The Toda Rarefaction Problem

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Abstract

In the Toda shock problem (see [7], [11], [8], and also [3]) one considers a driving particle moving with a fixed velocity $2a$ and impinging on a one-dimensional semi-infinite lattice of particles, initially equally spaced and at rest, and interacting with exponential forces. In this paper we consider the related Toda rarefaction problem in which the driving particle now moves away from the lattice at fixed speed, in analogy with a piston being withdrawn, as it were, from a container filled with gas.

We make use of the Riemann-Hilbert factorization formulation of the related inverse scattering problem. In the case where the speed $2|a|$ of the driving particle is sufficiently large ($|a| > 1$), we show that the particle escapes from the lattice, which then executes a free motion of the type studied, for example, in [5]. In other words, in analogy with a piston being withdrawn too rapidly from a container filled with gas, cavitation develops.

1. Introduction

The Toda rarefaction problem concerns the long-time behavior of the solution of the following initial-boundary-value problem,

$$\begin{aligned} \dot{x}_n &= y_n\,, \quad n \geqq 1\,, \\ \dot{y}_n &= e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}\,, \quad n \geqq 1\,, \end{aligned} \tag{1.1}$$

where $x_0(t)$ is prescribed by

$$\begin{aligned} x_0(t) &= 2at, \quad \text{for all } t\,, \\ x_n(0) &= 0, \quad \text{for } n \geqq 1\,, \\ y_n(0) &= 0, \quad \text{for } n \geqq 1\,, \end{aligned} \tag{1.2}$$

and the constant $a < 0$.

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If $a > 0$, equations (1.1) and (1.2) describe the so-called Toda shock problem, which has been analyzed in [7], [11], [8], [3] . It turns out that there is a critical value, $a = 1$. As $t \to \infty$ in the case $a > 1$, the solution develops periodic oscillations in the wake of the driving particle, $x_0 = 2at$. On the other hand, as $t \to \infty$ in the case $0 < a < 1$, the solution decays to a quiescent lattice. The name *Toda shock* arises from the fact that, for $a > 0$, equations (1.1) and (1.2) clearly describe a driving particle impinging at a fixed speed on a lattice $\{x_n, n \geqq 1\}$ at rest, in analogy with a piston compressing a container filled with gas. In the case $a < 0$, the piston is being withdrawn, as it were, at a fixed speed from the lattice and this gives rise to the name *Toda rarefaction.* On the basis of numerical computations, and also some analytical considerations (see below), we again expect critical behavior, with the critical value now given by $a = -1$. In this paper we restrict our attention to the case of strong rarefaction, $a < -1$.

Our main result, Theorem 1.1 below, shows that as $t \to \infty$, the system (1.1) and (1.2) behaves like the semi-infinite Toda lattice,

$$\begin{aligned} \dot{x}_n &= y_n, \quad n \geqq 1, \\ \dot{y}_n &= e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}, \quad n \geqq 2, \\ \dot{y}_1 &= -e^{x_1 - x_2} \end{aligned} \tag{1.3}$$

(cf. [5]). The intuition (see also Remark 2 below) behind this result is as follows. From [5] we learn that, for the initial conditions $x_n(0) = y_n(0) = 0$ in (1.3), the velocity $y_n(t)$ of the nth particle converges as $t \to \infty$ to -2. Hence, if we choose $a < -1$ in (1.1) and (1.2), we expect that the driving particle x_0 escapes from the lattice $\{x_n, n \geqq 1\}$, and as the interaction $e^{x_0(t)-x_1(t)}$ between the driver and the lattice decreases rapidly to zero, the system (1.1) and (1.2) is free to execute the motion (1.3).

As in [7], [11], and [8], we can convert (1.1) and (1.2) into an autonomous system by doubling up the lattice as follows. First, one changes the reference frame so that $x_0(t) = 0$ for all time. Then one simply sets $x_n = -x_{-n}$ and $y_n = -y_{-n}$, for $n < 0$. It turns out that the system (1.1) and (1.2) is equivalent to the following initial-value problem:

$$\begin{aligned} \dot{x}_n &= y_n, \quad n \in \mathbb{Z}, \\ \dot{y}_n &= e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}, \quad n \in \mathbb{Z}, \end{aligned} \tag{1.4}$$

with

$$\begin{aligned} x_n(0) &= 0, \quad n \in \mathbb{Z}, \\ y_n(0) &= -\mathrm{sgn}(n)\, 2a, \quad n \in \mathbb{Z}. \end{aligned} \tag{1.5}$$

If one uses the Flaschka transformation,

$$\begin{aligned} a_n &= -\frac{y_n}{2}, \\ b_n &= \frac{1}{2} e^{\frac{1}{2}(x_n - x_{n+1})}, \end{aligned} \tag{1.6}$$

one can rewrite system (1.4) and (1.5) as

$$\begin{aligned} \dot{a}_n &= 2(b_n^2 - b_{n-1}^2), \quad n \in \mathbb{Z}, \\ \dot{b}_n &= b_n(a_{n+1} - a_n), \quad n \in \mathbb{Z}, \end{aligned} \tag{1.7}$$

with initial conditions

$$\begin{aligned} a_n(0) &= \operatorname{sgn}(n)\, a, \\ b_n(0) &= \tfrac{1}{2}. \end{aligned} \tag{1.8}$$

In our analysis we will in fact consider the more general problem with initial data

$$\begin{aligned} a_n(0) &= a_n^0, \\ b_n(0) &= b_n^0, \end{aligned} \tag{1.9}$$

such that $a_n^0 - \operatorname{sgn}(n)\, a$ and $b_n^0 - \frac{1}{2}$ decay to zero at $\pm\infty$ faster than any polynomial, and such that a_n, b_n satisfy the symmetries

$$a_n = -a_{-n}, \quad b_{-n} = b_{n-1}, \tag{1.10}$$

at time $t = 0$ and hence for all t.

System (1.7) takes the form of a Lax-pair isospectral deformation

$$\frac{dL}{dt} = [B(L), L] = B(L)L - LB(L), \tag{1.11}$$

where the Lax operator L is the doubly infinite tridiagonal matrix

$$L = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & b_{-1} & a_0 & b_0 & & \\ & & b_0 & a_1 & b_1 & \\ & & & b_1 & a_2 & b_2 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & -b_{-1} & 0 & b_0 & & \\ & & -b_0 & 0 & b_1 & \\ & & & -b_1 & 0 & b_2 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Under the conditions (1.9) with $a < -1$, the spectrum of L (see Figure 1.1) consists of two continuous bands, together with a finite odd number ($2m + 1$, say) of L^2 eigenvalues,

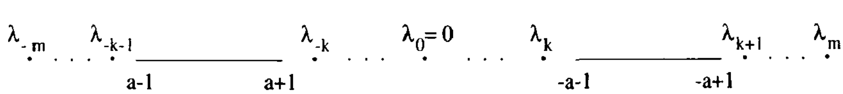

Figure 1.1. The spectrum of the Lax operator L.

$2k+1$ of which lie in the gap $(a+1, -a-1)$. The symmetry (1.10) of a_n, b_n implies that eigenvalues come in pairs ($\lambda_j = -\lambda_{-j}$) and that $\lambda_0 = 0$ is always an eigenvalue.

For simplicity, in this paper we will restrict our analysis to the case where the Lax operator L is nonresonant (and hence generic) in the sense of inverse scattering theory (see Section 2). However, our techniques can also be used to analyze the resonant case (see Section 4.8). The following is the main result of this paper.

THEOREM 1.1. *Let $\{x_n(t)\}_{n=1}^{\infty}$ be the solution of* (1.1) *with the initial condition that*

$$x_n(0) \to 0, \quad \dot{x}_n(0) \to 0, \tag{1.12}$$

rapidly as $n \to \infty$. Suppose that the corresponding Lax operator L is nonresonant. Then, for fixed n, as $t \to \infty$,

$$\begin{aligned} x_n &= 2(\lambda_n + a)t + K_n + O(e^{-\tilde{\epsilon}_n t}), \quad \text{for } 1 \leqq n \leqq k, \\ x_n &= -2t + (2(n-k) - 1/2)\log t + K_n + O(1/t), \quad \text{for } n > k, \end{aligned} \tag{1.13}$$

where $\tilde{\epsilon}_n > 0$ is defined in (4.36b) *and K_n is given by* (4.39).

Remark 1. Theorem 1.1 follows immediately from the formulae (4.37), which give the asymptotic behavior of $b_n^2(t)$. Formulae (4.37) should be compared with the formulae of [5, p. 391].

Remark 2. Note that if $-1 < a < 0$, the gap $(a+1, -a-1)$ in the spectrum of L closes up and the analysis of the solution of the rarefaction problem changes radically. This constitutes additional a priori evidence that $a = -1$ is a critical value for the rarefaction problem, and indeed, numerical calculations show that for $-1 < a < 0$ all the particles eventually move at the speed of the piston x_0, forming a regular lattice behind x_0 with spacing $\Delta x_n = x_n - x_{n+1}$, which converges to $2\log(1+a)$ as $t \to \infty$ (see also [8]).

Remark 3. In the resonant case formula (1.13) changes depending on the nature of the resonance (see Section 4.8). The results of Section 4.8 should also be compared with the calculations of [5, p. 391].

A simple computation (see the remark preceding equations (2.9) in Section 2) shows that for the pure shock initial data (1.8), the Lax operator L is nonresonant and $m = k = 0$.

Remark 4. In Theorem 1.1 the eigenvalues $\lambda_1, \ldots, \lambda_k \in (0, -a-1)$ play a distinguished role. From the mathematical point of view, the special character of these eigenvalues is rather subtle and emerges only after many detailed calculations. However, from the physical point of view, the explanation is clear. As noted above, the piston moves to the left with velocity $2a$, escaping from the bulk of the particles which move with velocity -2. (For this connection see also the discussion of [5, pp. 388 and 389]). In addition, the first k particles move with velocity $2(\lambda_n + a)$, $1 \leqq n \leqq k$, and as $0 < \lambda_n < -a - 1$, this means that these particles are moving in the vacuum behind the piston and ahead of the bulk of the gas. On the other hand, particles moving with velocity $2(\lambda_n + a)$ with $\lambda_n < 0$ would escape from the container, which surely cannot happen, whereas particles moving with velocity $2(\lambda_n + a)$, $\lambda_n > -a - 1$, would move into the bulk of the gas and not be seen.

Remark 5. In this paper we have restricted our attention to the symmetric case where the coefficients satisfy (1.10). The techniques introduced below can also be used to analyze the nonsymmetric case, where $a_n^0 - \operatorname{sgn}(n)a$ and $b_n^0 - \frac{1}{2}$ decay rapidly to zero as $n \to \pm\infty$, but (1.10) may fail (see Section 4.9).

Observe that changing $t \to -t$ in (1.1) and (1.2) is equivalent to changing $a \to -a$. Thus the solution of the rarefaction problem at time $-t$ is the same as the solution of the Toda shock problem at time t. In particular, the behavior of (1.1) and (1.2) as $t \to -\infty$ is precisely given by the solution of the Toda shock problem as in [11]. This leads to the following intriguing situation. It is well-known that on $\mathbb{Z}$ there are three basic types of spectral problems for which the spectral/inverse-spectral theory is completely understood, namely, scattering-type situations in which the operator coefficients converge as $n \to \pm\infty$, situations in which the coefficients are periodic functions of it, and half-line scattering problems in which the coefficients converge as $n \to +\infty$. The techniques to solve the first kind of problem involve integral equations such as the Fadeev-Marchenko equation, and, more recently, Riemann-Hilbert techniques and singular integral equations (see [1], and also [2]). The techniques to solve the second kind of problem involve the theory of Riemann surfaces and the associated theory of theta functions together with the Abel map. On the other hand, the techniques to solve the third kind of inverse problem are equivalent to the techniques used in the classical theory of orthogonal polynomials. The striking feature of problem (1.1) and (1.2) is that for any (finite) t, we solve the problem by the first set of techniques (in particular, we use the Riemann-Hilbert approach), but as $t \to -\infty$ and the solution develops oscillations, we obtain the asymptotic states by using techniques of the second kind; on the other hand, as $t \to \infty$, we obtain the asymptotic state using techniques of the third kind. Thus the solution of (1.1) and (1.2) as t runs from $-\infty$ to $+\infty$ involves the full range of scattering and inverse scattering theory on $\mathbb{Z}$.

To prove Theorem 1.1 we use a Laplace-type method for Riemann-Hilbert problems, modeled on the steepest-descent-type method introduced by Deift and Zhou [6], to analyze the long-time behavior of the modified Korteweg-de Vries equation. The techniques in [6] have also been applied to the Toda lattice in the context of the stability problem in [9].

In the text we find it convenient to analyze the particles $\{x_n, n < 0\}$ and recover the long-time behavior of the original system $\{x_n, n > 0\}$ from the symmetries $x_n = -x_{-n}$ and $y_n = -y_{-n}$. However, symmetry is not an essential element in the method (indeed, see Remark 5 above). Moreover, it will be clear from Section 4.9 that the method can be easily modified to analyze the long-time behavior of $\{x_n, n > 0\}$ directly.

In Section 2, we review basic facts from scattering theory and we pose the inverse scattering problem as a Riemann-Hilbert problem. In Section 3, we introduce a model Riemann-Hilbert problem and show, modulo some estimates, that as $t \to \infty$, the solution of the full problem converges to the solution of the model problem. Finally, in Section 4, we solve the model problem and obtain en route the estimates needed in Section 3.

The steepest-descent method applied to the problem at hand proceeds by a sequence of transformations that convert the original Riemann-Hilbert problem into an equivalent Riemann-Hilbert problem with jump matrix v_{equiv} of the form

$$v_{\text{equiv}} = v_{\text{model}} + v_{\text{error}} ,$$

where v_{model} denotes the jump matrix for an explicitly solvable Riemann-Hilbert problem and v_{error} contains only terms that are exponentially small as $t \to \infty$. Solving the Riemann-Hilbert problem corresponding to v_{model}, which turns out to be equivalent to a classical problem in the theory of orthogonal polynomials, then yields the asymptotic behavior of the lattice up to exponential errors, as in Theorem 1.1.

The sequence of transformations consists of the following steps:

Step 0. We use the direct scattering problem to derive the vector Riemann-Hilbert problem for $\mu(z)$ described by conditions 1 through 4 following (2.18).

Step 1. The asymptotic condition (2.13) for $\mu(z)$ is not convenient, as it contains implicit information about the solution of the Riemann-Hilbert problem. Using the symmetry property $\mu(z) = \mu(z^{-1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, however, the Riemann-Hilbert problem for μ is converted into a standard Riemann-Hilbert problem (Theorem 2.1) for a matrix Q with $Q(z) \to I$ as $z \to \infty$.

Step 2. The jump matrix for Q across the circle $|z| = 1$ is oscillatory and, in particular, does not converge as $t \to \infty$. This difficulty is remedied by introducing $U(z)$ in (3.1), which satisfies the jump conditions (3.2) through (3.6). Observe that $\lambda < 0$ on $|z| = 1$ and $\lambda > 0$ on $[z_1, z_2] \cup [z_2^{-1}, z_1^{-1}]$, and hence the jump matrices in (3.2) through (3.4) diagonalize as $t \to \infty$. Also note that the residues in (3.5) and (3.6) corresponding to $\lambda_i \geqq 0$ converge as $t \to \infty$. On the other hand, those corresponding to $\lambda_i < 0$ diverge exponentially.

Step 3. Whereas the jump matrices for U across $[z_1, z_2] \cup [z_2^{-1}, z_1^{-1}]$ trivialize as $t \to \infty$, the jump matrix across $|z| = 1$ converges to

$$\begin{pmatrix} Rz^{2n} & 0 \\ 0 & -\bar{R}z^{-2n} \end{pmatrix}.$$

The factors R and $\bar{R}$ can be removed by using the solution δ of the scalar Riemann-Hilbert problem (3.8). This leads to the Riemann-Hilbert problem (3.10) through (3.12) for G.

Step 4. As in Step 2, the residues for G in (3.12) with $\lambda_i < 0$ corresponding to the points $\zeta_{-1}, \zeta_{-1}^{-1}, \ldots, \zeta_{-m}, \zeta_{-m}^{-1}$ in the z-plane *diverge* exponentially as $t \to \infty$. Finally, and perhaps paradoxically, the diverging residues in (3.12) corresponding to these $2m$ points $\zeta_{-1}, \zeta_{-1}^{-1}, \ldots, \zeta_{-m}, \zeta_{-m}^{-1}$ can be replaced through a sequence of m similar transformations, $G \to \cdots \to Y$. This is done by the jump conditions (3.42) across the $2m$ circles $K_{-1}, L_{-1}, \ldots, K_{-m}, L_{-m}$ ($\zeta_{-i} \in \operatorname{int} K_{-i}$, $\zeta_{-i}^{-1} \in \operatorname{int} L_{-i}$, $i = 1, \ldots, m$), which now *converge* exponentially to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as $t \to \infty$.

The Riemann-Hilbert problem (3.39) through (3.44) is the desired Riemann-Hilbert problem with jump matrix described schematically as v_{equiv} above. Note that residues at $\zeta_1, \zeta_1^{-1}, \ldots, \zeta_m, \zeta_m^{-1}$ corresponding to $\lambda_i > 0$ remain (see (3.43) and (3.44)): Only those corresponding to $\lambda_i < 0$ are removed and replaced by (3.42).

Step 5. (The final step). Setting the (exponentially small) off-diagonal term in (3.40) to zero and omitting the (exponentially small) jump (3.42), we end up with the model Riemann-Hilbert problem (3.47) and (3.48) for M. This model problem can be solved explicitly, and we have indeed

$$v_{\text{equiv}} = v_{\text{model}} + v_{\text{error}} = v_{\text{model}} + \text{exponentially small terms}\,.$$

2. Inverse Scattering

In this section, we introduce a number of facts concerning the scattering theory of the Lax operator L for the Toda rarefaction problem, and we formulate the inverse scattering problem as a Riemann-Hilbert factorization problem. We consider operators satisfying the general asymptotic conditions (1.9). A general reference for this section is [10] (see also [11, Appendix A.1]). In these two references the case $a > 1$ is considered, but the same formulae hold, mutatis mutandis, when $a < -1$.

