Lyapunov functionals and L^1-stability for discrete velocity Boltzmann equations

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Abstract

We devise Lyapunov functionals and prove uniform L^1 stability for one-dimensional semilinear hyperbolic systems with quadratic nonlinear source terms. These systems encompass a class of discrete velocity models for the Boltzmann equation. The Lyapunov functional is equivalent to the L^1 distance between two weak solutions and non-increasing in time. They result from computations of two point interactions in the phase space. For certain models with only transversal collisional terms there exist generalizations for three and multi-point interactions.

1 Introduction

In this article we devise Lyapunov functionals and prove uniform L^1 stability for the Cauchy problem for semilinear hyperbolic systems with quadratic source terms,

$$\begin{aligned} \partial_t f_i + v_i \partial_x f_i &= \sum_{j,k=1}^{N} B_i^{jk} f_j f_k \\ f_i(x,0) &= f_{i0}(x) \end{aligned} \tag{1.1}$$

This system encompasses certain one-space dimensional discrete velocity models in kinetic theory of gases (see section 2). In this context, $f_i(x,t)$ stands for the number of particles moving with velocity v_i, $i = 1, \dots, N$ and $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. The collision operator is of the general form

$$Q_i(f) = \sum_{j,k=1}^{N} B_i^{jk} f_j f_k$$

and the system is assumed strictly hyperbolic: $v_i \neq v_j$ for $i \neq j$. Precise assumptions on the interaction coefficients B_i^{jk} will be placed in the sequel. We are interested in positive weak solutions of (1.1) of class L^1.

The study of discrete velocity approximations of the Boltzmann collision operator goes back to works of Carleman, Broadwell [10], Gatignol [15]. In the one-space dimensional context, there exist a number of global existence results for small, [23], or large L^1-data, [11, 25, 26], as well as

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studies concerning the asymptotic behavior, [1, 2], and uniform bounds for solutions emanating from $L^1 \cap L^\infty$ data, [3], [2], and results for initial-boundary value problems [8]. For global existence results in several space variables we refer to [4, 21, 18] and the survey article [19].

Bony [3, 6] introduced the following functional in the theory of discrete velocity models:

$$\mathcal{Q}(t) = \sum_{m,n} \int_{\mathbb{R}} \int_{\mathbb{R}} (v_m - v_n) \mathrm{sgn}(y - x) |f_m(x,t)| |f_n(y,t)| dy dx \,, \tag{1.2}$$

where $\mathrm{sgn}(x)$ equals -1 for $x < 0$, 0 for $x = 0$ and $+1$ for $x > 0$, to the study of one-dimensional discrete velocity Boltzmann equations. Study of the evolution of $\mathcal{Q}$ leads to uniform integrability of the transversal source terms in space-time, what plays a central role in the existence and asymptotic analysis of [3]. A continuous version of Bony's functional has been proposed by Cercignani [12, 14] for the full Boltzmann equation with a truncated collision kernel. The functional $\mathcal{Q}$ measures the potential interactions between particles with different velocities, and has some similarities with the potential of interaction functional introduced by Glimm [16] to the study of quasilinear hyperbolic systems. (It also has the important difference that $\mathcal{Q}$ is not positive).

The objective of the present work is to introduce a new functional measuring the L^1 distance between two weak solutions f and $\bar{f}$ of (1.1). This functional reduces to the Bony functional (1.2) when one of f or $\bar{f}$ is zero, and has certain analogies to the Liu-Yang functional [22] that was recently devised for the stability of small BV solutions for systems of conservation laws (see also Bressan-Liu-Yang [9], Hu-LeFloch [17]). The functional provides information regarding the long-time response of solutions, and will be used, in particular, to establish uniform L^1-stability for solutions to (1.1) of *small L^1-mass*. For the special case of the Broadwell system, there is an alternative functional, consisting of only positive terms and accounting only for forward interactions (see below), and the smallness assumption can be precisely quantified.

We proceed to explain the results. Consider first (1.1) under the structural hypotheses:

1. B_i^{jk} satisfy symmetry, sign and boundedness conditions:

$$\begin{aligned} & B_i^{jk} = B_i^{kj} \\ & B_i^{jk} \leq 0 \quad \text{if } j = i \ \text{ or } k = i, \quad B_i^{jk} \geq 0 \quad \text{if } j \neq i \ \text{ and } k \neq i. \\ & |B_i^{jk}| \leq B^*, \quad \text{for some positive constant } B^*. \end{aligned} \tag{1.3}$$

2. The system is strictly hyperbolic,

$$v_1 < v_2 < \cdots < v_N. \tag{1.4}$$

3. Conservation of mass and momentum: There exist weights $\nu_i \geq 1$, $(i = 1, \cdots, N)$ such that

$$\sum_{i=1}^N \nu_i B_i^{jk} = 0, \quad \sum_{i=1}^N v_i \nu_i B_i^{jk} = 0, \quad \text{for fixed } j, k. \tag{1.5}$$

4. Any existing quadratic interactions are assumed to satisfy:

$$\begin{aligned} &\text{if} \quad B_i^{ii} < 0 \quad \text{then there is a sequence of indices } i = i_1, i_2, \dots, i_r \text{ so that} \\ &\qquad B_{i_{k+1}}^{i_k i_k} > 0 \text{ for } k = 1, \dots, r-1\,, \quad B_{i_r}^{i_r i_r} = 0\,. \end{aligned} \tag{1.6}$$

In section 2 we discuss how models of kinetic theory of gases fit into the above framework. For the moment, note that (1.5) reflects conservation of mass and momentum and that certain non-strictly hyperbolic models of kinetic theory can be accomodated by the above assumptions. This is due to the presence of the weights ν_i; their role is discussed in section 2. Finally, the structural hypothesis (1.6) is always satisfied for kinetic theory models.

To outline the approach, let $\mathcal{L}(t)$ be the L^1-distance between two solutions f and $\bar{f}$,

$$\mathcal{L}(t) \equiv \sum_m \int_{\mathbb{R}} \nu_m |f_m(x,t) - \bar{f}_m(x,t)| dx,$$

and let δ_i stand for

$$\delta_i(x,t) = \operatorname{sgn}(f_i(x,t) - \bar{f}_i(x,t)).$$

A direct calculation (see Proposition 3.8) shows that the time derivative of $\mathcal{L}$,

$$\begin{aligned} \frac{d\mathcal{L}(t)}{dt} &+ \int_{\mathbb{R}} \sum_{n,i,\, n\neq i} \nu_i B_i^{nn} \Big(1 - \frac{\delta_i}{\delta_n}\Big) |f_n - \bar{f}_n| (f_n + \bar{f}_n)\, dx \\ &\leq O(1) \int_{\mathbb{R}} \sum_{m,n,m\neq n} |f_m - \bar{f}_m| (f_n + \bar{f}_n) dx\,, \end{aligned} \tag{1.7}$$

consists of two types of terms: (i) Terms accounting for interactions among particles moving with the same velocity; these appear to the left of the inequality and turn out to be positive. Such a property holds for the much simpler class of contractive relaxation systems (*e.g.* [20, 28]) and it is remarkable that, due to the conservation laws in (1.5), it holds also for discrete Boltzmann type operators. (ii) Terms that are due to interactions between transversally moving particles, which contribute the term to the right of the inequality in (1.7).

To control the terms on the right, we introduce a quadratic functional $\mathcal{Q}_d(t)$ of the form

$$\mathcal{Q}_d(t) = \sum_{m,n} \int_{\mathbb{R}} \int_{\mathbb{R}} \big(\operatorname{sgn}(y-x)\big)(v_m - v_n)\nu_m \nu_n |f_m - \bar{f}_m|(x,t)(f_n + \bar{f}_n)(y,t)\, dx dy\,. \tag{1.8}$$

$\mathcal{Q}_d$ accounts for the potential of (forward and backward) interactions of transversally moving particles between f or $\bar{f}$ and the difference $|f - \bar{f}|$. A second calculation (Proposition 3.8) shows

$$\frac{d\mathcal{Q}_d}{dt}(t) + c\Lambda_d(f,\bar{f})(t) \leq C ||f + \bar{f}||_{L^1(\mathbb{R})} \Big(\Lambda_s(f,\bar{f})(t) + \Lambda_d(f,\bar{f})(t)\Big), \tag{1.9}$$

where c, C are positive constants and

$$\Lambda_d(f,\bar{f})(t) \quad = \quad \sum_{m,n,\, m\neq n} \int_{\mathbb{R}} \nu_m \nu_n |f_m - \bar{f}_m|(x)(f_n + \bar{f}_n)(x) dx,$$

$$\Lambda_s(f,\bar{f})(t) \;=\; \sum_{m,n,\,m\neq n} \int_{\mathbb{R}} \nu_m B_m^{nn}\Big(1-\frac{\delta_m}{\delta_n}\Big)|f_n-\bar{f}_n|(f_n+\bar{f}_n)$$

Setting $\mathcal{H}(t) = \mathcal{L}(t) + K\,\mathcal{Q}_d(t)$, we have the following theorem:

Theorem 1.1. *Suppose the system (1.1) satisfies the assumptions (1.4) - (1.6) and let* $f, \bar{f} \in C(\mathbb{R}_+; (L^1(\mathbb{R}))^N)$ *be two mild solutions of (1.1) corresponding to initial data* $f_0,\ \bar{f}_0 \geq 0$,

$$f_0, \bar{f}_0 \in \big[(L^\infty \cap L^1)(\mathbb{R})\big]^N \ \textit{such that}\ ||f_0||_{L^1(\mathbb{R})} \ll 1 \ \textit{and}\ ||\bar{f}_0||_{L^1(\mathbb{R})} \ll 1.$$

Then, for an appropriate choice of K*, the functional* $\mathcal{H}(t) = \mathcal{L}(t) + K\,\mathcal{Q}_d(t)$ *is equivalent to the* L^1 *distance between* f *and* $\bar{f}$ *and satisfies*

$$\frac{d\mathcal{H}(t)}{dt} + c\Big(\Lambda_s(f,\bar{f})(t) + \Lambda_s(f,\bar{f})(t)\Big) \leq 0\,, \tag{1.10}$$

for some positive constant c*. Moreover,*

$$||f(\cdot,t) - \bar{f}(\cdot,t)||_{L^1(\mathbb{R})} \leq C||f_0(\cdot) - \bar{f}_0(\cdot)||_{L^1(\mathbb{R})},$$

where C *is a positive constant independent of time* t*.*

The theorem above is a statement on the asymptotic response of (1.1). (Note that, for quadratic models (1.1), given L^∞ bounds it is easy to establish L^1-stability estimates with a constant that depends on time. L^∞ bounds are available for data of class $L^1 \cap L^\infty$, see [2], [4].) The functional inequality (1.10) is valid for data of small initial L^1-mass, while the functional $\mathcal{H}(t)$ has the properties

1. $\mathcal{H}$ is equivalent to the L^1-distance, i.e., for some constant $C_0 > 0$,
   $\frac{1}{C_0}||f(\cdot,t) - \bar{f}(\cdot,t)||_{L^1(\mathbb{R})} \leq \mathcal{H}(t) \leq C_0||f(\cdot,t) - \bar{f}(\cdot,t)||_{L^1(\mathbb{R})}$
2. $\mathcal{H}$ is non-increasing in time t.

The functionals in (1.2), (1.8) account for both forward and backward interactions of transversally moving particles. In this sense they are different from the Glimm functional or the Liu-Yang functional, as the latter compute the potential of only forward interactions. The functional $\mathcal{Q}_d(t)$ is not positive, and thus not a Lyapunov functional itself.

