Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE @ DIRECT•
Applied Mathematics Letters 17 (2004) 309–316

Applied Mathematics Letters
www.elsevier.com/locate/aml

Upwinding Sources at Interfaces in Conservation Laws

TH. KATSAOUNIS
Department of Applied Mathematics, University of Crete
GR-71409 Heraklion, Greece
and
Institute of Applied and Computational Mathematics
FORTH, GR-71110 Heraklion, Greece

B. PERTHAME
Département de Mathématiques et Applications
École Normale Supérieure, CNRS UMR 8553
45, rue d'Ulm, 75230 Paris Cedex 05, France

C. SIMEONI
Department of Applied Mathematics, University of Crete
GR-71409 Heraklion, Greece
and
Institute of Applied and Computational Mathematics
FORTH, GR-71110 Heraklion, Greece
and
Département de Mathématiques et Applications
École Normale Supérieure, CNRS UMR 8553
45, rue d'Ulm, 75230 Paris Cedex 05, France

(Received and accepted March 2003)

Communicated by P. Markowich

Abstract— Hyperbolic conservation laws with source terms arise in many applications, especially as a model for geophysical flows because of the gravity, and their numerical approximation leads to specific difficulties. In the context of finite-volume schemes, many authors have proposed to upwind sources at interfaces, the U.S.I. method, while a cell-centered treatment seems more natural. This note gives a general mathematical formalism for such schemes. We define consistency and give a stability condition for the U.S.I. method. We relate the notion of consistency to the "well-balanced" property, but its stability remains open, and we also study second-order approximations, as well as error estimates. The general case of a nonuniform spatial mesh is particularly interesting, motivated by two-dimensional problems set on unstructured grids. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Conservation laws, Upwinding source terms, Finite-volume schemes, Error estimates, Second-order approximations.

This work was partially supported by the ACI (Ministère de la Recherche, France): Modélisation de processus hydrauliques à surface libre en présence de singularités (http://www-rocq.inria.fr/m3n/CatNat/), and by HYKE European programme HPRN-CT-2002-00282 (http://www.hyke.org).
The authors would like to thank F. Bouchut and Th. Gallouët for helpful discussions.

 Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
doi: 10.1016/S0893-9659(04)00012-6

1. INTRODUCTION

We consider the Cauchy problem for a scalar conservation law, in one-space dimension,

$$\partial_t u + \partial_x A(u) + B(x,u) = 0, \qquad t \in \mathbb{R}_+, \quad x \in \mathbb{R}, \tag{1}$$

$$u(0,x) = u_0(x) \in L^p(\mathbb{R}), \qquad 1 \le p \le +\infty, \tag{2}$$

with $u(t,x) \in \mathbb{R}$ and we set $a(u) = A'(u) \in C^1(\mathbb{R})$. We focus our analysis on the source terms given by

$$\begin{gathered} B(x,u) = z'(x)b(u), \qquad b \in C^1(\mathbb{R}), \quad \frac{a(u)}{b(u)} \in L^\infty(\mathbb{R}), \\ z' \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}), \qquad \text{for some } 1 \le p < +\infty. \end{gathered} \tag{3}$$

This is suggested by the usual application of hyperbolic conservation laws as simple mathematical models for geophysical flows: in the case of the Saint-Venant equations for shallow water, for instance, $z(x)$ describes the bottom topography (see e.g., [1,2]).

Equation (1) is endowed with the family of entropy inequalities, in the distributional sense,

$$\partial_t S(u) + \partial_x \eta(u) + S'(u)B(x,u) \le 0, \qquad \eta'(u) = S'(u)a(u), \tag{4}$$

for any pair of a convex entropy function S and corresponding entropy flux η (refer to [3–5]) and we are interested in computing the entropy solutions which means that we wish to use solvers (denoted by $\mathcal{A}$ below) satisfying the E condition [6] even though it does not appear as a necessary condition for our results. With the assumptions mentioned above, Kružkov's method [7] can be applied to prove existence and uniqueness of the entropy solution to (1),(2), in the functional space $C(\mathbb{R}_+; L^1(\mathbb{R})) \cap L^\infty((0,T) \times \mathbb{R})$. This relies on the L^1-control of the space translations and the contraction property of problem (1),(2).

