$

(v^0, F^0, Z^0, w^0) is as above and S_h is the solution operator $(v^0, F^0, Z^0, w^0) \mapsto (v, F, Z, w)$ defined by the equations

$$\frac{1}{h}(v_i - v_i^0) = \partial_\alpha g_{i\alpha}(F, Z, w; F^0) \tag{3.5}$$

$$\frac{1}{h}(F_{i\alpha} - F_{i\alpha}^0) = \partial_\alpha v_i \tag{3.6}$$

$$\frac{1}{h}(Z_{k\gamma} - Z_{k\gamma}^0) = \partial_\alpha(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}^0 v_i) \tag{3.7}$$

$$\frac{1}{h}(w - w^0) = \partial_\alpha\big((\operatorname{cof} F^0)_{i\alpha} v_i\big) \tag{3.8}$$

The right hand side of (3.5) is

$$\begin{aligned} g_{i\alpha}(F, Z, w; F^0) &= \frac{\partial G}{\partial F_{i\alpha}} + \frac{\partial G}{\partial Z_{k\gamma}}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}^0 + (\operatorname{cof} F^0)_{i\alpha}\frac{\partial G}{\partial w} \\ &= DG(F, Z, w) \circ D\Phi(F^0)\,. \end{aligned} \tag{3.9}$$

The solvability and properties of the map S_h are discussed in lemmas 1 and 2. Throughout this discussion we use the notation (v, F, Z, w) for the iterates of the map.

Lemma 1 *Given $(v^0, F^0, Z^0, w^0) \in L^2 \times L^p \times L^q \times L^r$ there exists $(v, F, Z, w) \in L^2 \times L^p \times L^q \times L^r$ which minimizes the functional*

$$Q(v, F, Z, w) = \int \frac{1}{2}|v - v^0|^2 + G(F, Z, w)$$

on the weakly closed affine subspace $\mathcal{C}$ defined by the weak form of equations (3.6), (3.7), (3.8), *i.e. the set $\mathcal{C} \subset L^2 \times L^p \times L^q \times L^r$ of (v, F, Z, w) such that for all $\phi \in C^\infty(\mathbb{T}^3)$:*

$$\begin{aligned} \int \phi\frac{1}{h}(F_{i\alpha} - F_{i\alpha}^0)\,dx &= -\int v_i\partial_\alpha\phi\,dx \\ \int \phi\frac{1}{h}(Z_{k\gamma} - Z_{k\gamma}^0)\,dx &= -\int \epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}^0 v_i\,\partial_\alpha\phi\,dx \\ \int \phi\frac{1}{h}(w - w^0)\,dx &= -\int (cof F^0)_{i\alpha} v_i\,\partial_\alpha\phi\,dx\,. \end{aligned} \tag{3.10}$$

The minimizer satisfies the Euler-Lagrange equation (3.5) *in the sense of distributions, i.e.*

$$\int \phi\frac{1}{h}(v_i - v_i^0)\,dx = -\int g_{i\alpha}(F, Z, w; F^0)\partial_\alpha\phi\,dx \tag{3.11}$$

for all smooth ϕ. Furthermore the constraints

$$\partial_\alpha Z_{\alpha i} = 0 \qquad \partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0 \tag{3.12}$$

are preserved by the map S_h. In fact if F^0 is a gradient then so is F, and thus we can assert the existence of a $W^{1,p}$ function $y : \mathbb{T}^3 \to \mathbb{R}^3$ such that $\partial_\alpha y_i = F_{i\alpha}$.

Proof of lemma 1. The subset $\mathcal{C}$ of the reflexive Banach space $L^2 \times L^p \times L^q \times L^r$ is weakly closed and the functional Q is co-ercive, weakly lower semicontinuous and not identically equal to $+\infty$, so it admits a minimizer. We now derive the Euler-Lagrange equation, giving sufficient detail to see where the growth conditions (H4) are used. For $i = 1, 2, 3$ let $\phi_i : \mathbb{T}^3 \to \mathbb{R}^3$ be a smooth function and consider the variation generated by this,

$$\epsilon(\delta v_i, \delta F_{i\alpha}, \delta Z_{k\gamma}, \delta w) = \epsilon\big(\phi_i, h\partial_\alpha \phi_i, h(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F^0_{j\beta}\partial_\alpha\phi_i), h(\operatorname{cof} F^0_{i\alpha}\partial_\alpha\phi_i)\big).$$

Such variations automatically satisfy (3.10). The fact that (v, F, Z, w) is minimizing implies that

$$\begin{aligned}\tilde{Q}(\epsilon) = Q\big(v_i + \epsilon\phi_i,\, F_{i\alpha} + \epsilon h \partial_\alpha\phi_i,\, Z_{k\gamma} + \epsilon h \partial_\alpha(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F^0_{j\beta}\phi_i), w + \epsilon h\partial_\alpha(\operatorname{cof} F^0_{i\alpha}\phi_i)\big)\\ \geq Q(v, F, Z, w) = \tilde{Q}(0).\end{aligned}$$

