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REMARKS ON THE CONTRIBUTIONS OF CONSTANTINE M. DAFERMOS TO THE SUBJECT OF CONSERVATION LAWS*

Dedicated to Professor Constantine M. Dafermos on the occasion of his 70th birthday

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Constantine M. Dafermos has done extensive research at the interface of partial differential equations and continuum physics. He is a world leader in nonlinear hyperbolic conservation laws, where he introduced several fundamental methods in the subject including the methods of relative entropy, generalized characteristics, and wave-front tracking, as well as the entropy rate criterion for the selection of admissible wave fans. He has also made fundamental contributions on the mathematical theory of the equations of thermomechanics as it pertains in modeling and analysis of materials with memory, thermoelasticity, and thermoviscoelasticity. His work is distinctly characterized by an understanding of the fundamental issues of continuum physics and their role in developing new techniques of mathematical analysis.

1 Biographic Remarks

Constantine M. Dafermos was born in Athens in 1941. He received a Diploma in Civil Engineering from the National Technical University of Athens in 1964 and a Ph.D. in Mechanics from the Johns Hopkins University in 1967. He was a postdoctoral fellow in the Department of Mechanics of Johns Hopkins in 1967–68 and was appointed as an Assistant Professor in the Department of Theoretical and Applied Mechanics of Cornell University from 1968–71. In 1971, he was appointed as an Associate Professor in the Division of Applied Mathematics at Brown University, where he spent his entire career, first as an Associate Professor (1971–76),

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then as a Professor (1976–87), and finally as the Alumni-Alumnae University Professor (1987–Present). To his students, postdocs, and friends, his name has become synonymous with the Division of Applied Mathematics of Brown University. He has served in the Editorial Boards of 14 professional journals and has held Honorary Doctorates from several institutions. He was elected a Correspondent Member of the Academy of Athens (1988–) and an Honorary Professor of Academia Sinica (2004–). He was awarded the SIAM W.T. and Idalia Reid Prize in 2000 for his broad contribution in the area of differential equations and control theory, and the Cataldo e Angiola Agostinelli Prize from the Accademia dei Lincei in 2011 for his work on the foundations of Mechanics and on rigorous methods of Mathematical Analysis. He was elected to be a Fellow of the American Academy of Arts and Sciences in 2001 and a Foreign Member of the Accademia Nazionale dei Lincei in 2011.

This article aims at presenting some contributions of Professor Dafermos to the subject of conservation laws and the equations of thermomechanics. It is not intended as a comprehensive review of his work, and the choice of topics naturally reflects the views of the authors. We have chosen to neglect a lot of the technically impressive work and focused on bringing up some of the core ideas that have staying power. Our hope is that the selected topics will pinpoint the fundamental link between Mechanics and Analysis that is so characteristic of Constantine's work.

2 Hyperbolic Conservation Laws

Hyperbolic systems of conservation laws are first-order quasilinear systems in d-space dimensions of the form

$$\partial_t U + \sum_\alpha \partial_\alpha F_\alpha(U) = 0\,, \tag{1}$$

describing the evolution of $U : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^n$, where the state vector U takes values in $\mathbb{R}^n$ and the fluxes $F_\alpha : \mathbb{R}^n \to \mathbb{R}^n$, $\alpha = 1, \cdots, d$, are smooth maps. Hyperbolicity means that, for all $U \in \mathbb{R}^n$ and $\nu \in \mathcal{S}^{d-1}$, the $n \times n$ matrix

$$\Lambda(U, \nu) := \sum_\alpha \nu_\alpha \nabla F_\alpha(U) \tag{2}$$

has n real eigenvalues $\lambda_1(U, \nu) \le \cdots \le \lambda_n(U, \nu)$ and complete sets of right $\big\{r_i(U, \nu)\big\}_{i=1,\cdots,n}$ and left $\big\{l_i(U, \nu)\big\}_{i=1,\cdots,n}$ eigenvectors.

Several systems of continuum physics share this basic structure with prime examples: the equations of compressible gas dynamics and the equations of elasticity. Accordingly, the mathematical study of (1) has been developed in intimate linkage with the subject of mechanics, and has influenced the development of efficient computational schemes. Due to the nonlinear nature of the wave speeds, weak waves can become steeper and break down in finite time, with shock waves emerging as a result of compressive effects. The class of classical solutions is thus inadequate for studying well-posedness beyond the time of shock wave formation, and global solutions have to be understood in a weak sense.

