Kinetic decomposition for kinetic models of BGK type

Seok Hwang

Department of Mathematics
University of Wisconsin-Madison

1 Introduction

Consider the scalar conservation law

$$\partial_t u + \partial_x F(u) = 0, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+ \tag{1.1}$$

where $F(u)$ is a smooth function. We are concerned here with kinetic models of BGK type which describe (1.1) at the microscopic scale $\varepsilon \to 0$. To this end we introduce a scalar function, $f^\varepsilon(x,t,\xi)$, which can be viewed as a microscopic description for the density of particles located at $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ with speed $\xi \in \mathbb{R}$. Then our kinetic model evolves according to

$$\partial_t f^\varepsilon(x,t,\xi) + a(\xi)\partial_x f^\varepsilon(x,t,\xi) = -\frac{1}{\varepsilon}(f^\varepsilon(x,t,\xi) - \mathcal{M}(u^\varepsilon(x,t),\xi)) \tag{1.2}$$

Here, $\varepsilon > 0$ and

$$u^\varepsilon(x,t) = \int_\xi f^\varepsilon d\xi$$

denotes the local density of particles at a given (x,t) location and $\mathcal{M}(u,\xi)$ is a maxwellian("equilibrium function").

This model is introduced in [9] for the special choice of Maxwellian function $\mathcal{M}(u,\xi) = \mathbb{1}(u,\xi)$. They showed the convergence by BV argument in the multidimensional case and by compensated compactness arguments in the one-dimensional case. Also, Bouchut [1] provide a general framework for such BGK models. Finally, this class of models incorporates various discrete velocity models of BGK type (See [4],[7] and [10] for a survey).

Such models are L^1 contractions, [1, 4, 5, 7, 9], for which the proof of convergence follows the general outline of Kruzhkov theory [5]. In this paper, we will show convergence by using a different methodology developed in [2]. The approach that is developed here will show the relation between relaxation approximations of BGK type and the kinetic formulation of scalar conservation laws developed in [6]. This relationship is achieved through a decomposition of the entropy dissipation via duality to arrive at an approximate transport equation:

(1.3)
$$\partial_t \mathbb{1}(u^\varepsilon, s) + F'(s)\partial_x \mathbb{1}(u^\varepsilon, s) = \partial_s m^\varepsilon - \partial_t \int_\xi I^\varepsilon \partial_s \mathcal{M}(s,\xi)d\xi - \partial_x \int_\xi a(\xi) I^\varepsilon \partial_s \mathcal{M}(s,\xi)d\xi$$

where

$$\mathbb{1}(u,\xi) = \begin{cases} \mathbb{1}_{0<\xi<u} & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -\mathbb{1}_{u<\xi<0} & \text{if } u < 0 \end{cases} . \tag{1.4}$$

is the usual Maxwellian associated with the kinetic formulation of scalar conservation laws, $\int_\xi I^\varepsilon \partial_s \mathcal{M}(s,\xi)d\xi$, $\int_\xi a(\xi) I^\varepsilon \partial_s \mathcal{M}(s,\xi)d\xi \to 0$ in $L^2_{loc}(\mathbb{R}_x \times \mathbb{R}^+_t; L^2(\mathbb{R}_s))$ are error terms and m^ε is a family of positive bounded measures that captures in the limit the behavior on shocks. Convergence is obtained via the averaging lemma of Perthame-Souganidis [8]. We prove that, as $\varepsilon \to 0$, $u^\varepsilon \to u$ where u is an entropy solution of (1.1) and that the limiting behavior of (1.3) encoding the entropy dissipation leads to the transport equation

$$\partial_t \mathbb{1}(u,s) + F'(s)\partial_x \mathbb{1}(u,s) = \partial_s m \;\text{ in }\; D'_{x,t,s}. \tag{1.5}$$

This paper shows how to relate the given kinetic model with the averaging lemma in order to get the convergence following a technique developed in [2] for relaxation approximation of Jin-Xin type [3] and diffusion-dispersion approximation. One important difference is that here the kinetic decomposition is achieved through extensions of entropies, and the approach applies to models like (1.2) where all equilibrium entropies can be extended to entropies for the kinetic system (see also Bouchut [1]).

