On the combinatorial complexity of Euclidean Voronoi cells and convex hulls of d-dimensional spheres

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Abstract

In this paper we show an equivalence relationship between additively weighted Voronoi cells in $\mathbb{R}^d$, power diagrams in $\mathbb{R}^d$ and convex hulls of spheres in $\mathbb{R}^d$. An immediate consequence of this equivalence relationship is a tight bound on the complexity of : (1) a single additively weighted Voronoi cell in dimension d; (2) the convex hull of a set of d-dimensional spheres. In particular, given a set of n spheres in dimension d, we show that the worst case complexity of both a single additively weighted Voronoi cell and the convex hull of the set of spheres is $\Theta(n^{\lceil \frac{d}{2} \rceil})$. The equivalence between additively weighted Voronoi cells and convex hulls of spheres permits us to compute a single additively weighted Voronoi cell in dimension d in worst case optimal time $O(n \log n + n^{\lceil \frac{d}{2} \rceil})$.

Keywords: computational geometry; combinatorial geometry; Voronoi diagrams; Power diagrams; Möbius diagrams; convex hulls; spheres.

1 Introduction

Let $\mathcal{E} = \{P_0, \ldots, P_n\}$ be a set of weighted points in $\mathbb{R}^d$. We note $P_i = (p_i, \omega_i)$, where $p_i \in \mathbb{R}^d$ is called the *center* of P_i and $\omega_i \in \mathbb{R}$ the *weight* of P_i, $i = 0, \ldots, n$. We define the additively weighted distance $\delta_+(\cdot, \cdot)$ of a point $p \in \mathbb{R}^d$ from a weighted point P_i to be $\delta_+(p, P_i) = \|p - p_i\| - \omega_i$, where $\| \cdot \|$ denotes the L_2-norm in $\mathbb{R}^d$. We can then assign each point in $\mathbb{R}^d$ to the weighted point P_i that is closest to p with respect to the distance $\delta_+(\cdot, \cdot)$. This assignment subdivides the space into j-dimensional cells, $0 \leq j \leq d$. The collection of all j-cells is called the *additively weighted Voronoi diagram* $V_+(\mathcal{E})$ of the set $\mathcal{E}$. The additively weighted Voronoi diagram does not change if we translate all weights ω_i by the same constant quantity. We thus assume without loss of generality that $\forall i$, $\omega_i \geq 0$. In this case the weighted points P_i are spheres in $\mathbb{R}^d$ centered at p_i, with radius ω_i. In the sequel, spheres will refer to weighted points with non-negative weights.

The additively weighted Voronoi diagram is a generalization to the usual Voronoi diagram for points, which can be obtained from the additively weighted Voronoi diagram if all the weights ω_i are equal. Another generalization of the point Voronoi diagram is the *power diagram*, where the distance metric $\delta_P(\cdot, \cdot)$ used is defined as $\delta_P(p, P_i) = \|p - p_i\|^2 - \omega_i^2$. A detailed description of the various variations of Voronoi diagrams, their properties, algorithms for their construction and their applications can be found in the survey paper by Aurenhammer and Klein [4], or the book by Okabe, Boots, Sugihara and Chiu [9].

Consider a set $\mathcal{E}$ of spheres. We call Π a *supporting hyperplane* of the set $\mathcal{E}$ if it has non-empty intersection with $\mathcal{E}$, and $\mathcal{E}$ is contained in one of the closed halfspaces limited by Π. We call $\mathcal{H}$ a *supporting halfspace* of the set $\mathcal{E}$ if it contains all the spheres in $\mathcal{E}$ and is limited by a supporting hyperplane Π of $\mathcal{E}$. The intersection of all the supporting halfspaces of $\mathcal{E}$ is called the *convex hull* $CH(\mathcal{E})$ of $\mathcal{E}$. The definition of convex hulls given above is general, i.e., it does not depend on the type of geometric objects considered. In the case of points there exist worst case optimal, as well as output sensitive algorithms for the construction of convex hulls. Erickson [7] gives a nice overview of the various algorithms for the computation of convex hulls of point sets. Convex hull algorithms for non-linear objects are very limited; the interested reader can refer to the paper by Nielsen and Yvinec [8] for an overview of the results for convex hulls of non-linear objects.

In this paper we focus on the combinatorial properties of Voronoi diagrams and convex hulls. In particular, we are interested in the worst case combinatorial complexity of a single additively weighted Voronoi cell and the convex hull of a set of spheres. Aurenhammer [2]

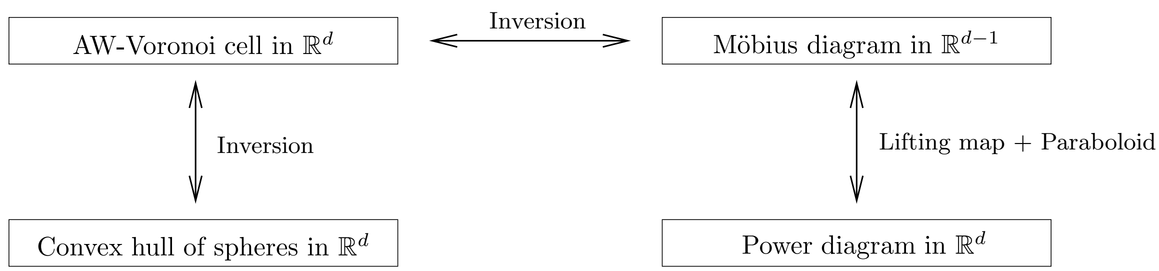

Figure 1: Equivalence relationships between the various Voronoi diagrams. "AW-Voronoi" stands for "Additively weighted Voronoi".

proved that the worst case complexity of the power diagram for a set of n spheres in dimension d is $\Theta(n^{\lceil \frac{d}{2} \rceil})$. As a consequence, he proved that the worst case complexity of the additively weighted Voronoi diagram is $O(n^{\lfloor \frac{d}{2} \rfloor + 1})$, which is tight in odd dimensions. The complexity for a single additively weighted Voronoi cell or the convex hull of a set of spheres is $O(n^{\lceil \frac{d}{2} \rceil})$ (see [2] and [5], respectively), which was known to be worst case optimal only for even d. Will [10] was the first to show that a 3-dimensional additively weighted Voronoi cell has complexity $\Omega(n^2)$. Boissonnat et al. [5] provide an example of $2n+1$ spheres in $\mathbb{R}^3$ whose convex hull has complexity $\Theta(n^2)$. They also conjecture that the worst case complexity of the convex hull in any dimension is $\Theta(n^{\lceil \frac{d}{2} \rceil})$.