Now consider $\lim_{\epsilon\to 0}(\tilde{Q}(\epsilon) - \tilde{Q}(0))/\epsilon$; we claim that this limit exists and is equal to

$$P = \int \phi_i(v_i - v_i^0)\, dx + h\int \partial_\alpha\phi_i g_{i\alpha}(F, Z, w; F^0)\, dx. \tag{3.13}$$

To see this consider the difference between P and $(\tilde{Q}(\epsilon) - \tilde{Q}(0))/\epsilon$; applying the mean value theorem we are led to consider the expression

$$\int \partial_\alpha\phi_i\Big(g_{i\alpha}(F + \epsilon_*\delta F, Z + \epsilon_*\delta Z, w + \epsilon_*\delta w; F^0) - g_{i\alpha}(F, Z, w; F^0)\Big), \tag{3.14}$$

for $\epsilon_* = \epsilon_*(x) \in [0, \epsilon]$. To apply the dominated convergence theorem consider the integrability properties of $g_{i\alpha}$. For $p' = \frac{p}{p-1}$ the dual exponent of p, hypothesis (H4) implies that

$$\begin{aligned}|g_{i\alpha}|^{p'} &\leq C\left(\left|\frac{\partial G}{\partial F_{i\alpha}}\right|^{\frac{p}{p-1}} + |F^0|^{\frac{p}{p-1}}\left|\frac{\partial G}{\partial Z_{k\gamma}}\right|^{\frac{p}{p-1}} + |F^0|^{\frac{2p}{p-1}}\left|\frac{\partial G}{\partial w}\right|^{\frac{p}{p-1}}\right)\\ &\leq C'\left(|F^0|^p + \left|\frac{\partial G}{\partial F_{i\alpha}}\right|^{\frac{p}{p-1}} + \left|\frac{\partial G}{\partial Z_{k\gamma}}\right|^{\frac{p}{p-2}} + \left|\frac{\partial G}{\partial w}\right|^{\frac{p}{p-3}}\right)\\ &\leq C''\big(|F^0|^p + |F|^p + |Z|^q + |w|^r + 1\big)\end{aligned} \tag{3.15}$$

and thus $|g_{i\alpha}(F+\epsilon_*\delta F, Z+\epsilon_*\delta Z, w+\epsilon_*\delta w; F^0)|^{p'}$ is dominated by a fixed integrable function, independent of ϵ for ϵ less than some ϵ_0.

Thus to conclude, consider a sequence $\epsilon_i \to 0$; the integrand of equation (3.14) has limit zero almost everywhere so the dominated convergence theorem ensures that the integral has limit zero so that the quantity in (3.13) must therefore be zero by the minimizing property. This implies the weak form of (3.5). $\square$

Lemma 2 *Write* $\Theta = (v, F, Z, w)$ *and define* $\eta(v, F, Z, w) = \frac{1}{2}|v|^2 + G(F, Z, w)$, *then if* G *is strictly convex as in hypothesis (H1) there exists* $c > 0$ *such that*

$$\int \Big\{ \eta(\Theta) + c|\Theta - \Theta^0|^2 \Big\}\, dx \le \int \Big\{ \eta(\Theta^0) \Big\}\, dx.$$

Corollary 3 *The iterates* $\Theta^J = (v^J, F^J, Z^J, w^J)$ *satisfy the energy dissipation inequality, for* $J \ge 1$,

$$\frac{1}{h}\Big(\eta(\Theta^J) - \eta(\Theta^{J-1})\Big) - \partial_\alpha \Big(g_{i\alpha}(F^J, Z^J, w^J; F^{J-1}) v_i^J \Big) \le 0 \tag{3.16}$$

in the sense of distributions. There exists a number E, *determined by the initial data, such that*

$$\sup_J (|v^J|^2_{L^2_{dx}} + \int G(\Xi^J)\, dx) + \sum_{J=0}^{\infty} (|v^J - v^{J-1}|^2_{L^2_{dx}} + |\Xi^J - \Xi^{J-1}|^2_{L^2_{dx}}) \le E. \tag{3.17}$$

Proof of lemma 2. Use the formula

$$\begin{aligned}
&\eta(\Theta) - \eta(\Theta^0) - D\eta(\Theta)(\Theta - \Theta^0) = \\
&\qquad - \int_0^1 \int_0^1 (1-s) D^2\eta(\Theta - \tau(1-s)(\Theta - \Theta^0))(\Theta - \Theta^0, \Theta - \Theta^0)\, ds\, d\tau.
\end{aligned}$$