2 BGK model

We consider the following kinetic model of BGK type :

$$\begin{aligned} \partial_t f^\varepsilon(x,t,\xi) + a(\xi)\partial_x f^\varepsilon(x,t,\xi) &= -\frac{1}{\varepsilon}(f^\varepsilon(x,t,\xi) - \mathcal{M}(u^\varepsilon(x,t),\xi)) \\ \text{with } u^\varepsilon &= \int_\xi f^\varepsilon d\xi \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}$ and $\xi \in \mathbb{R}$.

It is assumed that $a(\xi)$ is uniformly bounded and that $\mathcal{M}(u^\varepsilon, \xi)$ is smooth functions that satisfy $\mathcal{M}(0, \xi) = 0$,

$$\begin{aligned} \mathcal{M}(u, \cdot) &\in L^1_\xi, \quad \mathcal{M}(\cdot, \xi) \text{ is strictly increasing }, \\ u &= \int \mathcal{M}(u, \xi) d\xi \\ F(u) &= \int a(\xi) \mathcal{M}(u, \xi) d\xi \end{aligned} \tag{2.2}$$

From (2.1),(2.2) we have

$$\partial_t u^\varepsilon(x,t) + \partial_x \int_\xi a(\xi) f^\varepsilon(x,t,\xi) d\xi = 0 \tag{2.3}$$

In [9] they showed the existence of a global solution for (2.1) with $\mathcal{M}(u,\xi) = \mathbb{1}(u,\xi)$ in $L^\infty(\mathbb{R}_t^+; L^1(\mathbb{R}_x \times \mathbb{R}_\xi))$ by using a fixed point theorem. Similarly, in our case, the existence of solution for (2.1) follows by the same argument.

The model (2.1) also posseses an analog of the H-theorem. If we multiply the equation (2.1) by $(\mathcal{M}^{-1}(f,\xi) - u)$, integrate over $\xi \in \mathbb{R}$ and denote by

$$\mu(f,\xi) = \int_0^f \mathcal{M}^{-1}(g,\xi) dg$$

then $\mu(\cdot, \xi)$ is convex and we have

$$\begin{aligned} \partial_t \int_\xi \mu(f,\xi) + \partial_x \int_\xi a(\xi) \mu(f,\xi) \quad &- \quad u(\partial_t u + \partial_x \int_\xi a(\xi) f) \\ &+ \quad \frac{1}{\varepsilon} \int_\xi (\mathcal{M}^{-1}(f,\xi) - u)(f - \mathcal{M}(u,\xi)) = 0 \end{aligned} \tag{2.4}$$

The third term is vanishes due to the conservation law, the last term is positive due to the monotonicity assumption. Assume that $\partial_u \mathcal{M}(u,\xi) \geq c$ and $f^\varepsilon(x,0,\xi) \in L^2(\mathbb{R}_x \times \mathbb{R}_\xi)$, then (2.4) yields the bound

(2.5)
$$\int_{x,t} \int_\xi c|f - \mathcal{M}(u,\xi)|^2 dx dt \leq \int_{x,t} \int_\xi (\mathcal{M}^{-1}(f,\xi) - u)(f - \mathcal{M}(u,\xi)) d\xi dx dt = O(\varepsilon)$$

stating that the Maxwellians are enforced in the fluid limit $\varepsilon \to 0$.

In the sequel we use the limiting case of the averaging lemma proved in Perthame-Souganidis [8] :

Theorem 1 *Let $\{f_n\}, \{g_{i,n}\}$ be two sequences of solutions to the transport equation*

$$\partial_t f_n + a(\xi)\cdot\nabla_x f_n = \partial_t \partial_\xi^k g_{0,n} + \sum_{i=1}^d \partial_{x_i}\partial_\xi^k g_{i,n} \tag{2.6}$$

where $k \in N$. Assume that $a(\xi) \in C^\infty(\mathbb{R})$ satisfies the non-degeneracy condition: for $R > 0$

$$\omega(\beta) = \sup_{\alpha\in\mathbb{R},\omega\in S^{d-1}} \int_{\{|\xi|\le R\}} \Big(|\alpha + \frac{a(\xi)\cdot\omega}{\beta}|^2 + 1\Big)^{-1} d\xi \to 0\,, \qquad as\ \ \beta \to 0. \tag{2.7}$$

If $\{f_n\}$ is bounded in $L^q(\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R})$, for some $1 < q < \infty$, and $\{g_{i,n}\}$ is precompact in $L^q(\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R})$, then the average

$$\int_{\mathbb{R}} \psi(\xi) f_n(t,x,\xi)d\xi \qquad \text{is precompact in } L^q(\mathbb{R}^d\times\mathbb{R}^+)\,,$$

for any $\psi \in C_c^\infty(\mathbb{R})$.