In comparison with the homogeneous case, the analytical properties of equation (1) are modified by the source term. In particular, a significant difference is the occurrence of nonconstant stationary solutions, resulting from the balance between source term and internal forces. By integrating the stationary equation associated with (1), according to (3), we obtain an algebraic relation for smooth steady-state solutions,

$$D(u(x)) + z(x) = C^{\mathrm{st}}, \qquad D'(u) = \frac{a(u)}{b(u)}. \tag{5}$$

It is now accepted that suitable numerical approaches to problem (1),(2) should preserve the steady-state solutions (5), or some discrete versions at least, with enough accuracy. There are several advantages to do so. In many applications, one wishes to compute small perturbations of such steady states; schemes which do not preserve steady states exhibit a bad time decay towards these solutions. A better stability condition is generally obtained (the source term may not affect the CFL condition, as seen below). One wishes to keep the physical property that the value $u(x)$ of the steady states is only determined by $z(x)$ values (in the case D is monotonic). These properties are not satisfied by centered schemes applied to the source term and thus other methods have been proposed in the literature. All of them are based on the principle of upwinding sources at interfaces (the U.S.I. method). This idea of U.S.I. schemes was first used in the context of reacting flows (see [8–10], for instance), then the source does not depend on z and a conclusion was that it is enough to introduce a correction of the flux solver to take into account the source. In the context of gravity driven flows, many authors arrived at the conclusion that it is necessary to split and thus upwind the source itself at interfaces, see [1,11–19]. This yields interesting questions from a theoretical point of view.

In this note, we formalize the U.S.I. method in full generality and we isolate the new mathematical concepts that are needed for the convergence analysis. We first present the formalism of the U.S.I. method, and define a general notion of consistency and a possible notion of stability. The interest here is for nonuniform meshes, since stability is not always true, as we show on a counterexample. Then, we indicate the relations with well-balanced schemes and we also propose appropriate second-order extensions.

2. THE U.S.I. METHOD

In this section, we consider the finite-volume method for treating numerically hyperbolic systems of conservation laws, and the consistency analysis presented here extends to systems even though we restrict our notations to the scalar case. This method is robust and presents the advantage to be conservative (refer, e.g., to [4,20]). We set up a mesh on $\mathbb{R}$ made up of cells $\mathcal{C}_i = [x_{i-1/2}, x_{i+1/2})$, with center x_i, cell interfaces $x_{i+1/2}$ and nonuniform length Δx_i, for $i \in \mathbb{Z}$. We set $h = \sup_{i\in\mathbb{Z}} \Delta x_i$. Then, we construct a piecewise constant approximation of the function $z(x)$ on the mesh, whose coefficients are $z_i = (1/\Delta x_i)\int_{\mathcal{C}_i} z(x)\,dx$. We also introduce a time-step Δt and we set $t_n = n\Delta t$, $n \in \mathbb{N}$; therefore, we have to consider an additional restrictions on the ratio $\Delta t/\Delta x_i$, the usual CFL condition, to guarantee numerical stability. In this framework, the discrete unknowns are expected to be approximations of the cell-averages of the solution, $u_i^n \approx (1/\Delta x_i)\int_{\mathcal{C}_i} u(t_n, x)\,dx$, $i \in \mathbb{Z}$ (the conservative quantities are cell-centered), while the numerical fluxes are defined at the interfaces of the mesh. On the other hand, departing from discrete data, we reconstruct the piecewise constant function $u^h(t,x) = u_i^n$, $t \in [t_n, t_{n+1})$, $x \in \mathcal{C}_i$. The general fully explicit, three points, U.S.I. scheme for equation (1) reads

$$\frac{\Delta x_i}{\Delta t}\left(u_i^{n+1} - u_i^n\right) + \left(A_{i+1/2}^n - A_{i-1/2}^n\right) + \mathcal{B}_{i-1/2}^{n,+} + \mathcal{B}_{i+1/2}^{n,-} = 0. \tag{6}$$

The numerical fluxes are usual consistent approximations of the analytical flux function,

$$A_{i+1/2}^n = \mathcal{A}\left(u_i^n, u_{i+1}^n\right), \qquad \mathcal{A} \in C^1, \quad \mathcal{A}(u,u) = A(u), \tag{7}$$

and they can be chosen in the general class of E-schemes (see [5], for instance). Because of the choice of source terms in form (3), the function $z(x)$ is defined up to a constant. Therefore, without loss of generality, we assume that the source term is discretized by

$$\begin{gathered}\mathcal{B}_{i+1/2}^{n,\pm} = \mathcal{B}^\pm\left(\Delta x_i, \Delta x_{i+1}; u_i^n, u_{i+1}^n, z_{i+1} - z_i\right), \qquad \mathcal{B}^\pm \in C^2, \\ \mathcal{B}^\pm(h,k;u,v,0) = 0, \qquad \frac{\partial \mathcal{B}^\pm}{\partial u}(h,k;u,v,0) = \frac{\partial \mathcal{B}^\pm}{\partial v}(h,k;u,v,0) = 0.\end{gathered} \tag{8}$$