Notice that, at least formally for the moment,

$$\begin{aligned}
\frac{1}{h} D\eta(\Theta)(\Theta - \Theta^0) &= \frac{1}{h}\Big(\eta_v(v - v^0) + \eta_F(F - F^0) + \eta_Z(Z - Z^0) + \eta_w(w - w^0) \Big) \\
&= v_i \partial_\alpha g_{i\alpha} + \frac{\partial G}{\partial F_{i\alpha}} \partial_\alpha v_i + \frac{\partial G}{\partial Z_{k\gamma}} \partial_\alpha \big(\epsilon_{\alpha\beta\gamma} \epsilon_{ijk} F^0_{j\beta} v_i \big) + \frac{\partial G}{\partial w} \partial_\alpha \big((\operatorname{cof} F^0)_{i\alpha} v_i \big)
\end{aligned}$$

(where all the derivatives of G are evaluated at F, Z, w)

$$\begin{aligned}
&= v_i \partial_\alpha g_{i\alpha} + \Big(\frac{\partial G}{\partial F_{i\alpha}} + \frac{\partial G}{\partial Z_{k\gamma}} \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} F^0_{j\beta} + \frac{\partial G}{\partial w} (\operatorname{cof} F^0)_{i\alpha} \Big) \partial_\alpha v_i \\
&= v_i \partial_\alpha g_{i\alpha} + g_{i\alpha}\, \partial_\alpha v_i \\
&= \partial_\alpha (g_{i\alpha}(F, Z, w; F^0) v_i)
\end{aligned}$$

so that the integral of the first order term vanishes under periodic boundary conditions, and strict convexity of G implies the stated result.

To validate the above identities for solutions with the regularity of lemma 1, note that (3.5) and (3.6) imply that $\partial_\alpha g_{i\alpha} \in L^2$ and $\partial_\alpha v_i \in L^p$. By Poincarè's inequality $v_i \in W^{1,p}$ and from (3.15) we have $g_{i\alpha} \in L^{p'}$. We then employ the following product rule which follows from a density argument: let $p \geq 1$ and $q, r \geq p'$ the dual exponent of p. (Here q, r need not necessarily be as in the hypothesis (H2).) If $f \in W^{1,p}$ is a scalar valued function and $h \in L^q$ is a vector valued function with $\operatorname{div} h \in L^r$, then

$$\operatorname{div}(fh) = f \operatorname{div} h + \nabla f \cdot h$$

Use of the product rule, (2.4) and the regularity of the iterates: $v_i \in W^{1,p}$, $g_{i\alpha} \in L^{p'}$ and $\partial_\alpha g_{i\alpha} \in L^2$ validates the third and fifth equalities and completes the proof. $\square$

4 Proof of the main theorem.

We now study the behavior of the discretization (3.1) as the time step $h \to 0$. We use the notation $Q_T = [0,T] \times \mathbb{T}^3$ and let $Q_\infty, \overset{\circ}{Q}_\infty$ be as in section 3. Let v^J and $\Xi^J = \big(F^J, Z^J, w^J\big)$, defined on the torus $\mathbb{T}^3$, be the iterates constructed from the minimization process, $J = 0, 1, 2, \dots$. The iterates F^J are gradients, so we construct functions $y^J : \mathbb{T}^3 \to \mathbb{R}^3$ such that $\partial_\alpha y_i^J = F_{i\alpha}^J$. By selecting the integration constants appropriately (and choosing y^{-1} by extrapolation), the iterates y^J satisfy the identities

$$\frac{1}{h}(y^J - y^{J-1}) = v^J \, . \tag{4.1}$$

Construct the time-continuous, piecewise linear interpolates V^h, Ξ^h given by (suppressing the explicit dependence on x of the iterates)

$$\begin{aligned} V^h(t) &= \sum_{J=1}^{\infty} \chi^J(t)\big(v^{J-1} + \frac{t - h(J-1)}{h}(v^J - v^{J-1})\big) \\ \Xi^h(t) &= \big(F^h, Z^h, W^h\big)(t) \\ &= \sum_{J=1}^{\infty} \chi^J(t)\big(\Xi^{J-1} + \frac{t - h(J-1)}{h}(\Xi^J - \Xi^{J-1})\big), \end{aligned} \tag{4.2}$$

and the piecewise constant interpolates v^h, ξ^h by

$$
\begin{aligned}
v^h(t) &= \sum_{J=1}^{\infty} \chi^J(t) v^J \\
\xi^h(t) = (f^h, z^h, w^h)(t) &= \sum_{J=1}^{\infty} \chi^J(t) \Xi^J ,
\end{aligned}
\tag{4.3}
$$

where χ^J is the characteristic function of the interval $I_J := [(J-1)h, Jh)$. Finally, construct the piecewise linear approximation of the motion

$$
Y^h(t) = \sum_{J=1}^{\infty} \chi^J(t) \big(y^{J-1} + \frac{t - h(J-1)}{h} (y^J - y^{J-1}) \big)
\tag{4.4}
$$

and note the identities

$$
\partial_t Y_i^h = v_i^h \, , \qquad \partial_\alpha Y_i^h = F_{i\alpha}^h \, .
\tag{4.5}
$$