For certain special systems, such as the Broadwell model

$$\begin{aligned} &\partial_t f_1 - \partial_x f_1 = f_2^2 - f_1 f_3, \\ &\partial_t f_2 = -\frac{1}{2}(f_2^2 - f_1 f_3) \\ &\partial_t f_3 + \partial_x f_3 = f_2^2 - f_1 f_3\,, \end{aligned} \tag{1.11}$$

or models with transversal interactions ($B_i^{kl} = 0$ for k, l with $v_k = v_l$)), it is possible to define alternative functionals, where the interaction potential is positive and accounts for only forward interactions. For the Broadwell system, the interaction potential

$$\begin{aligned}\mathcal{Q}_d(t) \quad &\equiv \quad \int_{\mathbb{R}^2} \mathbf{1}_{x<y}\Big(\big(2f_2 + 2\bar{f}_2 + f_3 + \bar{f}_3\big)(x)\big(|f_1 - \bar{f}_1| + 2|f_2 - \bar{f}_2|\big)(y)\Big)dxdy \\ &+ \quad \int_{\mathbb{R}^2} \mathbf{1}_{x<y}\Big(\big(2|f_2 - \bar{f}_2| + |f_3 - \bar{f}_3|\big)(x)\big(f_1 + \bar{f}_1 + 2f_2 + 2\bar{f}_2\big)(y)\Big)dxdy\,.\end{aligned}$$

gives rise to a sharper Lyapunov inequality (see Section 3.2), and yields a uniform L^1-stability Theorem 3.11 where the smallness assumption on the data is quantified.

Finally, there exist some interesting models with wave speeds $v_i = v_i(x,t)$. The linearization of conservation laws leads for instance to such models in divergence form and an analog of the Glimm functional is then constructed by Schatzman [24]. Consider

$$\begin{aligned}&\partial_t f_i + \partial_x\big(v_i(x,t)f_i\big) = Q_i(f), \\ &f_i(x,0) = f_{i0}(x),\end{aligned} \tag{1.12}$$

where $Q_i(f) = \sum_{j,k} B_i^{jk}(x,t)f_jf_k$, $i = 1, \cdots, N$, and (1.12) is strictly hyperbolic in the sense that $v_i(x,t)$ are real-valued C^1 functions satisfying $v_i(x,t) < v_j(x,t)$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ and $i < j$. Analogous estimations are then established for (1.12) provided the source consists of only transversal terms. In return, one can relax the assumptions on signs of the data and collision coefficients and no conservation law is assumed for the source.

Theorem 1.2. *Suppose that the coefficients in (1.12) satisfy hypotheses* (3.39)-(3.40)*, and let* $f, \bar{f} \in C(\mathbb{R}_+; (L^1(\mathbb{R}))^N)$ *be two mild solutions corresponding to initial data*

$$f_0, \bar{f}_0 \in \big[(L_+^\infty \cap L^1)(\mathbb{R})\big]^N \text{ with } ||f_0||_{L^1(\mathbb{R})} \ll 1 \text{ and } ||\bar{f}_0||_{L^1(\mathbb{R})} \ll 1.$$

Then, f*,* $\bar{f}$ *satisfy the inequalities* (3.56)-(3.58) *and*

$$||f(\cdot,t) - \bar{f}(\cdot,t)||_{L^1(\mathbb{R})} \le C||f_0(\cdot) - \bar{f}_0(\cdot)||_{L^1(\mathbb{R})},$$

where C *is a positive constant independent of time* t*.*

The functionals $\mathcal{Q}$ and $\mathcal{Q}_d$ consist of sums of two-point distribution functions in the phase space. For the system (1.12), we introduce in section 3.3 functionals consisting of three-point and multi-point distribution functions (see (3.52), (3.53)) and obtain various Lyapunov inequalities.

This paper is organized as follows. In Section 2, we review the basics of a one-dimensional discrete model for the Boltzmann equation and give an outline of the global existence theory in $L^1 \cap L^\infty$. In Section 3, we explicitly construct the nonlinear functionals, study their time-variation, and finally prove L^1 stability. This is done consecutively, first for general discrete velocity models (1.1), then for the Broawell system (1.11), and finally for semilinear quadratic systems with transversal terms (1.12).

2 Preliminaries

2.1 Discrete velocity Boltzmann equations

We present a review of the basics for discrete velocity Boltmann equations and refer to [10, 15, 11, 21, 19] and [2] for further details. The discretization of the velocity space in the kinetic theory of gases allows to replace the Boltzmann equation by a system of semilinear hyperbolic equations. There is a set of preselected velocities $V_1, \ldots, V_N \in \mathbb{R}^3$ and a set of admitted binary collisions $(k,l) \to (i,j)$. The pre-collisional V_k, V_l and post-collisional velocities V_i, V_j satisfy microscopic conservation laws of mass, momentum and energy,

$$\begin{aligned} V_i + V_j &= V_k + V_l, \\ |V_i|^2 + |V_j|^2 &= |V_k|^2 + |V_l|^2 \end{aligned} \tag{2.1}$$

The interaction coefficients A_{ij}^{kl} are positive constants measuring the relative strengths of the collisions $(k,l) \to (i,j)$; if a collision $(k,l) \to (i,j)$ does not occur one sets $A_{ij}^{kl} = 0$. Typical assumptions for the interaction coefficients are symmetry

$$A_{ij}^{kl} = A_{ij}^{lk} = A_{ji}^{kl}\,, \tag{2.2}$$

and microreversibility (or detailed balance)

$$A_{ij}^{kl} = A_{kl}^{ij}. \tag{2.3}$$

The latter is sometimes relaxed to semi-detailed balance (see [15]) but we will not insist on that here. The kinetic function $f_i(X,t)$ describes the density of particles at the point $(X,t) \in \mathbb{R}^3 \times \mathbb{R}$ moving with velocity V_i and is governed by the discrete velocity Boltzmann equation

$$\partial_t f_i + V_i \cdot \nabla_X f_i = \sum_{j,k,l} (A_{ij}^{kl} f_k f_l - A_{kl}^{ij} f_i f_j), \tag{2.4}$$

for $i = 1, \ldots, N$.

Next, we briefly review some properties of the collision operator

$$Q_i(f) := \sum_{j,k,l} (A_{ij}^{kl} f_k f_l - A_{kl}^{ij} f_i f_j)\,, \quad i = 1, \ldots, N\,.$$

Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be any measurable function. Then we have

$$\partial_t \left(\sum_i \phi(V_i) f_i \right) + \mathrm{div}_X \left(\sum_i V_i \phi(V_i) f_i \right) = \sum_{i,j,k,l} \phi(V_i)(A_{ij}^{kl} f_k f_l - A_{kl}^{ij} f_i f_j).$$

In view of (2.2) and (2.3) the right hand side may be rearranged as

$$\begin{aligned}\sum_{i,j,k,l} \phi(V_i)(A_{ij}^{kl} f_k f_l - A_{kl}^{ij} f_i f_j) &= \frac{1}{4} \sum_{i,j,k,l} \Big(\phi(V_i) + \phi(V_j) - \phi(V_k) - \phi(V_l)\Big) (A_{ij}^{kl} f_k f_l - A_{kl}^{ij} f_i f_j) \\ &= \frac{1}{2} \sum_{i,j,k,l} \Big(\phi(V_i) + \phi(V_j) - \phi(V_k) - \phi(V_l)\Big) A_{ij}^{kl} f_k f_l \\ &= \frac{1}{4} \sum_{i,j,k,l} \Big(\phi(V_i) + \phi(V_j) - \phi(V_k) - \phi(V_l)\Big) A_{ij}^{kl} (f_k f_l - f_i f_j).\end{aligned} \tag{2.5}$$

For the choice of $\phi(V)$ equal to one of the collisional invariants $1, V^1, V^2, V^3$ or $|V|^2$ solutions of (2.4) satisfy macroscopic conservation laws of mass, momentum and energy. Moreover, by multiplying (2.4) by $1 + \log f_i$, we have the H-theorem

$$\begin{aligned}&\partial_t \left(\sum_i f_i \log f_i \right) + \mathrm{div}_X \left(\sum_i V_i f_i \log f_i \right) = \sum_{i,j,k,l} A_{ij}^{kl} (\log f_i)(f_k f_l - f_i f_j) \\ &= \frac{1}{4} \sum_{i,j,k,l} A_{ij}^{kl} (\log f_i + \log f_j - \log f_k - \log f_l)(f_k f_l - f_i f_j) \\ &= -\frac{1}{4} \sum_{i,j,k,l} A_{ij}^{kl} \log \left(\frac{f_k f_l}{f_i f_j} \right) (f_k f_l - f_i f_j) \leq 0.\end{aligned} \tag{2.6}$$

We consider now the description of one dimensional motions of a dilute gas. Let $D \in \mathbb{R}^3$ be the direction of motion and consider the ansatz

$$f_i(X,t) = \bar{f}_i(D \cdot X, t), \quad x = D \cdot X$$

Then $\bar{f}(x,t)$ satisfies a system of the form

$$\partial_t f_i + v_i \partial_x f_i = \sum_{j,k,l} A_{ij}^{kl} (f_k f_l - f_i f_j), \tag{2.7}$$

where the projected velocities $v_i = V_i \cdot D$ satisfy microscopic conservation of mass and momentum

$$v_i + v_j = v_k + v_l, \tag{2.8}$$

but not, in general, microscopic conservation of energy. In view of (2.8), $\bar{f}$ satisfies macroscopic conservations of mass and momentum (in the direction of motion)

$$\begin{aligned}\partial_t \sum_i f_i + \partial_x \sum_i v_i f_i = 0\,, \\ \partial_t \sum_i v_i f_i + \partial_x \sum_i v_i^2 f_i = 0\,,\end{aligned} \tag{2.9}$$

and the H-theorem

$$\partial_t \sum_i f_i \log f_i + \partial_x \sum_i v_i f_i \log f_i + \frac{1}{4} \sum_{i,j,k,l} A_{ij}^{kl} \log \left(\frac{f_k f_l}{f_i f_j} \right) (f_k f_l - f_i f_j) = 0\,. \tag{2.10}$$

From the viewpoint that the one-dimensional model (2.7) describes a one-dimensional motion of a three-dimensional discrete velocity model (2.4), the one-dimensional model does not have to satisfy conservation of energy and the system does not need to be strictly hyperbolic - even if the original system has distinct velocities. The loss of strict hyperbolicity causes various difficulties with the types of estimates pursued here. Following Beale [2], we wish to reduce the system by combining the densities f_i and $f_{i'}$ for which the projected velocities v_i and $v_{i'}$ coincide, $v_i \equiv v_{i'}$. The equations for f_i, $f_{i'}$ read

$$\begin{aligned} \partial_t f_i + v_i \partial_x f_i &= Q_i(f) = \sum_{j,k,l} A_{ij}^{kl}(f_k f_l - f_i f_j) && (2.11) \\ \partial_t f_{i'} + v_{i'} \partial_x f_{i'} &= Q_{i'}(f) = \sum_{j',k',l'} A_{i'j'}^{k'l'}(f_{k'} f_{l'} - f_{i'} f_{j'}) && (2.12) \end{aligned}$$

We place structural hypotheses on the system so that, when for two indices i and i' the projected velocities coincide $v_i \equiv v_{i'}$, we can identify the corresponding collision operators $Q_i(f) \equiv Q_{i'}(f)$ so that the equations (2.11) and (2.12) coincide. This dictates certain restrictions on the interaction coefficients A_{ij}^{kl}, see [2] for the precise hypotheses. Then, if the initial data are the same for particles moving with the same velocities, then we can identify the densities f_i and $f_{i'}$ and replace them by one equation that is counted ν_i times (where ν_i is the number of projected velocities that coincide with the velocity v_i). The system (2.7) can be put into the form of (1.1)

$$\partial_t f_i + v_i \partial_x f_i = \sum_{j,k=1}^{N} B_i^{jk} f_j f_k \tag{2.13}$$

with

$$B_i^{kl} = \sum_{j=1}^{N} A_{ij}^{kl} - \frac{1}{2} \sum_{m,n=1}^{N} A_{mn}^{kl} \delta_{ik} - \frac{1}{2} \sum_{m,n=1}^{N} A_{mn}^{kl} \delta_{li}$$

and δ_{ik} is the Kronecker symbol. It is clear that (1.4) and (1.4) are satisfied. Hypothesis (1.5) reflects in this setting the conservation laws of mass and momentum. Most of the usual examples of discrete kinetic theory fit under the above framework.