The dependency only upon $z_{i+1} - z_i$ is natural, because the problem is unchanged when adding a constant to z. Also, the last condition is natural when problem (1)–(3) becomes homogeneous, namely, $z'(x) = 0$, since the scheme (6)–(8) reduces to the usual finite-volume approximation for scalar conservation laws. Such a discretization is also upwinded, in the sense that $\mathcal{B}_{i+1/2}^-$ represents the contribution of the waves coming from the left of the interface $x_{i+1/2}$ and moving towards the cell $\mathcal{C}_i$ with nonpositive velocity, while $\mathcal{B}_{i+1/2}^+$ represents the waves moving forward from the right of the interface $x_{i+1/2}$ with nonnegative velocity.

When the spatial mesh is not regular, a general notion of consistency for the U.S.I. method is not obvious to formulate.

DEFINITION 1. *A U.S.I. scheme (6)–(8) is consistent with equation (1) if, locally uniformly in (u,h,k), it holds,*

$$\lim_{\lambda\to 0} \frac{\mathcal{B}^+(h,k;u,u,\lambda) + \mathcal{B}^-(h,k;u,u,\lambda)}{\lambda} = b(u). \tag{9}$$

The following result constitutes a first stage for the convergence analysis of the U.S.I. method and extends the classical Lax-Wendroff theorem [21]. We note that it is valid for systems as well.

THEOREM 1. *Consider a U.S.I. scheme (6)–(8), which satisfies the consistency condition (9). We assume, for various constants C: the CFL condition $\Delta t \leq Ch$, that $\|u^h\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R})} \leq C$ and that $u^h \longrightarrow u$ in $L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$, as $h \to 0$. We also assume either that, for all bounded subsets Ω of $\mathbb{R}_+ \times \mathbb{R}$, it holds*

$$\sum_{(n,i)\in K_\Omega} (\Delta x_i + \Delta x_{i+1}) \left|u^n_{i+1} - u^n_i\right| \longrightarrow 0, \qquad \text{as } h \to 0, \tag{10}$$

where K_Ω denotes the set of indices such that $(t_n, x_i) \in \Omega$; or weak regularity of the mesh, namely

$$\exists\, \alpha, \beta > 0, \qquad \text{so that } \alpha \Delta x_{i+1} \leq \Delta x_i \leq \beta \Delta x_{i+1}, \quad \forall\, i \in \mathbb{Z}. \tag{11}$$

Then u is a weak solution to the initial values problem (1),(2).

The proof of Theorem 1 applies standard arguments for homogeneous systems of conservation laws, adapted for dealing with the source term. We refer to [22] for details. We point out that the definition of consistency (9) for the source term does not imply that the consistency error vanishes, in finite-volume sense. Indeed, due to the choice of a nonuniform spatial mesh, the space-step Δx_i could be very different from the length $|x_{i+1} - x_i|$ of an interfacial interval. Therefore, standard techniques for uniform mesh (see e.g., [5,20]) do not apply in this case and the additional hypothesis (10) or (11) is required to control the spatial variations of the numerical solution in comparison with the cells of the mesh. Two examples illustrate the optimality of Theorem 1.

EXAMPLE 1. We consider $A = \mathcal{A} = 0$, $b(u) = 1$, and $z'(x) = C^{\mathrm{st}}$ in equation (1). We set up $\Delta x_i = h$ for i even, $\Delta x_i = h/2$ for i odd, and the discretization for the source term is given by

$$\mathcal{B}^-(u, v, \lambda) = \mathcal{B}^+(u, v, \lambda) = \frac{\lambda}{2},$$

independent of Δx_i, Δx_{i+1}. This is a consistent scheme and the (explicit) discrete solution converges weakly but not strongly to the solution of (1),(2). With an additional term $b(u)$, it does not converge to the correct limit.