Remark 1 *The non-degeneracy condition (2.7) is equivalent to for all $R > 0$*

$$meas\{\xi \in B_R \mid \alpha + a(\xi)\cdot\omega = 0\} = 0, \quad \forall\alpha\in\mathbb{R},\ \omega\in S^{d-1}, \tag{2.8}$$

where $B_R = \{|\xi| \le R\}$. The condition (2.8) can be interpreted geometrically, and means that the curve $\xi \mapsto a(\xi)\cdot\omega + \alpha$ is not locally contained in any hyperplane.

Let $H(f,\xi), Q(f,\xi)(= a(\xi)H(f,\xi)) \in C^1(\mathbb{R}\times\mathbb{R})$ be an entropy pair for the BGK model (2.1) that extends the entropy pair $\eta(u) - q(u)$ of (1.1) :

$$\begin{cases} \int_\xi H(\mathcal{M}(u,\xi),\xi)d\xi & = \ \eta(u) \\ \int_\xi a(\xi)H(\mathcal{M}(u,\xi),\xi)d\xi & = \ q(u) \end{cases} \tag{2.9}$$

$H - Q$ have the property that (smooth) solutions of (2.1) satisfy

$$\partial_t \int_\xi H(f,\xi)d\xi + \partial_x \int_\xi Q(f,\xi)d\xi = -\frac{1}{\varepsilon}\int_\xi H_f(f,\xi)(f - \mathcal{M}(u,\xi))d\xi \tag{2.10}$$

and when $\varepsilon \to 0$ they are expected to converge to the entropy dissipation of the limit conservation laws.

From (2.9) we have

$$\begin{cases} \int_\xi \frac{\partial H}{\partial f}(\mathcal{M}(u,\xi),\xi)\frac{\partial\mathcal{M}(u,\xi)}{\partial u}d\xi & = \ \eta'(u) \\ \int_\xi a(\xi)\frac{\partial H}{\partial f}(\mathcal{M}(u,\xi),\xi)\frac{\partial\mathcal{M}(u,\xi)}{\partial u}d\xi & = \ q'(u) \end{cases} \tag{2.11}$$

If we choose

$$\frac{\partial H}{\partial f}(\mathcal{M}(u,\xi),\xi) = \eta'(u),$$

then (2.11) holds and since $\mathcal{M}(\cdot,\xi)$ is increasing and by (2.1),(2.2), we have

$$\frac{\partial H}{\partial s}(s,\xi) = \eta'(\mathcal{M}^{-1}(s,\xi)) \tag{2.12}$$

where $\mathcal{M}\circ\mathcal{M}^{-1}(s,\xi) = s$ for a fixed ξ.

Now we state the main theorem of this section:

Theorem 2 *Let $f^\varepsilon(x,0,\xi) \in L^2(\mathbb{R}_x\times\mathbb{R}_\xi)\cap L^1(\mathbb{R}_\xi;L^\infty(\mathbb{R}_x))$ and $\partial_u\mathcal{M}(u,\xi)\geq c$ for some $c>0$.*
Then we have the following approximate transport equation:

$$\begin{aligned}
&\partial_t \int_\xi 1\!\!1(f^\varepsilon(x,t,\xi),\mathcal{M}(s,\xi))\frac{\partial\mathcal{M}(s,\xi)}{\partial s}d\xi\\
&+\partial_x \int_\xi a(\xi)1\!\!1(f^\varepsilon(x,t,\xi),\mathcal{M}(s,\xi))\frac{\partial\mathcal{M}(s,\xi)}{\partial s}d\xi\\
&= \partial_s \int_\xi \frac{1}{\varepsilon}[1\!\!1(\mathcal{M}^{-1}(f^\varepsilon(x,t,\xi),\xi),s) - 1\!\!1(u^\varepsilon(x,t),s)]\\
&\quad (f^\varepsilon(x,t,\xi)-\mathcal{M}(u^\varepsilon(x,t),\xi))d\xi\\
&=: \partial_s m^\varepsilon(x,t,s) \quad \text{in} \quad D'_{x,t,s}
\end{aligned} \tag{2.13}$$

where m^ε is a family of bounded positive measures.