Next we consider hypothesis (1.6). For a kinetic theory model quadratic terms arise as follows:

- If $B_i^{jj} > 0$ with $i \neq j$ then there is a collision $(j,j) \to (i,i')$. For a nontrivial collision, (2.8) implies that either $v_i < v_j = v_j < v_{i'}$ or $v_{i'} < v_j = v_j < v_i$.

- If $B_i^{ii} < 0$ then there is a collision $(i,i) \to (j,j')$. For a nontrivial collision, (2.8) implies $v_j \neq v_{j'}$ and, if we denote by v_j the smallest outgoing speed, $v_j < v_i = v_i < v_{j'}$.

Suppose now that $B_i^{ii} < 0$ and set $i_1 = i$. Then there is a nontrivial collision $(i_1, i_1) \to (i_2, i_{2'})$ with $v_{i_2} < v_{i_1} < v_{i_{2'}}$. Consider the equation for the balance of f_{i_2}. There, the interaction coefficient

$B_{i_2}^{i_1 i_1'} > 0$. If $B_{i_2}^{i_2 i_2} = 0$ then we are finished here, (1.6) is justified. If not, then $B_{i_2}^{i_2 i_2} < 0$ and there is a nontrivial collision $(i_2, i_2) \to (i_3, i_{3'})$ with $v_{i_3} < v_{i_2} < v_{i_{3'}}$. We consider the equation for the balance of f_{i_3}, note that $B_{i_3}^{i_2 i_2'} > 0$ and repeat the previous step. Since the velocities v_{i_k} are strictly decreasing in each step, the process necessarily terminates and (1.6) is justified.

A well studied paradigm of a one-dimensional discrete velocity model is the one proposed by Broadwell [10]. The Broadwell model describes particles moving with a set of six velocities and colliding with equal probabilities. It reads

$$
\begin{aligned}
\partial_t f_1^+ + \partial_x f_1^+ &= \frac{1}{2}(f_2^+ f_2^- + f_3^+ f_3^-) - f_1^+ f_1^- , \\
\partial_t f_1^- - \partial_x f_1^- &= \frac{1}{2}(f_2^+ f_2^- + f_3^+ f_3^-) - f_1^+ f_1^- , \\
\partial_t f_2^+ + \partial_y f_2^+ &= \frac{1}{2}(f_1^+ f_1^- + f_3^+ f_3^-) - f_2^+ f_2^- , \\
\partial_t f_2^- - \partial_y f_2^- &= \frac{1}{2}(f_1^+ f_1^- + f_3^+ f_3^-) - f_2^+ f_2^- , \\
\partial_t f_3^+ + \partial_z f_3^+ &= \frac{1}{2}(f_1^+ f_1^- + f_2^+ f_2^-) - f_3^+ f_3^- , \\
\partial_t f_3^- - \partial_z f_3^- &= \frac{1}{2}(f_1^+ f_1^- + f_2^+ f_2^-) - f_3^+ f_3^- ,
\end{aligned}
$$

where $f_1^\pm$, $f_2^\pm$ and $f_3^\pm$ are densities of particles moving with velocities ± 1 in the direction of the x, y, and z axes respectively. One then considers one-dimensional motions of particles, depending on x but independent of the y and z coordinates, and under the ansatz $f_2^+ = f_2^- = f_3^+ = f_3^-$. If we set

$$
f_1^+ = f_1, \quad f_1^- = f_3, \quad f_2^+ = f_2^- = f_3^+ = f_3^- = f_2 ,
$$

then (2.14) reduces to the one-dimensional system

$$
\begin{aligned}
\partial_t f_1 - \partial_x f_1 &= f_2^2 - f_1 f_3, \\
\partial_t f_2 &= -\frac{1}{2}(f_2^2 - f_1 f_3), \\
\partial_t f_3 + \partial_x f_3 &= f_2^2 - f_1 f_3.
\end{aligned}
$$

It is easy to check that the one-dimensional Broadwell model is of the general form of the system (1.1) and satisfies (1.4)-(1.6).

2.2 Existence theory

Next we discuss the existence theory for the Cauchy problem of (1.1). There are two venues for defining weak solutions of (1.1). First, using the notion of mild solution:

Definition 2.3. *$f = (f_1, \cdots, f_N) \in C([0,T]; (L^1(\mathbb{R}))^N)$ is a mild solution of (1.1) with data $f_0 \in (L^1(\mathbb{R}))^N$ if $Q_i(f) \in L^1(\mathbb{R} \times [0,T])$ and for $t \in [0,T]$ and a.e $x \in \mathbb{R}$, $f(x,t)$ satisfies the integral equation,*

$$
f_i(x,t) = f_{i0}(x_i - v_i t) + \int_0^t Q_i(f)(x - v_i(t-\tau), \tau) d\tau, \tag{2.14}
$$

for $i = 1, \cdots, N$.

A second possibility is to define f as a weak solution:

Definition 2.4. $f = (f_1, \cdots, f_N) \in C([0,T]; (L^1(\mathbb{R}))^N)$ *is a weak solution of (1.1) with data* $f_0 \in (L^1(\mathbb{R}))^N$ *if* $Q_i(f) \in L^1(\mathbb{R} \times [0,T])$ *and for any test function* $\varphi \in C_c^\infty(\mathbb{R} \times [0,\infty))$ *we have*

$$\iint f_i \big(\partial_t \varphi + v_i \partial_x \varphi\big) dx dt + \int f_{i0}(x) \varphi(x,0) dx = - \iint Q_i(f) \varphi dx dt \tag{2.15}$$

for $i = 1, \cdots, N$.

In fact, for solutions f of class $C([0,T]; (L^1(\mathbb{R}))^N)$ with $Q_i(f) \in L^1(\mathbb{R} \times [0,T])$ the two notions of solution are equivalent. Obviously, a mild solution is also a weak solution. To see that a weak solution is also a mild solution, rewrite first (2.15) in the equivalent form

$$\iint f_i(y + v_i \tau, \tau)(\partial_t \psi)(y,\tau) dy d\tau + \int f_{i0}(y) \psi(y,0) dy = - \iint Q_i(f)(y + v_i \tau, \tau) \varphi dy d\tau \,,$$

where $\varphi(x,t) = \psi(x - v_i t, t)$. Fix $t > 0$, $\delta \in (0,t)$ and take the test function $\psi(y,\tau) = a(y) b_\delta(\tau)$ where $a \in C_c^\infty(\mathbb{R})$ and $b_\delta \in C_c^\infty([0,\infty))$ is selected so that it takes the value 1 on $[0, t-\delta]$, decreases linearly on $(t-\delta, t)$, and takes the value 0 on (t, ∞). Taking the limit $\delta \to 0$, we deduce

$$\int_{\mathbb{R}} f_i(y + v_i t, t) a(y) dy - \int_{\mathbb{R}} f_{i0}(y) a(y) dy = \int_{\mathbb{R}} \int_0^t Q_i(f)(y + v_i \tau, \tau) a(y) d\tau dy \,,$$

from where (2.14) follows.

We state the main global existence result for (1.1).

Theorem 2.5. [2, 3] *Suppose that (1.1) satisfies the hypotheses* (1.4)-(1.6) *and let* $f_0 \geq 0$ *with* $f_0 \in \big(L^1(\mathbb{R}) \cap L_+^\infty(\mathbb{R})\big)^N$. *There exists a unique, nonnegative mild solution* f *of (1.1) with*

$$f \in C\big([0,T]; (L^1(\mathbb{R})^N\big) \cap \big(L^\infty(\mathbb{R} \times [0,T])\big)^N$$

for any $T > 0$, *and* $Q_i(f) \in L^1(\mathbb{R} \times \mathbb{R}_+)\big)$, $i = 1, \ldots, N$. *Moreover, if* f_0 *is of class* C^1, *then* f *is of class* C^1 *in* (x,t).

Variants of this theorem are proved by Tartar [26], Beale [2] and Bony [3]. The reader is also referred to [23, 25, 11, 21, 6, 7] for further existence and asymptotic behavior results.

Outline of the proof. It is instructive to outline the proof of Theorem 2.5 following the ideas of [3]. One starts with estimates for C^1 solutions, in terms of the L^1-norm $\mu := \int_{\mathbb{R}} \sum_j \nu_j f_{0j}(x) dx$ of the data.

Step 1. One first shows using Proposition 3.6 and (3.12) that the transversal terms satisfy the L^1-bound

$$\int_0^\infty \int_{\mathbb{R}} f_m(x,t) f_n(x,t) dx dt \leq C \mu^2 < \infty \,, \quad \text{for } m \neq n. \tag{2.16}$$

Step 2. Next, it is shown that

$$B_i^{jj} \neq 0 \quad \text{implies} \int_0^\infty \int_{\mathbb{R}} f_j^2(x,t)dxdt \leq C(\mu + \mu^2) < \infty \tag{2.17}$$

Suppose that $B_i^{jj} \neq 0$ with $i \neq j$. Then $B_i^{jj} > 0$. Consider the balance equation for f_i. If $B_i^{ii} = 0$ then then by integrating in space-time we conclude (2.17). If $B_i^{ii} < 0$ then using (1.6) we again conclude (2.17).

Step 3. The previous steps indicate that $Q_i(f) \in L^1([0,\infty) \times \mathbb{R})$. Bony [3] uses this fact to establish that solutions f are in L^∞, with an explicit bound depending on the L^1-mass of the data. We refer to [3] for the proof of the estimate. Once these estimates are established for smooth solutions, the existence of mild solutions for data $f_0 \in L^1 \cap L^\infty$ follows by a standard density argument. Uniqueness for mild solutions of class L^∞ is trivial. As weak and mild solutions coincide, uniqueness is inherited for the class of bounded weak solutions. □

Since (1.1) has quadratic nonlinearities it is easy to see that

$$\|f(\cdot,t) - \bar{f}(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|f_0 - \bar{f}_0\|_{L^1(\mathbb{R})}\, e^{Mt}$$

where M depends on $\max_k \|f_k(x,t)\|_{L^\infty(\mathbb{R}\times(0,t))}$. Thus the L^∞-bounds imply L^1-stability with a constant depending exponentially in time.

The estimates (2.16)-(2.17) give also information on asymptotic behaviour. By integrating (1.1), we write

$$f_i(y + v_i t, t) = f_{i0}(y) + \int_0^t Q_i(f)(y + v_i s, s)ds \tag{2.18}$$

If we formally set $F_{i\infty}(y) := f_{i0}(y) + \int_0^\infty Q_i(f)(y + v_i s, s)ds$, then

$$\|f_i(\cdot + v_i t, t) - F_{i\infty}\|_{L^1(\mathbb{R})} \leq \int_t^\infty \int_{\mathbb{R}} |Q_i(f)|(y + v_i s, s)dyds \to 0$$

as $t \to \infty$ and thus $f_i(x,t) \to F_{i\infty}(x - v_i t)$ in $L^1(\mathbb{R})$ and a.e. Hence, the leading term in the asymptotic response of f_i is a traveling wave. If the collision model contains the quadratic interaction $(i,i) \to (j,j')$ then the coefficient $B_i^{ii} < 0$ and (2.17) implies that $\int_0^\infty \int_{\mathbb{R}} f_i^2 dxdt < \infty$ and $F_{i\infty}(y) = 0$ a.e. In other words, the leading traveling wave in the asymptotic behavior of a field which self-interacts is trivial. We refer to [2, 4, 5, 26, 27] for further results on asymptotic behavior.