EXAMPLE 2. We consider $A = \mathcal{A} = 0$, $b(u) = 1$, and $z'(x) = C^{\mathrm{st}}$ in equation (1). We set up $\Delta x_i = h$ for i even, $\Delta x_i = h/2$ for i odd, and the discretization for the source term is given by

$$\mathcal{B}^-_{i+1/2}(u, v, \lambda) = \frac{\lambda \Delta x_i}{(\Delta x_i + \Delta x_{i+1})}, \qquad \mathcal{B}^+_{i+1/2}(u, v, \lambda) = \frac{\lambda \Delta x_{i+1}}{(\Delta x_i + \Delta x_{i+1})}.$$

This is a consistent scheme and, again on the explicit solution, one can see that it converges strongly.

This second example enters a general compactness framework, that completes Theorem 1, using the stability conditions (we do not write the more general condition here, but look for simplicity)

$$\begin{aligned}
\mathcal{B}^+_{i-1/2} &= \frac{\Delta x_i}{\Delta x_i + \Delta x_{i-1}}\, b^+\left(\Delta x_{i-1}, \Delta x_i; u_{i-1}, u_i, z_i - z_{i-1}\right)(z_i - z_{i-1}), \\
\mathcal{B}^-_{i+1/2} &= \frac{\Delta x_i}{\Delta x_i + \Delta x_{i+1}}\, b^-\left(\Delta x_i, \Delta x_{i+1}; u_i, u_{i+1}, z_{i+1} - z_i\right)(z_{i+1} - z_i).
\end{aligned} \tag{12}$$

Then, consistency condition (9) can be rewritten as follows:

$$hb^-(h, k; u, u, 0) + k\, b^+(h, k; u, u, 0) = (h + k) b(u). \tag{13}$$

It implies, by opposition to the more general condition (9), that the truncation error vanishes (for nonuniform grids, otherwise, the two conditions are equivalent). Another statement in the same direction is as follows.

THEOREM 2. *Assume that $z'' \in L^1(\mathbb{R})$, $b^-, b^+ \in C^1$, and $\mathcal{A}$ is an ordered scheme for the fluxes (i.e., $\mathcal{A}(u,v) \le \min(A(u), A(v))$, for $u < v$, and $\mathcal{A}(u,v) \ge \max(A(u), A(v))$, for $u > v$; refer to [23,24]). Then, a U.S.I. scheme with the stability condition (12) satisfies, for $T = n\Delta t$,*

$$\sum_{i\in\mathbb{Z}} \left|u_{i+1}^n - u_i^n\right| \le \|u_0\|_{TV(\mathbb{R})} + C\left(T, \|z'\|_{\mathrm{BV}(\mathbb{R})}\right).$$

A specific convergence result which does not involve BV, neither weak BV, bounds is given in [16] for uniform grids. But no general theory is available in the "weak BV" framework (see [8]), and especially for two-dimensional problems set on unstructured grids. We conjecture that either the form (12) is enough or, at least, the "well-balanced" property that we relate now to the above theory.

3. WELL-BALANCED SCHEMES

We now assume that the function D in (5) is strictly monotonic, unbounded from below and above (conditions that are not satisfied in the case of systems). For all z', this ensures the existence of a unique Lipschitz continuous steady state, or a discrete steady state

$$D(u_i) + z_i = C^{\mathrm{st}}, \qquad \forall\, i \in \mathbb{Z}. \tag{14}$$

Following [13], the "well-balanced" schemes are the U.S.I. schemes (6)–(8) for which the above discrete stationary solutions are preserved.

LEMMA 1. *The U.S.I. method (6)–(8) for problem (1),(2) is well balanced if and only if, for all u, v, z_-, z_+ such that $D(u) + z_- = D(v) + z_+$, we have the equalities*

$$\mathcal{A}(u,v) - A(u) + \mathcal{B}^-(u,v,z_+ - z_-) = 0, \qquad A(v) - \mathcal{A}(u,v) + \mathcal{B}^+(u,v,z_+ - z_-) = 0.$$

In particular, we can see that the mesh size Δx_i, Δx_{i+1} does not appear in this scheme that satisfies this property.