Proof. Fix a test function $\varphi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$ and regard $\eta'(s)\in C_c^\infty(\mathbb{R})$ as a test function also. We have the formula

$$\eta(u)-\eta(0) = \int_s 1\!\!1(u,s)\eta'(s)ds$$

Then we have the following:

$$\begin{aligned}
\int_\xi H(f(x,t,\xi),\xi)d\xi &= \int_\xi\int_s 1\!\!1(f(x,t,\xi),s)\frac{\partial H(s,\xi)}{\partial s}dsd\xi\\
&= \int_\xi\int_s 1\!\!1(f(x,t,\xi),s)\eta'(\mathcal{M}^{-1}(s,\xi))dsd\xi\\
&= \int_\xi\int_s 1\!\!1(f(x,t,\xi),\mathcal{M}(s,\xi))\frac{\partial\mathcal{M}(s,\xi)}{\partial s}\eta'(s)dsd\xi\\
\int_\xi \partial_f H(f,\xi)(f-\mathcal{M}(u,\xi))d\xi &= \int_\xi [\eta'(\mathcal{M}^{-1}(f,\xi))-\eta'(u)](f-\mathcal{M}(u,\xi))d\xi\\
&= \int_\xi\int_s [1\!\!1(\mathcal{M}^{-1}(f,\xi),s)-1\!\!1(u,s)]\eta''(s)(f-\mathcal{M}(u,\xi))d\xi
\end{aligned}$$

which implies that

$$-\int_{x,t}\int_\xi \frac{1}{\varepsilon}H_f(f,\xi)(f-\mathcal{M}(u,\xi))\varphi d\xi dxdt =< \partial_s m^\varepsilon, \varphi(x,t)\otimes\eta'(s) >$$

where

$$m^\varepsilon = \frac{1}{\varepsilon}\int_\xi [\mathbb{1}(\mathcal{M}^{-1}(f,\xi),s)-\mathbb{1}(u,s)](f-\mathcal{M}(u,\xi))d\xi$$

Then we have

$$\begin{aligned} &< \partial_t \int_\xi \mathbb{1}(f^\varepsilon(x,t,\xi),\mathcal{M}(s,\xi))\frac{\partial\mathcal{M}(s,\xi)}{\partial s}d\xi \\ &\quad +\partial_x \int_\xi a(\xi)\mathbb{1}(f^\varepsilon(x,t,\xi),\mathcal{M}(s,\xi))\frac{\partial\mathcal{M}(s,\xi)}{\partial s}d\xi,\ \varphi\otimes\eta' > \\ &\quad =< \partial_s m^\varepsilon,\ \varphi\otimes\eta'(s) > \end{aligned} \tag{2.14}$$

Since the subspace generated by the direct sum test functions $\varphi\otimes\eta'$ is dense in $C_c^\infty(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R})$, the bracket (2.14) is extended to test function $\theta(x,t,s)\in C_c^\infty(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R})$. So, we obtain the approximate transport equation (2.13).

Next, we turn to the properties. Since $\mathcal{M}^{-1}(f,\xi)\geq u$ if $f\geq\mathcal{M}(u,\xi)$ and $\mathbb{1}(\cdot,\xi)$ is increasing function, it follows that $m^\varepsilon\geq 0$.

For $\theta(x,t,\xi)\in C_0$ with $|\theta|\leq 1$, we have

$$\begin{aligned} &| < m^\varepsilon,\theta > | \\ &\quad = |\int_{x,t,s}\int_\xi \frac{1}{\varepsilon}[\mathbb{1}(\mathcal{M}^{-1}(f,\xi),s)-\mathbb{1}(u,s)](f-\mathcal{M}(u,\xi))\theta d\xi dxdtds| \\ &\quad \leq C\int_{x,t,\xi}(\int_s \frac{1}{\varepsilon}|\mathbb{1}(\mathcal{M}^{-1}(f,\xi),s)-\mathbb{1}(u,s)|)|(f-\mathcal{M}(u,\xi))|dsd\xi dxdt \\ &\quad \leq C\int_{x,t,\xi}\frac{1}{\varepsilon}|\mathcal{M}^{-1}(f,\xi)-u||f-\mathcal{M}(u,\xi)|d\xi dxdt \\ &\quad = O(1) \end{aligned}$$