In the case where $a < -1$, the spectrum of L consists of the bands $[a-1, a+1]$ and $[-a-1, -a+1]$, together with a finite number of eigenvalues, as in Figure 1.1. If we use the Joukowski transformation

$$\lambda - a = \frac{z + z^{-1}}{2}, \tag{2.1}$$

we have a 2:1 map from the z-plane to the λ-plane such that the union of the unit circle (corresponding to the left band), two bands (corresponding to the right λ-band) and several points (corresponding to the eigenvalues) is mapped onto the spectrum (see Figure 2.1). We denote the spectrum of L in the z-plane by Σ. Also we denote its continuous part by Σ_c. Thus $\Sigma \setminus \Sigma_c$ consists of the z inverse images of the eigenvalues of L.

It is also convenient to define $\tilde{z}$ through

$$\lambda + a = \frac{\tilde{z} + \tilde{z}^{-1}}{2}. \tag{2.2}$$

Note that (2.1) and (2.2) determine $\tilde{z}$ as a multivalued function of z through the relation $z + z^{-1} + 4a = \tilde{z} + \tilde{z}^{-1}$.

We follow [10] and define the Jost functions as follows: Let $f^+(n,z)$ and $f^-(n,\tilde{z})$ be such that

$$\begin{aligned} Lf^+(n,z) &= \lambda f^+(n,z), \\ Lf^-(n,\tilde{z}) &= \lambda f^-(n,\tilde{z}), \\ f^+(n,z) &\sim z^n, \quad \text{as } n \to \infty,\ 0 < |z| \leqq 1, \\ f^-(n,\tilde{z}) &\sim \tilde{z}^{-n}, \quad \text{as } n \to -\infty,\ 0 < |\tilde{z}| \leqq 1. \end{aligned} \tag{2.3}$$

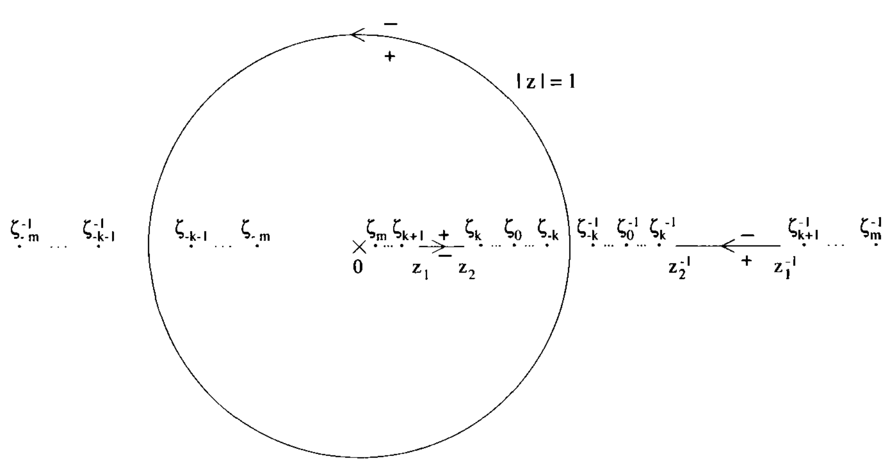

Figure 2.1. The inverse image of the spectrum of L in the z-plane.

We define the reflection and transmission coefficients by

$$T^+(z)f^-(\tilde{z}) = R^+(z)f^+(z) + f^+(z^{-1}) \tag{2.4}$$

on the unit z-circle, and by

$$T^-(\tilde{z})f^+(z) = R^-(\tilde{z})f^-(\tilde{z}) + f^-(\tilde{z}^{-1}) \tag{2.5}$$

on $\{|\tilde{z}| = 1\}$.

We next state a few analyticity results (for proofs see [10] and [11]).

Let $B_n^+ = \prod_{k=n}^{\infty}(2b_k)^{-1}$ and $B_n^- = \prod_{k=-\infty}^{n-1}(2b_k)^{-1}$. We can write

$$\begin{aligned} f^+(n,z) &= B_n^+ z^n v_n(z), \\ f^-(n,\tilde{z}) &= B_n^- \tilde{z}^{-n} u_n(\tilde{z}), \end{aligned} \tag{2.6}$$

where $v_n(z) = 1 + \sum_{k=1}^{\infty} v_{n,k} z^k$ and $u_n(\tilde{z}) = 1 + \sum_{k=1}^{\infty} u_{n,k}\tilde{z}^k$, where the series converge uniformly in $|z| \leqq 1$ and $|\tilde{z}| \leqq 1$, respectively. In particular, $f^+(n,z)$ and $f^-(n,\tilde{z})$ are analytic in $|z| < 1$ and $|\tilde{z}| < 1$, respectively.

We also have relations for the Wronskian $W(\lambda)$ of f^+ and f^- defined by

$$W(\lambda) = b_n(f^+(n,z)f^-(n+1,\tilde{z}) - f^+(n+1,z)f^-(n,\tilde{z})). \tag{2.7}$$

Indeed,

$$\begin{aligned} W(\lambda) &= \frac{z^{-1} - z}{2T^+(z)}, \quad |z| = 1, \\ \bar{W}(\bar{\lambda}) &= \frac{\tilde{z}^{-1} - \tilde{z}}{2T^-(\tilde{z})}, \quad |\tilde{z}| = 1, \\ R^+(z) &= -\frac{\bar{W}(\lambda)}{W(\lambda)}, \quad |z| = 1, \\ R^-(\tilde{z}) &= -\frac{\bar{W}(\lambda)}{W(\lambda)}, \quad |\tilde{z}| = 1. \end{aligned} \tag{2.8}$$

Also $R^+(z^{-1}) = \bar{R}^+(z)$. From (2.8) we see that $|R^+(z)| = 1$, and hence we obtain $(R^+(z))^{-1} = \bar{R}^+(z) = R^+(z^{-1})$ for $|z| = 1$.

It follows from the properties of the Wronskian that the transmission coefficients T^- and T^+ not only are defined on $|z| = 1$ and $|\tilde{z}| = 1$, respectively, but can also be extended meromorphically via equation (2.8). For example, T^+ is meromorphic in $\{|z| < 1\}\setminus[z_1, z_2]$, where z_1, z_2 are the inverse images of $-a \pm 1$ under the map given by (2.1) such that $|z_1| < 1$ and $|z_2| < 1$. They have at most a finite number of poles, which in fact correspond to the eigenvalues of the Lax operator. In general, however, the reflection coefficient $R^{\pm}(z)$ has no meromorphic extension. We also note that $T^{\pm}(0) = 1/B_n^+ B_n^-$. Finally, it is easy to check that

$W(\lambda) = \bar{W}(\bar{\lambda})$ for $|z| \leqq 1$, and thus all sides in the two first lines of (2.8) are equal there.

On the circles $\{|z| = 1\}$ and $\{|\tilde{z}| = 1\}$, W is nonzero except possibly at the points corresponding to $\lambda = a \pm 1$ and $\lambda = -a \pm 1$. Generically, however, the matrix L is not at resonance, which means that generically $W(a\pm 1)$ and $W(-a\pm 1)$ are nonzero, and hence generically $T^+(z = \pm 1) = 0$ and $T^-(\tilde{z} = \pm 1) = 0$. These conditions are satisfied for a dense open set of matrices L, and as mentioned in Section 1, we will restrict ourselves to this case. The methods in the nonresonant case extend directly to the resonant case (see Section 4.8).

Remark. In the pure shock case (1.8), the eigenfunctions $f^{\pm}$ of L are simple combinations of powers of z and $\tilde{z}$, and we obtain $W = \frac{1}{2}(\tilde{z}^{-1} - 2a - z)$. If we use (2.1) and (2.2), we see that

$$\begin{aligned} W = 0 \Leftrightarrow \tilde{z} = (2a + z)^{-1} &\Leftrightarrow 4a + z + z^{-1} \\ &= (2a + z)^{-1} + (2a + z) \Leftrightarrow 4a\lambda = 0\,. \end{aligned}$$

Thus $W = 0 \Leftrightarrow \lambda = 0$, and hence L is nonresonant and $m = k = 0$.

The evolution of the scattering coefficients with time is given by

$$\begin{aligned} R^+(z,t) &= R^+(z,0)e^{(z-z^{-1})t}, \quad |z| = 1\,, \\ |T^+(z,t)|^2 &= |T^+(z,0)|^2 e^{(z-z^{-1})t}, \quad z \in [z_1, z_2]\,. \end{aligned} \tag{2.9}$$

Remark. Note that T^+ is not well-defined on $z \in [z_1, z_2]$, as it has a jump across $[z_1, z_2]$. However, by $W(\lambda) = \bar{W}(\bar{\lambda})$ and the first relation of (2.8), we see that $|T^+|$ is indeed well-defined on $[z_1, z_2]$.

Next we define the following row-vector-valued function in the z-plane:

$$\mu(n,z) := \begin{cases} (f^+(n,z)z^{-n}, T^+(z)f^-(n,\tilde{z})z^n) \\ \quad \text{for } 0 < |z| < 1, z \notin [z_1, z_2], \\ (T^+(z^{-1})f^-(n,\tilde{z})z^{-n}, f^+(n,z^{-1})z^n) \\ \quad \text{for } |z| > 1, z \notin [z_2^{-1}, z_1^{-1}]. \end{cases} \tag{2.10}$$

Note that

$$\mu(n,0) = \lim_{z\to 0} \mu(n,z) = (B_n^+, \quad (B_n^+)^{-1}) \tag{2.11}$$

and

$$\mu(n,\infty) = \lim_{z\to\infty} \mu(n,z) = ((B_n^+)^{-1}, \quad B_n^+)\,. \tag{2.12}$$

Observe that $\mu(n,\cdot)$ is meromorphic in $\mathbb{C}\setminus\Sigma$. Standard computations show that

$$\mu(n,z) \to (1,1) \quad \text{as } n \to +\infty,\ z \notin \Sigma .$$

We now adopt the following convention. For each segment of an oriented contour its + side is to the left (according to the given orientation), and its − side is to the right as one traverses the segment in the direction of the orientation (see Figure 2.1). Accordingly, we denote by μ_+ and μ_- the nontangential limits of μ on an oriented contour from the + side and − side of the contour, respectively.

Caveat: The ± signs of the contour should not be confused with the ± signs occurring, for example, in $f^{\pm}$ and $T^{\pm}$.

From the scattering relation (2.4) at time t, we obtain for $|z| = 1$,

$$\mu_+(n,z) = \mu_-(n,z)\begin{pmatrix} 1 & R^+(z,0)z^{2n}e^{t(z-z^{-1})} \\ -\bar{R}^+(z,0)z^{-2n}e^{-t(z-z^{-1})} & 0 \end{pmatrix}. \tag{2.13}$$

Across $[z_1, z_2]$, we have from (2.5) at time t,

$$\mu_+(n,z) = \mu_-(n,z)\begin{pmatrix} 1 & -\frac{\tilde{z}-\tilde{z}^{-1}}{z-z^{-1}}|T^+(z,0)|^2 z^{2n}e^{(z-z^{-1})t} \\ 0 & 1 \end{pmatrix}, \tag{2.14}$$

whereas across $[z_2^{-1}, z_1^{-1}]$

$$\mu_+(n,z) = \mu_-(n,z)\begin{pmatrix} 1 & 0 \\ \frac{\tilde{z}-\tilde{z}^{-1}}{z-z^{-1}}|T^+(z^{-1},0)|^2 z^{-2n}e^{-(z-z^{-1})t} & 1 \end{pmatrix}. \tag{2.15}$$

Remark. In (2.14) and (2.15) we specify $\tilde{z}$ (there are two possibilities) by the condition $\mathrm{Im}(\tilde{z}) \geqq 0$.

To obtain (2.13–2.15) we use the relations (2.8) above and also the equality of the first two lines in (2.8) for $|z| \leqq 1$.

We denote the poles of T^+ in $|z| < 1$ by ζ_i, $i = 0, \pm 1, \dots, \pm m$. As noted above these are the inverse images under the transformation (2.1) of the eigenvalues λ_i, $i = 0, \dots, \pm m$. The residues at these points can be easily calculated (again see [10], [11]), and one obtains

$$\begin{aligned} \mathrm{Res}_{\zeta_i}\, \mu(n,z) &= \left(0,\quad -c_i\zeta_i^{n+1}e^{(\zeta_i-\zeta_i^{-1})t}f^+(\zeta_i)\right), \\ \mathrm{Res}_{\zeta_i^{-1}}\, \mu(n,z) &= \left(c_i\zeta_i^{n-1}e^{(\zeta_i-\zeta_i^{-1})t}f^+(\zeta_i),\quad 0\right), \end{aligned} \tag{2.16}$$

where $c_i = ||f^+(\zeta_i)||^{-2} = (\sum_{n=-\infty}^{\infty}(f^+(\zeta_i,n))^2)^{-1}$ are the norming constants at time $t = 0$.

An alternative way (see, e.g., [1]) to express the conditions at the poles is the following:

$$
\begin{aligned}
\operatorname{Res}_{\zeta_i} \mu(n,z) &= \lim_{z\to\zeta_i} \mu(n,z) \begin{pmatrix} 0 & \gamma_i e^{(\zeta_i-\zeta_i^{-1})t} \\ 0 & 0 \end{pmatrix}, \\
\operatorname{Res}_{\zeta_i^{-1}} \mu(n,z) &= \lim_{z\to\zeta_i^{-1}} \mu(n,z) \begin{pmatrix} 0 & 0 \\ -\gamma_i \zeta_i^{-2} e^{(\zeta_i-\zeta_i^{-1})t} & 0 \end{pmatrix},
\end{aligned}
\tag{2.17}
$$

where

$$\gamma_i = -c_i \zeta_i^{2n+1}. \tag{2.18}$$

We have now arrived at the following vector Riemann-Hilbert factorization problem.

1. $\mu(n,\cdot)$ is meromorphic in $\mathbb{C}\setminus\Sigma_c$.
2. $\mu(n,\cdot)$ satisfies the jumps (2.13–2.15) in Σ_c.
3. The poles of $\mu(n,\cdot)$ on $\Sigma\setminus\Sigma_c$ are determined by (2.17).
4. $\mu(n,\cdot)$ satisfies (2.12) as $z\to\infty$.

The asymptotic condition (2.12) is not convenient, as it contains implicit information about the solution of the inverse problem. As we now show, if we use a certain symmetry argument (see [9]) we can reduce our problem to a (2×2)-matrix Riemann-Hilbert problem with the same jump and pole conditions but also with a standard condition at ∞.

THEOREM 2.1. *Let $Q(z)$ be a (2×2)-matrix-valued function satisfying the following Riemann-Hilbert problem:*

1. *$Q(z)$ is meromorphic in $\mathbb{C}\setminus\Sigma_c$.*
2.

$$
Q_+(z) = \begin{cases}
Q_-(z) \begin{pmatrix} 1 & R^+(z,0) z^{2n} e^{(z-z^{-1})t} \\ -\bar{R}^+(z,0) z^{-2n} e^{-(z-z^{-1})t} & 0 \end{pmatrix}, & \text{when } |z| = 1 \\
Q_-(z) \begin{pmatrix} 1 & -\frac{z-\bar{z}^{-1}}{z-z^{-1}} |T^+(z,0)|^2 z^{2n} e^{(z-z^{-1})t} \\ 0 & 1 \end{pmatrix}, & \text{when } z \in [z_1, z_2] \\
Q_-(z) \begin{pmatrix} 1 & 0 \\ \frac{z-\bar{z}^{-1}}{z-z^{-1}} |T^+(z^{-1},0)|^2 z^{-2n} e^{-(z-z^{-1})t} & 1 \end{pmatrix}, & \text{when } z \in [z_2^{-1}, z_1^{-1}].
\end{cases}
$$

3. *Q has poles at ζ_i and ζ_i^{-1}, $i = 0,\ldots,\pm m$, with*

$$
\begin{aligned}
\operatorname{Res}_{\zeta_i} Q(z) &= \lim_{z\to\zeta_i} Q(z) \begin{pmatrix} 0 & \gamma_i e^{(\zeta_i-\zeta_i^{-1})t} \\ 0 & 0 \end{pmatrix}, \\
\operatorname{Res}_{\zeta_i^{-1}} Q(z) &= \lim_{z\to\zeta_i^{-1}} Q(z) \begin{pmatrix} 0 & 0 \\ -\gamma_i \zeta_i^{-2} e^{(\zeta_i-\zeta_i^{-1})t} & 0 \end{pmatrix}.
\end{aligned}
$$

4. *$Q(z) \to I$ as $z \to \infty$.*
Then $\mu(n,0,t)$ can be recovered from $Q(0)$ as follows: If

$$Q(0) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then

$$\mu(n,0,t) = \left(\left(\frac{1+C(t)}{D(t)}\right)^{1/2}, \ \left(\frac{D(t)}{1+C(t)}\right)^{1/2} \right), \tag{2.19}$$

from which $B_n^+(t)$, and hence $b_n(t)$, can be recovered.