The approximates (V^h, F^h, Z^h, W^h) and (v^h, f^h, z^h, w^h) are uniformly bounded in $L^\infty(L^2) \oplus L^\infty(L^p) \oplus L^\infty(L^q) \oplus L^\infty(L^r)$. Therefore, there exists a subsequence in h and limit points $y : Q_\infty \to \mathbb{R}^3$ and $(v, \Xi) : Q_\infty \to \mathbb{R}^{22}$ with

$$
\begin{aligned}
y &\in W^{1,\infty}(L^2) \cap L^\infty(W^{1,p}), \\
(v, \Xi) = (v, F, Z, w) &\in L^\infty(L^2) \oplus L^\infty(L^p) \oplus L^\infty(L^q) \oplus L^\infty(L^r)
\end{aligned}
\tag{4.6}
$$

for all $T > 0$, and such that along the said subsequence

$$
\begin{aligned}
Y^h &\to y \qquad \text{strongly in } L^2_{loc}(\mathbb{T}^3) \text{ and a.e.} \\
(V^h, v^h, \Xi^h, \xi^h) &\rightharpoonup (v, v, \Xi, \Xi)
\end{aligned}
\tag{4.7}
$$

weak$\star$ in $L^\infty_{loc}\big(\mathbb{R}; [L^2]^2 \oplus [L^p \oplus L^q \oplus L^r]^2(\mathbb{T}^3)\big)$, and

$$
v_i = \partial_t y_i \, , \qquad F_{i\alpha} = \partial_\alpha y_i .
\tag{4.8}
$$

Weak limits of the geometric constraints

As mentioned previously a crucial point is the preservation of the geometric constraints (2.10) under weak convergence. It is well known that $\operatorname{cof} \nabla y$ and $\det \nabla y$ satisfy weak continuity properties in functional spaces ([2], [4]). The next lemma is a modest extension of these weak continuity properties to the case of functions having regularity typical for solutions of wave equations:

Lemma 4 (Weak continuity of minors in 3+1 dimensions) *Let y^n be a sequence bounded in the space $W^{1,\infty}([0,\infty);L^2(\mathbb{T}^3)) \cap L^\infty([0,\infty),W^{1,p}(\mathbb{T}^3))$, such that $(y^n, cof F^n, det F^n)$, converges in the weak$\star$ topology on $W^{1,\infty}(L^2)\cap L^\infty(W^{1,p}) \times L^\infty(L^q) \times L^\infty(L^r)$ to (y,Z,w). If $p \geq 2, q \geq \frac{p}{p-1}$ and $q > \frac{4}{3}$ then $Z = cof F$ and $w = det F$.*

Proof of lemma 4. The proposed weak$\star$ convergence hinges on the fact that the present assumptions on y^n allow us to write cofactor and determinant as divergences of bilinear quantities in which one of the factors converges strongly. To see this, recall that since y^n converges weakly in $W^{1,2}_{loc}(Q_\infty)$ it converges strongly in $L^r_{loc}(Q_\infty)$ for $r<4$ by Rellich's theorem. Therefore:

$$\begin{aligned}(\operatorname{cof} F^n)_{i\alpha} &= \frac{1}{2}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F^n_{j\beta}F^n_{k\gamma} = \frac{1}{2}\partial_\beta\Big(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}y^n_j F^n_{k\gamma}\Big)\\ &\rightharpoonup \frac{1}{2}\partial_\beta\Big(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}y_j F_{k\gamma}\Big) = (\operatorname{cof} F)_{i\alpha}\end{aligned} \tag{4.9}$$

where the second and last equalities follow from [2, Lemma 6.1] since $p \geq 2$, while the convergence, in the sense of distributions follows since $F^n \rightharpoonup F$ weak$\star$ in $L^\infty(L^p)$ and $p \geq 2 > \frac{4}{3}$, the dual exponent to 4. Similarly,

$$\begin{aligned}\det F^n &= \frac{1}{3}(\operatorname{cof} F^n)_{i\alpha}F^n_{i\alpha} = \frac{1}{3}\partial_\alpha\Big(y^n_i(\operatorname{cof} F^n)_{i\alpha}\Big)\\ &\rightharpoonup \frac{1}{3}\partial_\alpha\Big(y_i(\operatorname{cof} F)_{i\alpha}\Big) = \det F\end{aligned} \tag{4.10}$$

where the second and last equalities follow from [2, Lemma 6.1] since $p \geq 2$ and $q \geq \frac{p}{p-1}$ while the convergence, in the sense of distributions, follows since $\operatorname{cof} F^n \rightharpoonup \operatorname{cof} F$ weak$\star$ in $L^\infty(L^q)$ and $q > \frac{4}{3}$, the dual exponent to 4. These distributional limits are the same as the weak$\star$ limits. □

Notice that if for example $p \geq 4, q \geq 2$ then $q \geq \frac{p}{p-1}$ and $q > \frac{4}{3}$, so that the result applies, and in particular it applies to the present situation with p, q, r as in (H2).