The main result of our paper is a tight bound on the worst case combinatorial complexity of an additively weighted Voronoi cell in any dimension d. This is done by showing an equivalence between additively weighted Voronoi cells and a new type of Voronoi diagrams called *Möbius diagrams*. Möbius diagrams are generalizations of both power diagrams and multiplicatively weighted Voronoi diagrams. They are invariant under Möbius transformations, hence their name, and also generalizations of *affine diagrams* introduced by Aurenhammer (cf. [3]). We show that the problem of computing a Möbius diagram in $\mathbb{R}^{d-1}$ is equivalent to computing a power diagram in $\mathbb{R}^d$. We also present a relationship between additively weighted Voronoi cells and convex hulls of spheres, which permits us to obtain a worst case tight bound on the combinatorial complexity of the convex hull of a set of spheres in dimension d. In particular, both complexities, that of an additively weighted Voronoi cell and that of the convex hull of spheres, are shown to be $\Theta(n^{\lceil \frac{d}{2} \rceil})$ in the worst case. In view of our result the algorithm presented in [5] for the construction of the convex hull of a set of spheres is optimal for any d, and it gives us a way to optimally construct a single additively weighted Voronoi cell in any dimension.

The rest of the paper is structured as follows. In Section 2 we introduce Möbius diagrams. In Section 3 we show that the worst case complexity of a single additively weighted Voronoi cell in dimension d is $\Theta(n^{\lceil \frac{d}{2} \rceil})$, where n is the number of weighted points. In Section 4 we show that the worst case complexity of the convex hull of a set of n spheres in dimension d is $\Theta(n^{\lceil \frac{d}{2} \rceil})$. Section 5 discusses how to optimally construct an additively weighted Voronoi cell in dimension d. Finally, Section 6 is devoted to conclusions and open problems.

2 Möbius diagrams

Let $\mathcal{F} = \{Q_1, \ldots, Q_n\}$ be a set of *doubly weighted points* of $\mathbb{R}^{d-1}$, where $Q_i = (p_i, \lambda_i, \mu_i)$, p_i is a point of $\mathbb{R}^{d-1}$ and λ_i, μ_i are real numbers. For a point $x \in \mathbb{R}^{d-1}$, the distance from x to the doubly weighted point Q_i is defined as $\delta_M(x, Q_i) = \lambda_i(x-p_i)^2 - \mu_i$, where $y^2 = y \cdot y = \|y\|^2$. We can then assign each point x of $\mathbb{R}^{d-1}$ to the doubly weighted point Q_i that is closest to x with respect to the distance $\delta_M(\cdot,\cdot)$. The subdivision induced by this assignment is called the *Möbius diagram* $V_M(\mathcal{F})$ of $\mathcal{F}$. A cell of the Möbius diagram is called a Möbius cell.

Let $\mathcal{M} = \{M_1, \ldots, M_n\}$ be a set of weighted points of $\mathbb{R}^{d-1}$, where $M_i = (m_i, \mu_i)$, $m_i \in \mathbb{R}^{d-1}$ and $\mu_i \in \mathbb{R}$. The multiplicatively weighted distance of a point $x \in \mathbb{R}^{d-1}$ from a weighted point M_i is defined as $\delta_*(x, M_i) = \mu_i \|x - m_i\|$. By assigning every point $x \in \mathbb{R}^{d-1}$ to its closest weighted point M_i with respect to the distance $\delta_*(\cdot,\cdot)$ we get a subdivision of the space called the *multiplicatively weighted Voronoi diagram* (cf. [2]).

The Möbius diagram, induced by the distance $\delta_M(\cdot,\cdot)$, is a generalization of both power diagrams and multiplicatively weighted Voronoi diagrams. In particular, if all λ_i are equal to some positive λ, the Möbius diagram coincides with the power diagram of the spheres with centers the p_i's and squared radii the quantities μ_i/λ. If all μ_i are equal and all λ_i are positive, then the Möbius diagram coincides with the multiplicatively weighted Voronoi diagram of the weighted points $M_i = (p_i, \sqrt{\lambda_i})$.

We now exhibit an equivalence between Möbius diagrams in $\mathbb{R}^{d-1}$ and power diagrams in $\mathbb{R}^d$. This is a generalization of the equivalence between multiplicatively weighted Voronoi diagrams and power diagrams shown by Aurenhammer [2]. If $x \in \mathbb{R}^{d-1}$ is closer to Q_i than to Q_j, we have for all $j > 0$,

$$\begin{aligned}
& \lambda_i(x-p_i)^2 - \mu_i \le \lambda_j(x-p_j)^2 - \mu_j \\
\iff\ & \lambda_i x^2 - 2\lambda_i p_i \cdot x + \lambda_i p_i^2 - \mu_i \\
& \qquad \le \lambda_j x^2 - 2\lambda_j p_j \cdot x + \lambda_j p_j^2 - \mu_j \\
\iff\ & (x - \lambda_i p_i)^2 + (x^2 + \frac{\lambda_i}{2})^2 - \rho_i^2 \\
& \qquad \le (x - \lambda_j p_j)^2 + (x^2 + \frac{\lambda_j}{2})^2 - \rho_j^2 \\
\iff\ & (y - c_i)^2 - \rho_i^2 \le (y - c_j)^2 - \rho_j^2
\end{aligned}$$

where $y = (x, x^2) \in \mathbb{R}^d$, $c_i = (\lambda_i p_i, -\frac{\lambda_i}{2}) \in \mathbb{R}^d$ and $\rho_i^2 = \lambda_i^2 p_i^2 + \frac{\lambda_i^2}{4} - \lambda_i p_i^2 + \mu_i$. Let Σ_i be the sphere of $\mathbb{R}^d$ centered at c_i of squared radius ρ_i^2, $i = 1, \ldots, n$. The above inequality shows that x is closer to Q_i than to Q_j in the distance $\delta_M(\cdot,\cdot)$, if and only if y belongs to the cell of Σ_i in the power diagram of the spheres Σ_j, $j = 1, \ldots, n$. Hence,