Proof: Let $v(z)$ be the jump matrix for Q. Note that for $z \in \Sigma_c$

$$v(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (v(z^{-1}))^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & |z| = 1, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v(z^{-1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & z \in [z_1, z_2] \cup [z_2^{-1}, z_1^{-1}]. \end{cases} \tag{2.20}$$

Furthermore,

$$v(\zeta_i) = -\zeta_i^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v(\zeta_i^{-1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i = 0, \dots, \pm m. \tag{2.20'}$$

Consider

$$H(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Q(z^{-1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows from (2.20) and the jump conditions $Q_+ = Q_- v$ that

$$H_+(z) = H_-(z) v(z)$$

on the contour Σ_c. Now note that Q is invertible. Indeed, because $\det v = 1$ on Σ and $\det Q$ clearly has no poles at all, and because $\det Q = 1$ at infinity, then by Liouville's theorem $\det Q = 1$ everywhere. Furthermore, the analysis of the poles of Q and H using (2.20′) shows that HQ^{-1} is an entire bounded function. Hence it is a constant, say A_0, by Liouville's theorem. As both H and Q must have a determinant equal to 1, A_0 has to be invertible. Hence,

$$Q(z) = A_0^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Q(z^{-1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we set $z = \infty$ and use $Q(\infty) = I$, we find

$$Q(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (Q(0))^{-1} Q(z^{-1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.21}$$

Next we note that

$$\mu(n,z,t) = \mu(n,z^{-1},t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, as both μ and Q satisfy the same jump and pole conditions and μQ^{-1} is bounded by (2.12), we must have

$$\mu(n,z,t) = \alpha Q(z), \tag{2.22}$$

where $\alpha = (\alpha_1, \alpha_2)$ is a constant vector. From (2.22),

$$\alpha Q(z) = \alpha Q(z^{-1})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and letting $z \to \infty$,

$$\alpha Q(0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \alpha;$$

that is,

$$(\alpha_1, \ \ \alpha_2)\begin{pmatrix} B-1 & A \\ D & C-1 \end{pmatrix} = 0. \tag{2.23}$$

At this point we note two facts: First, for all z, the determinant of Q is 1 (see above). On the other hand, $B + C = 0$, as can be seen by setting $z = 0$ in (2.21). Hence the matrix $\begin{pmatrix} B & A \\ D & C \end{pmatrix}$ has eigenvalues ± 1.

It now follows from (2.23) that

$$\alpha = g(1-C, \ \ A) = g'(D, \ \ 1-B),$$

for some constants g, g'. But $\alpha = \mu(n,\infty,t) = ((B_n^+)^{-1}, \ \ B_n^+)$, by (2.12). Hence $g, g', A, 1-C, D, 1-B$ are all nonzero. In particular, as $\det Q = 1$, $A = (1-C^2)/D$, and hence

$$g(1-C) = (B_n^+)^{-1},$$

$$g\frac{1-C^2}{D} = B_n^+,$$

which yields $g^2(1-C)^2(1+C) = D$ and $D/(1+C) = (B_n^+)^{-2}$. If we recall (2.11) we obtain (2.19).

Remark 1. It is not a priori clear that a matrix solution of the Riemann-Hilbert problem of Theorem 2.1 exists. The situation is similar to the Schrödinger case, where one can show (see [2, chap. 38]) that matrix solutions may not in fact exist if poles are present in the Riemann-Hilbert problem. However, we will show explicitly in the calculations that follow that for fixed n, a matrix solution $Q(t)$ of the Riemann-Hilbert problem indeed exists for $t \geqq T(n)$ for some $T(n) < \infty$.

Remark 2. Observe that condition (2.11) is a consequence of (2.12) and the symmetry

$$\mu(n,z,t) = \mu(n,z^{-1},t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

3. Reduction to a Model Problem

The goal in this section is to derive a Riemann-Hilbert problem for a matrix Y (see (3.39) through (3.45) below) that is equivalent to the original Riemann-Hilbert problem for Q, but from which the leading behavior of the driven lattice (1.7) and (1.9) can be readily deduced. A crucial part of the derivation is the replacement, rather paradoxically, of the exponentially growing terms at some of the residues of Q by exponentially decaying terms. Moreover, when these (exponentially small) terms are dropped from the Riemann-Hilbert problem, one obtains a model problem that can be solved explicitly and that yields the leading asymptotics for the full problem Y. The solution of the model problem is deferred to Section 4.

We write $R = R^+(z, t = 0)$,

$$\tau = -\frac{\tilde{z}(z) - (\tilde{z}(z))^{-1}}{z - z^{-1}} |T^+(z, t = 0)|^2,$$

and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the third Pauli matrix.

3.1. If we define $U(z)$ as follows:

$$U(z) = \begin{cases} e^{a\sigma_3 t} Q(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{-(a+z)\sigma_3 t}, & \text{when } |z| < 1, \\ e^{a\sigma_3 t} Q(z) e^{-(a+1/z)\sigma_3 t}, & \text{when } |z| > 1, \end{cases} \tag{3.1}$$

we end up with the following Riemann-Hilbert problem. The problem is to find a 2×2 matrix U meromorphic in the complement of Σ with poles at ζ_i and $\zeta_i^{-1}, i = 0, \ldots, \pm m$. The matrix must be equal to the identity at infinity and be such that

$$U_+(z) = U_-(z) \begin{pmatrix} Rz^{2n} & e^{2\lambda t} \\ 0 & -\bar{R}z^{-2n} \end{pmatrix} \tag{3.2}$$

on the unit circle,

$$U_+(z) = U_-(z) \begin{pmatrix} 1 & 0 \\ \tau e^{-2\lambda t} z^{2n} & 1 \end{pmatrix} \tag{3.3}$$

on the band $[z_1, z_2]$, and

$$U_+(z) = U_-(z) \begin{pmatrix} 1 & 0 \\ -\tau e^{-2\lambda t} z^{-2n} & 1 \end{pmatrix} \tag{3.4}$$

on the band $[z_2^{-1}, z_1^{-1}]$. Also

$$\operatorname{Res}_{\zeta_i} U(z) = \lim_{z\to\zeta_i} U(z)\begin{pmatrix} 0 & 0 \\ -c_i\zeta_i^{2n+1}e^{-2\lambda_i t} & 0 \end{pmatrix} \tag{3.5}$$

and

$$\operatorname{Res}_{\zeta_i^{-1}} U(z) = \lim_{z\to\zeta_i^{-1}} U(z)\begin{pmatrix} 0 & 0 \\ c_i\zeta_i^{2n-1}e^{-2\lambda_i t} & 0 \end{pmatrix}, \tag{3.6}$$

where the norming constants c_i are defined as before by $c_i = ||f^+(\zeta_i, t = 0)||^{-2}$.

3.2. The first step in the solution of this Riemann-Hilbert problem is to remove the R factor from the diagonal in (3.2). To do this, we need information on the winding number of R.

LEMMA 3.1.[1]

$$\frac{1}{2\pi i}\int_{|z|=1} d\log R = -2m.$$

Proof: Let Σ_1 be the unit circle oriented counterclockwise, and Σ_2 be a cycle around the band $[z_1, z_2]$ oriented clockwise, as shown in Figure 3.1.

From formulae (2.7) and (2.8) we see immediately that $W(z)$ has no poles in $\{0 < |z| < 1\} \setminus [z_1, z_2]$ and has simple zeros at the poles ζ_i, $i = 0, \dots, \pm m$ of T_+. Also, (2.7) and (2.6) show that $W(z)$ has a simple pole at $z = 0$. Thus

$$\frac{1}{2\pi i}\int_{\Sigma_1} d\log W + \frac{1}{2\pi i}\int_{\Sigma_2} d\log W = 2m.$$

Now, the symmetry $a_{-n} = -a_n, b_{-n} = b_{n-1}$ implies that if $Lh = \lambda h$, then $L\tilde{h} = -\lambda\tilde{h}$, where $\tilde{h}_n = (-1)^n h_{-n}$. This implies $f^-(z, n) = (-1)^n f^+(z', -n)$, where $z' = z'(z)$ is defined through $(z + z^{-1})/2 + a = \lambda$, $(z' + (z')^{-1})/2 + a = -\lambda$. If we insert this relation for the Jost functions into (2.7), we find that $W(z) = -W(z')$. Moreover, a simple calculation shows that as z goes around the unit circle counterclockwise, then z' goes around the band $[z_1, z_2]$ in the clockwise direction. Thus

$$\frac{1}{2\pi i}\int_{\Sigma_1} d\log W = \frac{1}{2\pi i}\int_{\Sigma_2} d\log W,$$

and hence $(1/2\pi i)\int_{\Sigma_1} d\log W = m$. If we recall from (2.8) that $R = -\bar{W}/W$, Lemma 3.1 follows immediately.

[1] Recall that we are in the generic case (see Section 2) and have $W \neq 0$ on $\{|z| = 1\}$ and on $\{|\tilde{z}| = 1\}$.

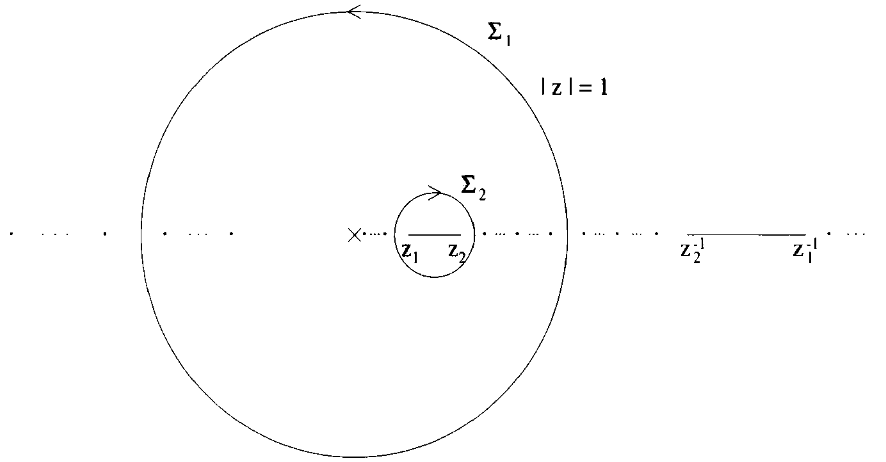

Figure 3.1.

We need the following result:

LEMMA 3.2. *Let*

$$\delta(z) = \exp\left(\frac{1}{2\pi i}\int_{|z|=1}\frac{\log(R(s)s^{2m})ds}{s-z}\right). \tag{3.7}$$

Then δ is analytic in the complement of the unit circle and continuous and nonzero up to the circle, whereas on the unit circle

$$\delta_+(z) = \delta_-(z)R(z)z^{2m} \tag{3.8}$$

and $\delta(\infty) = 1$.

Proof: The proof is immediate, as Rz^{2m} has no winding, and hence the function $\log(R(z)z^{2m})$ is smooth on $\{|z| = 1\}$.

Remark. From the uniqueness of solutions of (3.8) and the symmetries of $R_+(z)$ (in particular $R(z^{-1}) = (R(z))^{-1} = \bar{R}(z)$ on $|z| = 1$; see Section 2), together with $R_+(1) = -1$ (recall that L_0 is nonresonant), it follows easily that

$$\begin{aligned} \delta(z) &= -\delta(z^{-1}) \quad \text{on } \mathbb{C}\setminus\{|z| = 1\}, \\ \delta_+(z) &= -\bar{\delta}_-(z) \quad \text{on } |z| = 1. \end{aligned} \tag{3.9}$$

In particular $\delta(0) = -\delta(\infty) = -1$.

Let

$$G := \begin{cases} U\begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}\sigma_3, & \text{for } |z| < 1, \\ U\begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}, & \text{for } |z| > 1. \end{cases}$$

Then by using $\bar{R}(z) = (R(z))^{-1}$, we have

$$G_+ = G_- \begin{pmatrix} z^{2(n-m)} & |\delta_+(z)|^2 e^{2\lambda t} \\ 0 & z^{-2(n-m)} \end{pmatrix} \quad \text{on } |z| = 1, \tag{3.10}$$

and

$$G_+ = \begin{cases} G_- \begin{pmatrix} 1 & 0 \\ -\tau\delta^{-2}e^{-2\lambda t}z^{2n} & 1 \end{pmatrix} & \text{on } [z_1, z_2], \\ G_- \begin{pmatrix} 1 & 0 \\ -\tau\delta^{-2}e^{-2\lambda t}z^{-2n} & 1 \end{pmatrix} & \text{on } [z_2^{-1}, z_1^{-1}]. \end{cases} \tag{3.11}$$

However, at the poles

$$\begin{aligned} \operatorname{Res}_{\zeta_i} G &= \lim_{z\to\zeta_i} G(z) \begin{pmatrix} 0 & 0 \\ c_i\delta^{-2}(\zeta_i)\zeta_i^{2n+1}e^{-2\lambda_i t} & 0 \end{pmatrix}, \\ \operatorname{Res}_{\zeta_i^{-1}} G &= \lim_{z\to\zeta_i^{-1}} G(z) \begin{pmatrix} 0 & 0 \\ c_i\delta^{-2}(\zeta_i)\zeta_i^{2n-1}e^{-2\lambda_i t} & 0 \end{pmatrix}, \end{aligned} \tag{3.12}$$

because $\delta^2(\zeta_i^{-1}) = \delta^2(\zeta_i)$ by (3.9). The condition at ∞ remains

$$G(\infty) = I,$$

and also, if one notes that $Q(0) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

$$G(0) = \begin{pmatrix} -B & Ae^{2at} \\ -De^{-2at} & C \end{pmatrix}. \tag{3.13}$$

3.3. As seen from (3.12) the contribution from the negative eigenvalues (corresponding to ζ_i and ζ_i^{-1}, $i = -1, \ldots, -m$) grows exponentially. We now employ a procedure that will enable us to restate our problem so that the contribution from these eigenvalues becomes exponentially decaying.

The intuition for this procedure is as follows: First, observe that singularities of type (3.12) can be removed, provided we add an additional circle to the Riemann-Hilbert problem. For example, suppose Z solves a Riemann-Hilbert problem on a contour Σ_Z with a pole at some point z_0, with

$$\operatorname{Res}_{z_0} Z = \lim_{z \to z_0} Z(z) \begin{pmatrix} 0 & 0 \\ c_Z & 0 \end{pmatrix}, \tag{3.14}$$

for some constant c_Z. Then we note that

$$\hat{Z}(z) = Z(z) \begin{pmatrix} 1 & 0 \\ -\frac{c_Z}{z-z_0} & 1 \end{pmatrix}$$

is analytic in a neighborhood of z_0, and conversely if $\hat{Z}(z)$ is analytic in a neighborhood of z_0, then

$$Z(z) = \hat{Z}(z) \begin{pmatrix} 1 & 0 \\ \frac{c_Z}{z-z_0} & 1 \end{pmatrix}$$

has a singularity of type (3.14) at z_0. If we let K_{z_0} be a small clockwise-oriented circle that surrounds z_0 and does not intersect the rest of Σ_Z, then

$$\tilde{Z}(z) = \begin{cases} Z(z), & z \in \mathbb{C} \setminus \Sigma_Z,\ z \text{ outside } K_{z_0}, \\ Z(z) \begin{pmatrix} 1 & 0 \\ -\frac{c_Z}{z-z_0} & 1 \end{pmatrix}, & z \text{ inside } K_{z_0} \end{cases} \tag{3.15}$$

solves a Riemann-Hilbert problem on $\tilde{\Sigma}_Z = \Sigma_Z \cup K_{z_0}$ with the same jumps $\tilde{v} = v$ on Σ_Z, but on K_{z_0}:

$$\tilde{Z}_+ = \tilde{Z}_- \tilde{v} = \tilde{Z}_- \begin{pmatrix} 1 & 0 \\ \frac{c_Z}{z-z_0} & 1 \end{pmatrix}. \tag{3.16}$$

By the above remarks the Riemann-Hilbert problem for $\tilde{Z}$ is equivalent to the original Riemann-Hilbert problem for Z.

Second, observe that we are concerned with poles of type (3.14), where $c_z \sim e^{-2\lambda t}, \lambda < 0$, which is exponentially growing. The origin of such terms is in the fact that time enters into the problem in the form of conjugation of the jump matrices,

$$e^{\lambda t \sigma_3} \begin{pmatrix} a & b \\ c & d \end{pmatrix} e^{-\lambda t \sigma_3} = \begin{pmatrix} a & b e^{2\lambda t} \\ c e^{-2\lambda t} & d \end{pmatrix}.$$

The case above corresponds to $b = 0$ and $\lambda = \lambda_{-i} < 0$. However, if we could convert our basic jump matrices from lower-triangular form ($b = 0$) to upper-triangular form ($c = 0$), then time enters only through the factor $be^{2\lambda t}$, and for $\lambda = \lambda_{-i} < 0$ this is exponentially decreasing. So our goal is to turn jump matrices of type (3.16) into upper-triangular form.

One proceeds as follows: In the above Riemann-Hilbert problem for $\tilde{Z}$ on $\Sigma_Z \cup K_Z$, set

$$Z^\sharp(z) = \begin{cases} \tilde{Z}(z), & z \in \mathbb{C} \setminus \Sigma_Z,\ z \text{ outside } K_{z_0}, \\ \tilde{Z}(z)J^{-1}(z), & z \text{ inside } K_{z_0}, \end{cases}$$

where $J(z)$ is analytic and invertible inside K_{z_0}. Then on K_{z_0} we have

$$Z^\sharp_+(z) = Z^\sharp_-(z)v^\sharp(z), \tag{3.17}$$

where

$$v^\sharp(z) = J(z)\begin{pmatrix} 1 & 0 \\ \frac{c_Z}{z-z_0} & 1 \end{pmatrix}.$$

We want

$$v^\sharp(z) = \begin{pmatrix} 1 & \frac{c^\sharp}{z-z_0} \\ 0 & 1 \end{pmatrix},$$

for some constant $c^\sharp$. Then

$$J(z) = \begin{pmatrix} 1 & \frac{c^\sharp}{z-z_0} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{c_Z}{z-z_0} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{c^\sharp c_Z}{(z-z_0)^2} & \frac{c^\sharp}{z-z_0} \\ -\frac{c_Z}{z-z_0} & 1 \end{pmatrix}.$$

We see that, unfortunately, $J(z)$ is not analytic inside K_{z_0}. However, by conjugating $Z^\sharp(z)$ by a diagonal matrix, it turns out that we can remove those poles and still obtain a matrix $v^\sharp$ that is upper triangular.

