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On the combinatorial complexity of Euclidean Voronoi cells and convex hulls of d-dimensional spheres

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Abstract: In this paper we show an equivalence relationship between additively weighted Voronoi cells in $\mathbb{R}^d$, power diagrams in $\mathbb{R}^d$ and convex hulls of spheres in $\mathbb{R}^d$. An immediate consequence of this equivalence relationship is a tight bound on the complexity of : (1) a single additively weighted Voronoi cell in dimension d; (2) the convex hull of a set of d-dimensional spheres. In particular, given a set of n spheres in dimension d, we show that the worst case complexity of both a single additively weighted Voronoi cell and the convex hull of the set of spheres is $\Theta(n^{\lceil \frac{d}{2} \rceil})$. The equivalence between additively weighted Voronoi cells and convex hulls of spheres permits us to compute a single additively weighted Voronoi cell in dimension d in worst case optimal time $O(n \log n + n^{\lceil \frac{d}{2} \rceil})$.

Key-words: computational geometry, combinatorial geometry, Voronoi diagrams, convex hulls, spheres

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Sur la complexité combinatoire des cellules des diagrammes de Voronoï Euclidiens et des enveloppes convexes de sphères de $\mathbb{R}^d$

Résumé : Dans cet article, on établit une correspondance entre une cellule d'un diagramme de Voronoï additif, un diagramme de puissance et une enveloppe convexe de sphères. Comme conséquence immédiate de cette correspondance, on obtient des bornes exactes en toutes dimensions sur la complexité d'une cellule d'un diagramme de Voronoï additif et celle d'une enveloppe convexe de sphères. Plus précisément, étant données n sphères de $\mathbb{R}^d$, on montre que la complexité dans le cas le pire d'une cellule du diagramme de Voronoï additif et de l'enveloppe convexe des sphères est $\Theta(n^{\lceil \frac{d}{2} \rceil})$. La correspondance entre une cellule d'un diagramme de Voronoï additif et une enveloppe convexe de sphères conduit également à un algorithme qui permet de construire une cellule d'un diagramme de Voronoï additif en temps $O(n \log n + n^{\lceil \frac{d}{2} \rceil})$, ce qui est optimal en toutes dimensions dans le cas le pire.

Mots-clés : géométrie algorithmique, géométrie combinatoire, diagrammes de Voronoï, enveloppes convexes, sphères

1 Introduction

Let $\mathcal{E} = \{P_0, \ldots, P_n\}$ be a set of weighted points in $\mathbb{R}^d$. We note $P_i = (p_i, \omega_i)$, where $p_i \in \mathbb{R}^d$ is called the *center* of P_i and $\omega_i \in \mathbb{R}$ the *weight* of P_i, $i = 0, \ldots, n$. We define the (additively weighted) distance $\delta_+(\cdot,\cdot)$ of a point $p \in \mathbb{R}^d$ from a weighted point P_i to be $\delta_+(p, P_i) = \|p - p_i\| - \omega_i$, where $\|\cdot\|$ denotes the L_2-norm in $\mathbb{R}^d$. We can then assign each point in $\mathbb{R}^d$ to the weighted point P_i that is closest to p with respect to the distance $\delta_+(\cdot,\cdot)$. This assignment subdivides the space into j-dimensional cells, $0 \leq j \leq d$. The collection of all j-cells is called the *additively weighted Voronoi diagram* $V_+(\mathcal{E})$ of the set $\mathcal{E}$. The additively weighted Voronoi diagram does not change if we translate all weights ω_i by the same constant quantity. We thus assume without loss of generality that $\forall i, \omega_i \geq 0$. In this case the weighted points P_i are spheres in $\mathbb{R}^d$ centered at p_i, with radius ω_i. In the sequel, spheres will refer to weighted points with non-negative weights.

The additively weighted Voronoi diagram is a generalization to the usual Voronoi diagram for points, which can be obtained from the additively weighted Voronoi diagram if all the weights ω_i are equal. Another generalization of the point Voronoi diagram is the *power diagram*, where the distance metric $\delta_P(\cdot,\cdot)$ used is defined as $\delta_P(p, P_i) = \|p - p_i\|^2 - \omega_i^2$. A detailed description of the various variations of Voronoi diagrams, their properties, algorithms for their construction and their applications can be found in the survey paper by Aurenhammer and Klein [AK00], or the book by Okabe, Boots, Sugihara and Chiu [OBSC00].