Lemma 1. *Let $\mathcal{F}$ be a set of doubly weighted points in $\mathbb{R}^{d-1}$, let $\mathcal{P}$ be the paraboloid $x_d = x^2$ of $\mathbb{R}^d$ and let $\mathcal{C}$ be the CW-complex obtained by intersecting $\mathcal{P}$ with the power diagram of the spheres of $\mathbb{R}^d$ centered at c_i with squared radii ρ_i^2. There is an 1–1 correspondence between the k-dimensional faces of the Möbius diagram of $\mathcal{F}$ and the k-dimensional faces of $\mathcal{C}$, $k = 0, \ldots, d-1$.*

It follows that the combinatorial complexity of the Möbius diagram of n doubly weighted points in $\mathbb{R}^{d-1}$ is $O(n^{\lceil \frac{d}{2} \rceil})$. This bound is tight since Aurenhammer [2] has shown that it is tight for multiplicatively weighted Voronoi diagrams. To the best of our knowledge this is the first result on the combinatorial complexity of Möbius diagrams.

Theorem 1. *Let $\mathcal{F}$ be a set of n doubly weighted points in $\mathbb{R}^{d-1}$. The worst case complexity of the Möbius diagram $V_M(\mathcal{F})$ of $\mathcal{F}$ is $\Theta(n^{\lceil \frac{d}{2} \rceil})$.*

It is easy to verify that the bisectors of Möbius diagrams are spheres or hyperplanes. Moreover, we will show that Möbius diagrams are invariant under inversions, and thus under all Möbius transformations (hence their name). Other properties of spheres, such that the intersection of a sphere and a hyperplane is a sphere of lower dimension, also have interpretations in terms of Möbius diagrams. In the remainder of this section we show two such properties, which are used to prove our main combinatorial complexity results (cf. Theorems 2 and 3). In Subsection 3.1 we present another property of Möbius diagrams, namely, that given a set of spheres and a hyperplane in $\mathbb{R}^d$ the projection of the additively weighted Voronoi cell of the hyperplane onto itself is a Möbius diagram in $\mathbb{R}^{d-1}$. Finally, in Subsection 3.2 we show that Möbius diagrams in $\mathbb{R}^{d-1}$ can be understood as the intersection of a polyhedron and a sphere in $\mathbb{R}^d$. These two last properties are alternative, more geometric, ways to define Möbius diagrams.

Consider the standard inversion transformation $f(x; x_0)$ that maps a point $x \in \mathbb{R}^k$ to the point $x_0 + (x - x_0)/\|x - x_0\|^2 \in \mathbb{R}^k$. $f(x; x_0)$ maps spheres that pass through x_0 to hyperplanes and spheres that do not pass through x_0 to spheres. Moreover, it leaves hyperplanes that pass through x_0 invariant and maps hyperplanes that do not pass through x_0 to spheres. It can be easily verified that f is an involution, i.e. $f(f(x; x_0)) = x$. f is therefore 1–1 and $f^{-1}(x; x_0) = f(x; x_0)$.

Let $\mathcal{F} = \{Q_1, \ldots, Q_n\}$ be a set of doubly weighted points of $\mathbb{R}^d$, where $Q_i = (p_i, \lambda_i, \mu_i)$. Let x_0 be a point in $\mathbb{R}^d$ such that $x_0 \neq p_i$, $i > 0$. We can assume without loss of generality that x_0 coincides with the origin in $\mathbb{R}^d$. Let also x be a point in the Möbius cell of Q_i and let $y = f(x; x_0)$. Since x belongs to the Möbius cell of Q_i, we have, for all $j > 0$,

$$\begin{aligned}
& \lambda_i(x-p_i)^2 - \mu_i \le \lambda_j(x-p_j)^2 - \mu_j \\
\iff\ & \lambda_i(\frac{y}{y^2} - p_i)^2 - \mu_i \le \lambda_j(\frac{y}{y^2} - p_j)^2 - \mu_j \\
\iff\ & \lambda_i(\frac{1}{y^2} - 2\frac{y}{y^2} \cdot p_i + p_i^2) - \mu_i \\
& \qquad \le \lambda_j(\frac{1}{y^2} - 2\frac{y}{y^2} \cdot p_j + p_j^2) - \mu_j \\
\iff\ & (\lambda_i p_i^2 - \mu_i) y^2 - 2\lambda_i p_i \cdot y + \lambda_i \\
& \qquad \le (\lambda_j p_j^2 - \mu_j) y^2 - 2\lambda_j p_j \cdot y + \lambda_j \\
\iff\ & (\lambda_i p_i^2 - \mu_i)(y - \frac{\lambda_i p_i}{\lambda_i p_i^2 - \mu_i})^2 - \frac{\lambda_i \mu_i}{\lambda_i p_i^2 - \mu_i} \\
& \qquad \le (\lambda_j p_j^2 - \mu_j)(y - \frac{\lambda_j p_j}{\lambda_j p_j^2 - \mu_j})^2 - \frac{\lambda_j \mu_j}{\lambda_j p_j^2 - \mu_j}
\end{aligned}$$

Let $Q'_k = (\frac{\lambda_k p_k}{\lambda_k p_k^2 - \mu_k}, \lambda_k p_k^2 - \mu_k, \frac{\lambda_k \mu_k}{\lambda_k p_k^2 - \mu_k})$, $k > 0$. By the analysis above, we deduce that x belongs to the Möbius cell of Q_i if and only if y belongs to the Möbius cell of Q'_i. This observation implies that Möbius cells remain Möbius cells under inversion, which is not the case, e.g., for the usual Euclidean Voronoi diagram for points. Hence, Theorem 2. *The set of Möbius diagrams in $\mathbb{R}^d$ is closed under inversion.*