Several well-balanced schemes have been developed, either by taking the source term into account directly in the numerical fluxes (see e.g., [13,16,25]), or by making explicit use of discrete relations for the stationary solutions (see e.g., [15,17]). The simplest example, which does not extend to systems in practical situations, is to choose

$$\mathcal{B}^-_{i+1/2} = \mathcal{A}\left(u_i, u_{i+1}^-\right) - \mathcal{A}\left(u_i, u_{i+1}\right), \qquad D\left(u_{i+1}^-\right) + z_{i+1} = D\left(u_i\right) + z_i.$$

The following result, obtained by means of standard asymptotic expansions, guarantees the consistency with equation (1) for well-balanced schemes.

LEMMA 2. *A well-balanced scheme, in the sense of Lemma 1, verifies the consistency condition (9).*

Of course, well-balanced schemes do not satisfy the strong condition (12) on nonuniform grids. Therefore, a general notion of strong stability is still needed.

4. SECOND-ORDER U.S.I. SCHEMES

In practice, higher-order accuracy is needed. We propose two extensions of the finite-volume scheme (6)–(8) to second-order approximations. The difficulty is to take into account second-order accuracy on z also, which appears to be essential for practical computations.

We consider piecewise linear reconstructions of the discrete functions on the mesh, with numerical derivatives computed by applying an appropriate *slope limiter* technique (refer to [5]). For the sake of simplicity, we consider only first-order discretization in time, but it is easy to recover higher-order accuracy by applying Runge-Kutta methods, for instance.

A first approach to second-order schemes for the U.S.I. method reads

$$\frac{\Delta x_i}{\Delta t}\left(u_i^{n+1}-u_i^n\right)+\left(A_{i+1/2}^n-A_{i-1/2}^n\right)+\mathcal{B}_{i-1/2}^{n,+}+\mathcal{B}_{i+1/2}^{n,-}+\Delta x_i B_i^n=0, \tag{15}$$

where the numerical fluxes $A_{i+1/2}^n=\mathcal{A}(u_i^{n,+},u_{i+1}^{n,-})$ and $\mathcal{B}_{i+1/2}^{n,\pm}=\mathcal{B}^{\pm}(u_i^{n,+},u_{i+1}^{n,-},z_{i+1}^{-}-z_i^{+})$ are defined by means of the interfacial values

$$u_i^{n,-}=u_i^n-\frac{\Delta x_i}{2}u_i', \qquad u_i^{n,+}=u_i^n+\frac{\Delta x_i}{2}u_i', \qquad z_i^-=z_i-\frac{\Delta x_i}{2}z_i', \qquad z_i^+=z_i+\frac{\Delta x_i}{2}z_i'. \tag{16}$$

Because of the introduction of piecewise linear approximations of $z(x)$, the difference of interfacial values approximates the second derivative and the upwind discretization "overtakes" the source term (3). An additional centered term $B_i^n=z_i'b(u_i^n)$, which depends on the cell-averages, is thus necessary in (15) to recover the first derivative and to achieve second-order accuracy.

An alternative approach to obtain second-order extensions of the U.S.I. method is based on improving the consistency properties of the discrete source term. We consider piecewise constant approximations of the function $z(x)$ and piecewise linear reconstructions of the numerical solution on the mesh, whose interfacial values are used to define $A_{i+1/2}^n=\mathcal{A}(u_i^{n,+},u_{i+1}^{n,-})$ and $\mathcal{B}_{i+1/2}^{n,\pm}=\mathcal{B}^{\pm}(u_i^{n,+},u_{i+1}^{n,-},z_{i+1}-z_i)$ in the finite-volume scheme (6), as suggested by the special form of the source term (3) given by the product of functions with different orders of derivative. To recover second-order accuracy, we assume an improved consistency condition, as in Definition 1: there exists a constant $C_{\mathcal{B}}$ such that

$$\left|\frac{\mathcal{B}^+(u,u,\lambda)+\mathcal{B}^-(u,u,\lambda)}{\lambda}-b(u)\right|\le C_{\mathcal{B}}\lambda^2. \tag{17}$$

Both second-order methods illustrated above are strictly related, as formally verified by means of standard asymptotic expansions on the numerical functions and simple algebraic calculations with the differences of discrete interfacial values (16).

Following a remark by Bouchut [26], the property of preserving steady-state solutions is also satisfied by the above second-order schemes, when the piecewise linear reconstruction is performed on $D(u_i^n)$ and z_i also. Indeed, "symmetric" limiters will then preserve the discrete relation (14). The second-order scheme (15) is validated by the numerical results obtained for the steady-state solutions of the Saint-Venant equations in [2].