Consider a set $\mathcal{E}$ of spheres. We call Π a *supporting hyperplane* of the set $\mathcal{E}$ if it has non-empty intersection with $\mathcal{E}$, and $\mathcal{E}$ is contained in one of the closed halfspaces limited by Π. We call $\mathcal{H}$ a *supporting halfspace* of the set $\mathcal{E}$ if it contains all the spheres in $\mathcal{E}$ and is limited by a supporting hyperplane Π of $\mathcal{E}$. The intersection of all the supporting halfspaces of $\mathcal{E}$ is called the *convex hull* $CH(\mathcal{E})$ of $\mathcal{E}$. The definition of convex hulls given above is general, i.e., it does not depend on the type of geometric objects considered. In the case of points there exist worst case optimal, as well as output sensitive algorithms for the construction of convex hulls. Erickson [Eri99] gives a nice overview of the various algorithms for the computation of convex hulls of point sets. Convex hull algorithms for non-linear objects are very limited; the interested reader can refer to the paper by Nielsen and Yvinec [NY98] for an overview of the results for convex hulls of non-linear objects.

In this paper we focus on the combinatorial properties of Voronoi diagrams and convex hulls. In particular, we are interested in the worst case combinatorial complexity of a single Voronoi cell and the convex hull of a set of spheres. Aurenhammer [Aur87] proved that the worst case complexity of the power diagram for a set of n spheres in dimension d is $\Theta(n^{\lceil \frac{d}{2} \rceil})$. As a consequence, he proved that the worst case complexity of the additively weighted Voronoi diagram is $O(n^{\lfloor \frac{d}{2} \rfloor + 1})$, which is tight in odd dimensions. The complexity for a single additively weighted Voronoi cell or the convex hull for a set of spheres is $O(n^{\lceil \frac{d}{2} \rceil})$ (see [Aur87] and [BCD+96], respectively), which is worst case optimal only for even d. Will [Wil99] was the first to show that a 3-dimensional additively weighted Voronoi cell has complexity $\Omega(n^2)$. Boissonnat et al. [BCD+96] provide an example of $2n + 1$ spheres in $\mathbb{R}^3$ whose convex hull has complexity $\Theta(n^2)$. They also conjecture that the worst case complexity of the convex hull in any dimension is $\Theta(n^{\lceil \frac{d}{2} \rceil})$.

The main result of our paper is a tight bound on the worst case combinatorial complexity of an additively weighted Voronoi cell in any dimension d. This is done by showing an equivalence between additively weighted Voronoi cells and a new type of Voronoi diagram called the *multiplicatively weighted power diagram*, or MW-power diagram for short. MW-power diagrams are generalizations of both power diagrams and multiplicatively weighted Voronoi diagrams. We show that the problem of computing a MW-power diagram in $\mathbb{R}^{d-1}$ is equivalent to computing a power diagram in $\mathbb{R}^d$. We also present a relationship between additively weighted Voronoi cells and convex hulls of spheres, which permits us to provide a worst case tight bound on the complexity of the convex hull of a set of spheres in dimension d. In particular, both complexities, that of an additively weighted Voronoi cell and that of the convex hull of spheres, are shown to be $\Theta(n^{\lceil \frac{d}{2} \rceil})$ in the worst case. In view of our result the algorithm presented in [BCD+96] for the construction

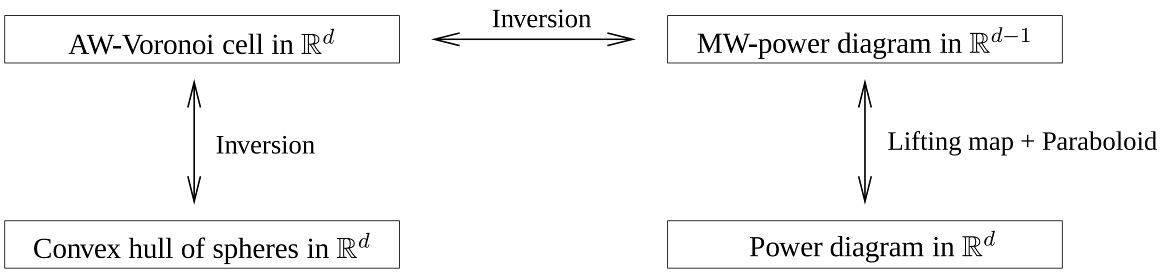

Figure 1: Equivalence relationships between the various Voronoi diagrams.

of the convex hull for a set of spheres is optimal for any d, and it gives us a way to optimally construct a single additively weighted Voronoi cell in any dimension.