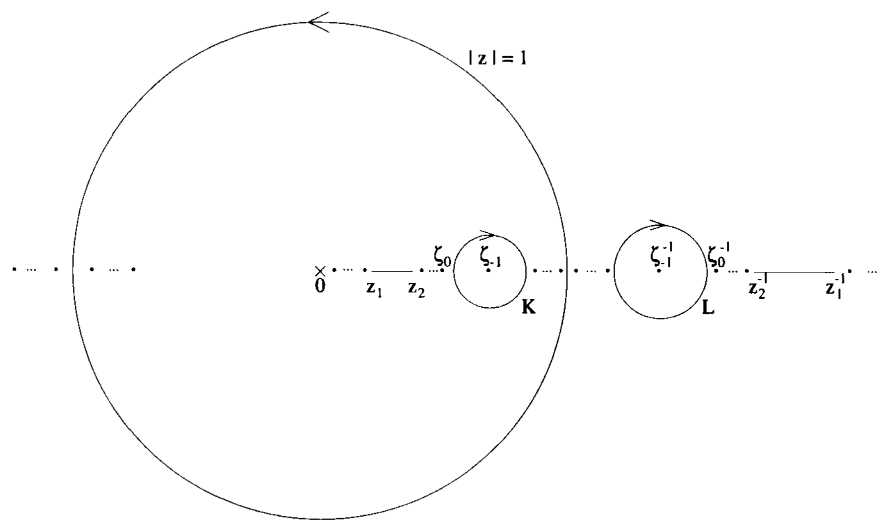

Figure 3.2.

We proceed as follows: For technical reasons we must consider pairs of poles, say $\zeta_{-1}, \zeta_{-1}^{-1}$ simultaneously (see Figure 3.2). Let $q = c_{-1}\delta(\zeta_{-1})^{-2}\zeta_{-1}^{2n+1}e^{-2\lambda_{-1}t}$. Set

$$\tilde{G}(z) := \begin{cases} G(z), & z \in \mathbb{C} \setminus \Sigma,\ z \text{ outside } K \text{ or } L, \\ G(z)\begin{pmatrix} 1 & 0 \\ -\frac{q}{z-\zeta_{-1}} & 1 \end{pmatrix}, & z \text{ inside } K, \\ G(z)\begin{pmatrix} 1 & 0 \\ \frac{-q\zeta_{-1}^{-2}}{z-\zeta_{-1}^{-1}} & 1 \end{pmatrix}, & z \text{ inside } L. \end{cases} \tag{3.18}$$

Then, as above, $\tilde{G}$ solves a Riemann-Hilbert problem on $\tilde{\Sigma} = (\Sigma \setminus \{\zeta_{-1}, \zeta_{-1}^{-1}\}) \cup K \cup L$ with the same jumps $\tilde{v} = v$ on Σ and

$$\begin{aligned} \tilde{v}(z) &= \begin{pmatrix} 1 & 0 \\ \frac{q}{z-\zeta_{-1}} & 1 \end{pmatrix}, \quad z \in K, \\ \tilde{v}(z) &= \begin{pmatrix} 1 & 0 \\ \frac{q\zeta_{-1}^{-2}}{z-\zeta_{-1}^{-1}} & 1 \end{pmatrix}, \quad z \in L. \end{aligned} \tag{3.19}$$

Set

$$G^\sharp := \begin{cases} \tilde{G}(z)\begin{pmatrix} d_1(z) & 0 \\ 0 & (d_1(z))^{-1} \end{pmatrix}, & |z| < 1,\ z \text{ outside } K, \\ \tilde{G}(z)J_K^{-1}(z)\begin{pmatrix} d_1(z) & 0 \\ 0 & (d_1(z))^{-1} \end{pmatrix}, & z \text{ inside } K, \\ \tilde{G}(z)\begin{pmatrix} d_1^{-1}(z) & 0 \\ 0 & d_1(z) \end{pmatrix}, & |z| > 1,\ z \text{ outside } L, \\ \tilde{G}(z)J_L^{-1}(z)\begin{pmatrix} d_1^{-1}(z) & 0 \\ 0 & d_1(z) \end{pmatrix}, & z \text{ inside } L, \end{cases} \tag{3.20}$$

where $d_1(z), J_K(z), J_L(z)$ will be determined below. On K, we have

$$G_+^\sharp(z) = G_-^\sharp(z)v^\sharp(z), \tag{3.21}$$

where

$$v^\sharp(z) = \begin{pmatrix} d_{1-}^{-1} & 0 \\ 0 & d_{1-} \end{pmatrix} J_K \tilde{v} \begin{pmatrix} d_{1+} & 0 \\ 0 & d_{1+}^{-1} \end{pmatrix}, \tag{3.22}$$

and similarly on L

$$G_+^\sharp(z) = G_-^\sharp(z)v^\sharp(z), \tag{3.23}$$

where

$$v^\sharp = \begin{pmatrix} d_{1-} & 0 \\ 0 & d_{1-}^{-1} \end{pmatrix} J_L \tilde{v} \begin{pmatrix} d_{1+}^{-1} & 0 \\ 0 & d_{1+} \end{pmatrix}. \tag{3.24}$$

We require

$$v^\sharp(z) = \begin{pmatrix} 1 & \frac{c_K^\sharp}{z-\zeta_{-1}} \\ 0 & 1 \end{pmatrix}, \quad z \in K, \tag{3.25}$$

and

$$v^\sharp(z) = \begin{pmatrix} 1 & \frac{c_L^\sharp}{z-\zeta_{-1}^{-1}} \\ 0 & 1 \end{pmatrix}, \quad z \in L, \tag{3.26}$$

for some constants $c_K^\sharp$, $c_L^\sharp$. Thus inside K,

$$J_K(z) = \begin{pmatrix} \frac{d_{1-}}{d_{1+}} - \frac{q c_K^\sharp d_{1-} d_{1+}}{(z-\zeta_{-1})^2} & \frac{c_K^\sharp d_{1-} d_{1+}}{z-\zeta_{-1}} \\ \frac{-q d_{1+}}{d_{1-}(z-\zeta_{-1})} & \frac{d_{1+}}{d_{1-}} \end{pmatrix}, \tag{3.27}$$

and inside L,

$$J_L(z) = \begin{pmatrix} \frac{d_{1+}}{d_{1-}} - \frac{q \zeta_{-1}^{-2} c_L^\sharp d_{1+}^{-1} d_{1-}^{-1}}{(z-\zeta_{-1}^{-1})^2} & \frac{c_L^\sharp d_{1-}^{-1} d_{1+}^{-1}}{z-\zeta_{-1}^{-1}} \\ \frac{-q \zeta_{-1}^{-2} d_{1-}}{d_{1+}(z-\zeta_{-1}^{-1})} & \frac{d_{1-}}{d_{1+}} \end{pmatrix}. \tag{3.28}$$

Choose

$$d_1(z) := \begin{cases} \frac{z-\zeta_{-1}}{z-\zeta_{-1}^{-1}} & \text{outside } K \text{ and } L, \\ \frac{1}{z-\zeta_{-1}^{-1}} & \text{inside } K, \\ z-\zeta_{-1} & \text{inside } L. \end{cases} \tag{3.29}$$

With these definitions, J_K and J_L become

$$J_K(z) = \begin{pmatrix} \frac{1-q c_K^\sharp d_1^2(z)}{z-\zeta_{-1}} & c_K^\sharp d_1^2 \\ -q & z-\zeta_{-1} \end{pmatrix} \tag{3.30}$$

and

$$J_L(z) = \begin{pmatrix} \frac{1-q c_L^\sharp \zeta_{-1}^{-2} d_1^{-2}}{z-\zeta_{-1}^{-1}} & \frac{c_L^\sharp}{d_1^2(z)} \\ -q\zeta_{-1}^{-2} & z-\zeta_{-1}^{-1} \end{pmatrix}. \tag{3.31}$$

Then $J_K(z)$ and $J_L(z)$ are analytic and invertible inside K and L provided we choose

$$c_K^\sharp = \frac{1}{q d_1^2(\zeta_{-1})}, \qquad c_L^\sharp = \frac{d_1^2(\zeta_{-1}^{-1})}{q\zeta_{-1}^{-2}}. \tag{3.32}$$

The result of these computations is that we have a Riemann-Hilbert problem for $G^\sharp$ on $\Sigma \cup K \cup L$ that is equivalent to the original Riemann-Hilbert problem for G on Σ, with jumps

$$\begin{aligned}
v^\sharp(z) &= \begin{pmatrix} z^{2(n-m)} d_1^2(z) & |\delta_+(z)|^2 e^{2\lambda t} \\ 0 & z^{-2(n-m)} d_1^{-2}(z) \end{pmatrix}, && |z| = 1, \\
v^\sharp(z) &= \begin{pmatrix} 1 & 0 \\ -\tau\delta^{-2} d_1^2 e^{-2\lambda t} z^{2n} & 1 \end{pmatrix}, && z \in [z_1, z_2], \\
v^\sharp(z) &= \begin{pmatrix} 1 & 0 \\ -\tau\delta^{-2} d_1^{-2} e^{-2\lambda t} z^{-2n} & 1 \end{pmatrix}, && z \in [z_2^{-1}, z_1^{-1}], \\
v^\sharp(\zeta_i) &= \begin{pmatrix} 0 & 0 \\ c_i \delta^{-2}(\zeta_i) d_1^2(\zeta_i) \zeta_i^{2n+1} e^{-2\lambda_i t} & 0 \end{pmatrix}, && i \neq -1, \\
v^\sharp(\zeta_i^{-1}) &= \begin{pmatrix} 0 & 0 \\ c_i \delta^{-2}(\zeta_i) d_1^{-2}(\zeta_i^{-1}) \zeta_i^{2n-1} e^{-2\lambda_i t} & 0 \end{pmatrix}, && i \neq -1, \\
v^\sharp(z) &= \begin{pmatrix} 1 & \frac{q^{-1} d_1^{-2}(\zeta_{-1})}{z-\zeta_{-1}} \\ 0 & 1 \end{pmatrix}, && z \in K, \\
v^\sharp(z) &= \begin{pmatrix} 1 & \frac{q^{-1}\zeta_{-1}^2 d_1^2(\zeta_{-1}^{-1})}{z-\zeta_{-1}^{-1}} \\ 0 & 1 \end{pmatrix}, && z \in L.
\end{aligned} \tag{3.33}$$

Observe that the off-diagonal factors in $v^\sharp$ on K and L are now exponentially decreasing as $q^{-1} \sim e^{2\lambda_{-1} t}$. Apart from the diagonal factors d_1^2 and d_1^{-2} in $v^\sharp$ on $|z| = 1$, the Riemann-Hilbert problem for $G^\sharp$ is precisely what we want. We can remove these diagonal factors by solving a scalar Riemann-Hilbert problem on $\{|z| = 1\}$ as follows. Let $\Delta_1(z)$ solve

$$\begin{aligned}
&\Delta_1(z) \text{ analytic in } \mathbb{C} \setminus \{|z| = 1\}, \\
&\Delta_{1+}(z) = \Delta_{1-}(z) d_1^2 z^{-2} \text{ on } |z| = 1, \\
&\Delta_1(z) \to 1 \text{ as } z \to \infty.
\end{aligned} \tag{3.34}$$

This problem has a smooth solution as $(d_1 z^{-1})^2$ has no winding on the unit circle (see (3.29)).

One shows easily that

$$\begin{aligned} &\Delta_1(z) = \zeta_{-1}^2 \Delta_1(z^{-1}), \quad \text{for } |z| < 1, \\ (3.35)\qquad &\Delta_1(0) = \zeta_{-1}^2 = d_1(0), \\ &\bar{\Delta}_1(\bar{z}) = \Delta_1(z), \end{aligned}$$

and hence $\Delta_{1+} = \zeta_{-1}^2 \bar{\Delta}_{1-}$. Finally, set

$$G^{(1)}(z) := G^\sharp(z) \begin{pmatrix} \Delta_1^{-1}(z) & 0 \\ 0 & \Delta_1(z) \end{pmatrix}, \quad |z| \neq 1, \tag{3.36}$$

which solves a Riemann-Hilbert problem on $(\Sigma \setminus \{\zeta_{-1}, \zeta_{-1}^{-1}\}) \cup K \cup L$ with jumps and poles given by

$$\begin{aligned} v^{(1)}(z) &= \begin{pmatrix} z^{2(n-m+1)} & |\delta_+(z)|^2 |\Delta_{1+}(z)|^2 \zeta_{-1}^{-2} e^{2\lambda t} \\ 0 & z^{-2(n-m+1)} \end{pmatrix}, \quad |z| = 1, \\ v^{(1)}(z) &= \begin{pmatrix} 1 & 0 \\ -\tau\delta^{-2} d_1^2 \Delta_1^{-2} e^{-2\lambda t} z^{2n} & 1 \end{pmatrix}, \quad z \in [z_1, z_2], \\ (3.37)\qquad v^{(1)}(z) &= \begin{pmatrix} 1 & 0 \\ -\tau\delta^{-2} d_1^{-2} \Delta_1^{-2} e^{-2\lambda t} z^{-2n} & 1 \end{pmatrix}, \quad z \in [z_2^{-1}, z_1^{-1}], \\ v^{(1)}(\zeta_i) &= \begin{pmatrix} 0 & 0 \\ c_i \delta_i^{-2} d_1^2(\zeta_i)(\Delta_1(\zeta_i))^{-2} \zeta_i^{2n+1} e^{-2\lambda_i t} & 0 \end{pmatrix}, \quad i \neq -1, \\ v^{(1)}(\zeta_i^{-1}) &= \begin{pmatrix} 0 & 0 \\ c_i \delta_i^{-2} d_1^{-2}(\zeta_i^{-1}) \Delta_1^{-2}(\zeta_i^{-1}) \zeta_i^{2n-1} e^{-2\lambda_i t} & 0 \end{pmatrix}, \quad i \neq -1, \end{aligned}$$

and

$$\begin{aligned} v^{(1)}(z) &= \begin{pmatrix} 1 & \frac{c_{-1}^{-1} \delta_{-1}^2 \Delta_1^2(z) \zeta_{-1}^{-2n-1} d_1^{-2}(\zeta_{-1}) e^{2\lambda_{-1} t}}{z - \zeta_{-1}} \\ 0 & 1 \end{pmatrix}, \quad z \in K, \\ (3.37')\qquad v^{(1)}(z) &= \begin{pmatrix} 1 & \frac{c_{-1}^{-1} \delta_{-1}^2 \Delta_1^2(z) \zeta_{-1}^{-2n+1} d_1^2(\zeta_{-1}^{-1}) e^{2\lambda_{-1} t}}{z - \zeta_{-1}^{-1}} \\ 0 & 1 \end{pmatrix}, \quad z \in L. \end{aligned}$$

Here $\delta_i = \delta(\zeta_i)$.

Finally, replacing $G^{(1)}(z)$ by

$$G^{(1)}(z) \begin{pmatrix} 1 & c_{-1}^{-1} \delta_{-1}^2 \zeta_{-1}^{-2n-1} d_1^{-2}(\zeta_{-1}) e^{2\lambda_{-1} t} \frac{\Delta_1^2(z) - \Delta_1^2(\zeta_{-1})}{z - \zeta_{-1}} \\ 0 & 1 \end{pmatrix}$$

inside K and by

$$G^{(1)}(z)\begin{pmatrix} 1 & c_{-1}^{-1}\delta_{-1}^2\zeta_{-1}^{-2n+1}d_1^2(\zeta_{-1}^{-1})e^{2\lambda_{-1}t}\frac{\Delta_1^2(z)-\Delta_1^2(\zeta_{-1}^{-1})}{z-\zeta_{-1}^{-1}} \\ 0 & 1 \end{pmatrix}$$

inside L, we see that (3.37′) takes the form

(3.38)
$$\begin{aligned} v^{(1)}(z) &= \begin{pmatrix} 1 & \frac{c_{-1}^{-1}\delta_{-1}^2\zeta_{-1}^{-2n-1}d_1^{-2}(\zeta_{-1})\Delta_1^2(\zeta_{-1})e^{2\lambda_{-1}t}}{z-\zeta_{-1}} \\ 0 & 1 \end{pmatrix}, \quad z \in K, \\ v^{(1)}(z) &= \begin{pmatrix} 1 & \frac{c_{-1}^{-1}\delta_{-1}^2\zeta_{-1}^{-2n+1}d_1^2(\zeta_{-1}^{-1})\Delta_1^2(\zeta_{-1}^{-1})e^{2\lambda_{-1}t}}{z-\zeta_{-1}^{-1}} \\ 0 & 1 \end{pmatrix}, \quad z \in L. \end{aligned}$$

The result of the above construction is that we have changed a Riemann-Hilbert problem with an exponentially growing factor at two of the poles $\zeta_{-1}, \zeta_{-1}^{-1}$, into a problem of exactly the same type except that the residue factor at the two poles has been replaced by a jump matrix on small circles surrounding these poles with factors that are now exponentially decreasing. En route the entries of the jump matrices are changed by harmless factors of d_1 and Δ_1. If we repeat this construction $(m-1)$ times for $\zeta_i, \zeta_i^{-1}, i = -2, \ldots, -m$, we can change the Riemann-Hilbert problem into one again of the same form, but now all the exponential factors that originally were growing are replaced by jump matrices on circles $K_i, L_i, i = -1, \ldots, -m$ with factors that are now exponentially decreasing. Along the way harmless multiplication factors are introduced.

Remark. Note that although all the exponentially growing factors in the specification of the Riemann-Hilbert problem have been converted into exponentially decreasing factors, this does not mean that elements in the solution of the Riemann-Hilbert problem cannot grow exponentially. In particular, note that the (2 1) entry of $J_K(z)$ in (3.30), say, has exponential growth, as does the (1 2) entry in (3.60) below.