To secure uniqueness within a class of weak solutions, admissibility criteria are adopted with the aim to disqualify spurious solutions. The second law of thermodynamics suggests the so called *entropy admissibility criterion*, namely requiring solutions to satisfy the entropy inequalities of the form:

$$\partial_t \eta(U) + \sum_\alpha \partial_\alpha \, q_\alpha(U) \leq 0 \,, \tag{3}$$

where the entropy $\eta : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function, i.e., $\nabla^2\eta(U) \geq 0$, and $q = (q_1, \cdots, q_d) : \mathbb{R}^n \to \mathbb{R}^d$ is the associated entropy flux. Any Lipschitz solution of (1) is postulated to be admissible and to satisfy (3) as an equality. This is consistent with the mechanical idea that smooth processes must be time-reversible, and that (3) imposes restrictions only on non-smooth processes which are deemed irreversible. This is the case if and only if the entropy and entropy flux functions satisfy the equations

$$\nabla q_\alpha = \nabla \eta \cdot \nabla F_\alpha \,, \qquad \alpha = 1, \cdots, d \,, \tag{4}$$

or, in turn, if η satisfies the compatibility restrictions

$$\nabla^2 \eta \, \nabla F_\alpha = (\nabla F_\alpha)^\top \, \nabla^2 \eta \,, \qquad \alpha = 1, \cdots, d \,. \tag{5}$$

These are $\frac{n(n-1)}{2}\, d$ equations and thus, except for the cases $n = 2$, $d = 1$ and $n = 1$ with any d, system (5) is overdetermined and the existence of entropies is the exception rather than the rule. However, systems of thermomechanics are naturally equipped with (usually) one entropy and thus fit under this framework. For the systems of isentropic gas dynamics and for the equations of elasticity, equation (3) expresses the dissipation of the mechanical energy on shocks.

A natural function class for seeking weak solutions is the class of functions of bounded variation (BV, for short), whose distributional derivatives are locally finite Borel measures. This constitutes a very general class of functions where the Gauss-Green theorem holds. At the level of models in continuum physics, the equations in (1) naturally arise in their integral form

$$\int_\Omega U(x,t)\, \mathrm{d}x - \int_\Omega U(x,\tau) \mathrm{d}x + \int_\tau^t \int_{\partial\Omega} \sum_\alpha \nu_\alpha F_\alpha(x,s) \mathrm{d}\sigma \mathrm{d}s = 0 \,, \;\; \forall \Omega \,, \; \forall \tau < t, \tag{6}$$

expressing the evolution of the vector of conserved quantities U on any control volume Ω through the action of the boundary fluxes F_α (and possibly production terms that are neglected here). The class of BV functions gives meaning to the equivalence of expressions (6) and (1), and is a natural class for considering the evolution of shock fronts. The reader is referred to [17, Ch. I] for a presentation of the mathematical underpinnings of continuum physics and the regularity classes where the formal calculations used in deriving the equations of thermomechanics acquire mathematical precision.

3 The Second Law of Thermodynamics and Stability

In the 1960s–70s, there was a concerted effort to develop a rational theory of Continuum Mechanics and to systematize the models of thermomechanics using deductive reasoning and starting from general mechanical principles, such as the second law of thermodynamics (in the form of the Clausius-Duhem inequality), the principle of frame indifference, and the effect of material symmetries, [38]. It was noted that, in certain cases, the Clausius-Duhem inequality induces Lyapunov stability of equilibrium processes in a variety of materials. By contrast, in other situations, still consistent with the second law of thermodynamics, instabilities are observed, as for instance in phase transitions.

An efficient mathematical object for analyzing stability among thermomechanical theories and/or stability in the dynamical sense is what is presently called relative entropy, introduced in the works of Dafermos [13] and DiPerna [22]. This may be most easily explained in the framework of (1). Let $U(x,t)$ be a weak entropy solution in BV satisfying (1) and (3) in the sense of distributions, and let $\bar{U}(x,t)$ be a Lipschitz solution of (1) satisfying (3) as an equality. In order to compare the distance among two solutions to that of the initial and other data of the problem, the quantities

$$\text{relative entropy:} \qquad \eta(U|\bar{U}) = \eta(U) - \eta(\bar{U}) - \nabla\eta(\bar{U})\cdot(U-\bar{U}), \tag{7}$$