3 Lyapunov functionals and uniform L^1 stability estimates

In this section, we construct nonlinear functionals which are equivalent to the L^1 distance and non-increasing in time. We begin in Section 3.1 with a general discrete velocity Boltzmann equation. Then, in Section 3.2, we specialize to the well-known Broadwell model. In this case, the proposed functional contains only the forward interaction potential and is thus in closer analogy to the spirit of the Glimm and Liu-Yang functionals. In Section 3.3, we take up systems that only have transversal source terms. We allow then for solutions that may be negative and calculate two-point, three-point or multi-point interactions and establish more complicated Lyapunov type functionals.

3.1 General discrete velocity Boltzmann equations

We first consider the Cauchy problem for the discrete velocity Boltzmann equation

$$\partial_t f_i + v_i \partial_x f_i = \sum_{j,k} B_i^{jk} f_j f_k \,, \tag{3.1}$$

$i = 1, \ldots, N$, under the hypotheses (1.4), (1.4) and (1.5). Using the fact that 1 and v_i are collisional invariants, it follows from (1.5) that

$$\partial_t \left(\sum_{n=1}^{N} \nu_n f_n \right) + \partial_x \left(\sum_{n=1}^{N} v_n \nu_n f_n \right) = 0, \tag{3.2}$$

$$\partial_t \left(\sum_{m=1}^{N} (v_m - v_n) \nu_m f_m \right) + \partial_x \left(\sum_{m=1}^{N} v_m (v_m - v_n) \nu_m f_m \right) = 0. \tag{3.3}$$

Motivated by (3.2) and (3.3), Bony's functional for (3.1) is defined by

$$\begin{aligned} \mathcal{Q}(t) &= \sum_{m,n} \int_{\mathbb{R}} \int_{\mathbb{R}} \operatorname{sgn}(y-x)(v_m - v_n) \nu_m \nu_n f_m(x,t) f_n(y,t) dy dx, \\ &= \sum_{m,n} \Big[\int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{x<y} (v_m - v_n) \nu_m \nu_n f_m(x,t) f_n(y,t) dy dx \\ &\quad - \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{x>y} (v_m - v_n) \nu_m \nu_n f_m(x,t) f_n(y,t) dy dx \Big] \\ &=: I + II. \end{aligned} \tag{3.4}$$

From the conservation of mass and the positivity of solutions, it easily follows that

$$|\mathcal{Q}(t)| \le 2 \Big(\max_{m,n} |v_m - v_n| \Big) \Big(\int_{\mathbb{R}} \sum_m \nu_m f_{0m}(x) dx \Big)^2 < \infty. \tag{3.5}$$

We note that for m, n such that $m > n$, I and II denote respectively the forward and backward interaction potentials between the waves traveling with speeds v_m and v_n. By the choice of the weight $v_m - v_n$, this functional can be negative. For notational simplicity, we suppress from now on the t dependence and write

$$f(x,t) \equiv f(x), \qquad Q_n(f)(x,t) \equiv Q_n(x).$$

Define the instantaneous interaction production $\Lambda(f)$ by

$$\Lambda(f)(t) = \sum_{m,n,\, m>n} \int_{\mathbb{R}} \nu_m \nu_n f_m(x) f_n(x) dx, \tag{3.6}$$

The following Proposition shows uniform integrability of the transversal terms of the source.

Proposition 3.6. [3] *Assume that (3.1) satisfies* (1.4)-(1.5)*, and let f be a solution emanating from the initial datum f_0. Then $\mathcal{Q}(t)$ is non-increasing in time t, i.e.,*

$$\frac{d\mathcal{Q}(t)}{dt} \leq -4v_*^2 \Lambda(f)(t).$$

where $v_*^2 = \min_{m \neq n}(v_m - v_n)^2$.

Proof. We consider I in (3.4). Recall that

$$\partial_t f_m(x) + v_m \partial_x f_m(x) = Q_m(x), \tag{3.7}$$

$$\partial_t f_n(y) + v_n \partial_y f_n(y) = Q_n(y). \tag{3.8}$$

Then (3.7) and (3.8) give

$$\begin{aligned} \partial_t \Big(\mathbb{1}_{x<y} f_m(x) f_n(y) \Big) &+ (v_m \partial_x + v_n \partial_y) \Big(\mathbb{1}_{x<y} f_m(x) f_n(y) \Big) \\ &+ (v_m - v_n)\delta(x-y) f_m(x) f_n(y) = \mathbb{1}_{x<y} \big(Q_m(x) f_n(y) + f_m(x) Q_n(y) \big) \end{aligned} \tag{3.9}$$

and in turn

$$\begin{aligned} \partial_t \Big(\mathbb{1}_{x<y} \sum_{m,n} (v_m - v_n)\nu_m \nu_n f_m(x) f_n(y) \Big) &+ (v_m \partial_x + v_n \partial_y) \Big(\mathbb{1}_{x<y} \sum_{m,n} (v_m - v_n)\nu_m \nu_n f_m(x) f_n(y) \Big) \\ &+ \sum_{m,n} (v_m - v_n)^2 \nu_m \nu_n \delta(x-y) f_m(x) f_n(y) \\ = \mathbb{1}_{x<y} \sum_{m,n} (v_m - v_n)\nu_m \nu_n \big(Q_m(x) f_n(y) + f_m(x) Q_n(y) \big) &= 0 \end{aligned} \tag{3.10}$$

where we have used that, from (1.5),

$$\sum_m \nu_m Q_m(x) = 0 \, , \quad \sum_m v_m \nu_m Q_m(x) = 0 \, . \tag{3.11}$$

Integrating (3.10) over $\mathbb{R} \times \mathbb{R}$, we have

$$\frac{d}{dt} \left[\sum_{m,n} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{x<y} (v_m - v_n)\nu_m \nu_n f_m(x) f_n(y) dy dx \right] = -\sum_{m,n} \int_{\mathbb{R}} (v_m - v_n)^2 \nu_m \nu_n f_m(x) f_n(x) dx.$$

The term II is treated in the same way and gives

$$\frac{d}{dt} \left[-\sum_{m,n} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{y<x} (v_m - v_n)\nu_m \nu_n f_m(x) f_n(y) dy dx \right] = -\sum_{m,n} \int_{\mathbb{R}} (v_m - v_n)^2 \nu_m \nu_n f_m(x) f_n(x) dx.$$

Combining the above, we obtain the desired result. □

Remark 3.7. By Proposition 3.6 and (3.5) that the transversal terms are integrable in space-time, that is, for $m \neq n$,

$$\int_0^\infty \int_{-\infty}^\infty \nu_m \nu_n f_m(x,t) f_n(x,t) dx dt \leq C ||f_0||^2_{L^1(\mathbb{R})} < \infty, \quad \text{for some } C > 0. \tag{3.12}$$

Next, we study the L^1 stability of mild solutions. Let f and $\bar{f}$ be two solutions corresponding to initial data $f_0(x)$ and $\bar{f}_0(x)$ respectively. Define the nonlinear functionals

$$
\begin{aligned}
\mathcal{L}(t) &\equiv \sum_m \int_{\mathbb{R}} \nu_m |f_m(x) - \bar{f}_m(x)| dx, \\
\mathcal{Q}_d(t) &\equiv \sum_{m,n} \int_{\mathbb{R}} \int_{\mathbb{R}} \operatorname{sgn}(y-x)(v_m - v_n)\nu_m \nu_n |f_m(x) - \bar{f}_m(x)|(f_n(y) + \bar{f}_n(y))\, dxdy, \\
\mathcal{H}(t) &\equiv \mathcal{L}(t) + K\mathcal{Q}_d(t)\,,
\end{aligned}
$$

where the positive constant $K > 0$ is later appropriately selected. $\mathcal{L}(t)$ measures the L^1 distance between f and $\bar{f}$, while $\mathcal{Q}_d(t)$ is a generalization of the Bony functional $\mathcal{Q}(t)$. $\mathcal{Q}_d(t)$ measures the (forward and backward) interaction potentials between f and $|f-\bar{f}|$ and between $\bar{f}$ and $|f-\bar{f}|$. We study the time-variation of these nonlinear functionals. In the calculations there will enter the analogs of the instantaneous interaction production $\Lambda(f)(t)$ that take the form

$$
\begin{aligned}
&\Lambda_d(f,\bar{f})(t) = \sum_{m,n,\, m\neq n} \int_{\mathbb{R}} \nu_m \nu_n |f_m - \bar{f}_m|(x)(f_n + \bar{f}_n)(x) dx, \\
&\Lambda_s(f,\bar{f})(t) = \sum_{m,n,\, m\neq n} \int_{\mathbb{R}} \nu_m B_m^{nn}\Big(1 - \frac{\delta_m}{\delta_n}\Big)|f_n - \bar{f}_n|(f_n + \bar{f}_n) \\
&\Lambda(f,\bar{f})(t) = \Lambda_d(f,\bar{f})(t) + \Lambda_s(f,\bar{f})(t).
\end{aligned}
$$

All these functionals are positive and their role will be clarified in the sequel.

Proposition 3.8. *Assume that* (3.1) *satisfies (1.4) - (1.5). Let f and $\bar{f}$ be two solutions of* (3.1) *corresponding to initial data f_0 and $\bar{f}_0$ with $||f_0||_{L^1(\mathbb{R})} + ||\bar{f}_0||_{L^1(\mathbb{R})} \ll 1$. Then, for an appropriate choice of K, the functional $\mathcal{H}(t)$ is equivalent to the L^1 distance between f and $\bar{f}$ and satisfies*

$$
\begin{aligned}
\frac{d\mathcal{L}(t)}{dt} + \Lambda_s(f,\bar{f})(t) &\leq C_1 \Lambda_d(f,\bar{f})(t) \\
\frac{d\mathcal{Q}_d(t)}{dt} + c_2\Lambda_d(f,\bar{f})(t) &\leq C_2\left(||f||_{L^1(\mathbb{R})} + ||\bar{f}||_{L^1(\mathbb{R})}\right)\Lambda(f,\bar{f})(t) \\
\frac{d\mathcal{H}(t)}{dt} &\leq -c_3\Lambda(f,\bar{f})(t),
\end{aligned}
\tag{3.13}
$$

where C_1, C_2, c_2 and c_3 are positive constants which are independent of time t.

Proof. We consider the time-evolution of each functional separately. Let f and $\bar{f}$ be C^1 solutions of compact support corresponding to compactly supported C^1 data $f_0(x)$ and $\bar{f}_0(x)$ respectively. The case of L^1 solutions will follow by a standard density argument.

