Derivation of fluid equations for kinetic models with one conserved quantity

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Abstract

We survey the current state of the mathematical theory on fluid-dynamic limits for BGK systems and discrete velocity models of relaxation type when the limit is a scalar conservation law.

1 Introduction

The subject of fluid dynamic limits is motivated by the derivation of the compressible Euler equations for a mono-atomic gas as the zero mean-free-path limit of the Boltzmann equation. While the rigorous justification of the fluid-dynamic limit for the Boltzmann equation is a challenging open problem, it has prompted recent work on the derivation of hyperbolic systems of conservation laws from kinetic models in simpler situations, where the limits are scalar equations or systems of two conservation laws.

In this note we outline certain techniques employed in the rigorous justification of fluid limits to scalar conservation laws. Consider the kinetic model

$$
\begin{aligned}
&\partial_t f + a(\xi)\cdot\nabla_x f = \frac{1}{\varepsilon} C(f(\cdot),\xi) \\
&f(0,x,\xi) = f_0(x,\xi)
\end{aligned} \tag{1.1}
$$

where $x \in \mathbb{R}^d$ and $C(f,\xi)$ is a functional on $f(t,x,\cdot)$ (depending on ξ) that encodes the detailed properties of a collision process. The variable ξ may be continuous ($\xi \in \mathbb{R}$) or it may take discrete values; in the latter case (1.1) becomes a discrete velocity kinetic model. Both cases are treated simultaneously and we retain a common notation.

The collision operator C is assumed to satisfy the structural properties: The function $f \equiv 0$ is an equilibrium, *i.e.* $C(0(\cdot),\xi) = 0$, and

$$
\int_{\mathbb{R}} C(f,\xi)\, d\xi = 0 \tag{hyp1}
$$

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so that (1.1) is equipped with a macroscopic balance law

$$\partial_t \int f \, d\xi + \mathrm{div}_x \int a(\xi) f \, d\xi = 0 \tag{1.2}$$

for the "mass" $u := \int f \, d\xi$. Second, the equilibria of (1.1) are parametrized in terms of exactly one scalar parameter w, which is either the mass itself ($w = u$) or, more generally, a reparametrization of the mass $u = b(w)$. It is assumed that the equilibria of $C(f, \xi) = 0$ are

$$f_{eq} = \mathcal{M}(w, \xi) \qquad \text{where} \qquad \int_\xi \mathcal{M}(w,\xi) d\xi = u = b(w) \tag{hyp2}$$

where $\mathcal{M}(0, \xi) = 0$ and b is a strictly increasing function, so that the map $w \to u$ is invertible. (Note that whenever ξ is a discrete variable $d\xi$ should be understood as a counting measure.)

Let f^ε be a family of solutions to (1.1), $u^\varepsilon = \int f^\varepsilon$, and set $w^\varepsilon = b^{-1}\Big(\int f^\varepsilon \Big)$. Then, (1.2) gives the approximate conservation law

$$\partial_t b(w^\varepsilon) + \mathrm{div}_x \int_\xi a(\xi) \mathcal{M}(w^\varepsilon, \xi) = \mathrm{div}_x \Big(\int_\xi a(\xi) \big(\mathcal{M}(w^\varepsilon, \xi) - f^\varepsilon \big) \Big) \tag{1.3}$$

We are interested in the behavior of (1.3) as $\varepsilon \to 0$. It is conceivable that under appropriate conditions on the collision operator the kinetic equation (1.1) generates a stable process, so that its mass u^ε stabilizes and the kinetic function f^ε approaches as $\varepsilon \to 0$ the equilibrium "local Maxwellian". In analytical terms, if one were to provide conditions guaranteeing that

$$w^\varepsilon \to w \quad \text{a.e} \tag{a3}$$

and that Maxwellian distributions are enforced in the limit $\varepsilon \to 0$:

$$\int_\xi a(\xi)(f^\varepsilon - \mathcal{M}(w^\varepsilon, \xi)) \to 0 \quad \text{in } \mathcal{D}', \quad \text{where} \quad w^\varepsilon = b^{-1}\Big(\int_\xi f^\varepsilon \Big). \tag{a4}$$

then the limit $u = b(w)$ would satisfy the scalar conservation law

$$\partial_t b(w) + \mathrm{div}_x \int_\xi a(\xi) \mathcal{M}(w, \xi) = 0 \, . \tag{1.4}$$

In the sequel we review certain ideas from [11, 6, 10, 1, 4] developed, in the context of various kinetic and discrete-kinetic (of relaxation type) models, to justify this limiting process. All models treated (see (2.3), (2.5), (3.7), (4.1)) fit under the framework of (1.1) equipped with exactly one conservation law, (hyp1)-(hyp2). We begin in section 2 with a class of models that generate L^1 contractions, [11, 6, 10, 1], for which the proof of convergence follows the general outline of Kruzhkov theory [7]. Then in section 3 we review the kinetic formulation of of scalar conservation laws (see [8]) and indicate how this notion of solution naturally arises from the $\varepsilon \to 0$ behavior of a special BGK model [11, 8]. Finally, in section 4, we outline a methodology from [4] that is useful for proving convergence for models that are not L^1-contractive but are equipped instead with strong dissipation estimates (see [16]). This approach, motivated by the kinetic formulation, achieves a decomposition of the entropy dissipation via duality to arrive at an approximate transport equation; one then concludes via the averaging lemma.