Since Möbius transformations are the function product of up to four inversions [6], we deduce :

Corollary 1. *The set of Möbius diagrams in $\mathbb{R}^d$ is closed under Möbius transformations.*

Let $x = (x', x'')$, $x' \in \mathbb{R}^{d-1}$, $x'' \in \mathbb{R}$. Similarly, $p_i = (p_i', p_i'')$, $p_i' \in \mathbb{R}^{d-1}$, $p_i'' \in \mathbb{R}$. Consider a hyperplane $\Pi \in \mathbb{R}^{d-1}$. We can assume, without loss of generality, that Π is the hyperplane $x_d = 0$. Suppose that $x \in \Pi$, i.e., $x'' = 0$. Then x belongs to the Möbius cell of Q_i, if and only if for all $j > 0$:

$$\begin{aligned}
& \lambda_i(x-p_i)^2 - \mu_i \le \lambda_j(x-p_j)^2 - \mu_j \\
\iff\ & \lambda_i(x'-p_i')^2 + \lambda_i(x''-p_i'')^2 - \mu_i \\
& \qquad \le \lambda_j(x'-p_j')^2 + \lambda_j(x''-p_j'')^2 - \mu_j \\
\iff\ & \lambda_i(x'-p_i')^2 + \lambda_i p_i''^2 - \mu_i \\
& \qquad \le \lambda_j(x'-p_j')^2 + \lambda_j p_j''^2 - \mu_j
\end{aligned}$$

Hence, x' belongs to the Möbius cell of the doubly weighted point $(p_i', \lambda_i, \mu_i - \lambda_i p_i''^2)$ whose center p_i' is the projection of p_i on Π. More generally,

Theorem 3. *The intersection of a Möbius diagram in $\mathbb{R}^d$ with a hyperplane Π is a Möbius diagram in $\mathbb{R}^{d-1}$, defined over the projections on Π of the centers of the d-dimensional doubly weighted points.*

3 Additively weighted Voronoi cells

Let $\mathcal{E} = \{P_0, \dots, P_n\}$ be a set of weighted points of $\mathbb{R}^d$. We note $P_i = (p_i, \omega_i)$, where $p_i \in \mathbb{R}^d$ and $\omega_i \in \mathbb{R}$ is the weight of P_i, $i = 0, \dots, n$. Without loss of generality, we can assume that the ω_i are non-negative. Let $V_+(\mathcal{E})$ be the additively weighted Voronoi diagram of $\mathcal{E}$. We are interested in computing combinatorial complexity of the cell $V_+(P_i)$ of $V_+(\mathcal{E})$ that is associated with P_i. For concreteness, in the sequel, the cell we want to compute is combinatorial complexity of the cell $V_+(P_0)$ associated with P_0. We also assume that $V_+(P_i) \neq \emptyset$, $i \geq 0$, which geometrically means that no sphere is contained inside another (see [10, Proposition 1]).

3.1 The lower bound. For simplicity we consider the case where ω_0 is infinite. In this case, P_0 is a hyperplane and all P_i, $i > 0$, are spheres. Without loss of generality we assume that P_0 is the hyperplane $x_d = 0$. The points $x = (x', x'')$, $x' \in \mathbb{R}^{d-1}$, $x'' \in \mathbb{R}$, that are at equal distance from P_0 and P_i, $i > 0$, belong to the paraboloid

$$\begin{aligned}
& x'' = \|x - p_i\| - \omega_i \\
\iff\ & (x'' + \omega_i)^2 = (x - p_i)^2 \\
\iff\ & 2\,(p_i'' + \omega_i)x'' = (x' - p_i')^2 + p_i''^2 - \omega_i^2,
\end{aligned}$$

where $p_i = (p_i', p_i'')$, $p_i' \in \mathbb{R}^{d-1}$, $p_i'' \in \mathbb{R}$. Note that our assumption $V_+(P_i) \neq \emptyset$ implies $p_i'' + \omega_i > 0$. Suppose that $V_+(P_0) \cap V_+(P_i) \neq \emptyset$. The points x that are at equal distance from P_0, P_i, $i > 0$, must verify, for any $j > 0$:

$$\begin{aligned}
2\,(p_i'' + \omega_i)x'' &= (x' - p_i')^2 + p_i''^2 - \omega_i^2, \\
2\,(p_j'' + \omega_j)x'' &\le (x' - p_j')^2 + p_j''^2 - \omega_j^2.
\end{aligned}$$

Eliminating x'' we get

$$\frac{1}{p_i'' + \omega_i}\,(x'-p_i')^2 + p_i'' - \omega_i \le \frac{1}{p_j'' + \omega_j}\,(x'-p_j')^2 + p_j'' - \omega_j.$$

This shows that the vertical projection onto P_0 of the boundary of the cell $V_+(P_0)$ is the Möbius diagram of the doubly weighted points $Q_i = (p_i', \frac{1}{p_i''+\omega_i}, \omega_i - p_i'')$. In particular, we have an 1–1 correspondence between the k-dimensional faces of $V_+(P_0)$ and the k-dimensional faces of the Möbius diagram in $\mathbb{R}^{d-1}$ of the Q_i's, $k = 0, \dots, d-1$. Suppose that $p_i'' = \omega_i$, $i > 0$. Then the Möbius diagram of the Q_i's is actually a multiplicatively weighted Voronoi diagram of the weighted points $M_i = (p_i', (2\omega_i)^{-1/2})$. Since the worst case complexity of multiplicatively weighted Voronoi diagrams is $\Omega(n^{\lceil \frac{d}{2} \rceil})$, we conclude that the worst case complexity of $V_+(P_0)$ is $\Omega(n^{\lceil \frac{d}{2} \rceil})$ in this special case. Our argumentation can be applied to the general case by taking ω_0 sufficiently large instead of infinite. Aurenhammer [2] showed that the worst case combinatorial complexity of an additively weighted Voronoi cell is $O(n^{\lceil \frac{d}{2} \rceil})$. Hence,

Theorem 4. *Let $\mathcal{E}$ be a set of n weighted points in $\mathbb{R}^d$. The worst case complexity of a single additively weighted Voronoi cell in the additively weighted Voronoi diagram $V_+(\mathcal{E})$ of $\mathcal{E}$ is $\Theta(n^{\lceil \frac{d}{2} \rceil})$.*

The construction above also provides a Euclidean model for Möbius diagrams. A special case of this construction has been recently used in [1].