Step1. Computation of $\frac{d\mathcal{L}(t)}{dt}$. Note that f and $\bar{f}$ satisfy

$$
\partial_t f_i + v_i \partial_x f_i = Q_i \tag{3.14}
$$

$$
\partial_t |f_i - \bar{f}_i| + \partial_x v_i |f_i - \bar{f}_i| = (Q_i - \bar{Q}_i)\delta_i \tag{3.15}
$$

where we use the notation

$$\delta_i(x,t) = \mathrm{sgn}(f_i(x,t) - \bar{f}_i(x,t))\,.$$

We decompose the terms due to transversal interactions from the terms due to self-interactions,

$$Q_i = \sum_{m\neq n} B_i^{mn} f_m f_n + \sum_n B_i^{nn} f_n^2\,,$$

and use (1.4) to write

$$(Q_i - \bar{Q}_i)\delta_i = \sum_{m,n,m\neq n} B_i^{mn}\frac{\delta_i}{\delta_m}|f_m - \bar{f}_m|(f_n + \bar{f}_n) + \sum_n B_i^{nn}\frac{\delta_i}{\delta_n}|f_n - \bar{f}_n|(f_n + \bar{f}_n). \tag{3.16}$$

We note the identity

$$\begin{aligned}
&\sum_i\sum_n \nu_i B_i^{nn}\frac{\delta_i}{\delta_n}|f_n - \bar{f}_n|(f_n+\bar{f}_n)\\
&\quad= \sum_{i,n,\,i\neq n}\nu_i B_i^{nn}\frac{\delta_i}{\delta_n}|f_n - \bar{f}_n|(f_n+\bar{f}_n) + \sum_n \nu_n B_n^{nn}|f_n - \bar{f}_n|(f_n+\bar{f}_n)\\
&\quad= \sum_{i,n,\,i\neq n}\nu_i B_i^{nn}\frac{\delta_i}{\delta_n}|f_n - \bar{f}_n|(f_n+\bar{f}_n) - \sum_n\sum_{i,\,i\neq n}\nu_i B_i^{nn}|f_n - \bar{f}_n|(f_n+\bar{f}_n)\\
&\quad= \sum_{i,n,\,i\neq n}\nu_i B_i^{nn}\left(\frac{\delta_i}{\delta_n} - 1\right)|f_n - \bar{f}_n|(f_n+\bar{f}_n) \leq 0.
\end{aligned} \tag{3.17}$$

which follows from conservation of mass $(1.5)_1$ in the form $\nu_n B_n^{nn} = -\sum_{i,\,i\neq n}\nu_i B_i^{nn}$, and the fact that $B_i^{nn}\geq 0$ for $i\neq n$.

From (3.15) - (3.17) we obtain

$$\begin{aligned}
&\partial_t\sum_i \nu_i|f_i - \bar{f}_i| + \partial_x\sum_i \nu_i v_i|f_i - \bar{f}_i| + \sum_{n,i,\,n\neq i}\nu_i B_i^{nn}\Big(1 - \frac{\delta_i}{\delta_n}\Big)|f_n - \bar{f}_n|(f_n+\bar{f}_n)\\
&\qquad= \sum_i\sum_{m,n,m\neq n}\nu_i B_i^{mn}\frac{\delta_i}{\delta_m}|f_m - \bar{f}_m|(f_n+\bar{f}_n)
\end{aligned} \tag{3.18}$$

and, in turn,

$$\frac{d\mathcal{L}(t)}{dt} + \Lambda_s(f,\bar{f})(t) \leq C_1\Lambda_d(f,\bar{f})(t), \tag{3.19}$$

for some positive constant C_1.

Step2. Calculation of $\frac{d\mathcal{Q}_d(t)}{dt}$. From (3.14) and (3.15), we obtain

$$\begin{aligned}
&\partial_t\Big(1\!\!1_{x<y}|f_m - \bar{f}_m|(x)f_n(y)\Big) + \big(v_m\partial_x + v_n\partial_y\big)\Big(1\!\!1_{x<y}|f_m - \bar{f}_m|(x)f_n(y)\Big)\\
&\quad + (v_m - v_n)\delta(x-y)|f_m - \bar{f}_m|(x)f_n(y)\\
&\quad = 1\!\!1_{x<y}\Big[(Q_m - \bar{Q}_m)(x)\delta_m(x)f_n(y) + |f_m - \bar{f}_m|(x)Q_n(y)\Big]
\end{aligned} \tag{3.20}$$

and

$$
\begin{aligned}
&\partial_t\Big(\sum_{m,n}(v_m - v_n)\nu_m\nu_n \mathbb{1}_{x<y}|f_m - \bar{f}_m|(x)(f_n + \bar{f}_n)(y)\Big)\\
&\quad + \sum_{m,n}(v_m - v_n)\nu_m\nu_n\big(v_m\partial_x + v_n\partial_y\big)\Big(\mathbb{1}_{x<y}|f_m - \bar{f}_m|(x)(f_n + \bar{f}_n)(y)\Big)\\
&\quad + \sum_{m,n}(v_m - v_n)^2\nu_m\nu_n\delta(x-y)|f_m - \bar{f}_m|(x)(f_n + \bar{f}_n)(y) \qquad (3.21)\\
&= \mathbb{1}_{x<y}\sum_{m,n}(v_m - v_n)\nu_m\nu_n(Q_m - \bar{Q}_m)(x)\delta_m(x)(f_n + \bar{f}_n)(y)\\
&\quad + \mathbb{1}_{x<y}\sum_{m,n}(v_m - v_n)\nu_m\nu_n|f_m - \bar{f}_m|(x)\big(Q_n(y) + \bar{Q}_n(y)\big)
\end{aligned}
$$

The last term in (3.21) vanishes, due to the conservation of mass and momentum (3.11).

To estimate the first term in the right hand side of (3.21), we note that, from (3.16),

$$
\begin{aligned}
\sum_i (v_i - v_j)\nu_i\nu_j(Q_i - \bar{Q}_i)\delta_i &= \sum_i (v_i - v_j)\nu_i\nu_j \sum_{m,n,m\neq n} B_i^{mn}\frac{\delta_i}{\delta_m}|f_m - \bar{f}_m|(f_n + \bar{f}_n)\\
&\quad + \sum_i (v_i - v_j)\nu_i\nu_j \sum_n B_i^{nn}\frac{\delta_i}{\delta_n}|f_n - \bar{f}_n|(f_n + \bar{f}_n)\,,
\end{aligned} \qquad (3.22)
$$

and that, as in (3.17) but using now both conservations of mass and momentum in (1.5),

$$
\begin{aligned}
&\sum_i (v_i - v_j)\nu_i\nu_j \sum_n B_i^{nn}\frac{\delta_i}{\delta_n}|f_n - \bar{f}_n|(f_n + \bar{f}_n)\\
&\quad = \sum_{i,n,\,i\neq n}(v_i - v_j)\nu_i\nu_j B_i^{nn}\left(\frac{\delta_i}{\delta_n} - 1\right)|f_n - \bar{f}_n|(f_n + \bar{f}_n).
\end{aligned} \qquad (3.23)
$$

Note that $B_i^{nn} \geq 0$ for $i \neq n$ and that $\left(\frac{\delta_i}{\delta_n} - 1\right) \leq 0$. We can now estimate (3.21) as

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{1}_{x<y}\sum_{m,n}(v_m - v_n)\nu_m\nu_n|f_m - \bar{f}_m|(x)(f_n + \bar{f}_n)(y)\,dxdy\right) + v_*^2\Lambda_d(f,\bar{f})(t)\\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{1}_{x<y}\sum_n (f_n + \bar{f}_n)(y)\sum_m (v_m - v_n)\nu_m\nu_n\big[(Q_m - \bar{Q}_m)(x)\delta_m(x)\big]\\
&\leq C_2||f + \bar{f}||_{L^1(\mathbb{R})}\Big(\Lambda_s(f,\bar{f})(t) + \Lambda_d(f,\bar{f})(t)\Big).
\end{aligned} \qquad (3.24)
$$

We conclude, by using similar estimations for the remaining part of $Q_d(t)$, that

$$
\frac{d\mathcal{Q}_d(t)}{dt} + 2v_*^2\Lambda_d(f,\bar{f})(t) \;\; \leq \;\; C_2\left(||f||_{L^1(\mathbb{R})} + ||\bar{f}||_{L^1(\mathbb{R})}\right)\Lambda(f,\bar{f})(t)\,, \qquad (3.25)
$$

where $v_*^2 = \min_{m\neq n}(v_m - v_n)^2$.

Step 3. Calculation of $\frac{d\mathcal{H}(t)}{dt}$ By the definition of $\mathcal{H}(t)$,

$$
\mathcal{H}(t) \;\; = \;\; \sum_m \int_{\mathbb{R}} \nu_m|f_m(x) - \bar{f}_m(x)|\Big[1 + K(v_n - v_m)\int_{-\infty}^{x}\nu_n(f_n(y) + \bar{f}_n(y))dy
$$

$$- \quad K(v_n - v_m)\int_x^\infty \nu_n(f_n(y) + \bar f_n(y))dy\Big]\,dx.$$

Therefore, if

$$KC_3\big(||f||_{L^1(\mathbb{R})} + ||\bar f||_{L^1(\mathbb{R})}\big) < 1 \tag{3.26}$$

where $C_3 := \max_{m,n}\nu_n|v_m - v_n|$, then there exists $M > 0$ such that

$$\frac{1}{M}\mathcal{L}(t) \le \mathcal{H}(t) \le M\mathcal{L}(t),$$

and $\mathcal{H}(t)$ is equivalent to the L^1-distance $\mathcal{L}(t)$.

On the other hand, from (3.19) and (3.25), we have

$$\begin{aligned}\frac{d\mathcal{H}(t)}{dt} &= \frac{d\mathcal{L}(t)}{dt} + K\frac{d\mathcal{Q}_d(t)}{dt}\\ &\le \Big(-1 + KC_2\big(||f||_{L^1(\mathbb{R})} + ||\bar f||_{L^1(\mathbb{R})}\big)\Big)\Lambda_s(f,\bar f)(t)\\ &\quad + \Big(-2v_*^2K + KC_2\big(||f||_{L^1(\mathbb{R})} + ||\bar f||_{L^1(\mathbb{R})}\big) + C_1\Big)\Lambda_d(f,\bar f)(t)\end{aligned} \tag{3.27}$$

If the L^1 mass of the two solutions is sufficiently small,

$$||f||_{L^1(\mathbb{R})} + ||\bar f||_{L^1(\mathbb{R})} < \frac{2v_*^2}{C_1\max\{C_2,\, C_3\}} < 2v_*^2\,, \tag{3.28}$$

then K can be selected in the nonempty interval

$$\frac{C_1}{2v_*^2 - (||f||_{L^1(\mathbb{R})} + ||\bar f||_{L^1(\mathbb{R})})} < K < \frac{1}{\max\{C_2,\, C_3\}(||f||_{L^1(\mathbb{R})} + ||\bar f||_{L^1(\mathbb{R})})}\,. \tag{3.29}$$

Then (3.26) is fulfilled and there exists a (possibly small) positive constant c_3 so that, for the above choice of K,

$$\frac{d\mathcal{H}(t)}{dt} \le -c_3\Lambda(f,\bar f)(t).$$

□

From Proposition 3.8, we obtain L^1 stability of mild solutions.

Proof of Theorem 1.1

Let $f_0^{(k)}$ and $\bar f_0^{(k)}$ be C^1-approximations of two given initial data $f_0,\ \bar f_0 \in L^1$ such that

$$f_0^{(k)} \to f_0,\quad \bar f_0^{(k)} \to \bar f_0 \quad \text{in } L^1(\mathbb{R}) \quad \text{as } k \to \infty$$

Then we can construct C^1 solutions $f^{(k)}(x,t)$ and $\bar f^{(k)}(x,t)$ corresponding to two smooth initial data $f_0^{(k)}$ and $\bar f_0^{(k)}$ respectively. It follows that f^k is Cauchy in L^1 and

$$f^{(k)}(x,t) \to f(x,t),\quad \bar f^{(k)}(x,t) \to f(x,t) \quad \text{in } L^1(\mathbb{R}) \quad \text{as } k \to \infty.$$

Define

$$\mathcal{H}(t) = \mathcal{H}[f(\cdot,t),\bar{f}(\cdot,t)] \equiv \lim_{k\to\infty} \mathcal{H}[f^{(k)}(\cdot,t),\bar{f}^{(k)}(\cdot,t)].$$

Then by the two key-properties of $\mathcal{H}(t)$, we have

$$||f^{(k)}(\cdot,t) - \bar{f}^{(k)}(\cdot,t)||_{L^1(\mathbb{R})} \le C||f_0^{(k)}(\cdot) - \bar{f}_0^{(k)}(\cdot)||_{L^1(\mathbb{R})}, \quad \text{for some constant } C > 0.$$

Letting $k \to \infty$, we have

$$||f(\cdot,t) - \bar{f}(\cdot,t)||_{L^1(\mathbb{R})} \le C||f_0(\cdot) - \bar{f}_0(\cdot)||_{L^1(\mathbb{R})}.$$

□

3.2 The one-dimensional Broadwell model

Next, we consider the one-dimensional Broadwell model (1.11). In this case, a variant of $\mathcal{Q}_d(t)$ can be defined using only the forward part of the interaction potential. This is in accord with the approach of the Glimm potential, and in contrast to the interaction potential used in the previous subsection for general discrete velocity Boltzmann models.