The reader is referred to [2, 12] for general information on fluid dynamic limits and to [17] for a survey of the mathematical aspects of fluid limits. Apart from the aforementioned works, we refer to [13] for the subject of kinetic formulations and averaging lemmas, and to [3, 5, 9, 15] (and references therein) for the subject of relaxation approximations.

2 Kinetic models that generate L^1-contractions

In this section we supplement the model (1.1) with a hypothesis on the collision operator stating that

$$\text{(hyp3)} \qquad \int_{\mathbb{R}} \Big(C(f(\cdot),\xi) - C(\bar{f}(\cdot),\xi) \Big) \operatorname{sgn}(f-\bar{f})(\xi)\, d\xi \le 0\,,$$

for all $f(\cdot)$, $\bar{f}(\cdot)$. Note that (hyp3) is equivalent to requiring that the space-homogeneous version of (1.1) be a contraction in L^1_ξ.

This property is carried over to the space-inhomogeneous case and, as a consequence, (1.1) is endowed with a class of "kinetic entropies".

Theorem 1 *Under hypotheses* (hyp1)-(hyp3) *:*

(i) The kinetic model is a contraction in $L^1(\mathbb{R}^d_x \times \mathbb{R}_\xi)$

(ii) For all $\kappa \in \mathbb{R}$, *we have*

$$\text{(2.1)} \qquad \partial_t \int_\xi |f - \mathcal{M}(\kappa,\xi)| + div_x \int_\xi a(\xi)|f - \mathcal{M}(\kappa,\xi)| \le 0 \quad \text{in } \mathcal{D}'.$$

(iii) If for some a, b *it is* $\mathcal{M}(a,\xi) \le \mathcal{M}(b,\xi)$ *for all* ξ, *then the domain* $\prod_\xi [\mathcal{M}(a,\xi), \mathcal{M}(b,\xi)]$ *is positively invariant.*

Proof. Let f and $\bar{f}$ be two solutions. By subtracting the corresponding equations, multiplying by $\operatorname{sgn}(f-\bar{f})$ and using (hyp3), we obtain

$$\text{(2.2)} \quad \partial_t \int |f-\bar{f}| d\xi + \operatorname{div} \int a(\xi)|f-\bar{f}| = \frac{1}{\varepsilon} \int \big(C(f(\cdot),\xi) - C(\bar{f}(\cdot),\xi) \big) \operatorname{sgn}(f-\bar{f}) \le 0\,.$$

Hence, f and $\bar{f}$ satisfy the L^1-contraction property

$$\int_x \int_\xi |f-\bar{f}|(t,x,\xi)\, dx d\xi \quad \text{is decreasing in } t.$$

Moreover, since $\int_x \int_\xi (f-\bar{f})\, dx d\xi$ is a conserved quantity, we also have

$$\int_x \int_\xi (f-\bar{f})^+(t,x,\xi)\, dx d\xi \quad \text{is decreasing in } t,$$

resulting to the comparison principle:

$$\text{if} \quad f_0 \leq \bar{f}_0 \quad \text{then} \quad f(t) \leq \bar{f}(t)\,,\ t > 0\,.$$

A special class of solutions to (1.1) are the global Maxwellians $\mathcal{M}(\kappa, \xi)$. These may be used as comparison functions. For instance

$$\text{if } f_0(x,\xi) \leq \mathcal{M}(a,\xi)\,, \quad \text{for some } a \in \mathbb{R}, \text{ then } \quad f(t,x,\xi) \leq \mathcal{M}(a,\xi)$$

whence, part (iii) follows. Finally, if $\bar{f} = \mathcal{M}(\kappa, \xi)$ in (2.2) then

$$\int_\xi (\partial_t + a(\xi)\cdot\nabla_x)|f - \mathcal{M}(\kappa,\xi)|\, d\xi = \frac{1}{\varepsilon}\int C(f(\cdot),\xi)\text{sgn}\,(f - \mathcal{M}(\kappa,\xi)) \leq 0$$

which shows (2.1). □

We next present two specific models that satisfy hypotheses (hyp1)-(hyp3).

I. A discrete velocity model. Consider the system

$$\tag{2.3} \begin{aligned} \partial_t f_0 + a_0 \cdot \nabla_x f_0 &= \frac{1}{\varepsilon}\sum_{i=1}^{d}\big(f_i - h_i(f_0)\big) \\ \partial_t f_i + a_i \cdot \nabla_x f_i &= -\frac{1}{\varepsilon}\big(f_i - h_i(f_0)\big) \qquad i = 1, \ldots, d. \end{aligned}$$

for the evolution of $f = (f_0, f_1, \ldots, f_d)$ where $a_0, a_1, \ldots, a_d \in \mathbb{R}^d$. This model is developed in [6] as a relaxation approximation for the scalar multi-dimensional conservation law. Assume that

$$\tag{A} h_i(w) \ \text{ are strictly increasing, } i = 1, \ldots, d,$$

and let $u = f_0 + \sum_k f_k$. The Maxwellian functions are

$$\begin{gathered} f_{eq} = \mathcal{M}(w,j)_{j=0,1,\ldots,d} = (w, h_1(w), \ldots, h_d(w))\,, \\ \text{where } \ u = w + \sum_i h_i(w) =: b(w)\,. \end{gathered}$$