5. ERROR ESTIMATES FOR U.S.I. SCHEMES

Deriving error estimates that confirm the accuracy of second-order schemes faces the specific difficulty that, for transport equations, only $h^{1/2}$ rate of convergence is proved. In order to avoid this difficulty, we only discretize the source term, while keeping continuous the transport terms in some kind of extended semidiscrete versions of equation (1) with a linear flux,

$$\partial_t u+\partial_x u+B(x,u)=0, \qquad t\in\mathbb{R}_+, \quad x\in\mathbb{R}, \tag{18}$$

and

$$\partial_t u^h+\partial_x u^h+B^h\left(x,u^h\right)=0, \tag{19}$$

where $B^h(x,u^h)$ indicates an appropriate discretization of the source term (3) according to the U.S.I. method, by using the cell-averages $u_i^h(t)=(1/h)\int_{C_i}u^h(t,x)\,dx$, $i\in\mathbb{Z}$.

For simplicity, we use an uniform spatial mesh, i.e. $\Delta x_i=h, \forall i\in\mathbb{Z}$, and we obtain

$$B^h\left(x,u^h\right)=\sum_{i\in\mathbb{Z}}\frac{1}{h}\left[\mathcal{B}^+\left(u_{i-1}^h,u_i^h,z_i-z_{i-1}\right)+\mathcal{B}^-\left(u_i^h,u_{i+1}^h,z_{i+1}-z_i\right)\right]\mathbb{1}_{C_i}(x). \tag{20}$$

THEOREM 3. *We assume $z \in W^{2,p}$, $1 \le p < +\infty$, and we consider a numerical source term (20) satisfying the first-order consistency condition (9). Then, for all $t \in \mathbb{R}_+$, we have the first-order error estimate*

$$\left\|u(t) - u^h(t)\right\|_{L^p} \le C(t)\left(\left\|u_0 - u_0^h\right\|_{L^p} + h\|z\|_{W^{2,p}} + h\int_0^t \|u(s)\|_{W^{1,p}}\,ds\right).$$

The convergence properties of second-order schemes are notably affected by the technique used to construct piecewise linear approximations of the numerical functions, namely the choice of the *slope limiter*. Without appropriate hypotheses on the coefficients of such approximations, the proof of the error estimate fails and numerical evidence shows that the discretization (15) loses second-order accuracy. This leads to the use of discrete derivatives computed in the restricted class of slope limiters introduced in [27,28]. We refer to [29] for details and for the proof of these results.

THEOREM 4. *With the above restrictions, we assume $z \in W^{3,p}$, $1 \le p < +\infty$, and we consider the numerical source term in (18) defined as in (15). Then, for all $t \in \mathbb{R}_+$, we have the second-order error estimate*

$$\left\|u(t) - u^h(t)\right\|_{L^p} \le C(t)\left(\left\|u_0 - u_0^h\right\|_{L^p} + h^2\|z\|_{W^{3,p}} + h^2\int_0^t \|u(s)\|_{W^{2,p}}\,ds\right).$$