Let f and $\bar{f}$ be two solutions of (1.11) which for the time are taken to be C^1. We use the notation $f(x)$, $f(y)$ for the evaluation of f at the points (x,t), (y,t) respectively; the t dependence is mostly suppressed. From (1.11), we derive the conservations for the partial masses,

$$\partial_t(2f_2 + f_3)(x) + \partial_x f_3(x) = 0. \tag{3.30}$$

$$\partial_t(f_1 + 2f_2)(y) - \partial_y f_1(y) = 0, \tag{3.31}$$

In the sequel, we define a potential of interaction functional $\mathcal{Q}(t)$ in the form

$$\mathcal{Q}(t) \equiv \int_{\mathbb{R}^2} \mathbf{1}_{x<y}[2f_2(x) + f_3(x)][f_1(y) + 2f_2(y)]dydx,$$

The definition is motivated in the following lemma, which appears in Tartar [27] and is there attributed to Varadhan.

Proposition 3.9. *Along solutions f of (1.11), we have*

$$\frac{d\mathcal{Q}(t)}{dt} = -2\int_{\mathbb{R}} \big(f_1f_3 + f_1f_2 + f_2f_3\big)(x,t)\,dx.$$

Proof. We multiply (3.30) by $(f_1 + 2f_2)(y)$ and (3.31) by $(2f_2 + f_3)(x)$. Adding and multiplying the resulting identity by $\mathbf{1}_{x<y}$, we arrive at the identity

$$\begin{aligned} &\partial_t\Big[(f_1 + 2f_2)(y)\,(2f_2 + f_3)(x)\,\mathbf{1}_{x<y}\Big] \\ &\quad + \mathrm{div}_{(x,y)}\Big[\Big(f_3(x)(f_1 + 2f_2)(y), -(2f_2 + f_3)(x)f_1(y)\Big)\mathbf{1}_{x<y}\Big] \end{aligned}$$

$$+\delta(x-y)\Big(f_3(x)(f_1+2f_2)(y)+f_1(y)(2f_2+f_3)(x)\Big)=0$$

The last term is positive and provides some decay by dispersion. Integrating the above equation over R^2, we obtain

$$\frac{d\mathcal{Q}(t)}{dt}=-2\int_{\mathbb{R}}[f_1(x)f_3(x)+f_1(x)f_2(x)+f_2(x)f_3(x)]dx.$$

Next, we define certain nonlinear functionals:

$$\begin{aligned}
\mathcal{L}(t) &\equiv \int_{\mathbb{R}}\Big(|f_1-\bar f_1|+4|f_2-\bar f_2|+|f_3-\bar f_3|\Big)(x)dx,\\
\mathcal{Q}_d(t) &\equiv \int_{\mathbb{R}^2}\mathbf{1}_{x<y}\Big((2f_2+2\bar f_2+f_3+\bar f_3)(x)\big(|f_1-\bar f_1|+2|f_2-\bar f_2|\big)(y)\Big)dxdy\\
&+\int_{\mathbb{R}^2}\mathbf{1}_{x<y}\Big(\big(2|f_2-\bar f_2|+|f_3-\bar f_3|\big)(x)\big(f_1+\bar f_1+2f_2+2\bar f_2\big)(y)\Big)dxdy,\\
\mathcal{H}(t) &\equiv \mathcal{L}(t)+K\mathcal{Q}_d(t),
\end{aligned}$$

where K is positive constant to be determined later. We also define the instantaneous interaction productions as follows.

$$\Lambda(f,\bar f)(t)=\Lambda_s(f,\bar f)(t)+\Lambda_d(f,\bar f)(t), \tag{3.32}$$

$$\Lambda_s(f,\bar f)\equiv\int_{\mathbb{R}}\left(2-\frac{\delta_1(x)}{\delta_2(x)}-\frac{\delta_3(x)}{\delta_2(x)}\right)|f_2(x)-\bar f_2(x)|(f_2(x)+\bar f_2(x))dx, \tag{3.33}$$

$$\Lambda_d(f,\bar f)(t)\equiv\int_{\mathbb{R}}\sum_{m=1}^{3}|f_m(x)-\bar f_m(x)|\sum_{n\neq m}(f_n(x)+\bar f_n(x))dx. \tag{3.34}$$

The above functionals are all positive and for positive L^1-solutions $\mathcal{H}(t)$ is equivalent to the L^1 distance between f and $\bar f$. Furthermore, $\mathcal{L}(t)$ denotes the weighted L^1 distance while $\mathcal{Q}_d(t)$ represents the potential of interaction between particles.

Next, we study the time-evolution of $\mathcal{L}$ and $\mathcal{Q}_d$.

Lemma 3.10. *Let f and $\bar f$ be two solutions of (1.11) corresponding to initial data f^0 and $\bar f^0$ with $||f^0+\bar f^0||<2$. Then K can be selected so that the functionals satisfy the Lyapunov type estimates:*

$$\begin{aligned}
\frac{d\mathcal{L}(t)}{dt} &\le -\Lambda_s(f,\bar f)(t)+\Lambda_d(f,\bar f)(t),\\
\frac{d\mathcal{Q}_d(t)}{dt} &\le (-2+||f+\bar f||)\Lambda_d(f,\bar f)(t),\\
\frac{d\mathcal{H}(t)}{dt} &\le -C_1\Lambda(f,\bar f)(t),
\end{aligned}$$

where C_1 is a positive constant independent of time t.

Proof. First, we derive the equations for the differences $|f_i(x,t) - \bar{f}_i(x,t)|$, $1 \le i \le 3$, in the form

$$\begin{aligned}
\partial_t|f_1 - \bar{f}_1| - \partial_x|f_1 - \bar{f}_1| &= \frac{\delta_1}{\delta_2}|f_2 - \bar{f}_2|(f_2 + \bar{f}_2) - |f_1 - \bar{f}_1|\frac{f_3 + \bar{f}_3}{2} \\
&\quad - \frac{\delta_1}{\delta_3}|f_3 - \bar{f}_3|\frac{f_1 + \bar{f}_1}{2}, \\
\partial_t|f_2 - \bar{f}_2| &= -\frac{1}{2}|f_2 - \bar{f}_2|(f_2 + \bar{f}_2) + \frac{\delta_2}{2\delta_1}|f_1 - \bar{f}_1|\frac{f_3 + \bar{f}_3}{2} \\
&\quad + \frac{\delta_2}{2\delta_3}|f_3 - \bar{f}_3|\frac{f_1 + \bar{f}_1}{2} \\
\partial_t|f_3 - \bar{f}_3| + \partial_x|f_3 - \bar{f}_3| &= \frac{\delta_3}{\delta_2}|f_2 - \bar{f}_2|(f_2 + \bar{f}_2) - \frac{\delta_3}{\delta_1}|f_1 - \bar{f}_1|\frac{f_3 + \bar{f}_3}{2} \\
&\quad - |f_3 - \bar{f}_3|\frac{f_1 + \bar{f}_1}{2}.
\end{aligned} \tag{3.35}$$

Step 1. We consider the functionals separately. From (3.35) we have

$$\begin{aligned}
&\partial_t\Big(|f_1 - \bar{f}_1| + 4|f_2 - \bar{f}_2| + |f_3 - \bar{f}_3|\Big) + \partial_x\big(|f_3 - \bar{f}_3| - |f_1 - \bar{f}_1|\big) \\
&\qquad + \big(2 - \frac{\delta_1}{\delta_2} - \frac{\delta_3}{\delta_2}\big)|f_2 - \bar{f}_2|(f_2 + \bar{f}_2) \\
&= \big(2\frac{\delta_2}{\delta_1} - \frac{\delta_3}{\delta_1} - 1\big)|f_1 - \bar{f}_1|\frac{f_3 + \bar{f}_3}{2} + \big(2\frac{\delta_2}{\delta_3} - \frac{\delta_1}{\delta_3} - 1\big)|f_3 - \bar{f}_3|\frac{f_1 + \bar{f}_1}{2}
\end{aligned} \tag{3.36}$$

By the definitions of $\mathcal{L}$, Λ_s and Λ_d, we have

$$\begin{aligned}
\frac{d\mathcal{L}(t)}{dt} \quad &\le -\Lambda_s(f,\bar{f})(t) + \int_{\mathbb{R}} |f_1 - \bar{f}_1|(f_3 + \bar{f}_3) + |f_3 - \bar{f}_3|(f_1 + \bar{f}_1)\,dx \\
&\le -\Lambda_s(f,\bar{f})(t) + \Lambda_d(f,\bar{f})(t)\,.
\end{aligned} \tag{3.37}$$

Step 2. By a direct yet cumbersome calculation, we obtain from (3.30), (3.31) and (3.35) the identity

$$\begin{aligned}
\frac{d\mathcal{Q}_d(t)}{dt} \quad &= \quad -2\int_{\mathbb{R}} \sum_{m=1}^{3} |f_m - \bar{f}_m|(x) \left(\sum_{n\neq m} (f_n + \bar{f}_n)(x)\right) dx \\
&+ \quad \int_{\mathbb{R}^2} \mathbf{1}_{x<y} \left(\frac{\delta_1}{\delta_2}(y) - 1\right) (f_2 + \bar{f}_2)(y)|f_2 - \bar{f}_2|(y)|(f_3 + \bar{f}_3)(x)\,dxdy \\
&+ \quad \int_{\mathbb{R}^2} 2\mathbf{1}_{x<y} \left(\frac{\delta_1}{\delta_2}(y) - 1\right) (f_2 + \bar{f}_2)(y)|f_2 - \bar{f}_2|(y)(f_2 + \bar{f}_2)(x)\,dxdy \\
&+ \quad \int_{\mathbb{R}^2} \mathbf{1}_{x<y} \left(\frac{\delta_3}{\delta_2}(x) - 1\right) (f_2 + \bar{f}_2)(x)|f_2 - \bar{f}_2|(x)|(f_1 + \bar{f}_1)(y)\,dxdy \\
&+ \quad \int_{\mathbb{R}^2} 2\mathbf{1}_{x<y} \left(\frac{\delta_3}{\delta_2}(x) - 1\right) (f_2 + \bar{f}_2)(x)|f_2 - \bar{f}_2|(x)|(f_2 + \bar{f}_2)(y)\,dxdy \\
&+ \quad \int_{\mathbb{R}^2} \mathbf{1}_{x<y} \left(\frac{\delta_2}{\delta_1}(y) - 1\right) \frac{f_3 + \bar{f}_3}{2}(y)|f_1 - \bar{f}_1|(y)(f_3 + \bar{f}_3)(x)\,dxdy
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}^2} \mathbf{1}_{x<y} \left(\frac{\delta_2}{\delta_1}(y) - 1\right) (f_3+\bar f_3)(y)|f_1-\bar f_1|(y)(f_2+\bar f_2)(x)\,dxdy \\
&+ \int_{\mathbb{R}^2} \mathbf{1}_{x<y} \left(\frac{\delta_2}{\delta_1}(x) - \frac{\delta_3}{\delta_1}(x)\right) \frac{f_3+\bar f_3}{2}(x)|f_1-\bar f_1|(x)(f_1+\bar f_1)(y)\,dxdy \\
&+ \int_{\mathbb{R}^2} \mathbf{1}_{x<y} \left(\frac{\delta_2}{\delta_1}(x) - \frac{\delta_3}{\delta_1}(x)\right) (f_3+\bar f_3)(x)|f_1-\bar f_1|(x)(f_2+\bar f_2)(y)\,dxdy \\
&+ \int_{\mathbb{R}^2} \mathbf{1}_{x<y} \left(\frac{\delta_2}{\delta_3}(x) - 1\right) \frac{f_1+\bar f_1}{2}(x)|f_3-\bar f_3|(x)(f_1+\bar f_1)(y)\,dxdy \\
&+ \int_{\mathbb{R}^2} \mathbf{1}_{x<y} \left(\frac{\delta_2}{\delta_3}(x) - 1\right) (f_1+\bar f_1)(x)|f_3-\bar f_3|(x)(f_2+\bar f_2)(y)\,dxdy \\
&+ \int_{\mathbb{R}^2} \mathbf{1}_{x<y} \left(\frac{\delta_2}{\delta_3}(y) - \frac{\delta_1}{\delta_3}(y)\right) \frac{f_1+\bar f_1}{2}(y)|f_3-\bar f_3|(y)(f_3+\bar f_3)(x)\,dxdy \\
&+ \int_{\mathbb{R}^2} \mathbf{1}_{x<y} \left(\frac{\delta_2}{\delta_3}(y) - \frac{\delta_1}{\delta_3}(y)\right) (f_1+\bar f_1)(y)|f_3-\bar f_3|(y)(f_2+\bar f_2)(x)\,dxdy \\
&\le -2\int_{\mathbb{R}} \sum_{m=1}^{3} |f_m-\bar f_m|(x) \left(\sum_{n\neq m}(f_n+\bar f_n)(x)\right) \\
&+ \left(||f_1+\bar f_1||_{L^1(\mathbb{R})} + 2||f_2+\bar f_2||_{L^1(\mathbb{R})}\right) \int_{\mathbb{R}} (f_3+\bar f_3)(x)|f_1-\bar f_1|(x)dx \\
&+ \left(2||f_2+\bar f_2||_{L^1(\mathbb{R})} + ||f_3+\bar f_3||_{L^1(\mathbb{R})}\right) \int_{\mathbb{R}} (f_1+\bar f_1)(x)|f_3-\bar f_3|(x)dx \\
&\le (-2+||f+\bar f||)\Lambda_d(f,\bar f)(t).
\end{aligned}
$$