Clearly (hyp1) and (hyp2) are satisfied. To see (hyp3), note that

$$\begin{aligned} I &= \sum_{j=0}^{d}\Big(C(f,j) - C(\bar{f},j)\Big)\text{sgn}\,(f_j - \bar{f}_j) \\ &= \sum_{i=1}^{d}\Big(f_i - \bar{f}_i - (h_i(f_0) - h_i(\bar{f}_0))\Big)\Big(\text{sgn}\,(f_0 - \bar{f}_0) - \text{sgn}\,(f_i - \bar{f}_i)\Big) \leq 0\,, \end{aligned}$$

where the last inequality follows from (A).

Under (A) the model (2.3) is endowed with a globally defined entropy function

$$\partial_t\Big(\frac{1}{2}f_0^2+\sum_{i=1}^{d}\Psi_i(f_i)\Big)+\mathrm{div}\Big(a_0\frac{1}{2}f_0^2+\sum_{i=1}^{n}a_i\Psi_i(f_i)\Big)+\frac{1}{\varepsilon}\sum_{i=1}^{d}\phi_i(f_0,f_i)=0\,, \tag{2.4}$$

where

$$\Psi_i(f_i)=\int_0^{f_i}h_i^{-1}(\tau)\,d\tau\,,$$
$$\phi_i(f_0,f_i)=(f_0-h_i^{-1}(f_i))(h_i(f_0)-f_i)\,,$$

Ψ_i is positive and strictly convex, while ϕ_i satisfies $\phi_i\geq 0$ with $\phi_i=0$ if and only if f is a Maxwellian, *i.e.* if $f_j=\mathcal{M}(w,j)$ for some w. The identity provides control on the distance of solutions from equilibrium: If $\frac{dh_i}{dw}\geq c$ then $\phi_i\geq c(h_i(f_0)-f_i)^2$ and (2.4) leads to

$$\int_0^\infty\int_{\mathbb{R}^d}\sum_{i=1}^{d}(h_i(f_0)-f_i)^2\,dxdt\leq O(\varepsilon)\,.$$

II. A BGK model. Consider next the kinetic model of BGK type

$$\begin{aligned}\partial_t f+a(\xi)\cdot\nabla_x f&=-\frac{1}{\varepsilon}(f-\mathcal{M}(u,\xi))\\ \text{with }\ u&=\int_\xi f\end{aligned} \tag{2.5}$$

where $x\in\mathbb{R}^d$ and $\xi\in\mathbb{R}$. The model is introduced in [11] for the special choice of Maxwellian function $\mathcal{M}(u,\xi)=\mathbb{1}(u,\xi)$. The general case is developed in [10, 1]. It is assumed that $\mathcal{M}(u,\xi)$ is smooth and satisfies

$$\mathcal{M}(\cdot,\xi)\ \text{ is strictly increasing,}\qquad u=\int\mathcal{M}(u,\xi) \tag{B}$$

Then (hyp1) and (hyp2) are satisfied. The monotonicity of $\mathcal{M}$ states $\mathcal{M}(u,\xi)>\mathcal{M}(\bar u,\xi)$ iff $u>\bar u$, and, hence,

$$\begin{aligned}\int_\xi|\mathcal{M}(u,\xi)-\mathcal{M}(\bar u,\xi)|&=\mathrm{sgn}\,(u-\bar u)\Big(\int_\xi\mathcal{M}(u,\xi)-\mathcal{M}(\bar u,\xi)\Big)\\ &=|u-\bar u|=\Big|\int_\xi f-\bar f\Big|\leq\int_\xi|f-\bar f|\,,\end{aligned}$$

which is easily seen to give (hyp3).

The model also possesses an analog of the H-theorem. If we multiple (2.5) by $(\mathcal{M}^{-1}(f,\xi)-u)$, integrate over $\xi\in\mathbb{R}$ and denote by $\mu(f,\xi)=\int_0^f\mathcal{M}^{-1}(g,\xi)\,dg$ then $\mu(\cdot,\xi)$ is convex and we have

$$\begin{aligned}\partial_t\int\mu(f)+div_x\int a(\xi)\mu(f)-u\Big(\partial_t u+\mathrm{div}_x\int a(\xi)f\Big)&\\ +\frac{1}{\varepsilon}\int(\mathcal{M}^{-1}(f,\xi)-u)(f-\mathcal{M}(u,\xi))&=0\end{aligned} \tag{2.6}$$

The third term vanishes due to the conservation law and the last term is positive due to the monotonicity assumption. If we further assume that $\partial_u \mathcal{M} \geq c$ then the last equation yields the bound

$$\int_0^T \int_x \int_\xi c|f - \mathcal{M}(u,\xi)|^2 \leq \int_0^T \int_x \int_\xi (\mathcal{M}^{-1}(f,\xi) - u)(f - \mathcal{M}(u,\xi)) \leq O(\varepsilon) \tag{2.7}$$

stating that Maxwellians are enforced in the fluid limit $\varepsilon \to 0$.

