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EQUATIONS AUX DERIVEES PARTIELLES. Théorie d'existence pour le problème de Riemann pour les systèmes hyperboliques non-conservatifs, par Philippe G. LeFloch et Athanasios E. Tzavaras.

Résumé. Dans cette Note, nous nous s'intéressons aux systèmes hyperboliques nonlinéaires et non-conservatifs dans le cadre introduit par Dal Maso-LeFloch-Murat. Nous donnons un théorème d'existence de solutions faibles entropiques pour le problème de Riemann lorsque le système est strictement hyperbolique, sans hypothèse de vraie nonlinéarité des champs caractéristiques. Nous suivons l'approche nouvelle récemment introduite par Tzavaras pour les systèmes de lois de conservation. Nous construisons des solutions approchées, regulières et auto-semblables qui sont bornées uniformément en norme BV et convergent vers une solution du problème de Riemann. La structure des ondes élementaires de cette solution est étudiée, et nous expliquons comment donner un sens rigoureux aux produits non-conservatifs.

PARTIAL DIFFERENTIAL EQUATIONS. Existence theory for the Riemann problem for non-conservative hyperbolic systems.

Abstract. In this Note, we study nonlinear and non-conservative hyperbolic systems in the framework introduced by Dal Maso-LeFloch-Murat. We prove an existence theorem of entropy weak solutions for the Riemann problem when the system is strictly hyperbolic, without assuming the genuine nonlinearity of the characteristic fields. We follow the new approach recently introduced by Tzavaras for the systems of conservation laws. Self-similar, smooth, approximate solutions are constructed, that are uniformly bounded in the BV norm and converge to a solution of the Riemann problem. The structure of the elementary waves in this solution is studied, and we explain how to rigorously define the non-conservative products.

Version Abrégée en Français.

Nous considérons les systèmes hyperboliques nonlinéaires sous forme non-conservative:

$$\partial_t u + A(u)\, \partial_x u = 0, \qquad u(x,t) \in \mathcal{U},\ t>0,\ x \in \mathbb{R}, \tag{1}$$

où $\mathcal{U} \subset \mathbb{R}^N$ est borné, convexe et fermé, et où la matrice $A(u)$ a N valeurs propres réelles et distinctes $\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_N(u)$ et une base de vecteur propres $r_i(u)$, $i = 1, \cdots, N$. Les champs caractéristiques ne sont pas supposés être vraiment nonlinéaires. On résout ici le problème de Riemann, (1)-(2), avec

$$u(x,0) = \begin{cases} u_- & \text{for } x < 0, \\ u_+ & \text{for } x > 0 \end{cases} \tag{2}$$

où u_+ et u_- appartiennent à $\mathcal{U}$ et $|u_+ - u_-| << 1$.

On considère une régularisation par viscosité artificielle du problème de Riemann:

$$\begin{aligned} &-\xi\, \partial_\xi v^\varepsilon + A(v^\varepsilon)\, \partial_\xi v^\varepsilon = \varepsilon\, \partial_{\xi\xi} v^\varepsilon, \\ &\lim_{\xi \to \pm\infty} v^\varepsilon(\xi) = u_\pm, \end{aligned} \tag{3}$$

avec $\xi = x/t$. On décompose $\partial_\xi v^\varepsilon$,

$$\partial_\xi v^\varepsilon(\xi) = \sum_{k=1}^N a_k^\varepsilon(\xi)\, r_k\big(v^\varepsilon(\xi)\big), \tag{4}$$

de sorte que

$$\varepsilon\, \partial_\xi a_k^\varepsilon(\xi) + \big[\xi - \lambda_k(v^\varepsilon(\xi))\big]\, a_k^\varepsilon(\xi) = \varepsilon \sum_{m=1}^N \sum_{n=1}^N \big[\nabla l_k(v^\varepsilon(\xi))\, r_m(v^\varepsilon(\xi)) \cdot r_n(v^\varepsilon(\xi))\big]\, a_m^\varepsilon(\xi)\, a_n^\varepsilon(\xi) \tag{5}$$

et

$$u_+ - u_- = \int_{-\infty}^{\infty} \sum_{k=1}^N a_k^\varepsilon r_k(v^\varepsilon)\, d\zeta. \tag{6}$$