The rest of the paper is structured as follows. In Section 2 we introduce MW-power diagrams. In Section 3 we show that the worst case complexity of a single additively weighted Voronoi cell in dimension d is $\Theta(n^{\lceil \frac{d}{2} \rceil})$, where n is the number of weighted points. In Section 4 we show that the worst case complexity of the convex hull of a set of n spheres in dimension d is $\Theta(n^{\lceil \frac{d}{2} \rceil})$. Section 5 discusses how to optimally construct an additively weighted Voronoi cell in dimension d. Finally, Section 6 is devoted to conclusions and open problems.

2 Multiplicatively weighted power diagrams

Let $\mathcal{F} = \{Q_1, \ldots, Q_n\}$ be a set of additively/multiplicatively weighted (AM-weighted) points of $\mathbb{R}^{d-1}$, where $Q_i = (p_i, \lambda_i, \mu_i)$, p_i is a point of $\mathbb{R}^{d-1}$ and λ_i, μ_i are real numbers. For a point $x \in \mathbb{R}^{d-1}$, the MW-power distance from x to Q_i is defined by $\delta_{*P}(x, Q_i) = \lambda_i (x - p_i)^2 - \mu_i$, where $y^2 = y \cdot y = \|y\|^2$. We can then assign each point x of $\mathbb{R}^{d-1}$ to the AM-weighted point Q_i that is closest to x with respect to the MW-power distance. The subdivision induced by this assignment will be called the *multiplicatively weighted power (MW-power) diagram* $V_{*P}(\mathcal{F})$ of $\mathcal{F}$. The MW-power diagram, induced by the MW-power distance is a generalization of both power diagrams and multiplicatively weighted Voronoi diagrams (see [Aur87]). In particular, if all λ_i are equal to some positive λ, the MW-power diagram coincides with the power diagram of the spheres with centers the p_i's and squared radii the quantities μ_i/λ. If all μ_i are equal and all λ_i are positive, then the MW-power diagram coincides with the multiplicatively weighted Voronoi diagram.

We now exhibit an equivalence between MW-power diagrams in $\mathbb{R}^{d-1}$ and power diagrams in $\mathbb{R}^d$. This is a generalization of the equivalence between multiplicatively weighted Voronoi diagrams and power diagrams shown by Aurenhammer [Aur87]. If $x \in \mathbb{R}^{d-1}$ is closer to Q_i than to Q_j, we have for all $j > 0$,

$$\begin{aligned}
& \lambda_i (x - p_i)^2 - \mu_i \leq \lambda_j (x - p_j)^2 - \mu_j \\
\iff \; & \lambda_i x^2 - 2\lambda_i p_i \cdot x + \lambda_i p_i^2 - \mu_i \leq \lambda_j x^2 - 2\lambda_j p_j \cdot x + \lambda_j p_j^2 - \mu_j \\
\iff \; & (x - \lambda_i p_i)^2 + (x^2 + \frac{\lambda_i}{2})^2 - \lambda_i^2 p_i^2 - \frac{\lambda_i^2}{4} + \lambda_i p_i^2 - \mu_i \\
& \qquad \leq (x - \lambda_j p_j)^2 + (x^2 + \frac{\lambda_j}{2})^2 - \lambda_j^2 p_j^2 - \frac{\lambda_j^2}{4} + \lambda_j p_j^2 - \mu_j \\
\iff \; & (y - c_i)^2 - \rho_i^2 \leq (y - c_j)^2 - \rho_j^2
\end{aligned}$$

where $y = (x, x^2) \in \mathbb{R}^d$, $c_i = (\lambda_i p_i, -\frac{\lambda_i}{2}) \in \mathbb{R}^d$ and $\rho_i^2 = \lambda_i^2 p_i^2 + \frac{\lambda_i^2}{4} - \lambda_i p_i^2 + \mu_i$. Let Σ_i be the sphere of $\mathbb{R}^d$ centered at c_i of squared radius ρ_i^2, $i = 1, \ldots, n$. The above inequality shows that x is closer to Q_i than to Q_j in the MW-power distance if and only if y belongs to the cell of Σ_i in the power diagram of the spheres Σ_j, $j = 1, \ldots, n$. Hence,