$$\text{relative entropy flux:} \qquad q_\alpha(U|\bar{U}) = q_\alpha(U) - q_\alpha(\bar{U}) - \nabla\eta(\bar{U})\cdot\big(F_\alpha(U) - F_\alpha(\bar{U})\big) \tag{8}$$

are introduced. When η is strictly convex, the relative entropy provides some notion of the distance between the two solutions. A computation using (1), (3) and (5) gives

$$\partial_t\eta(U|\bar{U}) + \mathrm{div}q(U|\bar{U}) \leq -\sum_\alpha \nabla^2\eta(\bar{U})\partial_\alpha\bar{U}\cdot\left[F_\alpha(U) - F_\alpha(\bar{U}) - \nabla F_\alpha(\bar{U})(U-\bar{U})\right]. \tag{9}$$

Inequality (9) serves as a starting point in order to compare the distance between the two solutions. A typical result reads:

Theorem 1 Let U be an entropy weak solution, and let $\bar{U}$ be a Lipschitz solution of (1) defined for $t\in[0,T)$. Suppose that both solutions lie in a convex, compact set $\mathcal{D}$ in the state space. If system (1) is endowed with a strictly convex entropy η, then the following local stability estimate holds:

$$\int_{|x|<r} |U(x,t)-\bar{U}(x,t)|^2\mathrm{d}x \leq a\mathrm{e}^{bt}\int_{|x|<r+kt} |U_0(x)-\bar{U}_0(x)|^2\mathrm{d}x$$

for any $r>0$ and $t\in[0,T)$, with constants a and k depending on $\mathcal{D}$ and b also depending on the Lipschitz norm of $\bar{U}$.

The article of Dafermos [13] articulates the intimate relation of the method with the second law of thermodynamics. It is in fact stated at the level of the thermomechanical theory of thermoelastic nonconductors of heat and the mechanical ramifications of the methodology and connections of stability and various thermodynamic quantities are pointed out. The article of DiPerna [22] studies the relative dissipation measure $\partial_t\eta(U|\bar{U}) + \mathrm{div}q(U|\bar{U})$ between two weak solutions with regard to the question of uniqueness. The relative entropy provides a natural tool for studying limits from one thermomechanical theory to another at least in the realm of smooth processes. It has been applied in various directions, e.g. for asymptotic stability problems in conservation laws [6, 7], for relaxation or kinetic limits [1, 40], or for comparing entropic measure-valued solutions and strong solutions for conservation laws [3].

The preceding work rests on convexity of the entropy. This assumption is not natural in several systems of thermomechanics. For example, for the equations of elasticity, the stored energy needs to be invariant under rotations and the mechanical energy cannot be convex. Hyperbolicity of the elasticity system stipulates that the stored energy be rank-one convex. This degeneracy is associated to the existence of multiple zero characteristic speeds and also appears for the system of magnetohydrodynamics.

The presence of zero characteristic speeds of high-multiplicity is associated to constraints that get propagated by the evolution, the so called involutions. Dafermos [14] proposed a theory for hyperbolic conservation laws with involutions suitable for handling such situations. Assume that the fluxes in (1) satisfy the equations:

$$A_\alpha F_\beta(U) + A_\beta F_\alpha(U) = 0 \tag{10}$$

for given matrices $A_\alpha \in \mathbb{M}^{k\times N}$, $\alpha = 1, \cdots, d$. That implies the constraint

$$\sum_\beta A_\beta \partial_\beta U = 0 \tag{11}$$

propagates from the initial data to the solution. Such constraints are called involutions. In applications, equation (11) stands for the constraint of being a gradient in the context of elasticity, while it is the divergence free conditions for the magnetic field and the electric field (in vacuum) in the context of Maxwell's equations.

It can be checked that (11) has the following implications:

(a) For shock solutions, the amplitudes of the shocks are restricted to satisfy

$$\sum_\beta \nu_\beta A_\beta [U] = N(\nu)[U] = 0$$

that is the jumps lie at the kernel of the matrix

$$N(n) := \sum_\beta \nu_\beta A_\beta .$$

(b) It can be directly checked that

$$N(\nu) \sum_\alpha \nu_\alpha \nabla F_\alpha(U) = N(\nu)\Lambda(U,\nu) = 0.$$

This implies in particular that any characteristic eigenvector $r_j(U,\nu)$ associated to a nonzero eigenvalue lies in the kernel of $N(\nu)$.