En linéarisant (5), on trouve un système d'équations découplées:

$$\varepsilon\, \partial_\xi \varphi_k^\varepsilon + (\xi - \lambda_k(v^\varepsilon))\, \varphi_k^\varepsilon = 0,$$

donc

$$\varphi_k^\varepsilon(\xi) = \frac{e^{-\frac{1}{\varepsilon} g_k^\varepsilon(\xi)}}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon} g_k^\varepsilon(\zeta)}\, d\zeta}$$

où on a posé:

$$g_k^\varepsilon(\xi) = \int_{c_k}^{\xi} \big(s - \lambda_k(v^\varepsilon(s))\big)\, ds.$$

On démontre dans [11] qu'il existe, pour le problème (3), une solution v^ε qui est uniformément bornée dans BV (l'espace des fonctions à variation totale bornée) et satisfait au développement asymptotique:

$$a_k^\varepsilon(\cdot\,;\tau) = \tau_k\, \varphi_k^\varepsilon + \theta_k^\varepsilon(\cdot\,;\tau)$$

avec

$$|\theta_k^\varepsilon| \le C\, |\tau|^2 \sum_{j=1}^{N} \varphi_j^\varepsilon\,, \quad k = 1, ..., N,$$

où $|\tau| = |\tau_1| + ... + |\tau_N|$ et les fonctions a_k^ε satisfont aux conditions (4)–(6). La démonstration suit l'approche générale récemment développée par Tzavaras pour les systèmes de lois de conservation [19].

On démontre qu'une sous-suite de $\{v^\varepsilon\}$ converge vers une solution faible entropique du problème de Riemann (1)-(2). Lorsque le système possède uniquement des champs caractéristiques vraiment non-linéaires, la limite est solution du système non-conservatif (1) au sense introduit par Dal Maso-LeFloch-Murat dans [3]. La famille de chemins Lipschitziens correspondantes est celle suggérée par LeFloch dans [8] et construite à partir des ondes progressives. Pour les systèmes qui ne sont pas vraiment nonlinéaires, on introduit dans [11] une approche plus générale pour définir les produits non-conservatifs.

English Version.

1. Introduction. The objective of this work is to develop a mathematical theory of existence of entropy weak solutions to the Riemann problem for hyperbolic systems containing non-conservative products. We consider a non-conservative hyperbolic system in one space dimension:

$$\partial_t u + A(u)\, \partial_x u = 0, \qquad u(x,t) \in \mathcal{U},\ t > 0,\ x \in \mathbb{R}. \tag{1}$$

It is assumed that $\mathcal{U}$ is an open and convex subset of $\mathbb{R}^N$, and that, for every u, the matrix $A(u)$ has N real and distinct eigenvalues denoted by $\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_N(u)$ and so a basis of right eigenvectors denoted by $r_i(u)$. The characteristic fields are not assumed to be genuinely nonlinear nor linearly degenerate. We are interested in solving the Riemann problem associated with (1), i.e. the Cauchy problem with a piecewise constant initial data of the form:

$$u(0,x) = u_- \quad \text{for } x < 0, \qquad u_+ \quad \text{for } x > 0 \tag{2}$$

with u_- and u_+ given in $\mathcal{U}$ and satisfying $|u_+ - u_-| << 1$. Existence of solutions for the Riemann problem is well understood for conservative, strictly hyperbolic systems. Indeed, when (1) is composed of conservation laws in divergence form, i.e. $A = \nabla f$ for some flux-function f, one can search for weak solutions to

$$\partial_t u + \partial_x f(u) = 0 \tag{3}$$

in the sense of distributions. Lax [5] solved the Riemann problem for genuinely nonlinear systems and constructed a weak solution satisfying as admissibility criterion the so-called Lax shock inequalities. Based on what is now called Liu's entropy criterion, Liu [12] constructed a solution to the Riemann problem when the characteristic fields are not genuinely nonlinear. In both cases, the solution is self-similar, that is $u(x,t) = v(\xi)$ with $\xi = x/t$, so that (2)-(3) should read:

$$-\xi\,\partial_\xi v + \partial_\xi f(v) = 0, \qquad \lim_{\xi\to\pm\infty} v(\xi) = u_\pm. \tag{4}$$

More recently, again for conservative systems, Tzavaras [19] developed a complementary viewpoint and proved the existence of a solution to the problem (2)-(3) as the $\varepsilon \to 0$ limit of solutions to the following elliptic regularization of the Riemann operator:

$$-\xi\,\partial_\xi v^\varepsilon + \partial_\xi f(v^\varepsilon) = \varepsilon\,\partial_{\xi\xi} v^\varepsilon, \qquad \lim_{\xi\to\pm\infty} v(\xi) = u_\pm. \tag{5}$$

This approach for solving the Riemann problem, called self-similar viscous limits, was proposed by Dafermos [2]. It may be motivated by introducing an artificial viscosity designed so as to preserve the invariance of the Riemann problem under dilations of coordinates. See [2, 16, 17, 18, 19] for studies of various systems.