3.4. The result of these computations is that we have replaced the original Riemann-Hilbert problem on Σ with an equivalent Riemann-Hilbert problem for a matrix Y given by

(3.39)
$$\begin{gathered} Y_+ = Y_- J, \quad \text{on } (\Sigma \setminus \{\zeta_i, \zeta_i^{-1} : i = -1, \ldots, -m\}) \cup (\cup_{i=1}^m (K_i \cup L_i)), \\ \lim_{z\to\infty} Y = I, \end{gathered}$$

where

(3.40)
$$J = \begin{pmatrix} z^{2n} & |\delta_+|^2 |\Delta_{f+}|^2 \prod_{j=1}^m \zeta_{-j}^{-2} e^{2\lambda t} \\ 0 & z^{-2n} \end{pmatrix} \quad \text{on } |z| = 1,$$

$$J = \begin{pmatrix} 1 & 0 \\ -\tau\delta^{-2}d_f^2(\Delta_f)^{-2}e^{-2\lambda t}z^{2n} & 1 \end{pmatrix} \quad \text{on } [z_1, z_2],$$

(3.41)

$$J = \begin{pmatrix} 1 & 0 \\ -\tau\delta^{-2}(d_f\Delta_f)^{-2}e^{-2\lambda t}z^{-2n} & 1 \end{pmatrix} \quad \text{on } [z_2^{-1}, z_1^{-1}].$$

$$J = \begin{pmatrix} 1 & \frac{\Delta_f^2(\zeta_i)e^{2\lambda_i t}}{d_f^2(\zeta_i)c_i\delta_i^{-2}\zeta_i^{2n+1}(z-\zeta_i)} \\ 0 & 1 \end{pmatrix} \quad \text{on } K_i,\ i = -1, \dots, -m,$$

(3.42)

$$J = \begin{pmatrix} 1 & \frac{(d_f\Delta_f)^2(\zeta_i^{-1})e^{2\lambda_i t}}{c_i\delta_i^{-2}\zeta_i^{2n-1}(z-\zeta_i^{-1})} \\ 0 & 1 \end{pmatrix} \quad \text{on } L_i,\ i = -1, \dots, -m.$$

Furthermore, Y has poles at ζ_i, $i = 0, 1, \dots, m$, with residues determined by the matrices

(3.43)
$$\begin{pmatrix} 0 & 0 \\ c_i\zeta_i^{2n+1}\delta_i^{-2}d_f^2(\zeta_i)(\Delta_f(\zeta_i))^{-2}e^{-2\lambda_i t} & 0 \end{pmatrix}$$

and at $\zeta_i^{-1}, i = 0, \dots, m$, with residues determined by

(3.44)
$$\begin{pmatrix} 0 & 0 \\ c_i\zeta_i^{2n-1}\delta_i^{-2}(d_f(\zeta_i^{-1})\Delta_f(\zeta_i^{-1}))^{-2}e^{-2\lambda_i t} & 0 \end{pmatrix}.$$

Here d_f and Δ_f denote the final results of the above computations,

(3.45)
$$d_f = \prod_{j=1}^{m} d_j, \qquad \Delta_f = \prod_{j=1}^{m} \Delta_j,$$

where each d_i is defined by the analog of (3.29) and each Δ_i is defined in terms of d_i by the analog of (3.34). Also note that Δ_f solves the Riemann-Hilbert problem with jump $d_f^2 z^{-2m}$ on $\{|z| = 1\}$. Finally, note from (3.29), (3.35), and the analogous relations for $i = -2, \dots, -m$ that for $|z| < 1$ and outside $\cup_{i=-1}^{-m} K_i$, we have

(3.45a)
$$\frac{d_f^2(z)}{\Delta_f^2(z)} = \frac{1}{d_f^2(z^{-1})\Delta_f^2(z^{-1})}.$$

If we keep track of the above operations, we see that

(3.46)
$$Y(0) = G(0)\begin{pmatrix} \frac{d_f}{\Delta_f}(0) & 0 \\ 0 & \frac{\Delta_f}{d_f}(0) \end{pmatrix} = G(0),$$

as $d_i(0) = \Delta_i(0)$ for each i (cf. (3.35)).

3.5. We are now ready to introduce the model problem that will be solved in Section 4. Let M solve the Riemann-Hilbert problem on a contour $\hat{\Sigma} = \Sigma \backslash \{\zeta_i, \zeta_i^{-1} : i = -1, \dots, -m\}$

$$\begin{aligned} M_+ &= M_- \hat{J} \\ M(\infty) &= I, \end{aligned} \tag{3.47}$$

where $\hat{J}$ agrees with J in (3.41), (3.43), (3.44), and

$$\hat{J} = \begin{pmatrix} z^{2n} & 0 \\ 0 & z^{-2n} \end{pmatrix} \tag{3.48}$$

on the unit circle. In particular, M has no jumps across K_i and L_i, $-m \leqq i \leqq -1$.

It is well-known (see, e.g., [4]) and easy to verify that a Riemann-Hilbert problem on the circle $\{|z| = 1\}$ with jump matrix

$$\begin{pmatrix} z^{2n} & 0 \\ 0 & z^{-2n} \end{pmatrix}$$

does not have a matrix solution for any $n \neq 0$. However, in Section 4 we will show that the jumps across $[z_1, z_2] \cup [z_2^{-1}, z_1^{-1}]$ and the poles stabilize the Riemann-Hilbert problem in the sense that, if $n \leqq 0$, then the solution M of the above model problem exists. We use results from Section 4 to show in the rest of this section that M is indeed a good approximation to Y, and thus the solution of the model problem leads to the solution of the Toda rarefaction problem. However, if $n > 0$, then it is easy to see that a matrix solution M does not exist, and indeed the above Riemann-Hilbert problem is not the correct model problem for the asymptotic behavior of $\{x_n, n > 0\}$ as $t \to \infty$.

Set $n' \equiv -n \geqq 0$. Let $\sigma(z)$ be defined by

$$\sigma(z) = \begin{cases} |\delta_+|^2 |\Delta_{f+}|^2 \prod_{i=1}^{m} \zeta_{-i}^{-2} & \text{on } |z| = 1\,, \\ \dfrac{(\Delta_f(\zeta_i))^2 (d_f(\zeta_i))^{-2} c_i^{-1} \delta_i^2 \zeta_i^{-2n-1}}{z - \zeta_i}, & z \in K_i,\ i = -1, \dots, -m\,, \\ \dfrac{(\Delta_f(\zeta_i^{-1}) d_f(\zeta_i^{-1}))^2 c_i^{-1} \delta_i^2 \zeta_i^{-2n+1}}{z - \zeta_i^{-1}}, & z \in L_i, i = -1, \dots, -m\,. \end{cases} \tag{3.49}$$

Set $E = YM^{-1}$. Then $E(\infty) = I$. Note that E solves a Riemann-Hilbert problem without poles and with jumps only on the oriented contour Σ_E, where $\Sigma_E = \{|z| = 1\} \bigcup \cup_{i=-1}^{-m} (K_i \cup L_i)$. Note that $\det M(z) = 1$, which is proven as in the argument following (2.20).

On $\{|z| = 1\}$

$$\begin{aligned} E_+ &= E_- M_- \begin{pmatrix} 1 & \sigma(z) z^{-2n'} e^{2\lambda t} \\ 0 & 1 \end{pmatrix} M_-^{-1} \\ &= E_- \left(I + \sigma(z) z^{-2n'} e^{2\lambda t} \begin{pmatrix} -M_{11-} M_{21-} & (M_{11-})^2 \\ -(M_{21-})^2 & M_{11-} M_{21-} \end{pmatrix} \right), \end{aligned} \tag{3.50}$$

and similarly, for each $i = -1, \ldots, -m$,

$$E_+ = E_- \left(I + \sigma(z) e^{2\lambda_i t} \begin{pmatrix} -M_{11-}M_{21-} & (M_{11-})^2 \\ -(M_{21-})^2 & M_{11-}M_{21-} \end{pmatrix} \right), \quad \text{on } K_i, L_i. \tag{3.51}$$

It will be shown in (4.5), (4.25), and (4.35) of Section 4 that for any given small $\varepsilon > 0$, on $\{|z| = 1\}$,

$$\begin{aligned} M_{11-} &= \begin{cases} 1, & \text{for } n' = 0, \\ O(e^{\varepsilon t}), & \text{for } n' > 0, \end{cases} \\ M_{21-} &= \begin{cases} O(e^{2(1+a)t}), & \text{for } n' > k, \\ O(e^{-2\lambda_{n'} t}), & \text{for } n' \leqq k, \end{cases} \end{aligned} \tag{3.52}$$

with similar estimates for M_{11} and M_{21} on the circles $K_i, L_i, i = -1, \ldots, -m$. Recall that k is the number of positive eigenvalues in the gap (see Section 1).

With these estimates, we obtain the following lemma.

LEMMA 3.3. *We introduce*

$$\mu_{n'} := \begin{cases} \lambda_{n'}, & \text{for } n' \leqq k, \\ -(1+a), & \text{for } n' > k, \end{cases}$$

so we have

$$\begin{aligned} E(0) &= I + \begin{pmatrix} O(e^{(-2\mu_1 - 2\mu_{n'} + \varepsilon)t}) & O(e^{-2(\mu_1 - \varepsilon)t}) \\ O(e^{(-4\mu_1 - 2\mu_{n'} + 3\varepsilon)t}) & O(e^{-4(\mu_1 - \varepsilon)t}) \end{pmatrix}, \quad \text{for } n' > 0 \\ E(0) &= I + e^{-2\mu_1 t} \begin{pmatrix} O(1) & O(1) \\ O(1) & O(1) \end{pmatrix}, \quad \text{for } n' = 0. \end{aligned} \tag{3.53}$$

Remark. We will see below in (3.60) that M_{12-} blows up exponentially. Fortunately, this matrix element does not appear in the jump matrix for $E = YM^{-1}$, and this makes it easy to show, as above, that M is a good approximation for Y.

Proof of Lemma 3.3: Let v_E denote the jump matrix for E in (3.50) and (3.51). Let C_- denote the Cauchy operator

$$(C_- f)(z) = \lim_{w \to z_-} \int_{\Sigma_E} \frac{f(s)}{s - w} \frac{ds}{2\pi i} \tag{3.54}$$

where $w \to z_-$ means that w approaches z from the negative side of the contour of z. As is well-known, C_- is bounded from $L^2(\Sigma_E)$ to itself.

Define the operator C on (2×2)-matrix-valued functions f by

$$Cf = C_-(f(v_E - I)). \tag{3.55}$$

The operator C is bounded from $L^2(\Sigma_E)$ to itself and

$$\|C\|_{L^2\to L^2} = O(e^{-2(\mu_1-\varepsilon)t}), \tag{3.56}$$

as follows from (3.50) through (3.52) and the fact that C_- is bounded.

Now a simple algebraic computation (see, e.g., [1]) shows that for $z \in \mathbb{C}\setminus\Sigma_E$,

$$E(z) = I + \int_{\Sigma_E} \frac{\phi(s)(v_E - I)(s)}{s-z}\frac{ds}{2\pi i}, \tag{3.57}$$

where ϕ solves

$$(I - C)\phi = I$$

in $L^2(\Sigma_E)$. But then $\phi = (I-C)^{-1}I = I + \psi$, where

$$\begin{aligned} \|\psi\|_{L^2(\Sigma_E)} &= \|(I-C)^{-1}(CI)\|_{L^2(\Sigma_E)} \\ &\leqq \|(I-C)^{-1}\|_{L^2\to L^2}\|C\|_{L^2\to L^2}\|I\|_{L^2} \\ &= O(e^{-2(\mu_1-\varepsilon)t}), \end{aligned} \tag{3.58}$$

by (3.56). Write

$$\begin{aligned} E(z) &= I + R_1(z) + R_2(z), \quad \text{where} \\ R_1(z) &= \int_{\Sigma_E} \frac{(v_E(s) - I)}{s-z}\frac{ds}{2\pi i}, \\ R_2(z) &= \int_{\Sigma_E} \frac{\psi(s)(v_E(s) - I)}{s-z}\frac{ds}{2\pi i}. \end{aligned} \tag{3.59}$$

For z in a compact set away from Σ_E, $R_1(z)$ can be estimated using (3.50) through (3.52). To estimate $R_2(z)$, we again use (3.50) through (3.52), but in addition we need the estimates (3.58) on $\|\psi\|_{L^2(\Sigma_E)}$. If we insert these estimates in (3.59), we obtain (3.53).

In Section 4, we will see in (4.4), (4.20), (4.22), (4.28), and (4.32) that

$$\begin{aligned} &M_{11}(0) = 1 + O(1/t), \\ &M_{12}(0) = \begin{cases} O(e^{(2\mu_{n'}+\varepsilon)t}), & \text{for } n' > 0, \\ 0, & \text{for } n' = 0, \end{cases} \\ &M_{21}(0) \text{ is of order } e^{-2\mu_{n'}t}, \\ &M_{22}(0) = 1 + O(1/t). \end{aligned} \tag{3.60}$$

As $Y(0) = E(0)M(0)$, we then deduce

$$\begin{aligned} Y_{21}(0) &= M_{21}(0)(1 + O(e^{-2\mu_1 t})), \\ Y_{22}(0) &= M_{22}(0)(1 + O(e^{-2\mu_1 t})). \end{aligned} \tag{3.61}$$

We will use (3.61) at the end of the next section to deduce Theorem 1.1 from the solution of the model problem.

4. The Model Problem

4.1. In this section we solve the model problem of Section 3.5. At the end of the section we also discuss the modifications in the method that are needed when the Lax operator L is resonant and/or nonsymmetric. We define

$$\nu(z) := \begin{cases} -\frac{\tau(z)}{(z_2-z)^{1/2}}(\delta(z))^{-2}d_f^2(z)(\Delta_f(z))^{-2} & \text{on } [z_1, z_2], \\ \frac{\tau(z)}{(z-z_2^{-1})^{1/2}}(\delta(z)d_f(z)\Delta_f(z))^{-2} & \text{on } [z_2^{-1}, z_1^{-1}], \end{cases} \tag{4.1}$$
$$\nu_j := c_j\delta_j^{-2}d_f^2(\zeta_j)(\Delta_f(\zeta_j))^{-2}.$$

We will see below that ν is in fact finite and nonzero at z_2 and z_2^{-1}.

Solving our model problem amounts to solving two vector Riemann-Hilbert problems, one for each row of M. If we denote our unknown by (g, h) we have two different cases. We write $(\alpha, \beta) = \lim_{z\to\infty}(g, h)$, and we need to consider $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$.

The jump and pole conditions for (g, h) are (recall $n' = -n$):

$$\begin{aligned} g_+ &= g_- z^{-2n'} \quad \text{on } |z| = 1, \\ h_+ &= h_- z^{2n'} \quad \text{on } |z| = 1, \\ g_+ &= g_- + h(z)\nu(z)(z_2 - z)^{1/2}e^{-2\lambda t}z^{-2n'} \quad \text{on } [z_1, z_2], \\ g_+ &= g_- - h(z)\nu(z)(z - z_2^{-1})^{1/2}e^{-2\lambda t}z^{2n'} \quad \text{on } [z_2^{-1}, z_1^{-1}], \\ \operatorname{Res}_{\zeta_i} g &= h(\zeta_i)\nu_i e^{-2\lambda_i t}\zeta_i^{-2n'+1}, \quad m \geqq i \geqq 0, \\ \operatorname{Res}_{\zeta_i^{-1}} g &= h(\zeta_i^{-1})\nu_i e^{-2\lambda_i t}\zeta_i^{-2n'-1}, \quad m \geqq i \geqq 0. \end{aligned} \tag{4.2}$$

Note here that we have used the symmetries (3.9) and (3.45a).

4.2. We begin with the case $n = 0$ first.

Case 1. $\alpha = 1, \beta = 0$. Then clearly $(g(z), h(z)) = (1, 0)$ is the solution.

Case 2. $\alpha = 0$, $\beta = 1$. In this case h has neither jumps nor poles; thus it is entire and bounded, and hence constant. Indeed $h(z) = 1$. On the other hand, from equation (4.2) one concludes

$$\begin{aligned} g(z) &= \sum_{j=0}^{m}\left(\frac{\nu_j\zeta_j e^{-2\lambda_j t}}{z-\zeta_j} + \frac{\nu_j\zeta_j^{-1}e^{-2\lambda_j t}}{z-\zeta_j^{-1}}\right) \\ &\quad + \int_{z_1}^{z_2}\frac{\nu(s)(z_2-s)^{1/2}e^{-2\lambda t}}{s-z}\frac{ds}{2\pi i} + \int_{z_2^{-1}}^{z_1^{-1}}\frac{\nu(s)(s-z_2^{-1})^{1/2}e^{-2\lambda t}}{s-z}\frac{ds}{2\pi i}, \\ g(0) &= -2\nu_0 + O(e^{-2\mu_1 t}). \end{aligned} \tag{4.3}$$

It follows that for the matrix solution M of the model problem

$$M(0) = \begin{pmatrix} 1 & 0 \\ -2\nu_0(1 + O(e^{-2\mu_1 t})) & 1 \end{pmatrix}. \tag{4.4}$$

Also,

$$M = \begin{pmatrix} 1 & 0 \\ O(1) & 1 \end{pmatrix}, \tag{4.5}$$

on both $\{|z| = 1\}$ and the small circles K_i, L_i.