Resuming now the proof of our main theorem, it is necessary to show that the evolution of $\operatorname{cof} F^h$ and $\det F^h$ preserves the relations (2.10). First, we give a direct proof that the constraint equations (2.10) hold for maps y with fairly weak regularity:

Lemma 5 (Validation of weak formulation of geometric constraints (1.8)**)** *Let y be a measurable function $[0,\infty)\times\mathbb{T}^3 \to \mathbb{R}^3$ with regularity as in (4.6) with*

$p \geq 4$. *Introduce* $(v, F, cof F, det F)$ *defined as in* (2.3), (2.5). *Then the equations* (1.8) *hold in the sense of distributions.*

Proof of lemma 5. Let $y \in W^{1,\infty}([0,\infty); L^2) \cap L^\infty([0,\infty); W^{1,p})$. Extend y to a function defined for all t by putting $y(t,x) = y(0,x)$ for $\{t \leq 0\}$. Convolution with a function $\rho_\epsilon(t)\Pi_{i=1}^3 \rho_\epsilon(x_i)$, with $\rho_\epsilon = \epsilon^{-1}\rho(\frac{x}{\epsilon})$ where $\rho \in C_0^\infty(\mathbb{R})$ is positive and of integral one, gives a sequence of $y^\epsilon \in C^\infty(Q_\infty)$ such that for any $s < \infty$ and for all $T > 0$

$$\|y_t^\epsilon - y_t\|_{L^s([-T,T];L^2)} + \|y^\epsilon - y\|_{L^s([-T,T];W^{1,p})} \to 0.$$

Set $v = y_t$, $F = \nabla y$, $v^\epsilon = y_t^\epsilon$ and $F^\epsilon = \nabla y^\epsilon$. Then since the cofactor matrix is bilinear in the components of F, we have that $\operatorname{cof} F^\epsilon$ converges to $\operatorname{cof} F$ in $L^s(L^{p/2})$.

The approximates y^ϵ, v^ϵ, F^ϵ satisfy the identities

$$\partial_t(\operatorname{cof} F^\epsilon)_{k\gamma} = \frac{1}{2}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\partial_t\partial_\beta(F_{i\alpha}^\epsilon y_j^\epsilon) = \partial_\alpha\big(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}^\epsilon v_i^\epsilon\big) \tag{4.11}$$

$$\partial_t(\det F^\epsilon) = \partial_t\partial_\alpha\big(\frac{1}{3}(\operatorname{cof} F^\epsilon)_{i\alpha}y_i^\epsilon\big) = \partial_\alpha((\operatorname{cof} F^\epsilon)_{i\alpha}v_i^\epsilon) \tag{4.12}$$

and the convergence is strong enough to take the limit and obtain the same identities for y, v, F since $p \geq 4$. □

As a consequence, $y(t,x)$ constructed in (4.6) will satisfy the kinematic constraint equations (1.8).

Remark. The equations (1.8) are stable in the regularity framework (4.6) under the weak convergence (4.7). Consider the equation for the evolution of $\operatorname{cof} F$ expressed along the approximate solution Y^h. Due to lemma 5 we have the identity

$$\partial_t(\operatorname{cof} F^n)_{k\gamma} = \frac{1}{2}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\partial_t\partial_\beta(F_{i\alpha}^n y_j^n) = \partial_\alpha\big(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}^n v_i^n\big) \tag{4.13}$$

Using (4.9) and (4.7) we pass the first two terms to the weak limit; the third term converges to the right weak limit because of lemma 5 and (4.6), thus producing the equation (4.11). A similar observation holds for the equation for the evolution of $\det F$. Here the approximates satisfy

$$\partial_t(\det F^h) = \partial_t\partial_\alpha\big(\frac{1}{3}(\operatorname{cof} F^h)_{i\alpha}Y_i^h\big) = \partial_\alpha((\operatorname{cof} F^h)_{i\alpha}v_i^h) \tag{4.14}$$

which are again weakly stable under the convergence (4.7) (due to (4.9) and (4.10)).

The following lemma, which is central to the argument, concerns the behavior of the weak limits Z and w.

Lemma 6 (Recovery of minors and determinants in the limit)
Let the piecewise linear approximates $\big(F^h(t,x), Z^h(t,x), W^h(t,x)\big)$ *be defined as in* (4.2), *then*

$$\partial_t\big(Z^h - cof(F^h)\big) \rightharpoonup 0 \tag{4.15}$$

$$\partial_t\big(W^h - det(F^h)\big) \rightharpoonup 0 \tag{4.16}$$

in the sense of distributions on Q_∞ *as* $h \to 0$.