Dafermos introduced the cone of amplitudes

$$\mathcal{K} = \left\{ V \in R^N : N(\nu)V = 0 \ \text{ for some } \nu \in \mathcal{S}^{d-1} \right\} \tag{12}$$

and proved the following theorem.

Theorem 2 Consider system (1) and (10) with the initial data satisfying the constraint equation (11). For U and $\bar{U}$ as in Theorem 1, if the entropy η is strictly convex on the directions $V \in \mathcal{K}$, the weak and strong solution satisfy the stability estimate:

$$\int_{\mathbb{R}^d} |U(x,t) - \bar{U}(x,t)|^2 \,\mathrm{d}x \le a e^{bt} \int_{\mathbb{R}^d} |U_0(x) - \bar{U}_0(x)|^2 \,\mathrm{d}x \quad \text{ for } \ t \in [0,T).$$

For the equations of elasticity, the assumption of strict convexity of the entropy in the directions of the cone of amplitudes is equivalent to rank-one convexity of the elastic stored energy. Theorem 2 thus provides a weak-strong stability framework for smooth solutions of the elasticity system. The structure of the equations of thermomechanics has been a central interest of Dafermos and the interested reader may consult his book [17, Ch. V] for further information on the topic and for an account of the theory of conservation laws with contingent entropy pairs.

4 Wave Fan Admissibility Criteria

The Riemann problem for hyperbolic systems of conservation laws in one space dimension,

$$\partial_t U + \partial_x F(U) = 0\,, \tag{13}$$

$$U(x,0) = \begin{cases} U_l, & x<0, \\ U_r, & x>0, \end{cases} \tag{14}$$

plays an important role in the theory of conservation laws. It is the building block for applying the Glimm scheme [24], and it describes both the local and asymptotic behavior of general BV solutions.

Solutions of (13)–(14) are sought as functions $U = U\left(\frac{x}{t}\right)$ of the self-similar variable $\xi = \frac{x}{t}$ and generated by solving the boundary value problem:

$$-\xi U'(\xi) + F(U(\xi))' = 0\,, \tag{15}$$

$$U(-\infty) = U_l\,,\ U(\infty) = U_r\,. \tag{16}$$

For a solution $U(\xi)$ in BV of (15), the regularity of the function U of bounded variation and the Volpert chain rule imply that the domain $(-\infty,\infty)$ can be decomposed into the union of three disjoint sets $\mathcal{S}$, $\mathcal{C}$, and $\mathcal{W}$:

(i) $\mathcal{C}$ is a maximal open set in $\mathbb{R}$ where the measure U' vanishes; on each connected component of $\mathcal{C}$, U stays constant;

(ii) $\mathcal{S}$ is the (at most countable) set of points of jump discontinuity and, on each point $\xi \in \mathcal{S}$, U satisfies the Rankine-Hugoniot jump conditions

$$F(U(\xi+)) - F(U(\xi-)) = \xi[U(\xi+) - U(\xi-)]\,;$$

(iii) $\mathcal{W}$ is the set of points of continuity of U that lie in the support of the measure U'. On each point of $\mathcal{W}$, $U(\xi)$ coincides with a rarefaction wave.

The traditional approach for constructing the Riemann problem solutions proceeds by constructing the wave curves associated to each characteristic family, employing admissibility criteria to select the admissible shocks. Effective admissibility criteria for the selection of admissible shocks were proposed by Lax [31] for genuinely nonlinear wave speeds and by Liu [33, 34] when the wave speeds lose genuine nonlinearity in finitely many points, and serve to provide a unique solution of (13) for waves of moderate strength.

Dafermos pursued the idea of testing the admissibility of solutions at the level of the entire wave fan. He proposed two specific criteria of that character, the viscous wave fan admissibility criterion in [11] and the entropy rate admissibility criterion in [10], and tested their results against the results of the Lax, Liu, and entropy criteria.