REFERENCES

1. B. Perthame and C. Simeoni, A kinetic scheme for the Saint-Venant system with a source term, *Calcolo* **38** (4), 201–231, (2001).
2. Th. Katsaounis and C. Simeoni, Second order approximation of the viscous Saint-Venant system and comparison with experiments, In *Hyperbolic Problems: Theory, Numerics, Applications, (HYP2002)*, (Edited by T. Hou and E. Tadmor), Springer, (2003).
3. F. Bouchut and B. Perthame, Kružkov's estimates for scalar conservation laws revisited, *Trans. Amer. Math. Soc.* **350** (7), 2847–2870, (1998).
4. R. Eymard, T. Gallouët and R. Herbin, *Finite Volume Methods, Handbook of Numerical Analysis, Volume VIII*, (Edited by P.G. Ciarlet and J.L. Lions), North-Holland, Amsterdam, (2000).
5. E. Godlewski and P.A. Raviart, *Hyperbolic Systems of Conservation Laws, Mathématiques & Applications, Volume 3/4*, Ellipses, Paris, (1991).
6. S. Osher, Riemann solvers, the entropy condition, and difference approximations, *SIAM J. Numer. Anal.* **21** (2), 217–235, (1984).
7. S.N. Kružkov, First order quasilinear equations in several independent space variables, *Math. USSR Sb.* **10**, 217–243, (1970).
8. P.L. Roe, Upwind differencing schemes for hyperbolic conservation laws with source terms, In *Nonlinear Hyperbolic Problems, Lecture Notes in Math., Volume 1270*, (Edited by C. Carasso, P.A. Raviart and D. Serre), pp. 41–51, Springer-Verlag, Berlin, (1987).
9. M. BenArtzi, The generalized Riemann problem for reactive flows, *J. Comp. Phys.* **81**, 70–101, (1989).
10. F. Bereux and L. Sainsaulieu, A Roe-type Riemann solver for hyperbolic systems with relaxation based on time-dependent wave decomposition, *Numer. Math.* **77** (2), 143–185, (1997).
11. P. Cargo and A.Y. LeRoux, Un schéma équilibre adapté au modèle d'atmosphère avec termes de gravité, *C.R. Acad. Sci. Paris Sér. I Math.* **318** (1), 73–76, (1994).
12. L. Gosse, A priori error estimate for a well-balanced scheme designed for inhomogeneous scalar conservation laws, *C.R. Acad. Sci. Paris Sér. I Math.* **327** (5), 467–472, (1998).
13. J.M. Greenberg and A.Y. LeRoux, A well-balanced scheme for the numerical processing of source terms in hyperbolic equations, *SIAM J. Numer. Anal.* **33**, 1–16, (1996).
14. T. Gallouët, J.M. Hérard and N. Seguin, Some approximate Godunov schemes to compute shallow-water equations with topography, *AIAA-2001*, (2000).
15. R.J. LeVêque, Balancing source terms and flux gradients in high-resolution Godunov methods: The quasi-steady wave-propagation algorithm, *J. Comput. Phys.* **146** (1), 346–365, (1998).
16. R. Botchorishvili, B. Perthame and A. Vasseur, Equilibrium schemes for scalar conservation laws with stiff sources, *Math. Comp.* **72** (241), 191–157, (2003).
17. S. Jin, A steady-state capturing method for hyperbolic systems with geometrical source terms, *M2AN Math. Model. Numer. Anal.* **35** (4), 631–645, (2001).
18. A. Kurganov and L. Doron, Central-upwind schemes for the Saint-Venant system, *M2AN Math. Model. Numer. Anal.* **36** (3), 397–425, (2002).

19. N. Botta, R. Klein, S. Langenberg and S. Lützenkirchen, Well balanced finite volume methods for nearly hydrostatic flows (preprint) (2002).
20. E. Godlewski and P.A. Raviart, Numerical approximation of hyperbolic systems of conservation laws, In *Applied Mathematical Sciences, Volume 118*, Springer-Verlag, (1996).
21. P.D. Lax and B. Wendroff, Systems of conservations laws, *Comm. Pure Appl. Math.* **13**, 217–237, (1960).
22. B. Perthame and C. Simeoni, Convergence of the upwind interface source method for hyperbolic conservation laws, In *Hyperbolic Problems: Theory, Numerics, Applications, (HYP2002)*, (Edited by T. Hou and E. Tadmor), Springer, (2003).
23. R. Sanders, On convergence of monotone finite difference schemes with variable spatial differencing, *Math. Comp.* **40**, 499–518, (1983).
24. B. Perthame, *Kinetic Formulation of Conservation Laws*, Oxford University Press, (2002).
25. L. Gosse and A.Y. LeRoux, A well-balanced scheme designed for inhomogeneous scalar conservation laws, *C.R. Acad. Sci. Paris Sér. I Math.* **323** (5), 543–546, (1996).
26. F. Bouchut, An introduction to finite volume methods for hyperbolic systems of conservation laws with source, In *Problèmes Nonlinéaires Appliqués: Ecoulements peu Profonds à Surface Libre, Écoles CEA-EDF-INRIA*, Octobre 2002, pp. 44–49.
27. A. Harten and S. Osher, Uniformly high-order accurate nonoscillatory schemes I, *SIAM J. Numer. Anal.* **24** (2), 279–309, (1987).
28. C.W. Shu, High order ENO and WENO schemes for computational fluid dynamics. High-order methods for computational physics, In *Lect. Notes Comput. Sci. Eng., Volume 9*, pp. 439–582, Springer, Berlin, (1999).
29. Th. Katsaounis and C. Simeoni, First and second order error estimates for the upwind interface source method, In *Hyperbolic Problems: Theory, Numerics, Applications, (HYP2002)*, (Edited by T. Hou and E. Tadmor), Springer, (2003).