In general, (1) can not be written in divergence form and thus the theory of distributions can not be used. Instead we can rely on the notion of non-conservative products introduced by Dal Maso-LeFloch-Murat [3]. Essential to defining such products is the notion of a *family of Lipschitz continuous paths*: the non-conservative product is not uniquely defined unless additional information is available. The latter is gathered in the form of "paths" in the state space. When a family of paths is known, the Riemann problem is solved by extending Lax's approach (see [6] for the so-called Volpert paths and [3] for general paths); the existence of entropy weak solutions to the Cauchy problem is established in LeFloch-Liu [10] by extending the Glimm scheme to non-conservative systems.

On the other hand, the family of paths can not be determined from the sole knowledge of the hyperbolic system. To systems arising in continuum mechanics higher-order terms (viscosity, heat conduction, relaxation, etc) are often associated, that reflect more detailed modeling and, from a mathematical standpoint, provide a mechanism to regularize solutions. LeFloch suggested in [8] to construct the family of paths, using the traveling wave solutions associated to a viscous regularization of (1), such as

$$\partial_t u^\varepsilon + A(u^\varepsilon)\,\partial_x u^\varepsilon = \varepsilon\,\partial_{xx} u^\varepsilon. \tag{6}$$

Indeed, connecting trajectories of traveling waves provide a natural way to define a family of paths for the system (1) and, thus, authorize applying the theory by DalMaso-LeFloch-Murat to define the non-conservative product $A(u)\partial_x u$. In particular, this provides a practical approach to solve the Riemann problem.

A rigorous proof of the existence of traveling waves for (6) was established by Sainsaulieu [15]. We refer to Raviart-Sainsaulieu [13], and the references cited therein, for interesting works on non-conservative systems arising in multiphase flow theory. For a different approach to defining non-conservative products, see for instance Cauret-Colombeau-Leroux [1] and the references therein.

2. Main Results. The present work is aimed at rigorously understanding the non-conservative products induced by artificial vanishing viscosity approximations for the Riemann problem. We consider the problem

$$-\,\xi\,\partial_\xi v^\varepsilon + A(v^\varepsilon)\,\partial_\xi v^\varepsilon = \varepsilon\,\partial_{\xi\xi} v^\varepsilon, \tag{7a}$$

$$\lim_{\xi\to\pm\infty} v^\varepsilon(\xi) = u_\pm. \tag{7b}$$

Self-similar viscous limits penalize the whole wave-fan simultaneously, and the resulting admissibility criterion is called the viscous wave-fan admissibility criterion.

The main result is a proof that solutions of (7) exist and converge as $\varepsilon \to 0$ to a solution of the Riemann problem (1)-(2). This approach provides with, both, an existence proof and a meaning for the non-conservative product $A(v)\partial_\xi v$.

Let us introduce the following decomposition of the first order derivative of v^ε

$$\partial_\xi v^\varepsilon(\xi) = \sum_{k=1}^{N} a_k^\varepsilon(\xi)\, r_k\big(v^\varepsilon(\xi)\big). \tag{8}$$

The system (7) is equivalent to the problem consisting of the equations

$$\varepsilon\,\partial_\xi a_k^\varepsilon(\xi) + \big[\xi - \lambda_k(v^\varepsilon(\xi))\big]\, a_k^\varepsilon(\xi) = \varepsilon \sum_{m=1}^{N}\sum_{n=1}^{N} \big[\nabla l_k(v^\varepsilon(\xi))\, r_m(v^\varepsilon(\xi)) \cdot r_n(v^\varepsilon(\xi))\big]\, a_m^\varepsilon(\xi)\, a_n^\varepsilon(\xi) \tag{9}$$

and

$$u_+ - u_- = \int_{-\infty}^{\infty} \sum_{k=1}^{N} a_k^\varepsilon r_k(v^\varepsilon)\, d\zeta. \tag{10}$$