The fluid-dynamic limit for a kinetic BGK-model

We provide an outline of the convergence justification for the fluid dynamic limit of the BGK-model

$$\begin{aligned} \partial_t f^\varepsilon + a(\xi) \cdot \nabla_x f^\varepsilon &= -\frac{1}{\varepsilon}\big(f^\varepsilon - \mathcal{M}(u^\varepsilon, \xi)\big) \\ f^\varepsilon(0, x, \xi) &= f_o^\varepsilon(x, \xi) \end{aligned} \tag{2.8}$$

where $u^\varepsilon = \int f^\varepsilon$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}$. It is assumed that $a(\xi)$ is uniformly bounded and that the Maxwellians are smooth functions that satisfy $\mathcal{M}(0,\xi) = 0$,

$$\begin{aligned} &\mathcal{M}(u,\cdot) \in L^1_\xi\,, \quad \mathcal{M}(\cdot,\xi) \text{ is strictly increasing} \\ &u = \int_{\mathbb{R}} \mathcal{M}(u,\xi)\, d\xi\,. \end{aligned} \tag{a}$$

Then (hyp1) and (hyp2) and (hyp3) are fulfilled. Let $\omega(\tau)$ be a positive, increasing function denoting a modulus of continuity, $\limsup_{\tau \to 0+} \omega(\tau) = 0$.

Theorem 2 *Let $|a(\xi)| \leq M$ and assume the initial data satisfy*

$$\begin{aligned} &\mathcal{M}(a,\xi) \leq f_o^\varepsilon(x,\xi) \leq \mathcal{M}(b,\xi) \qquad \textit{for some } a < b \\ &\int_x \int_\xi |f_o^\varepsilon(x,\xi)|\, dx d\xi \leq C \\ &\int_x \int_\xi |f_o^\varepsilon(x+h,\xi) - f_o^\varepsilon(x,\xi)|\, dx d\xi \leq \omega(|h|) \quad \textit{for } h \in \mathbb{R}^d \end{aligned} \tag{2.9}$$

Then

$$u^\varepsilon = \int_\xi f^\varepsilon \to u \quad \textit{a.e. and in } L^p_{loc}((0,T) \times \mathbb{R}^d) \textit{ for } 1 \leq p < \infty \tag{2.10}$$

and $u \in C\big([0,T]; L^1(\mathbb{R}^d)\big) \cap L^\infty((0,T) \times \mathbb{R}^d)$ is an entropy solution:

$$\partial_t |u - k| + div_x(F(u) - F(k))sgn\,(u-k) \leq 0 \quad \textit{in } \mathcal{D}', \textit{ for } k \in \mathbb{R}, \tag{2.11}$$

where $F(u) = \int_\xi a(\xi)\mathcal{M}(u,\xi)\, d\xi$.

Proof. From the L^1 contraction property, the invariance under translations, and upon using Maxwellians as comparison functions we have

$$
\begin{aligned}
&\mathcal{M}(a,\xi) \le f^\varepsilon(t,x,\xi) \le \mathcal{M}(b,\xi) \qquad \text{for } a<b \\
&\int_x |u^\varepsilon(t,x)| \le \int_x\int_\xi |f^\varepsilon| \le \int_x\int_\xi |f_o^\varepsilon| \le C
\end{aligned} \tag{2.12}
$$

and

$$
\begin{aligned}
\int_x |u^\varepsilon(t,x+h) - u^\varepsilon(t,x)| &\le \int_x\int_\xi |f^\varepsilon(t,x+h,\xi) - f^\varepsilon(t,x,\xi)| \\
&\le \int_x\int_\xi |f_o^\varepsilon(x+h,\xi) - f_o^\varepsilon(x,\xi)| \le \omega(|h|)
\end{aligned} \tag{2.13}
$$

Using a lemma of Kruzhkov (see [7] and the appendix in [17]) we obtain that for $k>0$

$$
\int_x |u^\varepsilon(t+k,x) - u^\varepsilon(t,x)|\, dx \,. \le C\omega(k) \tag{2.14}
$$

From (2.12), (2.13) and (2.14), u^ε is precompact in $L^1_{loc}((0,T)\times\mathbb{R}^d)$ and, along a subsequence, $u^\varepsilon \to u$ a.e. By (2.14) and Fatou's lemma, $u \in C\big([0,T];L^1(\mathbb{R}^d)\big)$.