The last term is bounded, because of (2.5). □

We obtain compactness of approximate solutions by using the averaging lemma:

Corollary 3 *Let F satisfy*

$$meas\{\xi\in\mathbb{R} \mid F'(\xi)=\alpha\}=0, \quad \forall\alpha\in\mathbb{R}. \tag{2.15}$$

Let f^ε be a family of solutions to (2.1) generated by data satisfying $f^\varepsilon(x,0,\xi)\in L^2(\mathbb{R}_x\times\mathbb{R}_\xi)\cap L^1(\mathbb{R}_\xi;L^\infty(\mathbb{R}_x))$. Then $u^\varepsilon=\int_\xi\mathcal{M}(u^\varepsilon,\xi)d\xi$ converges to u in $L^p_{loc}(\mathbb{R}\times\mathbb{R}^+)$, $1\leq p<\infty$ and u is an entropy solution of (1.1).

Proof. Let $I^\varepsilon(x,t,s,\xi) := \mathbb{1}(f(x,t,\xi), \mathcal{M}(s,\xi)) - \mathbb{1}(\mathcal{M}(u,\xi), \mathcal{M}(s,\xi))$.
Then we can rewrite (2.13) by :
(2.16)

$$\partial_t \mathbb{1}(u^\varepsilon, s) + F'(s)\partial_x \mathbb{1}(u^\varepsilon, s) = \partial_s m^\varepsilon - \partial_t \int_\xi I^\varepsilon \partial_s \mathcal{M}(s,\xi)d\xi - \partial_x \int_\xi a(\xi) I^\varepsilon \partial_s \mathcal{M}(s,\xi)d\xi$$

Since $\int_\xi \partial_s \mathcal{M}(s,\xi)d\xi = 1$, we can take a bounded set $A \in \mathbb{R}$ such that

$$\int_{\mathbb{R}\setminus A} \partial_s \mathcal{M}(s,\xi)d\xi < \varepsilon^{1/2}$$

Then we have, for any compact subset K of $\mathbb{R}_x \times \mathbb{R}_t^+$,

$$\begin{aligned}
&\int_K \int_s | \int_\xi I^\varepsilon(x,t,\xi,s)\partial_s \mathcal{M}(s,\xi)d\xi|^2 dsdxdt \\
&\quad \le C \int_K \int_s \int_A |\mathbb{1}(f, \mathcal{M}(s,\xi)) - \mathbb{1}(\mathcal{M}(u,\xi), \mathcal{M}(s,\xi))|^2 d\xi dsdtdx + C\varepsilon \\
&\quad = C \int_K \int_J |f - \mathcal{M}(u,\xi)| d\xi dtdx + C\varepsilon \\
&\quad \le C|K \times J|^{\frac{1}{2}} (\int_K \int_{\mathbb{R}} (f - \mathcal{M}(u,\xi))^2 d\xi dtdx)^{\frac{1}{2}} + C\varepsilon \\
&\quad = O(\varepsilon^{\frac{1}{2}}) + O(\varepsilon)
\end{aligned}$$

So, $\int_\xi I^\varepsilon \partial_s \mathcal{M}(s,\xi)d\xi$, $\int_\xi a(\xi) I^\varepsilon \partial_s \mathcal{M}(s,\xi)d\xi \to 0$ in $L^2_{loc}(\mathbb{R}_x \times \mathbb{R}_t^+; L^2(\mathbb{R}_s))$, the latter coming from a similar argument since $a(\xi)$ is uniformly bounded. m^ε is bounded in measures and precompact in $W^{-1,p}_{loc}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, for $1 \le p < \frac{d+2}{d+1}$. By the averaging lemma (Theorem 3),

$$\int_\xi \mathbb{1}(u^\varepsilon, \xi)\psi(\xi)d\xi \quad \text{is precompact in } L^p_{loc},\ 1 < p < \frac{d+2}{d+1}$$

for $\psi(\xi) \in C_c^\infty(\mathbb{R})$.