4.3. We next proceed to the case $n \neq 0$. Without loss of generality, we will restrict ourselves to $n < 0$. The case $n > 0$ can be recovered from the symmetries (1.10). We write $n' = -n$. One observes that h has no jump across the real bands, nor does it have poles. Across the unit circle, on the other hand, $h_+ = h_- z^{2n'}$. Hence there exists a polynomial $q(z)$ such that

$$h(z) = \begin{cases} q(z) = \sum_{i=0}^{2n'} q_i z^i, & \text{for } |z| < 1, \\ \frac{q(z)}{z^{2n'}}, & \text{for } |z| > 1. \end{cases} \tag{4.6}$$

We first remove the jump at the unit circle by defining

$$b(z) := \begin{cases} g(z), & \text{if } |z| < 1, \\ \frac{g(z)}{z^{2n'}}, & \text{if } |z| > 1, \end{cases} \tag{4.7}$$

The jump relations for b are then given by

$$\begin{aligned} b_+ &= b_- + q(z)\nu(z)(z_2 - z)^{1/2} e^{-2\lambda t} z^{-2n'} \quad \text{on } [z_1, z_2], \\ b_+ &= b_- - q(z)\nu(z)(z - z_2^{-1})^{1/2} e^{-2\lambda t} z^{-2n'} \quad \text{on } [z_2^{-1}, z_1^{-1}], \end{aligned} \tag{4.8}$$

with

$$\begin{aligned} \operatorname{Res}_{\zeta_i} b &= q(\zeta_i)\nu_i e^{-2\lambda_i t} \zeta_i^{-2n'+1}, \\ \operatorname{Res}_{\zeta_i^{-1}} b &= q(\zeta_i^{-1})\nu_i e^{-2\lambda_i t} \zeta_i^{2n'-1}. \end{aligned} \tag{4.9}$$

for $i \geqq 0$. It is immediately verified that

$$b(z) = \int \frac{q(s)d\rho(s)}{s - z}, \tag{4.10}$$

with measure

$$\begin{aligned} d\rho(s) &= \nu(s)s^{-2n'} e^{-2\lambda t}(z_2 - s)^{1/2} \chi_{[z_1, z_2]} \frac{ds}{2\pi i} \\ &+ \nu(s)s^{-2n'} e^{-2\lambda t} \chi_{[z_2^{-1}, z_1^{-1}]} (s - z_2^{-1})^{1/2} \frac{ds}{2\pi i} \\ &- \sum_{j=0}^{m} \nu_j e^{-2\lambda_j t} \left(\zeta_j^{-2n'+1} \delta(\cdot - \zeta_j) + \zeta_j^{2n'-1} \delta(\cdot - \zeta_j^{-1}) \right). \end{aligned} \tag{4.11}$$

It remains to determine the polynomial $q(z) = \sum_{i=0}^{2n'} q_i z^i$. As

$$b(z) = \frac{\alpha}{z^{2n'}} + O(z^{-2n'-1}) \quad \text{as } z \to \infty,$$

we expand (4.10) as a power series in $1/z$ and are led to the following conditions:

$$\int s^j q(s)\,d\rho(s) = 0 \quad \text{for } 0 \leqq j \leqq 2n'-2 \tag{4.12}$$

and

$$\int s^{2n'-1} q(s)\,d\rho(s) = -\alpha. \tag{4.13}$$

Remark. Note that (4.12) is precisely the statement that the polynomial $q(s)$ is orthogonal to all polynomials of degree $\leqq 2n'-2$ with respect to the measure $d\rho$. It is at this point that we are making contact with the method of orthogonal polynomials and the solution procedure for the free semi-infinite Toda lattice as described in [5].

Furthermore,

$$g(0) = b(0) = \int s^{-1} q(s)\,d\rho(s) \tag{4.14}$$

and

$$q_{2n'} = \lim_{z\to\infty} h(z) = \beta. \tag{4.15}$$

The combined equations for $q_0, q_1, \ldots, q_{2n'-1}, g(0)$ can be written in matrix form

$$\begin{pmatrix} \int s^{-1} & \int s^0 & \cdots & \int s^{2n'-2} & \int s^{2n'-1} \\ \int s^0 & \int s^1 & \cdots & \int s^{2n'-1} & \int s^{2n'} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \int s^{2n'-1} & \int s^{2n'} & \cdots & \int s^{4n'-2} & \int s^{4n'-1} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{2n'-1} \\ \beta \end{pmatrix} = \begin{pmatrix} g(0) \\ 0 \\ \vdots \\ -\alpha \end{pmatrix}. \tag{4.16}$$

We proceed to solve this explicitly and to determine the long-time asymptotic behavior of the solution.

4.4. Case $\alpha = 0$, $\beta = 1$.
By Cramer's rule we obtain

$$\beta = 1 = g(0)\frac{\det\begin{pmatrix} \int s^0 & \cdots & \int s^{2n'-1} \\ \vdots & \ddots & \vdots \\ \int s^{2n'-1} & \cdots & \int s^{4n'-2} \end{pmatrix}}{\det\begin{pmatrix} \int s^{-1} & \cdots & \int s^{2n'-1} \\ \vdots & \ddots & \vdots \\ \int s^{2n'-1} & \cdots & \int s^{4n'-1} \end{pmatrix}}. \tag{4.17}$$

A straightforward calculation shows that the numerator of the fraction in the right-hand side of (4.17) can be written as

$$\begin{aligned} &\int d^{2n'}\rho\, s_1^0 s_2^1 \cdots s_{2n'}^{2n'-1} V(s_1, s_2, s_3, ..., s_{2n'}) \\ &= \frac{1}{(2n')!}\int d^{2n'}\rho\, V^2(s_1, s_2, s_3, \ldots, s_{2n'}), \end{aligned} \tag{4.18}$$

where V denotes the usual Vandermonde determinant. Similarly, the corresponding denominator can be written as

$$\frac{1}{(2n'+1)!}\int d^{2n'+1}\rho\, \frac{1}{s_1 \cdots s_{2n'+1}} V^2(s_1, \ldots, s_{2n'+1}). \tag{4.19}$$

We apply Lemma A.3 in the appendix to determine $g(0)$ by making the obvious choice $f(s_1, \ldots, s_{2n'+1}) = 1/(s_1 \cdots s_{2n'+1})$. Then

$$g(0) = \begin{cases} -2\nu_{n'}\left(\prod_{j=0}^{n'-1}(\zeta_{n'} - \zeta_j)(\zeta_{n'} - \zeta_j^{-1})(\zeta_{n'}^{-1} - \zeta_j)(\zeta_{n'}^{-1} - \zeta_j^{-1})\right) \\ \quad \times e^{-2\lambda'_n t}\left(1 + O(e^{-\eta t})\right), \qquad \text{if } n' \leqq k, \\ \\ -\frac{2}{\sqrt{\pi}}|T^+(z_2)|^2\left(\frac{d_f(z_2)}{\delta(z_2)\Delta_f(z_2)}\right)^2 (z_2^{-1} - z_2)^{-2} \\ \quad \times \left(\prod_{j=0}^{k}(z_2 - \zeta_j)(z_2 - \zeta_j^{-1})(z_2^{-1} - \zeta_j)(z_2^{-1} - \zeta_j^{-1})\right) \\ \quad \times \frac{(2(n'-k)-1)!}{2^{2(n'-k)-1}}\frac{e^{2(1+a)t}}{t^{2(n'-k)-1/2}}\left(1 + O(1/t)\right), \qquad \text{if } n' > k, \end{cases} \tag{4.20}$$

where $\eta = \min(\mu_{n'} - \mu_{n'-1}, \mu_{n'+1} - \mu_{n'})$.
This completes the computation of $M_{21}(0)$.

The coefficients of the polynomial $q(z)$ can be obtained by Cramer's rule and by (4.17). One obtains

$$q_i = (-1)^i \frac{\det \begin{pmatrix} \int s^0 & \cdots & \int s^{i-1} & \int s^{i+1} & \cdots & \int s^{2n'} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \int s^{2n'-1} & \cdots & \int s^{2n'+i-2} & \int s^{2n'+i} & \cdots & \int s^{4n'-1} \end{pmatrix}}{\det \begin{pmatrix} \int s^0 & \cdots & \int s^{2n'-1} \\ \vdots & \ddots & \vdots \\ \int s^{2n'-1} & \cdots & \int s^{4n'-2} \end{pmatrix}}$$

$$= (-1)^i \frac{\frac{1}{(2n')!} \int d^{2n'}\rho \; s_{i+1} \cdots s_{2n'} V^2(s_1, \ldots, s_{2n'})}{\frac{1}{(2n')!} \int d^{2n'}\rho \; V^2(s_1, \ldots, s_{2n'})}.$$

We apply parts (a) and (c) of Lemma A.3 and the remark that follows it to conclude that the numerator and the denominator of the above fraction are of the same order, and hence there exists a $q_i^\infty \in \mathbb{R}$ such that $q_i = q_i^\infty(1 + O(1/t))$, for $n' > k + 1$, and $q_i = q_i^\infty(1 + O(e^{-(\mu_{n'} - \mu_{n'-1})t}))$, for $n' \leqq k + 1$.

For $i = 0$, the situation is especially simple, as

$$q_0 = \frac{V_{2n'}(f)}{V_{2n'}} = \begin{cases} 1 + O(1/t), & \text{if } n' > k+1, \\ 1 + O(e^{-(\mu_{n'} - \mu_{n'-1})t}), & \text{if } n' \leqq k+1. \end{cases} \tag{4.21}$$

Here $f(s_1, \ldots, s_{2n'}) = \prod_{j=1}^{2n'} s_j$.

It follows from equation (4.6) that $h(0) = q_0$:

$$h(0) = \begin{cases} 1 + O(e^{-(\mu_{n'} - \mu_{n'-1})t}), & \text{for } n' \leqq k+1, \\ 1 + O(1/t), & \text{for } n' > k+1. \end{cases} \tag{4.22}$$

Finally, we have to provide estimates for $g(t)$ on the unit circle and on the circles K_j, L_j, $1 \leqq j \leqq m$.

If we observe that the coefficients q_i are bounded in time and we use the time decay of ρ (see (4.11)), we conclude from (4.12) and (4.13) that

$$\sum_{i=0}^{\min(k,n'-1)} \zeta_i^j q(\zeta_i)\rho(\{\zeta_i\}) + \zeta_i^{-j} q(\zeta_i^{-1})\rho(\{\zeta_i^{-1}\}) = O(e^{-2\mu_{n'}t}), \quad 0 \leqq j \leqq 2n' - 1. \tag{4.23}$$

We apply Cramer's rule to (4.23) to get

$$q(\zeta_i)\,\rho(\{\zeta_i\}) = O(e^{-2\mu_{n'}t}), \quad q(\zeta_i^{-1})\,\rho(\{\zeta_i^{-1}\}) = O(e^{-2\mu_{n'}t}) \tag{4.24}$$

for $0 \leqq i \leqq \min(k, n'-1)$. From equations (4.7) and (4.10) we then see that

(4.25) $$g(z) = O(e^{-2\mu_{n'}t})$$

uniformly on the unit circle and on the circles K_j, L_j. We now determine the first row of the matrix solution.

4.5. Case $\alpha = 1$, $\beta = 0$.

We proceed as in Section 4.4. Equation (4.16) now gives (recall $q_{2n'} = \beta$)

(4.26)
$$g(0)\det\begin{pmatrix} \int s^0 & \cdots & \int s^{2n'-1} \\ \vdots & \ddots & \vdots \\ \int s^{2n'-1} & \cdots & \int s^{4n'-2} \end{pmatrix} - \det\begin{pmatrix} \int s^{-1} & \cdots & \int s^{2n'-2} \\ \vdots & \ddots & \vdots \\ \int s^{2n'-2} & \cdots & \int s^{4n'-3} \end{pmatrix} = 0.$$

Hence

(4.27) $$g(0) = \frac{\int d^{2n'}\rho \, (\prod_{i=1}^{2n'} s_i)^{-1} V^2(s_1, \ldots, s_{2n'})}{\int d^{2n'}\rho \, V^2(s_1, \ldots, s_{2n'})}.$$

From parts (a) and (c) of Lemma A.3,

(4.28) $$g(0) = \begin{cases} 1 + O(e^{-(\mu_{n'} - \mu_{n'-1})t}), & \text{if } n' \leqq k+1 \\ 1 + O(\frac{1}{t}), & \text{if } n' > k+1. \end{cases}$$

For $0 \leqq i \leqq 2n' - 1$, let

$$f_1^{(i)}(s_1, \ldots, s_{2n'}) = \prod_{j=i+1}^{2n'} s_j,$$

$$f_2^{(i)}(s_1, \ldots, s_{2n'}) = \prod_{j=1}^{i} s_j^{-1},$$

$$f(s_1, \ldots, s_{2n'+1}) = \prod_{j=1}^{2n'+1} s_j^{-1}.$$

Then the ith coefficient of $q(z)$ is given by

(4.29) $$q_i = (-1)^i \frac{g(0) V_{2n'}(f_1^{(i)}) - V_{2n'}(f_2^{(i)})}{V_{2n'+1}(f)}.$$

Lemma A.3 and the remark following it imply

(4.30) $$q_i(t) = \begin{cases} O(e^{2\lambda_{n'}t}), & \text{for } 1 \leqq n' \leqq k, \\ O(e^{-2(1+a)t} t^{2(n'-k)-1/2}), & \text{for } n' > k. \end{cases}$$

Hence

$$q_i(t) = O(e^{(2\mu_{n'}+\varepsilon)t}), \qquad \text{for any } \varepsilon > 0. \tag{4.31}$$

Equations (4.28) and (4.31) yield

$$(g(0), \ h(0)) = (1 + O(1/t), \ \ O(e^{(2\mu_{n'}+\varepsilon)t})), \qquad \text{for any } \varepsilon > 0. \tag{4.32}$$

Finally, we have to find estimates for $g(z)$ on the unit circle and on the small circles K_j and L_j. Again we use (4.11) through (4.13) and (4.31) to obtain

$$\sum_{i=0}^{\min(k,n'-1)} \zeta_i^j \, q(\zeta_i)\,\rho(\{\zeta_i\}) + \zeta_i^{-j}\, q(\zeta_i^{-1})\,\rho(\{\zeta_i^{-1}\}) = O(e^{\varepsilon t}), \tag{4.33}$$

for $0 \leqq j \leqq 2n' - 1$, and consequently

$$\begin{aligned} q(\zeta_i)\,\rho(\{\zeta_i\}) &= O(e^{\varepsilon t}), \\ q(\zeta_i^{-1})\,\rho(\{\zeta_i^{-1}\}) &= O(e^{\varepsilon t}), \end{aligned} \tag{4.34}$$

for $0 \leqq i \leqq \min(k, n' - 1)$. By (4.7) and (4.10) we arrive at a uniform estimate for $g(z)$ on the unit circle and on the circles K_j, L_j:

$$g(z) = O(e^{\varepsilon t}). \tag{4.35}$$

4.6. We can now summarize our computations. The model problem defined by (3.47), (3.41), (3.43), (3.44), and (3.48) has a matrix solution $M_{n',t}(z)$ for $n' \geqq 0$ and $t > 0$ such that as $t \to \infty$, we have the following results and estimates:

When $n' = 0$, (4.4) holds. When $n' > 0$, the estimates for $M_{11}(0)$ and $M_{12}(0)$ as $t \to \infty$ are given by (4.28) and (4.32), the estimates for $M_{21}(0)$ are given by (4.20), and the estimates for $M_{22}(0)$ are given by (4.22), once we recall the definitions of g, h, and q. Estimates (3.60) follow.

Furthermore, the following estimates hold uniformly on the union of the unit circle and the small circles $K_j, L_j, j = 1, \ldots, m$:

$$M_{11}(z) = \begin{cases} 1, & \text{for } n' = 0 \quad \text{(see (4.4))}, \\ O(e^{\varepsilon t}), & \text{for } n' > 0 \quad \text{(see (4.35))}, \end{cases}$$

$$M_{21}(z) = O(e^{-2\mu_{n'} t}) \quad \text{(see (4.25))}.$$

Equation (3.52) is then proved. As we have noted at the end of Section 3, these estimates suffice to prove that the full problem for Y has a matrix solution for t sufficiently large and

$$\begin{aligned} Y_{21}(0) &= M_{21}(0)(1 + O(e^{-2\mu_1 t})), \\ Y_{22}(0) &= M_{22}(0)(1 + O(e^{-2\mu_1 t})). \end{aligned}$$

4.7. If we use

$$B^{+}_{-n'}e^{at} = \left(\frac{1+Y_{22}(0)}{-Y_{21}(0)}\right)^{1/2} \tag{4.36}$$

(which follows from (3.46), (3.13), (2.19), and (2.11)) and we recall (4.4), (4.20), and (4.22), we obtain the following as $t \to \infty$:

For $n' = 0$:

$$B^{+}_{0}e^{at} = F_0(1 + O(e^{-2\mu_1 t})), \tag{4.36a}$$

where $F_0 := \nu_0^{-1/2}$.

For $1 \leqq n' \leqq k$:

$$B^{+}_{-n'}e^{at} = F_{n'}\, e^{\mu_{n'}t}(1 + O(e^{-\tilde{\varepsilon}_{n'}t})), \tag{4.36b}$$

where

$$F_{n'} := \left(\nu_{n'} \prod_{j=0}^{n'-1}(\zeta_{n'} - \zeta_j)(\zeta_{n'} - \zeta_j^{-1})(\zeta_{n'}^{-1} - \zeta_j)(\zeta_{n'}^{-1} - \zeta_j^{-1})\right)^{-1/2},$$

$$\tilde{\varepsilon}_{n'} := \min(2\mu_1, \mu_{n'} - \mu_{n'-1}, \mu_{n'+1} - \mu_{n'}).$$

For $n' > k$:

$$B^{+}_{-n'}e^{at} = F_{n'}\frac{t^{n'-k-1/4}}{e^{(1+a)t}}(1 + O(1/t)), \tag{4.36c}$$

where

$$F_{n'} := \left\{\frac{|T^{+}(z_2)|^2}{\sqrt{\pi}\,(z_2 - z_2^{-1})^2}\left(\prod_{j=0}^{k}(z_2 - \zeta_j)(z_2 - \zeta_j^{-1})(z_2^{-1} - \zeta_j)(z_2^{-1} - \zeta_j^{-1})\right)\right.$$
$$\left. \times \left(\frac{d_f(z_2)}{\delta(z_2)\Delta_f(z_2)}\right)^2 \frac{(2(n'-k)-1)!}{2^{2(n'-k)-1}}\right\}^{-1/2}.$$

Finally, we derive the formulae for $b_n^2(t)$ and for $x_n(t)$. If we recall that $B_n^{+} = \prod_{k=n}^{\infty}(2b_k)^{-1}$, we obtain from (1.10) for $n \geqq 0$

$$b_n^2 = b_{-n-1}^2 = \frac{B_{-n}^2}{4B_{-n-1}^2}$$
$$= \begin{cases} C_n\, e^{-2(\lambda_{n+1}-\lambda_n)t}(1 + O(e^{-\min(\tilde{\varepsilon}_n, \tilde{\varepsilon}_{n+1})t})), & \text{for } 0 \leqq n \leqq k-1, \\ C_k\, e^{2(1+a+\lambda_k)t}\, t^{-3/2}(1 + O(1/t)), & \text{for } n = k, \\ \frac{(2(n-k)+1)(n-k)}{8t^2}(1 + O(1/t)), & \text{for } n > k, \end{cases} \tag{4.37}$$

where

$$C_n := \frac{F_n^2}{4F_{n+1}^2}. \tag{4.37'}$$

From the symmetry of the system (1.4) and (1.5) it follows that $x_0 = 0$ for all time. Hence, for $n \leqq 0$,

$$x_n(t) = x_n(t) - x_0(t) = \log\left(\prod_{j=n}^{-1}(2b_j)^2\right) = \log\left(\frac{B_0^+}{B_n^+}\right)^2. \tag{4.37''}$$

Therefore, we use $x_{-n}(t) = -x_n(t)$ to obtain

$$x_n(t) = \begin{cases} 2\lambda_n t + K_n + O(e^{-\tilde{\varepsilon}_n t}), \\ \quad \text{if } 1 \leqq n \leqq k, \\ -2(1+a)t + (2(n-k) - 1/2)\log t + K_n + O(1/t), \\ \quad \text{if } n > k, \end{cases} \tag{4.38}$$

where

(4.39)

$$K_n = \log\left\{\frac{\nu_0}{\nu_n}\left(\prod_{j=0}^{n-1}(\zeta_n - \zeta_j)(\zeta_n - \zeta_j^{-1})(\zeta_n^{-1} - \zeta_j)(\zeta_n^{-1} - \zeta_j^{-1})\right)^{-1}\right\},$$

if $1 \leqq n \leqq k$,

$$K_n = \log\left\{\frac{\nu_0\sqrt{\pi}\,|T^+(z_2)|^{-2}d_f^{-2}(z_2)\delta^2(z_2)\Delta_f^2(z_2)(z_2^{-1} - z_2)^2 2^{2(n-k)-1}}{\left(\prod_{j=0}^{k}(z_2 - \zeta_j)(z_2 - \zeta_j^{-1})(z_2^{-1} - \zeta_j)(z_2^{-1} - \zeta_j^{-1})\right)(2(n-k)-1)!}\right\},$$

if $n > k$.