Proof. From (3.7) and (4.13) we have

$$\partial_t(Z^h_{k\gamma}) = \partial_\alpha\big(\sum_J \chi^J(t)\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F^{J-1}_{j\beta}v^J_i\big)$$
$$\partial_t(\operatorname{cof} F^h)_{k\gamma} = \partial_\alpha\big(\sum_J \chi^J(t)\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F^h_{j\beta}v^J_i\big).$$

Subtracting these two formulae gives:

$$\partial_t(Z^h_{k\gamma} - (\operatorname{cof} F^h)_{k\gamma}) = \partial_\alpha\big(\sum_J \chi^J(t)\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\frac{t-h(J-1)}{h}(F^J - F^{J-1})_{j\beta}v^J_i\big).$$

Notice that $|(t-h(J-1))\chi^J(t)/h| \le 1$. Now let $\phi \in C^\infty_0(Q_\infty)$ and define

$$I_\phi \equiv \int_0^\infty \int_\Omega \phi(t,x)\partial_t(Z^h - \operatorname{cof} F^h)\, dxdt$$

then using corollary 3 we can estimate

$$|I_\phi| \le c\Big[\sum_J\Big(\int_{(J-1)h}^{Jh}\sup_x|\nabla\phi|dt\Big)^2\Big]^{\frac{1}{2}}\ \Big[\sum_J h\Big(\int_{\mathbb{T}^3}\big|(F^J - F^{J-1})_{j\beta}v^J_i\big|\,dx\Big)^2\Big]^{\frac{1}{2}}$$
$$\le c.E.h^{1/2}\|\nabla_x\phi\|_{L^2_{dt}(L^\infty_{dx})}$$

In a similar way we can compute

$$\partial_t(W^h - \det F^h) = -\partial_\alpha\big(\sum_j \chi^J(t)(\operatorname{cof} F^h - \operatorname{cof} F^{J-1})_{i\alpha}v^J_i\big). \tag{4.17}$$

Now for $t \in [h(J-1), hJ)$

$$\begin{aligned}
&(\operatorname{cof} F^h - \operatorname{cof} F^{J-1})_{i\alpha}(t)\\
&\quad= \frac{t-h(J-1)}{2h}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\big[(F^J_{j\beta} - F^{J-1}_{j\beta})F^h_{k\gamma} + F^{J-1}_{j\beta}(F^J_{k\gamma} - F^{J-1}_{k\gamma})\big].\\
&\quad= \frac{1}{2}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\big[(\delta F)^h_{j\beta}(t)F^h_{k\gamma}(t) + f^h_{j\beta}(t-h)(\delta F)^h_{k\gamma}(t)\big]
\end{aligned}$$

where we used the equations (4.2), (4.3), the notation

$$\delta F(t)=\sum_J \frac{t-h(J-1)}{h}\chi^J(t)(F^J-F^{J-1})$$

and the formula

$$F^h(t)=\sum_J \chi^J(t)F^{J-1}+\delta F^h=f^h(t-h)+\delta F^h(t)\,.$$

Now, as above, let $\phi\in C_0^\infty(Q_\infty)$ and define

$$J_\phi\equiv\int_0^\infty\int_\Omega \phi(t,x)\partial_t(W^h-\det F^h)\;dxdt.$$

To explain how these terms are estimated, drop the superscripts h, and notice that $|\delta F(t)|\le|F(t)|+|f(t-h)|$. Also notice that corollary 3 implies that $\delta F/h^{\frac12}$ is bounded in L^2_{dxdt} in terms of the initial data. We estimate the terms in J_ϕ first for $p\ge 6$:

$$\begin{aligned}
|\int\int(\delta F)Fv\nabla_x\phi\;dxdt| &= |\int\int(\delta F)^{1/2}(\delta F)^{1/2}Fv\nabla_x\phi\;dxdt|\\
&\le c\int\|(\delta F)^{1/2}\|_{L^4_{dx}}\|(\delta F)^{1/2}F\|_{L^4_{dx}}\|v\nabla_x\phi\|_{L^2_{dx}}\,dt\\
&\le c\int\|\delta F(t,x)\|_{L^2_{dx}}\|F\|_{L^6_{dx}}(\|F(t)\|_{L^6_{dx}}+\|f(t-h)\|_{L^6_{dx}})^{1/2}\\
&\qquad\times\;\|v\|_{L^2_{dx}}\|\nabla_x\phi(t,x)\|_{L^\infty_{dx}}\,dt\\
&\le cE^{3/2}h^{1/4}\|(\delta F)/h^{1/2}\|^{1/2}_{L^2_{dxdt}}\|\nabla_x\phi\|_{L^{4/3}_{dt}(L^\infty_{dx})}.
\end{aligned}$$

For $p>4$, $\delta F\in L^2_{loc}(Q_\infty)\cap L^\infty(L^p)$ so, since $\delta F\to 0$ in L^2_{dxdt}, interpolation implies that $\delta F\to 0$ in $L^4_{loc}(Q_\infty)$ and then Holder's inequality gives (4.16).