According to the viscous wave fan admissibility criterion, admissible solutions emerge as $\varepsilon \to 0$ limits of the elliptic regularization to the Riemann problem (15):

$$\begin{aligned} &-\xi U_\varepsilon' + F(U_\varepsilon)' = \varepsilon U_\varepsilon''\,, \qquad \xi \in (-\infty,\infty)\,, \\ &U(-\infty) = U_l\,,\ \ U(\infty) = U_r\,. \end{aligned} \tag{17}$$

The analytical task when applying this methodology lies in obtaining uniform BV-bounds for solutions of (17) and analyzing the structure of the limiting solution. By its very nature, the limiting process captures the entire admissible wave fan simultaneously. This approach was first tested for hyperbolic systems of two conservation laws with large Riemann data in [11, 19], for non-strictly hyperbolic systems in [30], and for hyperbolic-elliptic systems exhibiting change of phase in [36]. This method produced the first complete existence theory for the Riemann problem (for moderate waves) requiring only strict hyperbolicity but with no further assumptions on the wave speeds (cf. [39]). We refer the reader to [17, Sec 9.8] for the construction of the wave fan curves as limits of viscous wave fans, and to [2] for the corresponding construction via viscous limits. Composite shock and boundary layer waves that appear in boundary Riemann problems are also captured via viscous wave fans (cf. [29] and [8]).

We next discuss the implications of applying the entropy admissibility criterion (3) with a designated entropy pair $(\eta(U), q(U))$ on the selection of admissible wave fans. The entropy $\eta(U)$ can always be normalized so that $\eta(U_l) = \eta(U_r) = 0$. Let U be a BV solution of the Riemann problem (15)–(16). The entropy admissibility takes the form

$$-\xi\eta(U)' + q(U)' = -\mu \leq 0\,, \tag{18}$$

Using (4) and the Volpert chain rule, one can check that the dissipation measure μ is supported on the set of points of jump discontinuity $\mathcal{S}$ and that, for each $\xi \in \mathcal{S}$,

$$-\xi\big[\eta(U(\xi+) - \eta(U(\xi-))\big] + \big[q(U(\xi+) - q(U(\xi-))\big] \leq 0\,.$$

In addition, from (18), one obtains

$$\int_{-\infty}^{\infty} \eta\big(U(\xi)\big)\mathrm{d}\xi + q(U_r) - q(U_l) = \mathcal{P}_U\,, \tag{19}$$

where

$$\mathcal{P}_U := \sum_{\xi\in\mathcal{S}} \big[q(U(\xi+) - q(U(\xi-))\big] - \xi\big[\eta(U(\xi+) - \eta(U(\xi-))\big] \tag{20}$$

is the total entropy dissipation of the wave fan associated to U.

The entropy rate admissibility criterion [10] stipulates that the wave fan $U\big(\frac{x}{t}\big)$ is admissible if it achieves maximal rate of dissipation relative to all wave fans $V\big(\frac{x}{t}\big)$ with the same end-states $U(\pm\infty) = V(\pm\infty)$, that is,

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\int_{\mathbb{R}} \eta\big(U(\frac{x}{t})\big)\mathrm{d}x \leq \frac{\mathrm{d}^+}{\mathrm{d}t}\int_{\mathbb{R}} \eta\big(V(\frac{x}{t})\big)\mathrm{d}x \qquad \text{at } t=0.$$

Using

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\int_{\mathbb{R}} \eta\big(U(\frac{x}{t})\big)\mathrm{d}x\Big|_{t=0} = \int_{-\infty}^{\infty} \eta(U(\xi))\mathrm{d}\xi$$

and (19), the entropy rate criterion may be equivalently expressed as minimizing the total entropy dissipation $\mathcal{P}_V$ defined in (20) over all wave fans $V(\xi)$ with $V(\pm\infty) = U(\pm\infty)$.

Obviously, the criterion is more stringent than the second law of thermodynamics, as it requires that entropy dissipates at the maximum rate that is consistent with the conservation laws (15). It is well known that, for scalar (but nonconvex) conservation laws, the entropy criterion should be stipulated for all convex entropies in order to guarantee that shocks satisfy the Oleinik-E condition. By contrast, stipulating the entropy rate criterion for the entropy $\eta(u) = \frac{1}{2}u^2$ is equivalent to demanding the Oleinik E-condition at shocks [10]. For hyperbolic systems of conservation laws, wave fans of moderate strength satisfying the entropy rate criterion consist of shocks that satisfy the Liu shock conditions, [16]. The criterion has been discussed for adiabatic (nonisentropic) gas dynamics [28] and for problems of phase transitions [25, 26]. Apart from the fascinating issues raised with regard to the second law of thermodynamics, the entropy rate criterion offers an intriguing connection between the hyperbolic theory and the calculus of variations. Indeed, Dafermos [18] employed this procedure to obtain an alternative method for constructing the wave fans of the Riemann problem.