4.8. Discussion of the resonant case. Here we sketch the modifications in the calculations that are needed to analyze the long-time behavior of the Toda rarefaction problem in the case that the Lax operator L is at resonance.

Because of the λ-symmetry in the problem ($W(\lambda) = -W(-\lambda)$), there are only three cases to consider:

(a) resonance at the outer edges of the spectral bands; that is, $W(a-1) = 0$, $W(-a+1) = 0$;
(b) resonance at the inner edges of the spectral bands; that is, $W(a+1) = 0$, $W(-a-1) = 0$;
(c) L_0 is fully resonant; that is, $W(a-1) = W(a+1) = W(-a-1) = W(-a+1) = 0$.

In case (a), the long-time behavior has the same form as (1.13), but now formula (4.39) for the constant K_n must be modified. In cases (b) and (c), formula (1.13) must be replaced by

$$(1.13') \quad \begin{aligned} x_n &= 2(\lambda_n + a)t + K'_n + O(e^{-\tilde{\varepsilon}_n t}), && \text{for } 1 \leqq n \leqq k, \\ x_n &= -2t + (2(n-k) - 3/2)\log t + K'_n + O(1/t), && \text{for } n > k, \end{aligned}$$

where the constant K'_n can be computed explicitly.

From the above we see that up to order $O(1)$ only a resonance at the inner edge of the spectral bands leads to a change in the long-time behavior of the lattice. Moreover, up to order $O(1)$ the influence of the resonance is felt only in the bulk of the lattice, that is, for $n > k$, where the net effect is simply an addition of half an eigenvalue (i.e., k is replaced by $k + \frac{1}{2}$ in (1.13)).

In order to obtain these results we have to make the following modifications in the analysis:

(i) Winding of R. Denote by $2m_{\text{res}}$ the number of resonances; that is, in cases (a) and (b) $m_{\text{res}} = 1$, and in case (c) $m_{\text{res}} = 2$. With this notation the formula in Lemma 3.1 is replaced by

$$\frac{1}{2\pi i}\int_{|z|=1} d\log R = -2m - m_{\text{res}}.$$

The proof of this follows from the fact that W can vanish at most to first order at $z = 1$ or $z = -1$ (cf. [10] and [11]). If we use δ as before to remove R and $\bar{R}$ from the diagonal terms, we eventually arrive at a Riemann-Hilbert problem for Y, where the (1 1) entry of the jump matrix at $\{|z| = 1\}$ is given by $z^{2n-m_{\text{res}}}$ and the (2 2) entry is given by $z^{-2n+m_{\text{res}}}$. It follows that the jump matrix for the model problem is given by

$$\begin{pmatrix} z^{2n-m_{\text{res}}} & 0 \\ 0 & z^{-2n+m_{\text{res}}} \end{pmatrix}$$

on $\{|z| = 1\}$, and all other jump matrices have the same form as before. This means that the previous techniques apply, and in the case $n \leqq m_{\text{res}}/2$, the solution M of the model problem exists and yields a bona fide approximation of the solution of the full Riemann-Hilbert problem for Y. Note that as $m_{\text{res}}/2 > 0$, x_n can be determined directly for $n \leqq 0$ and hence by the symmetry $x_{-n} = -x_n$ for all $n \in \mathbb{Z}$.

In addition to the above replacement of $2n$ by $2n - m_{\text{res}}$, there are two other differences that play a role in the detailed calculation of the long-time behavior of the rarefaction problem.

(ii) In the nonresonant case the measure ρ has edge behavior $\rho(z) \sim \text{const}\,\sqrt{|z - z'|}dz$ for z near $z' \in \{z_1, z_2, z_1^{-1}, z_2^{-1}\}$. On the other hand, in the case that L is resonant at z', the behavior of $\rho(z)$ is given by $(\text{const}\,/\sqrt{|z - z'|})dz$. From the calculations in the appendix, it is clear that the different behavior at the edge has no new effect on the long-time behavior of the lattice up to order $O(1)$ if the resonance occurs at $z' = z_1$, z_1^{-1}. If the resonance occurs at z_2, z_2^{-1}, however, then the singularity at the edge produces the shift $k \to k + \frac{1}{2}$ in (1.13′).

(iii) As noted above in the resonant case, $W(z)$ vanishes precisely to first order at $z = 1$ and/or $z = -1$. In particular, this implies $R_+(1) = 1$ if and only if L is resonant at $z = 1$. In this case (3.9) is replaced by

$$\begin{aligned} \delta(z) &= \delta(z^{-1}) \quad \text{on } \mathbb{C}\setminus\{|z| = 1\}, \\ \delta_+(z) &= \bar{\delta}_-(z) \quad \text{on } |z| = 1, \end{aligned} \tag{3.9′}$$

and hence $\delta(0) = \delta(\infty) = 1$, which in turn implies, eventually, that

$$(B_n^+ e^{at})^2 = \frac{1 - Y_{22}(0)}{Y_{21}(0)} \tag{4.40}$$

(cf. formula (4.36)).

In the detailed calculation the replacement of $2n$ by $2n - m_{\text{res}}$ in the jump matrix for M on $\{|z| = 1\}$ (see (i)) and the modification (iii) above work in tandem to ensure that there is no further change in the asymptotic behavior of (1.13) beyond that which is produced by (ii). For example, if $m_{\text{res}} = 2$, then $Y_{22} \sim 1$, so that a critical cancellation takes place in the numerator of (4.40) to leading order. It is a matter of detailed calculation to see that this cancellation counterbalances the replacement of $2n$ by $2n - m_{\text{res}}$, that is, of n by $n - 1$, with no further effect on (1.13).

4.9. Discussion of the nonsymmetric case. Here we sketch the modifications that are needed in the case that $a_n - \text{sgn}(n)a$ and $b_n - \frac{1}{2}$ decay rapidly as $|n| \to \infty$, but the symmetry condition (1.10) is violated. In terms of the scattering theory of the operator L_0, the difference between this and the symmetric case is that the relation $W(-\lambda) = -W(\lambda)$ now fails. Denote by $\lambda_1, \ldots, \lambda_{m_+}$ the positive eigenvalues of L and by $\lambda_{-1}, \ldots, \lambda_{m_-}$ the negative eigenvalues of L. If $0 \in \sigma(L)$, then let $\lambda_0 = 0$. It is clear that generically $m_+ \neq m_-$ and $\lambda_i \neq -\lambda_{-i}$. For simplicity we will assume that L_0 is nonresonant at all four ends of the spectral bands $[a-1, a+1]$ and $[-a-1, -a+1]$. Because symmetry plays no crucial role in the preceding discussion (Section 4.8) it is easy to incorporate resonances if so desired.

In the paper we have used symmetry to

(i) deduce the behavior of $\{x_n, n \leqq 0\}$ from the formula (4.37″), which relies on the identity $x_0(t) \equiv 0$ and which is true in general only in the symmetric case,

(ii) compute the winding of R (cf. Lemma 3.1),

(iii) recover the long-time behavior of $\{x_n, n > 0\}$ via $x_{-n}(t) = -x_n(t)$.

We first address the question of how to recover $x_n(t)$ once $B_n^+(t)$ is determined from the solution of the Riemann-Hilbert problem (compare with (i)). From $x_n(t) = x_n(0) - 2\int_0^t a_n(s)ds$ together with

- $|a_n(s)| \leqq \|L(s)\| = \|L_0\|$ (recall that $s \mapsto L(s)$ is an isospectral deformation),
- $x_n(0) \to x_{\pm\infty}(0)$ for $n \to \pm\infty$,
- $a_n(s) \to \pm a$ for $n \to \pm\infty$,

it follows that $\lim_{n\to\pm\infty} x_n(t) = -2at + x_{\pm\infty}(0)$ for all times t. This in turn implies by a telescoping argument that $(B_n^+)^2 = e^{-2at+x_\infty(0)-x_n}$. Thus

$$x_n(t) = -\log\left(B_n^{+2} e^{2at-x_\infty(0)}\right) .$$

Remark 1. In the symmetric case we know that $x_0(t) \equiv 0$. This implies $0 = -\log(\nu_0^{-1}) + x_\infty(0)$; that is, by (4.1)

$$c_0 = \frac{\delta(\zeta_0)^2 \Delta_f(\zeta_0)^2}{d_f(\zeta_0)^2 e^{x_\infty(0)}} .$$

We know of no direct proof of this interesting relation.

We now indicate how to compute the asymptotics of B_n^+ for all $n \in \mathbb{Z}$, in view of (ii) and (iii). The effect of (ii) is that we can no longer establish a relation between the winding of R and the number of eigenvalues. Instead we simply set

$$m_W := \frac{1}{2\pi i}\int_{\Sigma_1} d\log W$$

Then

$$\frac{1}{2\pi i}\int_{\Sigma_1} d\log R = -2m_W.$$

If one follows the procedure of Section 3 and replaces the poles corresponding to the negative eigenvalues by jumps on small cicles K_i, L_i, one arrives at a model problem where the jump matrix across $|z| = 1$ is given by

$$\begin{pmatrix} z^{2(n-m_W+m_-)} & 0 \\ 0 & z^{-2(n-m_W+m_-)} \end{pmatrix} .$$

This implies that the model problem can be solved explicitly and yields a bona fide approximation to the full problem *if and only if* $n \leqq m_W - m_-$.

In order to obtain the time-asymptotic behavior of $\{x_n, n > m_W - m_-\}$ we consider initial conditions for the Toda equation $\hat{a}_n^0 := -a_{-n}^0$, $\hat{b}_n^0 := b_{-n-1}^0$. These initial conditions satisfy the same asymptotic conditions in n as before and it is easy to see that for all times $\hat{a}_n(t) = -a_{-n}(t)$, $\hat{b}_n(t) = b_{-n-1}(t)$. If we apply the above methods to the new problem we obtain a solution of the original problem for $-n \leqq m_{\hat{W}} - \hat{m}_-$, where

$$m_{\hat{W}} := \frac{1}{2\pi i} \int_{\Sigma_1} d \log \hat{W}$$

and $\hat{m}_-$ denotes the number of negative eigenvalues of $\hat{L}_0$. Thus $B_n^+ e^{at}$ can be evaluated asymptotically for $n \leqq m_W - m_-$ and also for $n \geqq -m_{\hat{W}} + \hat{m}_-$. we now employ the relation between L and $\hat{L}$. It is easy to check that $\hat{W}(\lambda) = -W(-\lambda)$, which implies in turn that

$$\hat{m}_- = m_+ \,. \tag{4.41}$$

Furthermore, we conclude (see Figure 3.1)

$$\begin{aligned} m_W + m_{\hat{W}} &= \frac{1}{2\pi i}\left(\int_{\Sigma_1} d\log W + \int_{\Sigma_1} d\log \hat{W}\right) \\ &= \frac{1}{2\pi i}\left(\int_{\Sigma_1} d\log W + \int_{\Sigma_2} d\log W\right) \\ &= \#\{\text{eigenvalues of } L\} - 1\,; \end{aligned}$$

that is,

$$m_W + m_{\hat{W}} = \begin{cases} m_+ + m_- - 1\,, & \text{if } 0 \notin \sigma(L)\,, \\ m_+ + m_-\,, & \text{if } 0 \in \sigma(L)\,. \end{cases} \tag{4.42}$$

From (4.41) and (4.42) it follows that

$$-m_{\hat{W}} + \hat{m}_- = \begin{cases} m_W - m_- + 1\,, & \text{if } 0 \notin \sigma(L)\,, \\ m_W - m_-\,, & \text{if } 0 \in \sigma(L)\,. \end{cases}$$

Therefore $B_n^+ e^{at}$ and hence $x_n(t)$ can be evaluated asymptotically for all $n \in \mathbb{Z}$.

Remark 2. As described above, the analysis of the nonsymmetric case is based on the observation that $(a_n, b_n) \mapsto (-a_{-n}, b_{-n-1})$ takes a solution of the Toda lattice into a second solution. Alternatively, in Section 2, we could have chosen $\tilde{z}$ rather than z as the basic variable and so constructed a different Riemann-Hilbert problem for the solution of the lattice. In this case one would find that the analogous model problem is stable for $n \geqq \tilde{m}_W - m_+$ where $\tilde{m}_W := \frac{1}{2\pi i}\int_{\{|\tilde{z}|=1\}} d\log W$.

It is easy to show that $\tilde{m}_W - m_+ = m_{\hat{W}} - \hat{m}_-$ and that these two approaches to evaluating the asymptotic behavior of the lattice for $n \geqq \tilde{m}_W - m_+$ are in fact equivalent. In the symmetric case, for example, the calculations in the paper proceed for $n \leqq 0$ by neglecting the contribution from the off-diagonal elements in the jump matrix across $\{|z| = 1\}$, where $\lambda < 0$. For $n \geqq 0$, the Riemann-Hilbert problem is reformulated in terms of the $\tilde{z}$-variable, and now the off-diagonal elements of the jump matrix across $\{|\tilde{z}| = 1\}$, where $\lambda > 0$, are neglected. In other words, the long-time asymptotics of $\{x_n, n \leqq 0\}$ is governed by the nonnegative spectrum of L, whereas the behavior of $\{x_n, n \geqq 0\}$ is governed by the nonpositive spectrum.

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Appendix

In this appendix we show how to evaluate the time asymptotic behavior of integrals of the type

$$\frac{1}{l!}\int f(s_1,\dots,s_l)V^2(s_1,\dots,s_l)\,d\rho(s_1)\cdots d\rho(s_l), \tag{A.1}$$

where ρ is the measure defined in (4.11), in terms of which the solution of the model problem is determined in Section 4.

We need the following two propositions.

PROPOSITION A.1. *Let r be a natural number and denote by $V(s_1,\dots,s_r)$ the Vandermonde determinant. Then*

$$\frac{1}{r!}\int_0^\infty\!\cdots\int_0^\infty e^{-(s_1+\cdots+s_r)}(s_1\cdots s_r)^{1/2}V^2(s_1,\dots,s_r)\,ds = \pi^{r/2}\frac{\prod_{j=1}^r(2j-1)!}{2^{r^2}}. \tag{A.2}$$

Proof: It is easy to see that the left-hand side of (A.2) equals the determinant of the following matrix:

$$\begin{pmatrix} \int_0^\infty e^{-s}s^{1/2}ds & \int_0^\infty e^{-s}s^{3/2}ds & \cdots & \int_0^\infty e^{-s}s^{r-1/2}ds \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^\infty e^{-s}s^{r-1/2}ds & \int_0^\infty e^{-s}s^{r+1/2}ds & \cdots & \int_0^\infty e^{-s}s^{2r-3/2}ds \end{pmatrix}.$$

Therefore

$$\frac{1}{r!}\int_0^\infty \cdots \int_0^\infty e^{-(s_1+\cdots+s_r)}(s_1\cdots s_r)^{1/2}V^2(s_1,\ldots,s_r)\,ds_1\cdots ds_r$$

$$= \pi^{r/2}\det\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\frac{3}{2} & \cdots & \frac{2r-1}{2}\frac{2r-3}{2}\cdots\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2r-1}{2}\cdots\frac{1}{2} & \frac{2r+1}{2}\cdots\frac{1}{2} & \cdots & \frac{4r-3}{2}\cdots\frac{1}{2} \end{pmatrix}$$

$$= \pi^{r/2}\det\begin{pmatrix} \frac{1}{2} & * & * & \cdots & * \\ 0 & \frac{3!}{2^3} & * & \cdots & * \\ 0 & 0 & \frac{5!}{2^5} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & \cdot & \cdot\ \cdot & \frac{(2r-1)!}{2^{2r-1}} \end{pmatrix}$$

$$= \pi^{r/2}\cdot\frac{\prod_{j=1}^r(2j-1)!}{2^{r^2}}.$$

Let γ be a C^1 function on $[z_1,z_2]^r \cup [z_2^{-1},z_1^{-1}]^r$. We define, for $r \in \mathbb{N}$,

$$\text{(A.3)}\qquad \begin{aligned} I_1(r,\gamma) &:= \frac{1}{r!}\int_{z_1}^{z_2}\cdots\int_{z_1}^{z_2}\gamma(s)\left(\prod_{j=1}^r((z_2-s_j)^{1/2}e^{-2\lambda(s_j)t}\right)V^2(s_1,\ldots,s_r)\,ds,\\ I_2(r,\gamma) &:= \frac{1}{r!}\int_{z_2^{-1}}^{z_1^{-1}}\cdots\int_{z_2^{-1}}^{z_1^{-1}}\gamma(s)\left(\prod_{j=1}^r((s_j-z_2^{-1})^{1/2}e^{-2\lambda(s_j)t}\right)V^2(s_1,\ldots,s_r)\,ds.\end{aligned}$$

PROPOSITION A.2. *Let γ, r, I_1, I_2 be as above and assume $\gamma(z_2,\ldots,z_2)\neq 0$, $\gamma(z_2^{-1},\ldots,z_2^{-1})\neq 0$. Then*

(a) $I_1(r,\gamma)$

$$= \gamma(z_2,\ldots,z_2)\pi^{r/2}\frac{\prod_{j=1}^r(2j-1)!}{2^{r^2}}\left(\frac{z_2}{z_2^{-1}-z_2}\right)^{r^2+r/2}\frac{e^{2(1+a)tr}}{t^{r^2+r/2}}(1+O(1/t)).$$

(b) $I_2(r,\gamma)$

$$= \gamma(z_2^{-1},\ldots,z_2^{-1})\pi^{r/2}\frac{\prod_{j=1}^r(2j-1)!}{2^{r^2}}\left(\frac{z_2^{-1}}{z_2^{-1}-z_2}\right)^{r^2+r/2}\frac{e^{2(1+a)tr}}{t^{r^2+r/2}}(1+O(1/t)).$$

Proof: (a) Recall that $\lambda(s) = (s+s^{-1})/2 + a$ with $\lambda(z_2) = -1-a$. On the interval $[z_1,z_2]$ we obtain

$$\text{(A.4)}\qquad -2\lambda(s)t = 2(1+a)t - (z_2^{-2}-1)(z_2-s)t - 2r(s)(z_2-s)^2t,$$

where $r(s) \geqq 0$ as $\lambda''(s) \geqq 0$. If we substitute $\kappa_j = (z_2^{-2} - 1)(z_2 - s_j)t$, standard methods show that the highest-order term of $I_1(r, \gamma)$ is given by the integral calculated in Proposition A.1.