The limiting weak equation

Let (V^h,Ξ^h) and (v^h,ξ^h) be the interpolates as given in (4.2) and (4.3). The discrete equations (3.10)-(3.11) take the weak form

$$\text{(4.18)}\quad \int\phi\left(\partial_t F^h_{i\alpha},\;\partial_t Z^h_{k\gamma},\;\partial_t W^h\right)\;dxdt$$
$$=-\int\partial_\alpha\phi\left(v^h_i(t),\;\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}f^h_{j\beta}(t-h)v^h_i,\;(\operatorname{cof}f^h(t-h))_{i\alpha}v^h_i(t)\right)\;dxdt$$

$$\text{(4.19)}\quad \int\phi\partial_t V^h_i\;dxdt=-\int(\partial_\alpha\phi)\,g_{i\alpha}(\xi^h\;;\;f^h(t-h))\;dxdt$$

with $\Xi^h=(F^h,Z^h,W^h)$ and $\xi^h=(f^h,z^h,w^h)$ (the explicit dependence on (t,x) is suppressed except when evaluation is at $(t-h)$ rather than at t).

The sequences $(V^h,\Xi^h)_{h>0}$, $(v^h,\xi^h)_{h>0}$ are bounded in $L^\infty(L^2\times L^p\times L^q\times L^r)$ and have the properties

$$
\begin{aligned}
\|V^h - v^h\|_{L^2_{loc}} \to 0 \qquad & \|F^h - f^h\|_{L^{\bar p}_{loc}} \to 0 \quad \text{for } \bar p < p \\
\|Z^h - z^h\|_{L^{\bar q}_{loc}} \to 0 \quad \text{for } \bar q < q \qquad & \|W^h - w^h\|_{L^{\bar r}_{loc}} \to 0 \quad \text{for } \bar r < r
\end{aligned}
\tag{4.20}
$$

This is a consequence of corollary 3, which implies for instance that

$$
\|F^h - f^h\|_{L^2(Q_\infty)} \le h^{\frac12}\Big(\sum_J \int_{\mathbb{T}^3} |F^J - F^{J-1}|^2 dx\Big)^{\frac12}
$$

and the stated result follows by interpolation, due to the uniform bound of the iterates F^J in $L^\infty(L^p)$. The rest of the statements are proved similarly.

There exists a subsequence of (V^h,Ξ^h) and (v^h,ξ^h) converging to the same weak limit $(v,\Xi)=(v,F,Z,w)$. This generates a Young measure $\boldsymbol{\nu}\in\mathcal{P}(\mathbb{R}^{1+3}\times\mathcal{B})$, where $\mathcal{B}$ is the target space

$$
\mathcal{B} = \mathbb{R}^3 \times \mathrm{Mat}^{3\times3} \times \mathrm{Mat}^{3\times3} \times \mathbb{R}.
$$

To be precise, ν is a probability measure on the product space of $\mathbb{R}^{1+3}$ with target space $\mathcal{B}$, such that its projection on the $\mathbb{R}^{1+3}$ is the Lebesgue measure $\mathcal{L}^{1+3}$ and its disintegration measure $\nu_{t,x}\in\mathcal{P}(\mathcal{B})$, the space of probability measures on $\mathcal{B}$. It is customary to view $\boldsymbol{\nu}$ as a (weakly measurable) map $\boldsymbol{\nu}:(t,x)\mapsto\nu_{t,x}$ where for a.e. $(t,x)\in Q_\infty$, $\nu_{t,x}\in\mathcal{P}(\mathcal{B})$; this map is usually an element in a space of type $L^p(\mathbb{R}^{1+3},\mathcal{C}_p')$, with $\mathcal{C}_p$ a subset of continuous functions of some polynomial growth.

In the present context, the Young measure will represent the weak limits of all functions ψ with polynomial growth at infinity of the type

$$
\psi(v,F,Z,w) \le c\Big(|v|^2 + |F|^p + |Z|^q + |w|^r\Big)^\theta \quad \text{for any } \theta < 1,
\tag{4.21}
$$

as $|v|$, $|F|$, $|Z|$, $|w|\to\infty$. Next, we show that both sequences (V^h,Ξ^h) and (v^h,ξ^h) admit the same Young measure representation. This is a typical property of time discretizations (see [13, 10]); in the present case technical difficulties arise from the diverse growth rates at infinity. Let ψ be any function satisfying the global Lipschitz condition

$$
\begin{aligned}
&|\psi(v,F,Z,w) - \psi(\hat v,\hat F,\hat Z,\hat w)| \le \\
&\qquad c\Big(\omega^{\frac12}|v-\hat v| + \omega^{\frac{\bar p-1}{\bar p}}|F-\hat F| + \omega^{\frac{\bar q-1}{\bar q}}|Z-\hat Z| + \omega^{\frac{\bar r-1}{\bar r}}|w-\hat w|\Big)
\end{aligned}
$$