Note that $a \le u^\varepsilon \le b$ is uniformly bounded and that (2.7) implies $|f^\varepsilon - \mathcal{M}(u^\varepsilon,\xi)| \to 0$ a.e (t,x,ξ). Using (2.12) we conclude

$$
\int_\xi |f^\varepsilon - \mathcal{M}(u^\varepsilon,\xi)| \to 0 \quad \text{a.e. } (t,x)
$$

Along a further subsequence, $f^\varepsilon \to \mathcal{M}(u,\xi)$ a.e., and passing to the limit in the kinetic entropies (2.1) we see that

$$
\partial_t \int_\xi |\mathcal{M}(u,\xi) - \mathcal{M}(k,\xi)| + \mathrm{div}_x \int_\xi a(\xi)|\mathcal{M}(u,\xi) - \mathcal{M}(k,\xi|\, d\xi \le 0
$$

in $\mathcal{D}'$. This is recast in the form (2.11) by using (a) and the property $\mathrm{sgn}\,(\mathcal{M}(u,\xi) - \mathcal{M}(k,\xi)) = \mathrm{sgn}\,(u-k)$. Since u is an entropy solution, it is unique and the whole family $u^\varepsilon \to u$ in L^p_{loc}, $1 \le p < \infty$. ☐

3 The connection with the kinetic formulation

Consider the scalar conservation law

$$
\begin{cases} \partial_t u + \mathrm{div} F(u) = 0 \\ u(x,0) = u_o(x) \end{cases} \tag{3.1}
$$

with data $u_o \in L^1 \cap L^\infty$. There are two equivalent notions of solution for this problem: The notion of Kruzhkov entropy solution [7] stating that u is an entropy solution of the initial value problem (3.1) if

$$
\text{ess } \lim_{t\to 0} \int |u(x,t) - u_o(x)| dx = 0 \tag{3.2}
$$

and u satisfies the entropy conditions

$$\partial_t \eta(u) + \text{div} q(u) \leq 0 \tag{3.3}$$

in $\mathcal{D}'$ for all entropy pairs $\eta - q$ with η convex. (Recall that entropy pairs $\eta - q$ are required to satisfy $q_i'(u) = \eta'(u) a_i(u)$, $i = 1, \ldots, d$, where $a_i = F_i'$.)

The second notion is the so called kinetic formulation of Lions-Perthame-Tadmor [8] and is based on the representation formulas for entropy pairs

$$\begin{aligned} \eta(u) - \eta(0) &= \int_\xi 1\!\!1(u,\xi) \eta'(\xi) d\xi \,, \\ q_j(u) - q_j(0) &= \int_\xi 1\!\!1(u,\xi) a_j(\xi) \eta'(\xi) d\xi \,, \end{aligned} \tag{3.4}$$

in terms of the kernel

$$1\!\!1_u(\xi) = 1\!\!1(u,\xi) = \begin{cases} 1\!\!1_{0<\xi<u} & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -1\!\!1_{u<\xi<0} & \text{if } u < 0 \end{cases} . \tag{3.5}$$

The kinetic formulation is equivalent to the Kruzhkov entropy solution and states that u is a solution of (3.1) if it takes the initial data as in (3.2) and there exist a positive bounded measure $m = m(t, x, \xi)$ on $\mathbb{R}_t^+ \times \mathbb{R}_x^d \times \mathbb{R}_\xi$ so that

$$\partial_t 1\!\!1(u,\xi) + a(\xi) \cdot \nabla_x 1\!\!1(u,\xi) = \partial_\xi m \tag{3.6}$$

Moreover, the measure m is supported on shocks.

This notion arises naturally as the $\varepsilon \to 0$ limit of the special BGK model

$$\begin{aligned} \partial_t f^\varepsilon + a(\xi) \cdot \nabla_x f^\varepsilon &= -\frac{1}{\varepsilon} \big(f^\varepsilon - 1\!\!1(u^\varepsilon, \xi) \big) \\ f^\varepsilon(0, x, \xi) &= 1\!\!1(u_o(x), \xi) \end{aligned} \tag{3.7}$$

where $\xi \in \mathbb{R}$. Note that (3.7) is of the form (1.1), for a special selection of the Maxwellian kernel. This allows to calculate the kinetic equation that the limiting f satisfies. Following [11, 8] we show:

Theorem 3 *As $\varepsilon \to 0$,*

$$u^\varepsilon \to u \quad a.e, \qquad f^\varepsilon \to \chi = 1\!\!1(u,\xi) \quad a.e.$$

and χ satisfies (3.6) *for some positive bounded measure m.*

Proof. As before (3.7) defines an L^1-contraction and $u^\varepsilon \to u$ a.e. Comparisons with the Maxwellians $1\!\!1(V, \xi)$ and $1\!\!1(-V, \xi)$, where $V = \sup |u_o(x)|$, give

$$\begin{gathered} -1 \leq f^\varepsilon \leq 1 \,, \qquad \text{supp}_\xi f^\varepsilon \subset [-V, V] \\ f^\varepsilon \geq 0 \ \text{ for } \xi > 0 \,, \quad f^\varepsilon \leq 0 \ \text{ for } \xi < 0. \end{gathered}$$

Introduce m^ε by

$$\partial_\xi m^\varepsilon = \frac{1}{\varepsilon}\big(\mathbb{1}(u^\varepsilon,\xi) - f^\varepsilon\big) = \begin{cases} > 0 & \text{for } \xi < u^\varepsilon \\ < 0 & \text{for } \xi > u^\varepsilon \end{cases}$$

The function

$$m^\varepsilon = \int_{-\infty}^{\xi} \frac{1}{\varepsilon}\big(\mathbb{1}(u^\varepsilon,\xi) - f^\varepsilon(\xi)\big)\, d\xi$$

satisfies $m^\varepsilon(-\infty) = 0$, $m^\varepsilon(+\infty) = 0$ by conservation, and $m^\varepsilon > 0$ for $\xi \in \mathbb{R}$.