3.2 Correspondence with Möbius diagrams. Let now $\mathcal{E} = \{P_0, \dots, P_n\}$ be our set of spheres, where $P_i = (p_i, \omega_i)$, $i \geq 0$. We can assume without loss of generality that p_0 coincides with the origin. Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ centered at the origin. Let x be a point on the boundary of the additively weighted Voronoi cell $V_+(P_0)$ of P_0. Let P_i be a sphere, such that x lies on the bisector of P_0 and P_i. We denote by $x_s = \psi(x)$, the radial projection of x onto $\mathbb{S}^{d-1}$. Clearly :

$$x = \psi^{-1}(x_s) = \delta_+(x, P_i)\, x_s.$$

It can easily be shown that (see [10, Proposition 4] for the case $\omega_0 = 0$) :

$$\delta_+(x, P_i) = \frac{\alpha_i}{2(\omega_i^* + x_s \cdot p_i)} - \omega_0,$$

where

$$\alpha_i = p_i^2 - (\omega_i^*)^2, \qquad \omega_i^* = \omega_i - \omega_0.$$

Note that $\alpha_i > 0$, since otherwise P_0 would be contained inside P_i and thus $V_+(P_0) = \emptyset$. It can also easily be shown that $\omega_i^* + x_s \cdot p_i > 0$ (cf. [10, Proposition 4]).

Suppose that $V_+(P_0) \cap V_+(P_i) \neq \emptyset$. Let $x \in \mathbb{R}^d$ be a point on the bisector of P_0, P_i and let x_s be its radial projection on $\mathbb{S}^{d-1}$. Since x is closer to P_i (and P_0) than to any other sphere P_j, we have, for any $j > 0$:

$$\begin{aligned}
& \delta_+(x, P_i) \le \delta_+(x, P_j) \\
\iff\ & \frac{p_i}{\alpha_i} \cdot x_s + \frac{\omega_i^*}{\alpha_i} \ge \frac{p_j}{\alpha_j} \cdot x_s + \frac{\omega_j^*}{\alpha_j} \\
\iff\ & x_s^2 - 2\,\frac{p_i}{\alpha_i} \cdot x_s - \frac{2\omega_i^*}{\alpha_i} \le x_s^2 - 2\,\frac{p_j}{\alpha_j} \cdot x_s - \frac{2\omega_j^*}{\alpha_j} \\
\iff\ & (x_s - \frac{p_i}{\alpha_i})^2 - \mu_i \le (x_s - \frac{p_j}{\alpha_j})^2 - \mu_j,
\end{aligned}$$

where

$$\mu_k = \frac{2\omega_k^* \alpha_k + p_k^2}{\alpha_k^2}, \qquad k = i, j.$$

Hence x belongs to the bisector of P_0, P_i if and only if x_s belongs to the power cell of the sphere Σ_i centered at $q_i = \frac{p_i}{\alpha_i}$ of squared radius μ_i. Therefore, the projection of the bisector of P_0, P_i on $\mathbb{S}^{d-1}$ coincides with the intersection of $\mathbb{S}^{d-1}$ with the power cell of Σ_i. Let $\mathcal{S} = \{\Sigma_1, \ldots, \Sigma_n\}$. Let x_0 be a point of $\mathbb{S}^{d-1}$ that is in the interior of a cell of the power diagram $V_P(\mathcal{S})$ of $\mathcal{S}$ in $\mathbb{R}^d$. The sphere $\mathbb{S}^{d-1}$ is mapped by $f(\cdot; x_0)$ onto a hyperplane Π. Without loss of generality, we can assume that $x_0 = (0, \ldots, 0, 1)$. Hence Π is the hyperplane $x_d = \frac{1}{2}$. By Theorem 2, the power diagram of $\mathcal{S}$ is mapped by $f(\cdot; x_0)$ to the Möbius diagram $V_M(\mathcal{S}')$ of another set $\mathcal{S}' \in \mathbb{R}^d$. More precisely, the cell of Σ_i in $V_P(\mathcal{S})$ is mapped to the cell of the doubly weighted point $\Sigma_i' = (q_i', \lambda_i', \mu_i')$ in $V_M(\mathcal{S}')$, where $\mathcal{S}' = \{\Sigma_1', \ldots, \Sigma_n'\}$, $q_i' = \frac{q_i - x_0}{\lambda_i'} + x_0$, $\lambda_i' = (q_i - x_0)^2 - \mu_i$ and $\mu_i' = \mu_i / \lambda_i'$. By Theorem 3, the intersection of Π with $V_M(\mathcal{S}')$ is a $(d-1)$-dimensional Möbius diagram $V_M(\mathcal{S}'')$ of a set $\mathcal{S}''$, the centers of which lie on Π. More precisely, a point $x_s \in \mathbb{S}^{d-1}$ lies in the power cell of some Σ_i if and only if the image by $f(\cdot; x_0)$ of x_s, which lies on Π, lies in the cell of the doubly weighted point $\Sigma_i'' = (q_i'', \lambda_i', \mu_i' - \lambda_i' h_i^2)$ in $V_M(\mathcal{S}'')$, where $\mathcal{S}'' = \{\Sigma_1'', \ldots, \Sigma_n''\}$, q_i'' is the projection of q_i' onto Π, and $h_i = \|q_i' - q_i''\| + \frac{1}{2}$. This shows :

Lemma 2. *Let $\mathcal{E}$ be a set of n spheres in $\mathbb{R}^d$, and let $\mathcal{S}''$ be the set of doubly weighted points in $\mathbb{R}^{d-1}$ that we get by the series of transformations described above. Then the k-dimensional faces of $V_+(P_0)$ are in* 1–1 *correspondence with the k-dimensional faces of the Möbius diagram $V_M(\mathcal{S}'')$ of $\mathcal{S}''$, $k = 0, \ldots, d-1$.*