for some $\bar p<p$, $\bar q<q$, $\bar r<r$, where

$$
\omega = |v|^2 + |\hat v|^2 + |F|^p + |\hat F|^p + |Z|^q + |\hat Z|^q + |w|^r + |\hat w|^r + 1
$$

accounts for the growth of the Lipshitz constant at infinity. This class contains all C^1 functions that together with their derivatives behave like polynomials of growth as in (4.22). If ϕ is a test function and ψ as above then (4.21) implies that

$$|K_\phi| = \Big| \int_{Q_T} \big[\psi(V^h, \Xi^h) - \psi(v^h, \xi^h)\big]\, \phi \; dxdt \Big| \to 0 \tag{4.22}$$

and thus the L^p-weak limits coincide and the two sequences generate the same Young measure. (To see that, note that K_ϕ is majorized by terms of the type

$$\Big| \int_{Q_T} |\omega^h|^{\frac{\bar p - 1}{\bar p}} |F^h - f^h|\, |\phi| \; dxdt \Big| \le \|\phi\|_{L^\infty_{dxdt}} \Big(\int_{Q_T} |\omega^h| \; dxdt \Big)^{\frac{1}{\bar p'}} \|F^h - f^h\|_{L^{\bar p}_{dxdt}}$$

which converge to zero as $h \to 0$ by (4.21) and the uniform bounds.) A similar property holds for the sequences $(v^h, \xi^h)_{h>0}$ and $\big(v^h(\cdot - h), \xi^h(\cdot - h)\big)_{h>0}$: they satisfy the same estimate as in (4.21) thus generating the same weak limits for functions ψ as above.

We now take the limit $h \to 0^+$ in (4.20) to obtain a measure-valued weak form of the elasticity equations (1.1). The growth conditions on G (see hypothesis (H4) and (3.15)) ensure L^1 precompactness of the quantities $g_{i\alpha}(\xi^h(t), f^h(t-h))$ and $g_{i\alpha}(\xi^h(t), f^h(t))$. Notice that, by (3.9) and (3.15),

$$\begin{aligned}
|g_{i\alpha}(F, Z, &w; F) - g_{i\alpha}(F, Z, w; \hat F)| \\
&\le \Big|\frac{\partial G}{\partial Z}\Big| |F - \hat F| + \Big|\frac{\partial G}{\partial w}\Big| \big(|F| + |\hat F|\big) |F - \hat F| \\
&\le c\Big[\big(|F|^p + |Z|^q + |w|^r + 1\big)^{\frac{p-2}{p}} + \big(|F| + |\hat F|\big)^{p-2} \Big] |F - \hat F|
\end{aligned}$$

From the property $\|f^h(\cdot) - f^h(\cdot - h)\|_{L^{\bar p}(Q_T)} = o(h)$ for $\bar p < p$, we conclude that

$$\int_{Q_T} \big(g_{i\alpha}(\xi^h(t), f^h(t-h)) - g_{i\alpha}(\xi^h(t), f^h(t)) \big) \phi \; dxdt \to 0$$

and thus they admit the same Young measure representation. We then obtain a weak measure-valued form of (2.9) by taking the limit in (4.20):

$$-\int v \partial_t \phi \; dxdt + \int v(x,0)\phi(x,0)\, dx = -\int \partial_\alpha \phi \, \langle \boldsymbol{\nu}, g_{i\alpha} \rangle \; dxdt. \tag{4.23}$$

Also consider the Young measure $\boldsymbol{\mu}$ generated by $(F^h)_{h>0}$ alone: this is a marginal of $\boldsymbol{\nu}$. The weak continuity of determinant and cofactor implies these functions commute with the Young measure: by the notation introduced in (2.2),

$$\Phi(\langle \mu_{t,x}, \lambda_F \rangle) = \langle \mu_{t,x}, \Phi(\lambda_F) \rangle = \langle \nu_{t,x}, (\lambda_F, \lambda_Z, \lambda_w) \rangle = (F, Z, w)(t,x) \text{ a.e.}$$

Here the notation $(\lambda_F, \lambda_Z, \lambda_w)$ is being used for the variables with respect to which the integrals over the Young measure are carried out. The limit function is $\Xi = (F, Z, w) = \Phi(F) = (F, \operatorname{cof} F, \det F)$.

In summary, we constructed a function $y : (0, \infty) \times \mathbb{T}^3 \to \mathbb{R}^3$

$$y \in W^{1,\infty}([0,T]; L^2) \cap L^\infty([0,T]; W^{1,p}), \tag{4.24}$$

such that $\partial_t y = v$, $\nabla_x y = F$, $\operatorname{cof}(\nabla_x y) = Z$, $\det(\nabla_x y) = w$ and they satisfy (2.6), the measure-valued form (4.24) and the weak form of the additional conservation laws (1.8) by lemmas 4,5 and 6 as asserted in the main theorem.

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