We may thus write the BGK-model in the form

$$\partial_t f^\varepsilon + a(\xi)\cdot\nabla_x f^\varepsilon = \partial_\xi m_\varepsilon$$

We multiply by ξ and integrate over $[0,T]\times\mathbb{R}^d\times\mathbb{R}$. Taking account of the compact support in ξ we obtain

$$\begin{aligned}\int_0^T\int_x\int_\xi m^\varepsilon &= -\int_x\int_\xi \xi f^\varepsilon(t,x,\xi)d\xi\, dx + \int_x\int_\xi \xi f_o(x,\xi)d\xi\, dx \\ &\le V\int_x\int_\xi |f^\varepsilon| + V\int_x\int_\xi |f_o| \le C\end{aligned}$$

Using the relations

$$u^\varepsilon \to u \ \text{ a.e.}, \quad f^\varepsilon - \mathbb{1}(u^\varepsilon,\xi) \to 0 \ \text{ in } \mathcal{D}', \quad \mathbb{1}(u^\varepsilon,\xi)\to\mathbb{1}(u,\xi) \ \text{ a.e.}$$

and the property (along subsequences)

$$m^\varepsilon \rightharpoonup m \quad \text{weak-}\star \text{ in measures}$$

we pass to the limit $\varepsilon \to 0$ in $\mathcal{D}'$ to arrive at (3.6). □

4 Kinetic decomposition of approximate solutions

Consider now the relaxation system

$$\text{(4.1)} \qquad \begin{cases} \partial_t u + \sum_{j=1}^{d} \partial_{x_j} v_j &= 0 \\ \\ \partial_t v_i + A_i^2 \partial_{x_i} u &= -\frac{1}{\varepsilon}(v_i - F_i(u)) \qquad i = 1,\cdots,d. \end{cases}$$

where $u, v_i : \mathbb{R}^d_x \times \mathbb{R}^+_t \longrightarrow \mathbb{R}$, and the flux $F = (F_1(u),\ldots,F_d(u))$ is a smooth function. Let $a_i = F_i'$ and assume that the constants A_i satisfy the subcharacteristic condition

$$\text{(4.2)} \qquad A_i > |a_i(u)| \qquad i = 1,\cdots,d, \quad u\in\mathbb{R}.$$

In particular, (4.2) implies F_i are globally Lipshitz.

The system (4.1) belongs to the class of relaxation systems proposed in [5], and solutions $(u^\varepsilon, v^\varepsilon)$ of (4.1) are formally expected to converge as $\varepsilon \to 0$ toward a weak solution (u, v), $v = F(u)$, of the scalar conservation law

$$\partial_t u + \mathrm{div} F(u) = 0\,. \tag{4.3}$$

The 1-d version of (4.1) is L^1-contractive (see [9]). If one considers the 1-d variant and introduces Riemann coordinates the system is diagonalized and fits under the structure discussed in section 2. By contrast, the multi-d version of (4.1) is not diagonalizable, the L^1-contraction property cannot be established by the previous method and L^∞ estimates are not available. In the sequel, we show how to deal with the convergence problem using an alternative approach, in the spirit of the kinetic formulation; we work on the natural L^2 stability-framework suggested by lemma 4.

The system (4.1) is written in the form of regularization by a wave operator

$$\begin{aligned} \partial_t u + \sum_{j=1}^d \partial_{x_j} F_j(u) &= \sum_{j=1}^d \partial_{x_j} (F_j(u) - v_j) \\ &= \varepsilon\Big(\sum_{j=1}^d A_j^2 \partial_{x_j x_j} u - \partial_{tt} u\Big)\,. \end{aligned} \tag{4.4}$$

Moreover, it satisfies a strong dissipation identity (see [16, 4] for the derivation):

$$\begin{aligned} &\partial_t\Big(\frac{1}{2}(u + \varepsilon u_t)^2 + \frac{1}{2}\varepsilon^2 (u_t)^2 + \varepsilon^2 \sum_{j=1}^d A_j^2 (\partial_{x_j} u)^2\Big) + \sum_{j=1}^d \partial_{x_j} Q_j(u) \\ + \quad & \varepsilon |u_t + \mathrm{div} F(u)|^2 + \varepsilon \sum_{j=1}^d (A_j^2 - F_j'(u)^2)(\partial_{x_j} u)^2 \\ = \quad & \sum_{j=1}^d \partial_{x_j} (\varepsilon A_j^2 u u_{x_j} + 2\varepsilon^2 A_j^2 u_t u_{x_j}). \end{aligned} \tag{4.5}$$

where $Q_j' = u F_j'$. Using the notation $u^\varepsilon \in_b X$ to denote sequences that are uniformly bounded in the norm of the Banach space X, we obtain from (4.5) and the subcharacteristic condition the bounds:

Lemma 4 *If* (4.2) *holds and the initial data satisfy the uniform bound*

$$\|u_0^\varepsilon\|_{L^2(\mathbb{R}^d)} + \varepsilon \|\partial_t u_0^\varepsilon\|_{L^2(\mathbb{R}^d)} + \varepsilon \sum_{j=1}^d \|\partial_{x_j} u_0^\varepsilon\|_{L^2(\mathbb{R}^d)} = O(1)\,, \tag{4.6}$$

then the solutions of (4.1) *satisfy the uniform estimates*

$$u^\varepsilon(x,t) \quad \in_b \quad L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d)) \tag{4.7}$$

$$\varepsilon \sum_{j=1}^{d} (\partial_{x_j} u^\varepsilon(x,t))^2 \quad \in_b \quad L^1(\mathbb{R}^d \times \mathbb{R}^+) \tag{4.8}$$

$$\varepsilon(\partial_t u^\varepsilon(x,t))^2 \quad \in_b \quad L^1(\mathbb{R}^d \times \mathbb{R}^+) \tag{4.9}$$

Next, we outline a duality type of argument, used to obtain an approximate kinetic equation from the entropy dissipation estimate of the relaxation system; one then concludes via the averaging lemma [13, 14].

Theorem 5 *Assume that F_i are globally Lipshitz and satisfy the genuine-nonlinearity condition*

$$meas\{\xi \mid \tau + a(\xi)\cdot\sigma = 0\} = 0 \quad \text{for } (\tau,\sigma) \in \mathbb{R}\times\mathbb{R}^d \text{ with } \tau^2 + |\sigma|^2 = 1\,. \tag{4.10}$$

Let A_i be selected so that (4.2) holds, and let u^ε be a family of solutions to (4.1) generated by data subject to the uniform bounds (4.6). *Then, along a subsequence if necessary, u^ε converges to u in $L^p_{loc}(\mathbb{R}^d \times \mathbb{R}^+)$, $1 < p < 2$, and u is a weak solution of (4.3).*

Proof. We give a sketch of the proof and refer to [4] for the details. Let $(u^\varepsilon, v^\varepsilon)$ be a family of solutions to (4.1). For $\eta - q$ an entropy-entropy flux pair, we compute the entropy dissipation

$$\begin{aligned}\partial_t\eta(u^\varepsilon) + \mathrm{div}q(u^\varepsilon) &= \varepsilon\Big(\sum_{j=1}^{d} A_j^2 \partial_{x_j x_j}\eta(u^\varepsilon)\Big) - \varepsilon\partial_{tt}\eta(u^\varepsilon)\\ &- \eta''(u^\varepsilon)\Big(\sum_{j=1}^{d} A_j^2(\partial_{x_j}u^\varepsilon)^2\Big) + \varepsilon\eta''(u^\varepsilon)(\partial_t u^\varepsilon)^2.\end{aligned} \tag{4.11}$$

Fix $\varphi(x,t) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ and regard $\eta'(\xi) \in C_c^\infty(\mathbb{R})$ as a test function in velocity space. Using (3.4), the notations

$$\begin{aligned}\chi^\varepsilon(x,t,\xi) &= 1\!\!1(u^\varepsilon,\xi)\\ G^\varepsilon(x,t) &= \varepsilon\Big(\sum_{j=1}^{d} A_j^2(\partial_{x_j}u^\varepsilon)^2 - (\partial_t u^\varepsilon)^2\Big)\,,\end{aligned} \tag{4.12}$$

the fact that $G^\varepsilon \in_b L^1(\mathbb{R}^d \times \mathbb{R}^+)$ (by lemma 4), and the Schwartz kernel theorem, we obtain from (4.11)

$$\begin{aligned}&< \partial_t\chi^\varepsilon + a(\xi)\cdot\nabla\chi^\varepsilon\,,\, \eta'(\xi)\varphi(x,t) >\\ &= \; < \sum_{j=1}^{d} \partial_{x_j}\Big(\varepsilon A_j^2 \partial_{x_j}\chi^\varepsilon\Big) - \partial_t\big(\varepsilon\partial_t\chi^\varepsilon\big)\,,\, \eta'(\xi)\varphi(x,t) >\\ &\quad + < \partial_\xi(\delta(u^\varepsilon - \xi)G^\varepsilon\,,\, \eta'(\xi)\varphi(x,t) > .\end{aligned} \tag{4.13}$$

Since the subspace generated by the direct sum test functions $\varphi \otimes \eta'$ is dense in $C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, the bracket (4.13) is extended to test functions $\theta(x,t,\xi) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ and we have

$$\partial_t \chi^\varepsilon + a(\xi)\cdot\nabla\chi^\varepsilon = \sum_{j=1}^d \partial_{x_j}\left(\varepsilon A_j^2 \partial_{x_j}\chi^\varepsilon\right) - \partial_t\left(\varepsilon\partial_t\chi^\varepsilon\right) + \partial_\xi \delta(u^\varepsilon - \xi)G^\varepsilon \quad \text{in} \quad D'_{x,t,\xi}. \tag{4.14}$$

$$\text{with} \quad \pi_j^\varepsilon = \varepsilon A_j^2 \partial_{x_j}\chi^\varepsilon\,, \quad \pi_0^\varepsilon = -\varepsilon\partial_t\chi^\varepsilon\,, \quad k^\varepsilon = \delta(u^\varepsilon - \xi)G^\varepsilon\,.$$