4 Convex hulls of spheres

Let $\delta_\varepsilon(x, \Pi)$ denote the signed distance of a point $x \in \mathbb{R}^d$ from a hyperplane Π. We define the distance $\delta_+(P, \Pi)$ of a weighted point $P = (p, \omega)$ from a hyperplane Π to be $\delta_+(P, \Pi) = \delta_\varepsilon(p, \Pi) - \omega$. Finally we define the distance $\delta_+(P, Q)$ between two weighted points $P = (p, \omega_P)$ and $Q = (q, \omega_Q)$ to be

$$\begin{aligned}
\delta_+(P, Q) &= \|p - q\| - \omega_P - \omega_Q \\
&= \delta_+(p, Q) - \omega_P = \delta_+(q, P) - \omega_Q.
\end{aligned}$$

Observe that, if P and Q are two spheres, $\delta_+(P, Q) > 0$ (resp. $= 0$) if and only if the two balls bounded by P and Q do not intersect (resp. are tangent). Let again $\mathcal{E} = \{P_0, \ldots, P_n\}$, $P_i = (p_i, \omega_i)$ be a set of spheres in $\mathbb{R}^d$, and suppose that $V_+(P_0) \neq \emptyset$. Let u_k be a point of a k-dimensional face of $V_+(P_0)$, $0 \le k \le d$. In particular, u_0 is a Voronoi vertex of $V_+(P_0)$ and u_d is a point in the interior of $V_+(P_0)$. The co-dimension $(d-k)$ of the face of $V_+(P_0)$ containing u_k is called the *Voronoi dimension* (V-dimension) of u_k. Let $\beta_k = \delta_+(u_k, P_0)$. The distance β_k may be positive, zero or negative, since u_k may lie on the exterior, boundary or interior of P_0, respectively. We call the weighted point $U_k = (u_k, \beta_k)$ the *Voronoi weighted point* associated with u_k. We use the term *Voronoi sphere* to refer to a Voronoi weighted point with non-negative weight. We define the V-dimension of U_k to be the V-dimension of u_k.

Let us consider the convex hull $CH(\mathcal{S})$ of a set $\mathcal{S}$ of spheres. We say that a supporting hyperplane Π of $\mathcal{S}$ has *convex hull dimension* (CH-dimension) k, if it is tangent to exactly k spheres of $\mathcal{S}$. Finally, a *face of $CH(\mathcal{S})$ of circularity* k, $0 \le k \le d-1$, is a maximal connected portion of the boundary of $CH(\mathcal{S})$, consisting of points where the supporting hyperplanes are tangent to a given set of $(d-k)$ spheres.

4.1 A special case.

We assume that $\omega_0 = 0$. Let $\Sigma_i = (c_i, \rho_i) = f(P_i; p_0)$, $i > 0$. Since $V_+(P_0) \neq \emptyset$, none of the spheres P_i pass through p_0 and thus the Σ_i are spheres with

$$c_i = \frac{p_i - p_0}{(p_i - p_0)^2 - \omega_i^2}, \qquad \rho_i = \frac{\omega_i}{(p_i - p_0)^2 - \omega_i^2}.$$

Let u_k, $k < d$, be a point of $V_+(P_0)$ of V-dimension $(d-k)$ and let $U_k = (u_k, \beta_k)$ be the corresponding

Voronoi sphere. Let $\Pi_k = f(U_k; p_0)$. Since U_k passes through p_0, Π_k is a hyperplane in $\mathbb{R}^d$. The normal of Π_k is chosen such that the points at positive distance to U_k map to points that are at positive distance to Π_k. Without loss of generality, let P_i, $i = 1, \ldots, d-k$, be the weighted points that define U_k along with P_0. By construction,

$$\begin{aligned} \delta_+(P_i, U_k) &= 0, \qquad 0 \le i \le d-k, \\ \delta_+(P_i, U_k) &> 0, \qquad i > d-k. \end{aligned}$$

The above relations are equivalent to

$$\begin{aligned} \delta_+(\Sigma_i, \Pi_k) &= 0, \qquad 1 \le i \le d-k, \\ \delta_+(\Sigma_i, \Pi_k) &> 0, \qquad i > d-k. \end{aligned}$$

Hence Π_k is a supporting hyperplane of the convex hull of the set of spheres $\mathcal{S} = \{\Sigma_1, \ldots, \Sigma_n\}$ of CH-dimension $(d-k)$ and conversely, a hyperplane Π of CH-dimension $(d-k)$ maps to a point of $V_+(P_0)$ of V-dimension $(d-k)$. In particular, this implies an 1–1 correspondence between the faces of $CH(\mathcal{S})$ of circularity k and the k-dimensional faces of $V_+(P_0)$.

4.2 The general case. In this subsection we want to show the equivalence of the previous subsection when $\omega_0 \ge 0$. In particular, we want to find a set of spheres the convex hull of which is combinatorially equivalent to the additively weighted Voronoi cell $V_+(P_0)$ of P_0.

Let $P_i' = (p_i, \omega_i - \omega_0)$, $i = 0, \ldots, n$ (see Fig. 2(top right)), and let $\Sigma_i = (c_i, \rho_i) = f(P_i'; p_0)$ (see Fig. 2(bottom left)). In this case :

$$c_i = \frac{p_i - p_0}{(p_i - p_0)^2 - (\omega_i - \omega_0)^2},$$

$$\rho_i = \frac{\omega_i - \omega_0}{(p_i - p_0)^2 - (\omega_i - \omega_0)^2}.$$

Note that the additively weighted Voronoi diagram does not change combinatorially, as well as geometrically, if we translate the weights by the same quantity, which implies that the Voronoi cells $V_+(P_0)$ and $V_+(P_0')$ are exactly the same. Let u_k, $k < d$ be a point of $V_+(P_0)$ of V-dimension $(d-k)$ and let $U_k = (u_k, \beta_k)$ be the corresponding Voronoi weighted point. In this case β_k may be positive as well as zero or negative. Let $U_k' = (u_k, \beta_k + \omega_0)$. Then :

$$\begin{aligned} \delta_+(P_0, U_k) = 0 \quad &\iff \quad \|p_0 - u_k\| - (\omega_0 + \beta_k) = 0 \\ &\iff \quad \delta_+(p_0, U_k') = 0. \end{aligned}$$