Let R be a large positive number and consider $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi = 1$ on $(-R, R)$ and $0 \le \psi \le 1$. Then

$$\begin{aligned}
\Big| u^\varepsilon - \int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)\psi(\xi)d\xi \Big| &= \Big| \int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)(1 - \psi(\xi))d\xi \Big| \\
&\le \int_R^\infty |\mathbb{1}(u^\varepsilon, \xi)|d\xi + \int_{-\infty}^{-R} |\mathbb{1}(u^\varepsilon, \xi)|d\xi = (u^\varepsilon - R)^+ + (u^\varepsilon + R)^-
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int (u^\varepsilon - R)^+ + (u^\varepsilon + R)^- dxdt &\le \int_{|u^\varepsilon|>R} |u^\varepsilon| \, dxdt \\
&\le \frac{1}{R} \int_0^T \int |u^\varepsilon|^2 dxdt \le \frac{C}{R}
\end{aligned}$$

We conclude that $\{u^\varepsilon\}$ is Cauchy in $L^1_{loc,x,t}$.

Since $u^\varepsilon \in L^\infty_{x,t}$, it follows that (along subsequences) $u^\varepsilon \to u$ in $L^p_{loc}, p < \infty$, and almost everywhere and that $u \in L^\infty_{x,t}$, Next we pass to the limit $\varepsilon \to 0$ in (2.16). Along a further subsequence $m^\varepsilon \rightharpoonup m$ weak-* in measure; it follows

$$\partial_t \mathbb{1}(u,s) + a(s)\partial_x \mathbb{1}(u,s) = \partial_s m \qquad \text{in} \quad D'_{x,t,s} \tag{2.17}$$

Since m is a positive measure, we conclude u is the entropy solution of (1.1). $\square$

3 Discrete BGK model

In this section, as a special case of BGK model, we consider the following discrete BGK model:

$$\partial_t f_i^\varepsilon(x,t) + a_i \partial_x f_i^\varepsilon(x,t) = -\frac{1}{\varepsilon}[f_i^\varepsilon - \mathcal{M}_i(u^\varepsilon(x,t))] \quad i = 1, \cdots, d. \tag{3.1}$$

where $x \in \mathbb{R}, \quad a_i \in \mathbb{R}$ and $f_i : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}, \quad i = 1, 2, \cdots, d$. The function u^ε is defined by

$$u^\varepsilon(x,t) = \sum_{i=1}^{d} f_i^\varepsilon(x,t) \tag{3.2}$$

Like the BGK-model, we assume that the function $\mathcal{M}_i$ in system (3.1) is smooth, strictly increasing and satisfies

$$\begin{aligned} u(x,t) &= \sum_{i=1}^{d} \mathcal{M}_i(u(x,t)), \\ F(u(x,t)) &= \sum_{i=1}^{d} a_i \mathcal{M}_i(u(x,t)) \end{aligned} \tag{3.3}$$

Then if we sum the d equations in (3.1), we have

$$\partial_t u^\varepsilon(x,t) + \partial_x\left(\sum_{i=1}^{d} a_i f_i^\varepsilon(x,t)\right) = 0 \tag{3.4}$$

This model (3.1) is also equipped with a globally defined entropy function. If we denote by

$$\mu_i(f_i) = \int_0^{f_i} \mathcal{M}_i^{-1}(g)dg$$

then $\mu_i(\cdot)$ is convex and we have

$$
\begin{aligned}
\partial_t \sum_{i=1}^d \mu_i(f_i) + \partial_x \sum_{i=1}^d a_i\mu_i(f_i) \quad &- \quad u(\partial_t u + \partial_x \sum_{i=1}^d a_i f_i) \\
&+ \quad \frac{1}{\varepsilon}\sum_{i=1}^d (\mathcal{M}_i^{-1}(f_i) - u)(f_i - \mathcal{M}_i(u)) = 0
\end{aligned}
\tag{3.5}
$$

Let $H_i(f_i), Q_i(f_i)(= a_iH_i) \in C^1(\mathbb{R})$ be an entropy pair for the kinetic model (3.1) that extends the entropy pair $\eta(u) - q(u)$ of (1.1) :

$$
\begin{cases}
\sum_{i=1}^d H_i(\mathcal{M}_i(u)) & = \eta(u) \\
\sum_{i=1}^d a_i H_i(\mathcal{M}_i(u)) & = q(u)
\end{cases}
\tag{3.6}
$$