Using again the estimates in lemma 4, it is a technical but straightforward matter to show that k^ε lies in a bounded set of the space of bounded measures $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ (the dual of $C_0(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, the continuous functions that vanish at infinity), and that the terms $\pi_j^\varepsilon(x,t,\xi)$, $j = 0,1,\ldots,d$ satisfy

$$\begin{aligned} \varepsilon\partial_{x_j}\mathbb{1}(u^\varepsilon,\xi) \to 0 \quad &\text{in } L^2_{x,t}(H_\xi^{-1}),\\ \varepsilon\partial_t\mathbb{1}(u^\varepsilon,\xi) \to 0 \quad &\text{in } L^2_{x,t}(H_\xi^{-1}). \end{aligned}$$

The Sobolev embedding theorem, in turn, implies $\mathcal{M}$ is compactly embedded in $W_{loc}^{-1,p}$, for $1 \le p < \frac{d+2}{d+1}$, and thus k^ε is precompact in $W_{loc}^{-1,p}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, for the same range of p. Thus $\chi^\varepsilon = \mathbb{1}(u^\varepsilon,\xi)$ satisfies the approximate transport equation

$$\partial_t \chi^\varepsilon + a(\xi)\cdot\nabla\chi^\varepsilon = \sum_{j=1}^d \partial_{x_j}\left(\bar g_j^\varepsilon + \partial_\xi g_j^\varepsilon\right) + \partial_t\left(\bar g_0^\varepsilon + \partial_\xi g_0^\varepsilon\right) + \partial_\xi k^\varepsilon \qquad \text{in} \quad D'_{x,t,\xi}, \tag{4.15}$$

where $\bar g_i^\varepsilon$, $g_i^\varepsilon \to 0$ in $L^2(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, k^ε is bounded in measures (not necessarily positive) and precompact in $W_{loc}^{-1,p}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ for $1 \le p < \frac{d+2}{d+1}$. From the averaging lemma in [14] we deduce that, for $\psi(\xi) \in C_c^\infty(\mathbb{R})$,

$$\int_\xi \mathbb{1}(u^\varepsilon,\xi)\psi(\xi)d\xi \quad \text{is precompact in } L^p_{loc},\ 1 < p < \frac{d+2}{d+1}.$$

Let R be a large positive number and consider $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi = 1$ on $(-R,R)$ and $0 \le \psi \le 1$. Then

$$\begin{aligned} \left|u^\varepsilon - \int_\mathbb{R} \mathbb{1}(u^\varepsilon,\xi)\psi(\xi)d\xi\right| &= \left|\int_\mathbb{R} \mathbb{1}(u^\varepsilon,\xi)(1-\psi(\xi))d\xi\right| \\ &\le \int_R^\infty |\mathbb{1}(u^\varepsilon,\xi)|d\xi + \int_{-\infty}^{-R} |\mathbb{1}(u^\varepsilon,\xi)|d\xi = (u^\varepsilon - R)^+ + (u^\varepsilon + R)^- \end{aligned}$$

Moreover,

$$\begin{aligned} \int (u^\varepsilon - R)^+ + (u^\varepsilon + R)^- dxdt &\le \int_{|u^\varepsilon|>R} |u^\varepsilon|\, dxdt \\ &\le \frac{1}{R}\int_0^T \int |u^\varepsilon|^2 dxdt \le \frac{C}{R} \end{aligned}$$

Hence $\{u^\varepsilon\}$ is Cauchy in $L^1_{loc,x,t}$ and, since $u^\varepsilon \in_b L^\infty(L^2)$, it follows that (along subsequences) $u^\varepsilon \to u$ in L^p_{loc}, $p < 2$, and almost everywhere and that $u \in L^\infty(L^2)$. Integration of (4.14) over ξ yields

$$\partial_t \int \chi^\varepsilon d\xi + \operatorname{div} \int a(\xi)\chi^\varepsilon d\xi = 0\,,$$

and, as $\varepsilon \to 0$, u satisfies the scalar conservation law.

Next, we pass to the limit $\varepsilon \to 0$ in (4.15). Note that

$$\begin{aligned} &\chi^\varepsilon = \mathbb{1}(u^\varepsilon, \xi) \to \chi = \mathbb{1}(u, \xi) && \text{a.e. and in } L^p_{loc,x,t}(L^p_\xi),\ 1 \le p < 2 \\ &k^\varepsilon = G^\varepsilon \delta(u^\varepsilon - \xi) \rightharpoonup k && \text{weak-}\star \text{ in } \mathcal{M}_{x,t,\xi} \end{aligned} \tag{4.16}$$

and χ satisfies

$$\partial_t \chi + a(\xi) \cdot \nabla \chi = \partial_\xi k \qquad \text{in} \quad D'_{x,t,\xi}. \tag{4.17}$$

□

Note that while (4.17) is of the general type of the kinetic formulation (3.6), it is not known whether the limiting k is positive. For the 1-d variant it is possible (using a different decomposition) to obtain (4.17) with k a positive measure (see [4]). This is in accord with the fact that the 1-d case is an L^1 contaction and produces in the limit an entropy solution for the conservation law.

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