Trivially, $\omega_0 + \beta_k = \|p_0 - u_k\| \ge 0$. Hence U_k' is a Voronoi sphere that passes through p_0, and corresponds to u_k in $V_+(P_0')$. Let $\Pi_k = f(U_k'; p_0)$. The orientation of Π_k is as in the previous subsection. Clearly,

$$\begin{aligned} \delta_+(P_i', U_k') &= 0, \qquad 0 \le i \le d-k, \\ \delta_+(P_i', U_k') &> 0, \qquad i > d-k, \end{aligned}$$

which in turns implies that :

$$\begin{aligned} \delta_+(\Sigma_i, \Pi_k) &= 0, \qquad 1 \le i \le d-k, \\ \delta_+(\Sigma_i, \Pi_k) &> 0, \qquad i > d-k. \end{aligned}$$

Let $R \in \mathbb{R}$ be a sufficiently large number such that $\rho_i + R \ge 0$, and let $\Sigma_i' = (c_i, \rho_i + R)$, $i > 0$. Finally, let Π_k' be the translation of Π_k by R in the opposite direction of its normal (see Fig. 2(bottom right)). Obviously :

$$\begin{aligned} \delta_+(\Sigma_i', \Pi_k') &= 0, \qquad 1 \le i \le d-k, \\ \delta_+(\Sigma_i', \Pi_k') &> 0, \qquad i > d-k, \end{aligned}$$

i.e., Π_k' is a supporting hyperplane of the set of spheres $\mathcal{S} = \{\Sigma_1', \ldots, \Sigma_n'\}$ of CH-dimension $(d-k)$. As in the preceding subsection, we can show, by means of the inverse transformation, that a supporting hyperplane of $\mathcal{S}$ of CH-dimension $(d-k)$, maps to a point of $V_+(P_0)$ of V-dimension $(d-k)$. Hence,

LEMMA 3. *Let $\mathcal{E} = \{P_0, \ldots, P_n\}$ be a set of $n+1$ spheres in $\mathbb{R}^d$, and let $\mathcal{S}$ be the set of n spheres that we get by the series of transformations described above. Then the k-dimensional faces of $V_+(P_0)$ are in 1–1 correspondence with the faces of $CH(\mathcal{S})$ of circularity k, $k = 0, \ldots, d-1$.*

An immediate consequence of the above lemma is that the worst case complexity of the convex hull of a set of spheres in dimension d is the same with the worst case complexity of an additively weighted Voronoi cell in dimension d, i.e.,

THEOREM 5. *Let $\mathcal{S}$ be a set of n spheres in $\mathbb{R}^d$. The worst case complexity of the convex hull $CH(\mathcal{S})$ of $\mathcal{S}$ is $\Theta(n^{\lceil \frac{d}{2} \rceil})$.*

It has been shown in [5] that the worst case complexity of the convex hull of a set of n d-dimensional spheres is $O(n^{\lceil \frac{d}{2} \rceil})$. It has also been shown that the worst case complexity of the convex hull of n spheres is $\Omega(n^{\lfloor \frac{d}{2} \rfloor})$. Our construction provides an alternative way to prove the upper bound in [5], and at the same time it gives us a tight lower bound. A corollary of Theorem 5 is that the algorithm presented in [5] for the construction of the convex hull of spheres in dimension d is optimal in any dimension.