If we multiply (3.1) by $\frac{\partial H}{\partial f_i}(f_i)$, we have

$$
\partial_t \sum_{i=1}^d H_i(f_i) + \partial_x \sum_{i=1}^d a_i H_i(f_i) = -\frac{1}{\varepsilon}\sum_{i=1}^d \frac{\partial H}{\partial f_i}(f_i)[f_i - \mathcal{M}_i(u)]
\tag{3.7}
$$

and when $\varepsilon \to 0$ they are expected to converge to the entropy dissipation of the limit conservation laws.

If we follow the argument in section 2, then we get the approximate transport equation for the discrete BGK model (3.1):

Theorem 4 *Let $\mathcal{M}_i,\ i = 1, \cdots, d$ be smooth, strictly increasing function with (3.3). Let $f_i^\varepsilon(x, 0) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ i = 1, \cdots, d$.*
Then we have

$$
\begin{aligned}
&\partial_t \sum_{i=1}^d 1\!\!1(f_i, \mathcal{M}_i)\mathcal{M}_i'(s) + \partial_x \sum_{i=1}^d a_i\, 1\!\!1(f_i, \mathcal{M}_i)\mathcal{M}_i'(s) \\
&= \quad \partial_s \sum_{i=1}^d \frac{1}{\varepsilon}(1\!\!1(f_i, \mathcal{M}_i(s)) - 1\!\!1(\mathcal{M}_i(u), \mathcal{M}_i(s)))(f_i - \mathcal{M}_i(u)) \\
&= \quad \partial_s m^\varepsilon(x, t, s)
\end{aligned}
\tag{3.8}
$$

where m^ε is a bounded family of positive measures.

Also, we obtain compactness of approximate solutions by using the averaging lemma:

Corollary 5 *Let F satisfy*

$$
\mathit{meas}\{\xi \in \mathbb{R} \mid F'(\xi) = \alpha\} = 0, \quad \forall \alpha \in \mathbb{R}.
\tag{3.9}
$$

Let f_i^ε be a family of solutions to (3.1) with the initial data $f_i^\varepsilon(x,0) \in L^1 \cap L^\infty$. Then $u^\varepsilon = \sum_{i=1}^d f_i^\varepsilon$ converges to u in $L^p_{loc}(\mathbb{R}\times\mathbb{R}^+), 1 \leq p < \infty$, and u is an entropy solution of (1.1).

References

[1] F. Bouchut, Construction of BGK models with a family of kinetic entropies for a given system of conservation laws. *J. Statist. Phys.* **95** (1999), 113–170.

[2] S. Hwang and A. Tzavaras, Kinetic decomposition of approximate solutions to conservation laws: applications to relaxation and diffusion-dispersion approximations, *Comm. Partial Differential Equations*, to appear.

[3] Jin, S.; Xin, Z. The relaxing schemes for systems of conservation laws in arbitrary space dimensions, *Comm. Pure Appl. Math.* **1995**, 48, 235-277.

[4] M.A. Katsoulakis and A.E. Tzavaras, Contractive relaxation systems and the scalar multidimensional conservation law, *Comm. Partial Differential Equations* **22** (1997), 195-233.

[5] S.N. Kruzhkov, First order quasilinear equations with several independent variables, *Math. USSR Sbornik* **10** (1970), 217-243.

[6] Lions, P.L.; Perthame, B.; Tadmor, E. A kinetic formulation of scalar multidimensional conservation laws, *J. AMS* **1994**, 7, 169-191.

[7] R. Natalini, A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws, *J.Differential Equations* **148** (1998), 292-317.

[8] B. Perthame and P.E. Souganidis, A limiting case for velocity averaging, *Ann.Scient.Ecole Normale Superieure* **31** (1998), 591-598.

[9] B.Perthame and E.Tadmor, A kinetic equation with kinetic entropy functions for scalar conservation laws, *Comm. Math. Phys.* **136** (1991), 501-517.

[10] A.E.Tzavaras, On the mathematical theory of fluid dynamic limits to conservation laws, In*Advances in Mathematical Fluid Mechanics*, J. Malek, J. Necas and M. Rokyta, eds.; Springer, New York, 2000, pp 192-222.