The linearized version of (9), consisting of the decoupled system of equations

$$\varepsilon\,\partial_\xi \varphi_k^\varepsilon + (\xi - \lambda_k(v^\varepsilon))\,\varphi_k^\varepsilon = 0, \tag{11}$$

has solutions constant multiples of

$$\varphi_k^\varepsilon(\xi) = \frac{e^{-\frac{1}{\varepsilon} g_k^\varepsilon(\xi)}}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon} g_k^\varepsilon(\zeta)}\, d\zeta}, \tag{12}$$

where

$$g_k^\varepsilon(\xi) = \int_{c_k}^{\xi} \big(s - \lambda_k(v^\varepsilon(s))\big)\, ds \tag{13}$$

and the c_k's are any constants. Due to their form, the functions φ_k^ε are strictly positive and uniformly bounded in L^1 with respect to the parameter ε.

We establish the following two theorems.

Theorem 1. *Assume that* (1) *is strictly hyperbolic and let $\varepsilon > 0$ and u_- be fixed. There exist N one-parameter families of solutions $v_i^\varepsilon(\cdot\,;\tau_i)$ satisfying* (7*a*) *and $\lim_{\xi\to-\infty} v_i^\varepsilon(\xi) = u_-$, $i = 1,\dots,N$, parametrized by τ_i and defined for $|\tau_i| \le \delta$ for some $\delta > 0$. For each fixed τ_i the function $v_i^\varepsilon(\cdot\,;\tau_i)$ is a classical solution of* (7*a*) *connecting u_- to some state $v_i^\varepsilon(\infty\,;\tau_i) = v_{i,+}^\varepsilon(\tau_i) \in \mathcal{U}$. Moreover*

$$\partial_\xi v_i^\varepsilon(\cdot\,;\tau_i) = \sum_{k=1}^{N} a_{i,k}^\varepsilon(\cdot\,;\tau_i)\, r_k(v_i^\varepsilon), \tag{14}$$

where $a_{i,k}^\varepsilon(\cdot\,;\tau_i)$ admits an asymptotic expansion in τ_i of the form

$$a_{i,k}^\varepsilon(\cdot\,;\tau_i) = \begin{cases} \tau_i\,\varphi_i^\varepsilon + \theta_{i,i}^\varepsilon(\cdot\,;\tau_i) & \text{for } k = i \\ \theta_{i,k}^\varepsilon(\cdot\,;\tau_i) & \text{for } k \neq i \end{cases} \tag{15}$$

with, for a constant C independent of ε,

$$|\theta_{i,k}^\varepsilon(\cdot\,;\tau_i)| \le C\,\Big(|\tau_i|^2\,\varphi_i^\varepsilon + |\tau_i|^3 \sum_{j=1}^{N} \varphi_j^\varepsilon\Big), \quad |\tau_i| \le \delta\,, \quad k = 1,\cdots,N. \tag{16}$$

As an extension of Theorem 1, it is also possible to construct $v^\varepsilon(\cdot,\tau)$, an N-parameter family of solutions to (7a), depending on the vector-parameter $\tau = (\tau_1,\dots,\tau_N)$. The corresponding amplitudes $a_k^\varepsilon = a_k^\varepsilon(\cdot\,;\tau)$ satisfy (9) and admit an asymptotic expansion of the form

$$a_k^\varepsilon(\cdot\,;\tau) = \tau_k\,\varphi_k^\varepsilon + \theta_k^\varepsilon(\cdot\,;\tau) \quad \text{with } |\theta_k^\varepsilon| \le C\,|\tau|^2 \sum_{j=1}^{N} \varphi_j^\varepsilon\,, \quad k = 1,\dots,N, \tag{17}$$

where $|\tau| = |\tau_1| + \dots + |\tau_N|$. Such a solution connects u_- to some vector $v^\varepsilon(\infty;\tau) = u_+(\tau)$.

We then fix both Riemann data and pursue the question of existence and uniform bounds for solutions to (7).