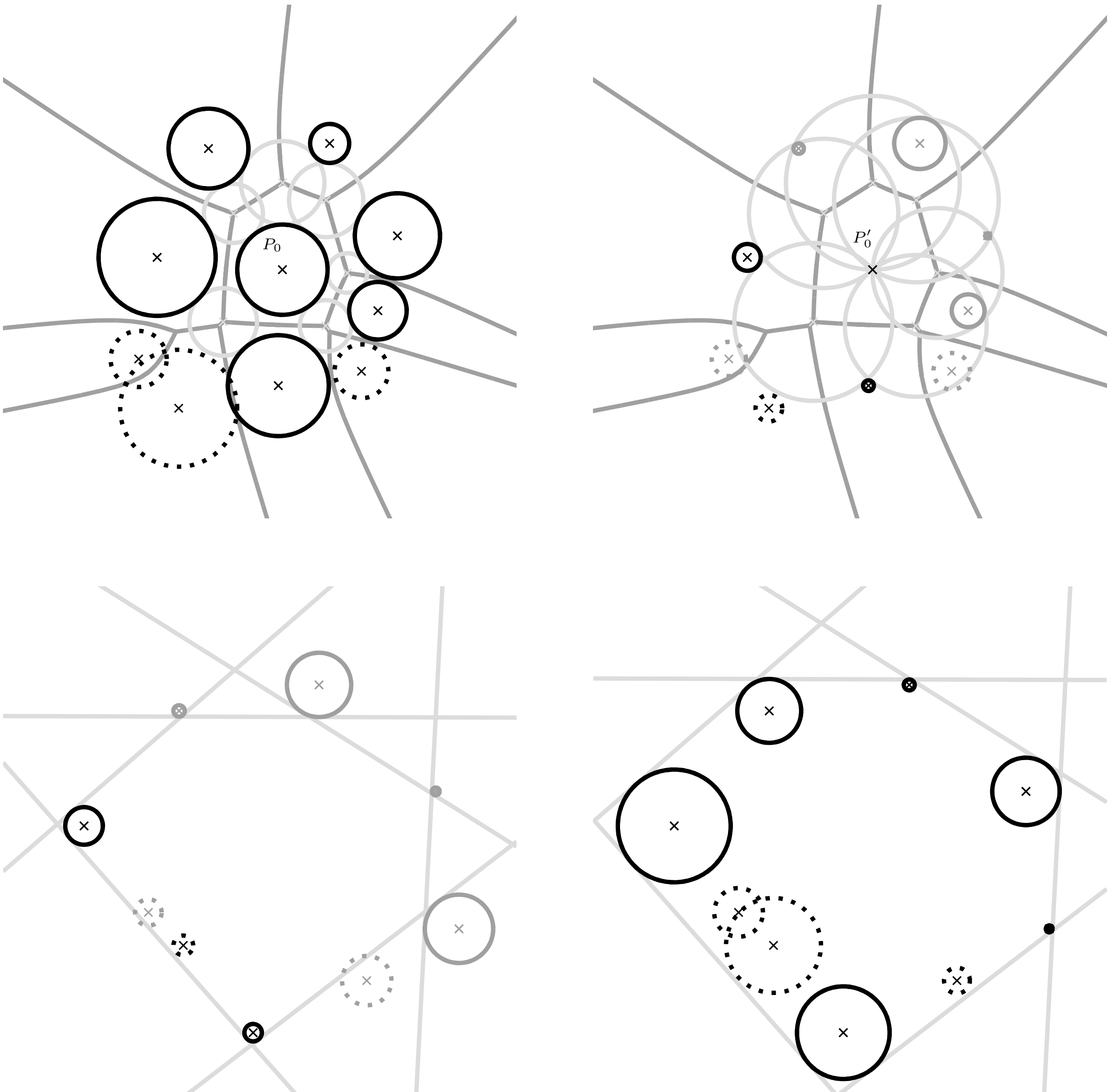

Figure 2: The equivalence relationship between additively weighted Voronoi cells and convex hulls of spheres in two dimensions. The input circles and their transforms are shown in black or dark gray. Voronoi circles and their transforms are shown in light gray. Black circles have positive weight. Dark gray circles have negative weight. Solid circles correspond to neighbors of P_0 in $V_+(\mathcal{E})$. Dotted circles are not neighbors of P_0 in $V_+(\mathcal{E})$. Top left: the set $\{P_0, \ldots, P_n\}$. The Voronoi circles U_k of V-dimension 0 are shown in light gray. The 1-dimensional Voronoi cells are shown in dark gray. Top right: the set $\{P'_0, \ldots, P'_n\}$. The circles U'_k are in light gray. The 1-dimensional Voronoi cells remain the same and P'_0 is a point. Bottom left: the set $\{\Sigma_1, \ldots, \Sigma_n\}$. The hyperplanes Π_k are in light gray. Bottom right: the set $\{\Sigma'_1, \ldots, \Sigma'_n\}$. The hyperplanes Π'_k are in light gray.

5 Computing a cell of an additively weighted Voronoi diagram

The algorithm of Aurenhammer [2] for the computation of the entire additively weighted Voronoi diagram suggests also an algorithm for the computation of a single additively weighted Voronoi cell. This algorithm runs in time $O(n^{\lfloor \frac{d}{2} \rfloor + 1})$ and it is worst case optimal only for odd d.

The construction described in Subsection 4.2 provides an alternative to the above algorithm of Aurenhammer for the computation of a single additively weighted Voronoi cell in any dimension. Suppose that we are given a set $\mathcal{E} = \{P_0, \ldots, P_n\}$ of weighted points in $\mathbb{R}^d$ and suppose we want to compute the additively weighted Voronoi cell $V_+(P_0)$ of $P_0 = (p_0, \omega_0)$. The first step is to decrease the weights of all P_i by ω_0. Then we invert all P_i's, $i > 0$, using p_0 as the pole of inversion. After the inversion we get a new set of n weighted points $\mathcal{S} = \{\Sigma_1, \ldots, \Sigma_n\}$. We enlarge the weights of all Σ_i by the same quantity R, so that they become non-negative. Finally, we use the algorithm in [5] to construct the convex hull $CH(\mathcal{S})$ of $\mathcal{S}$. The additively weighted Voronoi cell $V_+(P_0)$ of P_0 can now be constructed from $CH(\mathcal{S})$ in time proportional to its complexity. By Lemma 3 and Theorem 5 we conclude that the algorithm just described is worst case optimal in any dimension, i.e.,

THEOREM 6. *Let $\mathcal{E}$ be a set n of weighted points in $\mathbb{R}^d$. A single additively weighted Voronoi cell of $V_+(\mathcal{E})$ can be computed in worst case optimal time $O(n \log n + n^{\lceil \frac{d}{2} \rceil})$.*

Yet another worst case optimal algorithm is that suggested in Subsection 3.2. Assuming that P_0 is the origin, we first compute the set of spheres $\mathcal{S}$, such that the intersection of $V_P(\mathcal{S})$ with the unit sphere $\mathbb{S}^{d-1}$ coincides with the projection of $V_+(P_0)$ with $\mathbb{S}^{d-1}$. Then we invert $\mathcal{S}$ using a suitable point x_0 on $\mathbb{S}^{d-1}$, to get a set of doubly weighted points $\mathcal{S}'$. Let Π be the image of $\mathbb{S}^{d-1}$ under the inversion. The next step is to project the set $\mathcal{S}'$ on Π. This gives us another set of $(d-1)$-dimensional doubly weighted points $\mathcal{S}''$, the Möbius diagram of which can be computed as per Lemma 1. $V_+(P_0)$ can then be constructed from $V_M(\mathcal{S}'')$ in time proportional to its complexity.

6 Conclusion

In this paper we presented an equivalence relationship between additively weighted Voronoi cells in $\mathbb{R}^d$, convex hulls of spheres in $\mathbb{R}^d$, power diagrams in $\mathbb{R}^d$ and Möbius diagrams in $\mathbb{R}^{d-1}$. Using this equivalence, we proved tight bounds on the worst case complexity of a single additively weighted Voronoi cell and the convex hull for a set of spheres in dimension d. We also presented two worst case optimal algorithms for the construction of a single additively weighted Voronoi cell in any dimension.

The worst case complexity of the whole additively weighted Voronoi diagram in even dimensions $d > 2$ is still an open problem. It is also unknown what is the complexity of a single additively weighted Voronoi cell, the whole additively weighted Voronoi diagram or the convex hull of a set of spheres, if the spheres have a constant number of different radii.

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