(b) Proof as in (a).

The following lemma provides the formulae used in Section 4.

For $f : \mathbb{R}^l \to \mathbb{R}, \sigma$ a permutation of l letters, define $\sigma f : \mathbb{R}^l \to \mathbb{R}$ by

$$(\sigma f)(s_1, \ldots, s_l) = f(s_{\sigma(1)}, \ldots, s_{\sigma(n)}).$$

Let ρ be defined by (4.11). For $l > 0$ let

$$V_l(f) := \frac{1}{l!} \int \cdots \int f(s_1, \ldots, s_l) V^2(s_1, \ldots, s_l) d\rho(s_1) \cdots d\rho(s_l),$$
$$V_l := V_l(1_l),$$

where $1_l : \mathbb{R}^l \to \mathbb{R}$ denotes the function that is identically equal to 1.

LEMMA A.3. *Assume that $f : \mathbb{R}^l \to \mathbb{R}$ satisfies $\sigma f = f$ for all permutations σ of l letters and $f(s_1, \ldots, s_l) \neq 0$ for all $s_1, \ldots, s_l \in \mathbb{R}$.*

(a) *When $1 \leqq n' \leqq k + 1$,*

$$V_{2n'}(f) = f(\zeta_{n'-1}, \zeta_{n'-2}, \ldots, \zeta_0, \zeta_0^{-1}, \ldots, \zeta_{n'-1}^{-1}) V_{2n'} (1 + O(e^{-(\mu_{n'} - \mu_{n'-1})t})),$$
$$V_{2n'} = \prod_{j=0}^{n'-1} (\nu_j^2 e^{-4\lambda_j t}) V^2(\zeta_{n'-1}, \ldots, \zeta_{n'-1}^{-1}) (1 + O(e^{-(\mu_{n'} - \mu_{n'-1})t})).$$

(b) *When $1 \leqq n' \leqq k$,*

$$V_{2n'+1}(f) = -V_{2n'} \nu_{n'} e^{-2\lambda_{n'} t} \prod_{j=0}^{n'-1} (\zeta_{n'} - \zeta_j)(\zeta_{n'} - \zeta_j^{-1})(\zeta_{n'}^{-1} - \zeta_j)(\zeta_{n'}^{-1} - \zeta_j^{-1})$$
$$\times \left(f(\zeta_{n'}, \ldots, \zeta_{n'-1}^{-1}) \zeta_{n'} + f(\zeta_{n'-1}, \ldots, \zeta_{n'}^{-1}) \zeta_{n'}^{-1} \right) (1 + O(e^{-\eta t})),$$

where $\eta = \min(\mu_{n'} - \mu_{n'-1}, \mu_{n'+1} - \mu_{n'})$.

(c) *When $n' > k + 1$, and we write $r = n' - (k + 1)$,*

$$V_{2n'}(f) = f(z_2, \ldots, z_2, \zeta_k, \ldots, \zeta_k^{-1}, z_2^{-1}, \ldots, z_2^{-1}) V_{2n'} (1 + O(1/t)),$$
$$V_{2n'} = V_{2k+2} (z_2^{-1} - z_2)^{-r} \left(\prod_{j=0}^{k} (z_2 - \zeta_j)(z_2 - \zeta_j^{-1})(z_2^{-1} - \zeta_j)(z_2^{-1} - \zeta_j^{-1}) \right)^{2r}$$
$$\times \pi^r \left(\frac{\prod_{j=1}^{r} (2j-1)!}{2^{r^2}} \right)^2 \left(\frac{\nu(z_2)}{2\pi i} \right)^r \left(\frac{\nu(z_2^{-1})}{2\pi i} \right)^r \frac{e^{4(1+a)tr}}{t^{2r^2+r}} (1 + O(1/t)).$$

(d) *When $n' > k$, and we write $r = n' - (k+1)$,*

$$V_{2n'+1}(f) = -V_{2n'}\left(\prod_{j=0}^{k}(z_2 - \zeta_j)(z_2 - \zeta_j)^{-1}(z_2^{-1} - \zeta_j)(z_2^{-1} - \zeta_j^{-1})\right)$$

$$\times \frac{\pi^{-1/2}|T^+(z_2)|^2}{(z_2 - z_2^{-1})^2}\left(\frac{d_f(z_2)}{\delta(z_2)\Delta_f(z_2)}\right)^2 \frac{(2r+1)!}{2^{2r+1}}\frac{e^{2(1+a)t}}{t^{2r+3/2}}(1 + O(1/t))$$

$$\times \left(z_2 f(z_2,\dots,z_2,\zeta_k,\dots,\zeta_k^{-1},z_2^{-1},\dots,z_2^{-1})\right.$$

$$\left. + z_2^{-1} f(z_2,\dots,z_2,\zeta_k,\dots,\zeta_k^{-1},z_2^{-1},\dots,z_2^{-1})\right),$$

where z_2 is repeated $r+1$ (respectively, r) times in the first (respectively, second) function f, and z_2^{-1} is repeated r (respectively, $r+1$) times in the first (respectively, second) function f.

Remark. With the following argument we can use the above lemma to compute $V_l(f)$ even in the case where $f = \sigma f$ does not hold for all permutations of l letters. It is easy to see that $V_l(f) = V_l(\sigma f)$ for all $\sigma \in S^l$. Then define $\tilde{f} = \frac{1}{l!}\sum_{\sigma \in S^l} \sigma f$. Hence $V_l(f) = V_l(\tilde{f})$ and $\tilde{f}$ satisfies the hypothesis of Lemma A.3.

Proof of Lemma A.3: As $\sigma f = f$ and $\sigma(V^2) = V^2$ for all $\sigma \in S^l$, one obtains

$$V_l(f) = \int_{s_1 < \cdots < s_l} f(s_1, s_2, s_3, \dots, s_l) V^2(s_1, s_2, s_3, \dots, s_l)\, d^l\rho,$$

where V denotes the usual Vandermonde determinant. We begin with the case $l \leqq 2k + 2$.

We look for the configurations of the s_i's that give the leading term in t. It is immediate from the time dependence of the measure ρ that, for $l = 2n'$, $n' \leqq k+1$, the least decay is achieved at the configuration where $s_1 = \zeta_{n'-1}$, $s_2 = \zeta_{n'-2}$, $\dots$, $s_{n'} = \zeta_0$, and $s_{n'+1} = \zeta_0^{-1}$, $\dots$, $s_{2n'} = \zeta_{n'-1}^{-1}$. Similarly, in the case $l = 2n' + 1$, $1 \leqq n' \leqq k$, we obtain the leading order in t by choosing either $s_1 = \zeta_{n'}$, $\dots$, $s_{n'+1} = \zeta_0$, $s_{n'+2} = \zeta_0^{-1}$, $\dots$, $s_{2n'+1} = \zeta_{n'-1}^{-1}$ or $s_1 = \zeta_{n'-1}$, $\dots$, $s_{n'} = \zeta_0$, $s_{n'+1} = \zeta_0^{-1}$, $\dots$, $s_{2n'+1} = \zeta_{n'}^{-1}$. From the definition of ρ (4.11) (a) and (b) follow immediately.

In the case $l > 2k + 2$, we have to investigate the contribution of the configuration of Figure A.1, with r_1 variables in $[z_1, z_2]$ and r_2 variables in $[z_2^{-1}, z_1^{-1}]$. Note that $r_1 + r_2 = l - (2k+2)$. We evaluate $V_l^{(r_1)}(f)$, the contribution of Figure A.1 to $V_l(f)$, by separating the contribution of poles and bands.

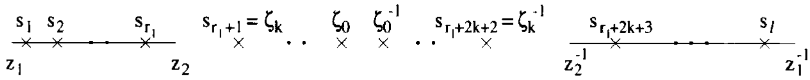

Figure A.1. Dominating configurations for Lemma A.3.

Observe that

$$V^2(s_1,\dots,s_{r_1},\zeta_k,\dots,\zeta_0,\zeta_0^{-1},\dots,\zeta_k^{-1},s_{r_1+2k+3},\dots,s_l)$$

$$= V^2(s_1,\dots,s_{r_1})\,V^2(\zeta_k,\dots,\zeta_k^{-1})\,V^2(s_{r_1+2k+3},\dots,s_l)$$

$$\times \prod_{\substack{1\leqq j\leqq r_1\\0\leqq i\leqq k}}(s_j-\zeta_i)^2(s_j-\zeta_i^{-1})^2 \prod_{\substack{l-r_2+1\leqq j\leqq l\\0\leqq i\leqq k}}(\zeta_i-s_j)^2(s_j-\zeta_i^{-1})^2 \prod_{\substack{0\leqq j\leqq r_1\\l-r_2+1\leqq i\leqq l}}(s_i-s_j)^2.$$

We then arrive at

$$V_l^{(r_1)}(f) = V_{2k+2}\,\frac{1}{r_1!}\int_{z_1}^{z_2}\frac{ds_1}{2\pi i}\cdots\int_{z_1}^{z_2}\frac{ds_{r_1}}{2\pi i}$$

$$\Bigg\{\prod_{\substack{1\leqq j\leqq r_1\\0\leqq i\leqq k}}(s_j-\zeta_i)^2(s_j-\zeta_i^{-1})^2\prod_{1\leqq j\leqq r_1}\nu(s_j)s_j^{-2n'}(z_2-s_j)^{1/2}e^{-2\lambda(s_j)t}$$

$$\times V^2(s_1,\dots,s_r)\,\frac{1}{r_2!}\int_{z_2^{-1}}^{z_1^{-1}}\frac{d\kappa_1}{2\pi i}\cdots\int_{z_2^{-1}}^{z_1^{-1}}\frac{d\kappa_{r_2}}{2\pi i}$$

$$\Bigg\{V^2(\kappa_1,\dots,\kappa_{r_2})\prod_{\substack{1\leqq j\leqq r_2\\0\leqq i\leqq k}}(\kappa_j-\zeta_i)^2(\kappa_j-\zeta_i^{-1})^2\prod_{\substack{1\leqq j\leqq r_1\\1\leqq i\leqq r_2}}(s_j-\kappa_i)^2$$

$$\times f(s_1,\dots,s_{r_1},\zeta_k,\dots,\zeta_k^{-1},\kappa_1,\dots,\kappa_{r_2})$$

$$\times\prod_{1\leqq j\leqq r_2}\nu(\kappa_j)\kappa_j^{-2n'}(\kappa_j-z_2^{-1})^{1/2}e^{-2\lambda(\kappa_j)t}\Bigg\}\Bigg\}.$$

We first apply Proposition A.2 (part (b)) for the integral on $[z_2^{-1},z_1^{-1}]^{r_2}$ and then Proposition A.2 (part (a)) for the integral on $[z_1,z_2]^{r_1}$. We recall that we are restricted to the nonresonant generic case where $\nu(z_2)\neq 0, \nu(z_2^{-1})\neq 0$ (see (A.8) below, together with (2.8) above and the remark that generically $W(-a-1)\neq 0$). We obtain

(A.5)

$$V_l^{(r_1)}(f) = V_{2k+2}\, f(z_2,\ldots,z_2,\zeta_k,\ldots,\zeta_k^{-1},z_2^{-1},\ldots,z_2^{-1}) \left(\frac{\nu(z_2^{-1})}{2\pi i}\right)^{r_2} \left(\frac{\nu(z_2)}{2\pi i}\right)^{r_1}$$
$$\times \prod_{i=0}^{k} \left((z_2-\zeta_i)^{r_1}(z_2-\zeta_i^{-1})^{r_1}(z_2^{-1}-\zeta_i)^{r_2}(z_2^{-1}-\zeta_i^{-1})^{r_2}\right)^2 (z_2^{-1}-z_2)^{2r_1r_2}$$
$$\times \pi^{\frac{r_1+r_2}{2}} 2^{-r_1^2-r_2^2} z_2^{2n'(r_2-r_1)} \left(\frac{z_2}{z_2^{-1}-z_2}\right)^{r_1^2+r_1/2} \left(\frac{z_2^{-1}}{z_2^{-1}-z_2}\right)^{r_2^2+r_2/2}$$
$$\times \prod_{j=1}^{r_1}(2j-1)! \prod_{j=1}^{r_2}(2j-1)! \frac{e^{2(1+a)t(r_1+r_2)}}{t^{r_1^2+r_1/2+r_2^2+r_2/2}}(1+O(1/t)).$$

Hence the leading behavior is given by the configuration where r_1 is chosen such that $r_1^2+r_1/2+r_2^2+r_2/2$ is minimal, with $r_1+r_2 = l-2k-2$. This corresponds to the following choices:

For $l = 2n' : r_1 = r_2 = n'-k-1$.

For $l = 2n'+1 : r_1 = n'-k-1$ and $r_2 = n'-k$, or $r_1 = n'-k$ and $r_2 = n'-k-1$.

If we recall the definition of r in parts (c) and (d) of Lemma A.3, respectively, we see that

$$V_l(f) = V_l^{(r)}(f)(1+O(1/t)), \tag{A.6}$$

if $l = 2n'$, and

$$V_l(f) = (V_l^{(r)}(f) + V_l^{(r+1)}(f))(1+O(1/t)), \tag{A.7}$$

if $l = 2n'+1$.

Relations (A.6) and (A.5) prove (c). The proof of (d) will follow from (A.5) and (A.7) once we evaluate $\nu(z_2)$ and $\nu(z_2^{-1})$. Using the definition of $\tilde{z}$ (see Section 2), we easily compute

$$\lim_{z\to z_2} \frac{\tilde{z}-\tilde{z}^{-1}}{(z_2-z)^{1/2}} = 2i(z_2^{-2}-1)^{1/2}.$$

From the definitions of ν and τ

$$\frac{\nu(z_2)}{2\pi i} = -\frac{1}{\pi}\frac{(z_2^{-2}-1)^{1/2}}{z_2^{-1}-z_2}|T^+(z_2)|^2 \frac{d_f^2(z_2)}{\delta^2(z_2)\Delta_f^2(z_2)}. \tag{A.8}$$

Similarly one derives a formula for $\nu(z_2^{-1})$. If the symmetries (3.9) and (3.45a) are used, one observes

$$\nu(z_2^{-1}) = z_2\nu(z_2).$$

The proof of Lemma A.3 is now complete.

Bibliography

[1] Beals, R., and Coifman, R., *Scattering and inverse scattering for first order systems* Comm. Pure Appl. Math. 37, 1984, pp. 39–90.

[2] Beals, R., Deift, P., and Tomei, C. *Direct and Inverse Scattering on the Line,* Mathematical Surveys and Monographs No. 28, American Mathematical Society, Providence, R.I., 1988.

[3] Bloch, A., and Kodama, Y., *The Whitham equation and shocks in the Toda lattice*, Proceedings of the NATO Advanced Study Workshop on Singular Limits of Dispersive Waves held in Lyons, July 1991, Plenum Press, New York, 1994.

[4] Clancey, K., and Gohberg, I. *Factorization of matrix functions and singular integral operators*, in: *Operator Theory: Advances and Applications*, vol. 3, Birkhäuser, Basel, 1981.

[5] Deift, P., Li, L. C., and Tomei, C., *Toda flows with infinitely many variables*, J. Funct. Anal. 64, 1985, pp. 358–402.

[6] Deift, P., and Zhou, X., *A steepest descent method for oscillatory Riemann-Hilbert problems: Asymptotics for the MKdV equation*, Ann. of Math. 137, 1993, pp. 295–368.

[7] Holian, B. L., Flaschka, H., and McLaughlin, D. W., *Shock waves in the Toda lattice: Analysis*, Phys. Rev. A 24, 1981, pp. 2595–2623.

[8] Kamvissis, S., *On the Toda shock problem*, Physica D 65, 1993, pp. 242–266.

[9] Kamvissis, S., *Long time behavior of the doubly infinite Toda lattice under initial data decaying at infinity*, Comm. Math. Phys. 153, 1993, pp. 479–519.

[10] Oba, R., *Doubly Infinite Toda Lattice with Antisymmetric Asymptotics*, Doctoral Dissertation, Courant Institute of Mathematical Sciences, New York University, 1988.

[11] Venakides, S., Deift, P., and Oba, R., *The Toda shock problem*, Comm. Pure Appl. Math. 44, 1991, pp. 1171–1242.

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