Theorem 2. *Assume* (1) *is strictly hyperbolic and let u_- be fixed. There exists r sufficiently small, such that for $\varepsilon > 0$ and for any u_+ satisfying $|u_+ - u_-| < r$ the problem* (7) *admits a solution v^ε with the properties :*

(1) *The family $\{v^\varepsilon\}_{\varepsilon>0}$ satisfies $\|v^\varepsilon(\xi) - u_-\|_{L^\infty} \leq \mu$ with some μ independent of ε.*
(2) *The solutions v^ε satisfy the representation formula*

$$\partial_\xi v^\varepsilon = \sum_{k=1}^{N} \left[\, \tau_{k,\varepsilon} \varphi_k^\varepsilon + \theta_k^\varepsilon(\cdot\,; \tau^\varepsilon) \right] r_k(v^\varepsilon), \tag{18}$$

where φ_k^ε is given by (11), *$a_k^\varepsilon(\cdot\,;\tau)$ of the form* (17) *satisfy* (9), *and τ_ε solves $S^\varepsilon(\tau_\varepsilon) = u_+$, where*

$$S^\varepsilon(\tau^\varepsilon) = u_- + \sum_{k=1}^{N} \int_{-\infty}^{\infty} \left(\tau_k^\varepsilon\, \varphi_k^\varepsilon(\zeta) + \theta_k^\varepsilon(\zeta;\, \tau)\right) r_k(v^\varepsilon(\zeta))\, d\zeta.$$

(3) *The family $\{\partial_\xi v^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^1(I\!R)$ and $\{v^\varepsilon\}_{\varepsilon>0}$ is of uniformly bounded (and small) variation.*

The method of proof for Theorems 1 and 2 follows Tzavaras [19] for conservative systems. Next we apply Theorem 2 and pass to the limit as $\varepsilon \to 0$.

Theorem 3. *Assume* (1) *is a strictly hyperbolic system and the initial jump $|u_r - u_l|$ is small.*

1) As $\varepsilon \to 0$, the solutions to the regularized Riemann problem (7) *converge toward an entropy weak solution of the Riemann problem* (1)-(2).

2) If, furthermore, the characteristic fields of the system (1) *are genuinely nonlinear, the non-conservative product $A(u)\,\partial_x u$ has a meaning in the sense of Dal Maso-LeFloch-Murat for the family of Lipschitz continuous paths constructed via trajectories of the traveling waves solutions associated with* (6).

Part 2 of Theorem 3 justifies the formal construction of the family of paths in [8]. A more general standpoint is necessary however when the characteristic fields are not genuinely nonlinear, as is explained in [19]: the non-conservative product in general should be defined as the limit of the uniformly bounded sequence of bounded Borel measures $A(v^\varepsilon)\,\partial_\xi v^\varepsilon$. This point of view was also recently suggested by Raymond [14] to obtain a more general definition of non-conservative products.

See LeFloch-Tzavaras [11] for the proofs of Theorems 1 to 3 and additional results on the structure of the elementary waves in Riemann solutions.

3. Applications.

Hyperbolic systems containing non-conservative products arise in a large number of applications in the physical sciences: multi-phase flows such as water-bubbles or gas-droplet mixtures, K-epsilon models for turbulence, elasto-plasticity models, two-temperature models in magnetohydrodynamics, etc. As it has been recognized for a long time, the presence of non-conservative products is a consequence of averaging procedures used in the derivation of such models. In the highly complex problems set by industry, various approximations in the basic conservation laws of continuum mechanics are necessary in order to obtain tractable models for numerical analysis and computational purposes. In the dynamics of elasto-plastic materials, the equations of state are often provided in an incremental form based on experiments. The use of non-conservative formulations may also be of interest, at times, for numerical computations.

Rigorous analysis of the convergence of finite difference schemes for nonconservative systems is a completely open issue. It is pointed out by Hou-LeFloch [4] that non-conservative difference schemes, containing numerical viscosity such as the Lax-Friedrichs or Godunov schemes say, do not converge to “the” correct solution in general, even in the case of conservation laws. In the case of conservation laws, nonconservative schemes converge to solutions of an equation containing a measure source-term, whose dramatic effect is to modify the jump relations. The numerical error between the exact solution and the limit may be estimated (for conservation laws) and turns out to be small in the L^1 norm if the shocks are of small amplitude. This makes the error difficult to detect in numerical computations with complex flow features and caution is necessary. Such difficulties do not arise with the Glimm scheme successfully treated in [10] nor with the front tracking schemes (Zhong-Hou-LeFloch [20]).

See LeFloch-Tzavaras [11] for a review and references on non-conservative systems arising in continuum mechanics.

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