Step 3. By the definition of $\mathcal{H}$, we have

$$
\begin{aligned}
\frac{d\mathcal{H}(t)}{dt} &= \frac{d\mathcal{L}(t)}{dt} + K\frac{d\mathcal{Q}_d(t)}{dt} \\
&\le -\Lambda_s(f,\bar f)(t) + [1+K(-2+||f+\bar f||)]\Lambda_d(f,\bar f)(t).
\end{aligned}
$$

Since $||f+\bar f|| < 2$, we can choose K sufficiently large so that

$$1+K(-2+||f+\bar f||) < 0.$$

We then have

$$\frac{d\mathcal{H}(t)}{dt} \le -C_1\Lambda(f,\bar f)(t),$$

where C_1 is a positive constant independent of time t. □

Proceeding as in the proof of Theorem 1.1, we have from Lemma 3.10 the following L^1 stability estimate.

Theorem 3.11. *Let f and $\bar f$ be two mild solutions of* (1.11) *subject to the hypotheses of Lemma 3.10. Then we have the uniform L^1 stability estimate*

$$\int_{\mathbb{R}} \big(|f_1-\bar f_1| + 4|f_2-\bar f_2| + |f_3-\bar f_3|\big)(x,t)dx \le C\int_{\mathbb{R}} \big(|f_1^0-\bar f_1^0| + 4|f_2^0-\bar f_2^0| + |f_3^0-\bar f_3^0|\big)(x)dx\,,$$

where C is a constant independent of time t.

3.3 Systems with transversal source terms

In this subsection, we consider the semilinear hyperbolic system

$$\partial_t f_i + \partial_x(v_i(x,t)f_i) = \sum_{j,k,j\neq k} B_i^{jk}(x,t) f_j f_k \qquad (3.38)$$
$$f_i(x,0) = f_{i0}(x),$$

$(x,t) \in \mathbb{R} \times \mathbb{R}_+$, $i = 1, \cdots, N$. We do not assume any conservation laws and any nonnegativity for the initial data but only the transversality of the source. We also admit variable wave speeds $v_i = v_i(x,t)$ and variable interaction coefficients $B_i^{jk} = B_i^{jk}(x,t)$. The assumptions (1.4)-(1.6) are replaced by

1. The interaction coefficients B_i^{jk} are bounded and only transversal terms enter the interaction:

$$\begin{aligned} |B_i^{jk}(x,t)| \leq B^*, \quad &\text{for some constant} B^*\,, \\ B_i^{kk} = 0\,, \quad &\text{for all } i,k\,. \end{aligned} \qquad (3.39)$$

2. The wave speeds are globally separated,

$$v_1(x,t) < v_2(x,t) < \cdots < v_N(x,t)\,,$$
$$\text{with} \quad v_* := \min_{i>j} \sup_{x\in\mathbb{R}\,,\,t>0} \big(v_i(x,t) - v_j(x,t)\big) > 0\,. \qquad (3.40)$$

Throughout, we suppress the t-dependence and write $f(x) \equiv f(x,t)$, $v_i(x) \equiv v_i(x,t)$ and so on. The equation for $|f_i|$ is

$$\partial_t |f_i| + \partial_x(v_i|f_i|) = \tilde{Q}_i(f) = Q_i(f)\mathrm{sgn} f_i = \sum_{j\neq k} B_i^{jk} f_j f_k \,\mathrm{sgn} f_i\,. \qquad (3.41)$$

Define the nonlinear functionals

$$L(f)(t) \equiv \sum_m \int_{\mathbb{R}} |f_m(x,t)|dx = ||f(\cdot,t)||_{L^1(\mathbb{R})}\,, \qquad (3.42)$$

$$\mathcal{Q}(f)(t) \equiv \sum_{m>n} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{x<y} |f_m(x,t)||f_n(y,t)|\, dxdy\,, \qquad (3.43)$$

$$F(f)(t) \equiv L(f)(t) + M\mathcal{Q}(f)(t), \qquad (3.44)$$

and the instanteneous interaction production

$$\Lambda(f)(t) = \sum_{m>n} \int_{\mathbb{R}} |f_m(x,t)||f_n(x,t)|dx\,. \qquad (3.45)$$

We study the evolution of these functionals for data f_0 with sufficiently small L^1-norm.

Proposition 3.12. *Let (3.38) satisfy the structural hypotheses* (3.39)-(3.40) *and* f *be a mild solution. Then*

(i) f *satisfies the inequalities*

$$\frac{dL(t)}{dt} \le b\Lambda(t)\,, \quad \frac{d\mathcal{Q}(t)}{dt} + v_*\Lambda(t) \le b\Lambda(t)L(t)$$

where b *is a (generic) constant depending on* B^* *and* N.

(ii) There exist constants M *(large) and* ϵ *(small) so that if* f_0 *satisfies*

$$L(0) = ||f_0||_{L^1(\mathbb{R})} < \epsilon, \quad \textit{and} \quad F(0) = L(0) + M\mathcal{Q}(0) < \epsilon\,, \tag{3.46}$$

then

$$\frac{d}{dt}\big(L + MQ\big)(t) \le -b\Lambda(t)\,, \quad L(t) \le \epsilon\,. \tag{3.47}$$

Proof. It follows from (3.41) that $L(t)$ satisfies

$$\frac{dL(t)}{dt} \le b\Lambda(t). \tag{3.48}$$

Consider now $\mathcal{Q}(t)$. The equations for $f_m(x)$ and $f_n(y)$ read

$$\partial_t|f_m|(x) + \partial_x(v_m|f_m|)(x) = \sum_{j,k} B_m^{jk}(f_jf_k)(x)\mathrm{sgn} f_m(x), \tag{3.49}$$

$$\partial_t|f_n|(y) + \partial_y(v_n|f_n|)(y) = \sum_{j,k} B_n^{jk}(f_jf_k)(y)\mathrm{sgn} f_n(y)\,. \tag{3.50}$$

If we multiply (3.49) by $|f_n(y)|$, (3.50) by $|f_m(x)|$, add the identities, and multiply the result by $\mathbb{1}_{x<y}$, we obtain

$$\begin{aligned}
\partial_t\Big(\mathbb{1}_{x<y}|f_m(x)||f_n(y)|\Big) &+ \mathrm{div}_{(x,y)}\Big(\big(v_m(x), v_n(y)\big)\,\mathbb{1}_{x<y}|f_m(x)||f_n(y)|\Big)\\
&+ \delta(y-x)\big(v_m(x) - v_n(y)\big)|f_m(x)||f_n(y)|\\
&\le b\mathbb{1}_{x<y}\left(|f_n(y)|\sum_{j>k}|f_j(x)||f_k(x)| + |f_m(x)|\sum_{j>k}|f_j(y)||f_k(y)|\right)
\end{aligned}$$

This identity is integrated over $\mathbb{R}^2$ and we add the contributions for the velocities with $m > n$. We then obtain

$$\frac{d\mathcal{Q}(t)}{dt} + v_*\Lambda(t) \le b\Lambda(t)L(t) \tag{3.51}$$

Note that (3.48) and (3.51) give

$$\frac{dF(t)}{dt} = \frac{d}{dt}\Big(L + M\mathcal{Q}\Big) \le \Lambda(t)\Big(b - M(v_* - bL(t))\Big)\,.$$

Select now $\epsilon < \frac{v_*}{2b}$ and $M > \frac{4b}{v_*}$ and let f_0 satisfy (3.46). For the solution f define

$$T := \sup_{t>0}\big\{L(s) < \epsilon \ \text{ for } s \in (0,t)\big\}$$

Clearly, $T > 0$. Moreover, for $t \in (0,T)$ we have $L(t) < \epsilon$ and $v_* - bL(t) > \frac{v_*}{2}$. This implies that F satisfies the differential inequality

$$\frac{dF(t)}{dt} \leq -b\Lambda(t)\,,$$

and that for $t \in [0,T]$

$$L(t) \leq \Big(L + M\mathcal{Q}\Big)(t) \leq \Big(L + M\mathcal{Q}\Big)(0) < \epsilon\,.$$

We conclude that $T = \infty$ and the inequalities follow. □

From the perspective of kinetic theory, (3.51) is interpreted as describing the evolution of a two-point distribution function. Analogous differential inequalities appear for the three-point or multi-point distribution functions. For a triplet of wave speeds $v_k > v_m > v_n$ define the triple interaction potential

$$T_3(t) := \sum_{k,m,n,\; k>m>n} \int_{\mathbb{R}^3} \mathbb{1}_{x<y}\mathbb{1}_{y<z}|f_k|(x,t)|f_m|(y,t)|f_n|(z,t)\,dxdydz\,. \tag{3.52}$$

More generally, for an n-tuple index $(k_1,\cdots,k_n)$ with $(k_1 > k_2 > \cdots > k_n)$, we define the multiple interaction potential $M^n(t)$ by

$$M_n(t) := \sum_{k_1>\ldots>k_n} \int_{\mathbb{R}^n} \Big(\prod_{i=1}^{n-1} \mathbb{1}_{x_i<x_{i+1}}\Big) \prod_{i=1}^{n} |f_{k_i}(x_i,t)|\,dx_1\ldots dx_n\,. \tag{3.53}$$

The evolution of T_3 and M_n obeys respectively

$$\begin{aligned} &\frac{d}{dt}T_3(t) + \Lambda_3(t) \leq b\Lambda(t)L^2(t) \\ &\frac{d}{dt}M_n(t) + \Lambda_n(t) \leq b\Lambda(t)L^{n-1}(t) \end{aligned} \tag{3.54}$$

where $\Lambda(t)$ and $L(t)$ as before, b dependes on B_* and N, while $\Lambda_n(t)$ is given by

$$\Lambda_n(t) := \sum_{k_1>\ldots>k_n} \sum_j \int_{\mathbb{R}^{n-1}} (v_{k_j}(x_j) - v_{k_{j+1}}(x_{j+1}))\Big(\prod_{i\neq j} \mathbb{1}_{x_i<x_{i+1}}\Big)\Big(f_{k_j}^2(x_j) \prod_{i,\; i\neq j,j+1} |f_{k_i}(x_i)|\Big) \prod_{i\neq j} dx_i \tag{3.55}$$