Lemma 1 *Let $\mathcal{F}$ be a set of additively/multiplicatively weighted points in $\mathbb{R}^{d-1}$, let $\mathcal{P}$ be the paraboloid $x_d = x^2$ of $\mathbb{R}^d$ and let $\mathcal{C}$ be the CW-complex obtained by intersecting $\mathcal{P}$ with the power diagram of the spheres of $\mathbb{R}^d$ centered at c_i with squared radii ρ_i^2. There is an 1–1 correspondence between the k-dimensional faces of the MW-power diagram of $\mathcal{F}$ and the k-dimensional faces of $\mathcal{C}$, $k = 0, \ldots, d-1$.*

It follows that the combinatorial complexity of the MW-power diagram of n weighted points in $\mathbb{R}^{d-1}$ is $O(n^{\lceil \frac{d}{2} \rceil})$. This bound is tight since Aurenhammer [Aur87] has shown that it is tight for multiplicatively weighted Voronoi diagrams.

Theorem 1 *Let $\mathcal{F}$ be a set of n additively/multiplicatively weighted points in $\mathbb{R}^{d-1}$. The worst case complexity of the multiplicatively weighted power diagram $V_{*P}(\mathcal{F})$ of $\mathcal{F}$ is $\Theta(n^{\lceil \frac{d}{2} \rceil})$.*

Consider now the standard inversion transformation $f(x; x_0)$ that maps a point $x \in \mathbb{R}^k$ to the point $x_0 + (x - x_0)/\|x - x_0\|^2 \in \mathbb{R}^k$. $f(x; x_0)$ maps spheres that pass through x_0 to hyperplanes and spheres that do not pass through x_0 to spheres. Moreover, it leaves hyperplanes that pass through x_0 invariant and maps hyperplanes that do not pass through x_0 to spheres. It can be easily verified that f is an involution, i.e. $f(f(x; x_0)) = x$. f is therefore 1–1 and $f^{-1}(x; x_0) = f(x; x_0)$.

Let $\mathcal{F} = \{Q_1, \ldots, Q_n\}$ be a set of AM-weighted points of $\mathbb{R}^d$, where $Q_i = (p_i, \lambda_i, \mu_i)$. Let x_0 be a point in $\mathbb{R}^d$ such that $x_0 \neq p_i$, $i > 0$. We can assume without loss of generality that x_0 coincides with the origin in $\mathbb{R}^d$. Let also x be a point in the MW-power cell of Q_i and let $y = f(x; x_0)$. Since x belongs to the MW-power cell of Q_i, we have, for all $j > 0$,

$$
\begin{aligned}
& \lambda_i (x - p_i)^2 - \mu_i \leq \lambda_j (x - p_j)^2 - \mu_j \\
\iff\ & \lambda_i (\frac{y}{y^2} - p_i)^2 - \mu_i \leq \lambda_j (\frac{y}{y^2} - p_j)^2 - \mu_j \\
\iff\ & \lambda_i (\frac{1}{y^2} - 2\frac{y}{y^2} \cdot p_i + p_i^2) - \mu_i \leq \lambda_j (\frac{1}{y^2} - 2\frac{y}{y^2} \cdot p_j + p_j^2) - \mu_j \\
\iff\ & (\lambda_i p_i^2 - \mu_i) y^2 - 2\lambda_i p_i \cdot y + \lambda_i \leq (\lambda_j p_j^2 - \mu_j) y^2 - 2\lambda_j p_j \cdot y + \lambda_j \\
\iff\ & (\lambda_i p_i^2 - \mu_i)(y - \frac{\lambda_i p_i}{\lambda_i p_i^2 - \mu_i})^2 - \frac{\lambda_i \mu_i}{\lambda_i p_i^2 - \mu_i} \leq (\lambda_j p_j^2 - \mu_j)(y - \frac{\lambda_j p_j}{\lambda_j p_j^2 - \mu_j})^2 - \frac{\lambda_j \mu_j}{\lambda_j p_j^2 - \mu_j}
\end{aligned}
$$