5 Generalized Characteristics

For smooth solutions to nonlinear hyperbolic equations, characteristics provide one of the principal tools of the classical theory for the study of analytical and geometric properties of solutions, since characteristics are carriers of waves of various types. However, as described earlier, solutions in general become discontinuous no matter how smooth the initial data functions are. Developing a theory of generalized characteristics as an efficient tool for the study of properties of discontinuous solutions is another fundamental contribution Dafermos has made; see [12, 15, 17] and the references cited therein.

Consider the one-dimensional strictly hyperbolic system

$$\partial_t U + \partial_x F(U) = 0, \qquad U \in \mathbb{R}^n. \tag{21}$$

Let $\lambda_1(U) < \lambda_2(U) < \cdots < \lambda_n(U)$ be the distinct eigenvalues.

In the classical theory, an i-characteristic, $i = 1, 2, \cdots, n$, of (21), associated with a classical solution $U(x,t)$, is a C^1-function $x = \xi(t)$ that is an integral curve of the ordinary differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_i(U(x,t)). \tag{22}$$

The existence-uniqueness theory for ordinary differential equations (22) implies that, for any fixed point $(\bar{x}, \bar{t})$ in the domain of a classical solution of (21), there exists a unique characteristic $x = \xi(t)$ of each characteristic family passing through $(\bar{x}, \bar{t})$.

For a discontinuous weak solution $U(x,t)$ in L^∞ of (21), the generalized i-characteristics of (21), associated with the solution $U(x,t)$, are defined in analogy to the classical case, as integral curves of (22) in the sense of Filippov [23]. More precisely, a generalized i-characteristic of (21), associated with the solution $U(x,t)$, on the time interval $[t_1, t_2] \subset [0, \infty)$, is a Lipschitz function $\xi : [t_1, t_2] \to (-\infty, \infty)$ which satisfies the differential inclusion:

$$\dot{\xi}(t) \in \Lambda_i(\xi(t), t), \qquad \text{a.e. on } [t_1, t_2], \tag{23}$$

where

$$\Lambda_i(\bar{x}, \bar{t}) := \bigcap_{\varepsilon > 0} \Big[\operatorname*{essinf}_{[\bar{x}-\varepsilon, \bar{x}+\varepsilon]} \lambda_i(U(x,\bar{t})), \ \operatorname*{esssup}_{[\bar{x}-\varepsilon, \bar{x}+\varepsilon]} \lambda_i(U(x,\bar{t})) \Big]. \tag{24}$$

The standard properties of solutions of differential inclusions immediately imply that, through any fixed point $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times [0, \infty)$, there are two generalized i-characteristics, associated with U and defined on $[0, \infty)$, the minimal $\xi_-(\cdot)$ and the maximal $\xi_+(\cdot)$, with

$$\xi_-(\cdot) \leq \xi_+(\cdot) \quad \text{for} \quad t \in [0, \infty).$$

The funnel-shaped region confined between the graphs of $\xi_-(\cdot)$ and $\xi_+(\cdot)$ comprises the set of points (x, t) that may be connected to $(\bar{x}, \bar{t})$ by a generalized i-characteristic associated with U.

In addition to classical i-characteristics, i-shocks that satisfy the Lax entropy condition are obvious examples of generalized i-characteristics. In fact, as shown by Dafermos [15], these are the only possibilities when the weak solution U is in BV.

Furthermore, assume that the i-characteristic family for (21) is genuinely nonlinear and the oscillation of $U \in BV$ is sufficiently small, Dafermos in [15] proved that, the minimal and the maximal backward i-characteristics, emanating from any point $(\bar{x}, \bar{t})$ of the upper half-plane, are shock free.

The method of generalized characteristics can be used to establish regularity and asymptotic behavior properties of BV solutions of genuinely nonlinear 2×2 systems of conservation laws, see [20, 37] and Chapter 12 in [17].

The detailed theory of generalized characteristics and its further applications to the study of analytical and geometric properties of the solutions in BV can be found in Dafermos [12, 15, 17] and the references cited therein.