We outline the proof of the first inequality in (3.54). Using (3.41) at three distinct points x, y, z, one obtains

$$\begin{aligned} &\Big(\partial_t + \partial_x v_k(x) + \partial_y v_m(y) + \partial_z v_n(z)\Big)\Big(|f_k(x)||f_m(y)||f_n(z)|\Big) \\ &= \Big(\tilde{Q}_k(x)|f_m(y)f_n(z)| + \tilde{Q}_m(y)|f_k(x)f_n(z)| + \tilde{Q}_n(z)|f_k(x)f_m(y)|\Big) \end{aligned}$$

whence

$$\begin{aligned} &\Big(\partial_t + \partial_x v_k(x) + \partial_y v_m(y) + \partial_z v_n(z)\Big)\Big(\mathbb{1}_{x<y}\mathbb{1}_{y<z}|f_k(x)||f_m(y)||f_n(z)|\Big) \\ &+\Big((v_k(x) - v_m(y))\delta(x-y)\mathbb{1}_{y<z} + (v_m(y) - v_n(z))\delta(y-z)\mathbb{1}_{x<y}\Big)|f_k(x)||f_m(y)||f_n(z)| \end{aligned}$$

$$= \mathbb{1}_{x<y}\mathbb{1}_{y<z}\Big(\tilde{Q}_k(x)|f_m(y)f_n(z)| + \tilde{Q}_m(y)|f_k(x)f_n(z)| + \tilde{Q}_n(z)|f_k(x)f_m(y)|\Big)$$

We sum over the contributions of all velocities $v_k > v_m > v_n$ and integrate over $\mathbb{R}^3$ to arrive at

$$\begin{aligned}
&\frac{dT^3(t)}{dt} + \sum_{k>m>n}\int_{y<z}(v_k(y)-v_m(y))|f_k(y)f_m(y)||f_n(z)|dydz \\
&+ \sum_{k>m>n}\int_{x<y}(v_m(y)-v_n(y))|f_k(x)||f_m(y)f_n(y)|dxdy \\
&= \sum_{k>m>n}\int_{\{x<y\}\cap\{y<z\}}\tilde{Q}_k(x)|f_m(y)f_n(z)| + \tilde{Q}_m(y)|f_k(x)f_n(z)| + \tilde{Q}_n(z)|f_k(x)f_m(y)|\;dxdydz \\
&\le b\Lambda(t)(L(t))^2
\end{aligned}$$

The second inequality in (3.54) follows from a similar though lengthier computation.

Consider now f, $\bar{f}$ two mild solutions and define the nonlinear functionals:

$$\begin{aligned}
\mathcal{L}(t) &:= \sum_m \int_{\mathbb{R}} |f_m(x)-\bar{f}_m(x)|dx, \\
\mathcal{Q}_d(t) &:= \sum_{m>n}\int_{\{x<y\}} |f_m-\bar{f}_m|(x)\big(|f_n|+|\bar{f}_n|\big)(y) + \big(|f_m|+|\bar{f}_m|\big)(x)|f_n-\bar{f}_n|(y)dxdy, \\
\mathcal{H}(t) &= \mathcal{L}(t) + M_1\mathcal{Q}_d(t) + M_2(\mathcal{Q}(t)+\bar{\mathcal{Q}}(t)),
\end{aligned}$$

where $\mathcal{Q}(t)$, $\bar{\mathcal{Q}}(t)$ are the interaction potenials for f and $\bar{f}$ as in Proposition 3.12, and M_1, M_2 are constants to be selected later.

Proposition 3.13. *Let (3.38) satisfy* (3.39)-(3.40)*, and let f, $\bar{f}$ be two mild solutions emanating from data f_0, $\bar{f}_0$. Then*
(i) f, $\bar{f}$ satisfy the differential inequalities

$$\begin{aligned}
\frac{d\mathcal{L}(t)}{dt} &\le b\,\Lambda_d(f,\bar{f})(t), \\
\frac{d\mathcal{Q}_d(t)}{dt} + v_*\Lambda_d(f,\bar{f})(t) &\le b\Lambda_d(f,\bar{f})(t)\Big(L(f)(t)+L(\bar{f})(t)\Big) + b\mathcal{L}(t)\Big(\Lambda(f)(t)+\Lambda(\bar{f})(t)\Big),
\end{aligned} \tag{3.56}$$

where b depends on B_* and N, Λ is defined in (3.45)*, and Λ_d is given by*

$$\Lambda_d(f,\bar{f})(t) = \sum_{k,l,\,k\neq l}\int_{\mathbb{R}} |f_k(x)-\bar{f}_k(x)|(|f_l|(x)+|\bar{f}_l|(x))dx \tag{3.57}$$

(ii) There are choices of the parameters ϵ (small) and M_1, M_2 (large) such that if the data are selected to satisfy $||f_0||_{L^1(\mathbb{R})} + ||\bar{f}_0||_{L^1(\mathbb{R})} < \epsilon \ll 1$, then we have

$$\frac{d\mathcal{H}(t)}{dt} + \frac{v_*}{2}\Lambda_d(f,\bar{f})(t) + c\Big(\Lambda(f)(t)+\Lambda(\bar{f})(t)\Big) \le 0\,, \tag{3.58}$$

for some constant $c>0$.

Proof. Recall that

$$\partial_t|f_m - \bar{f}_m| + \partial_x\big(v_m(x)|f_m - \bar{f}_m|\big) = \sum_{k\neq l} B_m^{kl}\left[(f_k - \bar{f}_k)f_l + (f_l - \bar{f}_l)\bar{f}_k\right]\delta_m =: R_m \quad (3.59)$$

$$\partial_t|f_n| + \partial_y(v_n(y)|f_n|) = \sum_{k\neq l} B_n^{kl} f_k f_l \mathrm{sgn}(f_n) = \tilde{Q}_n(f)(y), \quad (3.60)$$

$$\partial_t|\bar{f}_n| + \partial_y(v_n(y)|\bar{f}_n|) = \sum_{k\neq l} B_n^{kl} \bar{f}_k \bar{f}_l \mathrm{sgn}(\bar{f}_n) = \tilde{Q}_n(\bar{f})(y). \quad (3.61)$$

We have

$$\frac{d\mathcal{L}(t)}{dt} = \sum_m \int_{\mathbb{R}} R_m(x,t)dx \leq b\bar{\Lambda}(f,\bar{f})(t)\,. \quad (3.62)$$

Next, consider $\frac{d\mathcal{Q}_d(t)}{dt}$. From (3.59), (3.60) and (3.61), we obtain the identity

$$\begin{aligned}
&\Big(\partial_t + \partial_x v_m(x) + \partial_y v_n(y)\Big)\Big(\mathbb{1}_{x<y}|f_m - \bar{f}_m|(x)(|f_n| + |\bar{f}_n|)(y)\Big) \\
&\quad + (v_m(x) - v_n(y))\delta(y-x)|f_m - \bar{f}_m|(x)(|f_n| + |\bar{f}_n|)(x) \\
&= \mathbb{1}_{x<y}\Big(R_m(x)(|f_n| + |\bar{f}_n|)(y) + |f_m - \bar{f}_m|(x)\big(\tilde{Q}_n(f) + \tilde{Q}_n(\bar{f})\big)(y)\Big)
\end{aligned}$$

and from here

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} \mathbb{1}_{x<y}\sum_{m>n}\Big(|f_m - \bar{f}_m|(x)(|f_n| + |\bar{f}_n|)(y) + (|f_m| + |\bar{f}_m|)(x)|f_n - \bar{f}_n|(y)\Big)\,dxdy \\
&+ \int_{\mathbb{R}}\sum_{m>n}(v_m(x) - v_n(x))\Big(|f_m - \bar{f}_m|(x)(|f_n| + |\bar{f}_n|)(x) + (|f_m| + |\bar{f}_m|)(x)|f_n - \bar{f}_n|(x)\Big)dx \\
&= \sum_{m>n}\int_{\mathbb{R}^2}\mathbb{1}_{x<y}\Big(R_m(x)(|f_n| + |\bar{f}_n|)(y) + |f_m - \bar{f}_m|(x)\big(\tilde{Q}_n(f) + \tilde{Q}_n(\bar{f})\big)(y) \\
&\qquad + \big(\tilde{Q}_m(f) + \tilde{Q}_m(\bar{f})\big)(x)|f_n - \bar{f}_n|(y) + (|f_m| + |\bar{f}_m|)(x)R_n(y)\Big)dxdy \\
&\leq b\Lambda_d(f,\bar{f})(t)\big(L(t) + \bar{L}(t)\big) + b\mathcal{L}(t)\big(\Lambda(t) + \bar{\Lambda}(t)\big)
\end{aligned}$$

where we used the notation $\bar{L}(t) = L(\bar{f})(t)$, $\bar{\mathcal{Q}}(t) = \mathcal{Q}(\bar{f})(t)$ and $\bar{\Lambda}(t) = \Lambda(\bar{f})(t)$.

Hence, $\mathcal{Q}_d$ obeys the differential inequality

$$\frac{d\mathcal{Q}_d}{dt} + v_*\Lambda_d \leq b\Lambda_d(L + \bar{L}) + b\mathcal{L}(\Lambda + \bar{\Lambda})\,. \quad (3.63)$$

Combining with (3.62), we obtain

$$\frac{d}{dt}\Big(\mathcal{L} + M_1\mathcal{Q}_d\Big) + M_1\big(v_* - b(L + \bar{L})\big)\Lambda_d \leq b\Lambda_d + M_1 b\mathcal{L}(\Lambda + \bar{\Lambda})\,. \quad (3.64)$$

Proposition 3.12 yields

$$\frac{d}{dt}\Big(\mathcal{Q} + \bar{\mathcal{Q}}\Big) + v_*(\Lambda + \bar{\Lambda}) \leq b(\Lambda L + \bar{\Lambda}\bar{L}) \quad (3.65)$$

and that there exists a threshold ϵ_0 such that for $\epsilon < \epsilon_0$ and for $\|f_0\|_{L^1}$, $\|\bar{f}_0\|_{L^1}$ sufficiently small we have $(L + M\mathcal{Q})(t)$ and $(\bar{L} + M\bar{\mathcal{Q}})(t)$ are decreasing in time and $L(t)$, $\bar{L}(t) < \epsilon$. From (3.64) and (3.65) we deduce

$$\begin{aligned}\frac{d}{dt}\Big(\mathcal{L} + M_1\mathcal{Q}_d + M_2(\mathcal{Q} + \bar{\mathcal{Q}})\Big) + M_1\big(v_* - b(L+\bar{L})\big)\Lambda_d + M_2\big(v_* - b(L+\bar{L})\big)(\Lambda + \bar{\Lambda}) \\ \leq b\Lambda_d(L+\bar{L}) + M_1 b\mathcal{L}(\Lambda + \bar{\Lambda})\,.\end{aligned} \tag{3.66}$$

Since $L + \bar{L} < 2\epsilon$, by selecting ϵ even smaller (if necessary) and M_1, M_2 sufficiently large we have

$$\frac{d}{dt}\Big(\mathcal{L} + M_1\mathcal{Q}_d + M_2(\mathcal{Q} + \bar{\mathcal{Q}})\Big) + \frac{v_*}{2}\Lambda_d + c(\Lambda + \bar{\Lambda}) \leq 0$$

for some constant $c > 0$. □

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