6 Wave Front Tracking

It is well-known that, when the initial data function is a step function, a local solution of the Cauchy problem of one-dimensional hyperbolic conservation laws can be constructed as a superposition of solutions of the Riemann problem. In general, the solution of the Riemann problem may consist of constant states separated by shocks, contact discontinuities, and/or rarefaction waves. However, for the scalar case, if the scalar flux function is piecewise linear, then the constant states of the solution of the Riemann problem are separated exclusively by shocks and/or contact discontinuities. Based on this observation, Dafermos in [9] developed an analytical method, the method of wave front tracking, to construct entropy solutions for scalar conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \qquad u|_{t=0} = u_0 \tag{25}$$

with initial data u_0 of locally bounded variation.

Roughly speaking, front tracking consists in making a step function approximation to the initial data and a piecewise linear approximation to the flux function. The approximate initial data function defines a series of Riemann problems, one at each step, which can be easily solved. Since the solutions have finite speed of propagation, they are independent of each other until waves from neighboring cells interact. Front tracking then resolves this interaction in order to propagate the solution to larger times for the continuous and piecewise linear flux functions. More precisely, the method of wave front tracking developed in Dafermos [9] consists of the following four steps:

(i) Approximate f by a continuous, piecewise linear flux function f^ε;

(ii) Approximate initial data u_0 by a piecewise constant function u_0^δ;

(iii) Solve the initial value problem

$$\partial_t u + \partial_x f^\varepsilon(u) = 0, \qquad u|_{t=0} = u_0^\delta \tag{26}$$

exactly to obtain the solution $u^{\varepsilon,\delta}$;

(iv) As f^ε and u_0^δ approach f and u_0, respectively, the approximate solutions $u^{\varepsilon,\delta}$ converge to u, the entropy solution of the Cauchy problem (25), almost everywhere as $\varepsilon, \delta \to 0$.

Since u_0^δ is a piecewise constant function with a finite number of discontinuities, the solution initially consists of a number of non-interacting solutions of Riemann problems. Each solution is a piecewise constant function with discontinuities traveling at constant speed. Hence, at some later time $t_1 > 0$, two discontinuities from the neighboring Riemann problems interact.

For $t \geq t_1$, the solution can be constructed by solving the Cauchy problem for the same equation with initial data $u^{\varepsilon,\delta}(x, t_1)$ that of the same type of function as $u_0^\delta(x)$. This can be achieved by solving the Riemann problems at the discontinuities of $u^{\varepsilon,\delta}(x, t_1)$ as we did initially, so that the solution can be extended up to the next interaction at t_2. It is clear that we can continue this process for any number of interactions occurring at times t_n with

$$0 < t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_n \leq \cdots.$$

However, we cannot a priori be sure that $\lim t_n = \infty$, or in other words, that we can extend the solution up to any predetermined time. One would envisage that the number of discontinuities might grow for each interaction and that their number increases without bound at some finite time. Fortunately, this does not happen for the scalar case with convex/concave flux function, which guarantees the existence of a global approximate solution $u^{\varepsilon,\delta}(x, t)$ satisfying the entropy condition for (25) for fixed $\varepsilon > 0$ and $\delta > 0$. The function $u^{\varepsilon,\delta}(x, t)$ is a piecewise constant function of x for each t and takes values in a finite set. Moreover, there are only a finite number of interactions between the discontinuities (i.e., wave fronts) of $u^{\varepsilon,\delta}$. Furthermore, the total variations of the approximate solutions $u^{\varepsilon,\delta}(x, t)$ are uniformly bounded with respect to $\varepsilon, \delta > 0$, which yields (iv).

The analytical idea of front-tracking introduced in Dafermos [9] has been further developed for scalar conservation laws and applied for various numerical methods; see Holden-Risebro [27].

The method of wave front tracking was extended to genuinely nonlinear systems of two conservation laws by DiPerna [21] and then to genuinely nonlinear systems of any size, independently, by Bressan [4] and Risebro [35]. The method of wave front tracking has been also further developed to construct the standard Riemann semigroups, which has played an important role in establishing the uniqueness and L^1-stability of entropy solutions in BV for hyperbolic conservation laws. For more details, see Bressan [5] and Dafermos [17].

7 A Monumental Book

The book "Hyperbolic Conservation Laws in Continuum Physics" [17] is a comprehensive treatise which elucidates Dafermos' view of the linkage between Continuum Mechanics and Conservation Laws. In its three editions, the book has kept up to date with the most recent developments on the subject of Conservation Laws and is a definitive work that will be referred to by future generations of scientists.

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