Let $Q'_k = (\frac{\lambda_k p_k}{\lambda_k p_k^2 - \mu_k}, \lambda_k p_k^2 - \mu_k, \frac{\lambda_k \mu_k}{\lambda_k p_k^2 - \mu_k})$, $k > 0$. By the analysis above, we deduce that x belongs to the MW-power cell of Q_i if and only if y belongs to the MW-power cell of Q'_i. This observation implies also that MW-power cells remain MW-power cells under inversion, which is not the case, e.g., for the usual Euclidean Voronoi diagram for points. Hence,

Theorem 2 *The set of multiplicatively weighted power diagrams in $\mathbb{R}^d$ is closed under inversion.*

Let $x = (x', x'')$, $x' \in \mathbb{R}^{d-1}$, $x'' \in \mathbb{R}$. Similarly, $p_i = (p'_i, p''_i)$, $p'_i \in \mathbb{R}^{d-1}$, $p''_i \in \mathbb{R}$. Consider a hyperplane $\Pi \in \mathbb{R}^{d-1}$. We can assume, without loss of generality, that Π is the hyperplane $x_d = 0$. Suppose that $x \in \Pi$, i.e., $x'' = 0$. Then x belongs to the MW-power cell of Q_i, if and only if for all $j > 0$:

$$
\begin{aligned}
& \lambda_i (x - p_i)^2 - \mu_i \leq \lambda_j (x - p_j)^2 - \mu_j \\
\iff\ & \lambda_i (x' - p'_i)^2 + \lambda_i (x'' - p''_i)^2 - \mu_i \leq \lambda_j (x' - p'_j)^2 + \lambda_j (x'' - p''_j)^2 - \mu_j \\
\iff\ & \lambda_i (x' - p'_i)^2 + \lambda_i p''^2_i - \mu_i \leq \lambda_j (x' - p'_j)^2 + \lambda_j p''^2_j - \mu_j
\end{aligned}
$$

Hence, x' belongs to the MW-power cell of the AM-weighted point $(p'_i, \lambda_i, \mu_i - \lambda_i p''^2_i)$ whose center p'_i is the projection of p_i on Π. More generally,

Theorem 3 *The intersection of a multiplicatively weighted power diagram in $\mathbb{R}^d$ with a hyperplane Π is a multiplicatively weighted power diagram in $\mathbb{R}^{d-1}$, defined over the projections on Π of the centers of the d-dimensional additively/multiplicatively weighted points.*

3 Additively weighted Voronoi cells

Let $\mathcal{E} = \{P_0, \ldots, P_n\}$ be a set of additively weighted (or simply weighted) points of $\mathbb{R}^d$. We note $P_i = (p_i, \omega_i)$, where $p_i \in \mathbb{R}^d$ and $\omega_i \in \mathbb{R}$ is the weight of P_i, $i = 0, \ldots, n$. Without loss of generality, we can assume that the ω_i are non-negative. Let $V_+(\mathcal{E})$ be the additively weighted Voronoi diagram of $\mathcal{E}$. We are interested in computing the cell $V_+(P_i)$ of $V_+(\mathcal{E})$ that is associated with P_i. For concreteness, in the sequel, the cell we want to compute is the cell $V_+(P_0)$ associated with P_0. We also assume that $V_+(P_i) \neq \emptyset$, $i \geq 0$, which geometrically means that no sphere is contained inside another (see [Wil99, Proposition 1]).

3.1 The lower bound

For simplicity we consider the case where ω_0 is infinite. For ω_0 finite the same bound can also be obtained using the correspondence presented in the subsection that follows.

When ω_0 is infinite, P_0 is a hyperplane and all P_i, $i > 0$, are spheres. Without loss of generality we assume that P_0 is the hyperplane $x_d = 0$. The points $x = (x', x'')$, $x' \in \mathbb{R}^{d-1}$, $x'' \in \mathbb{R}$, that are at equal distance from P_0 and P_i, $i > 0$, belong to the paraboloid

$$\begin{aligned} x'' = \|x - p_i\| - \omega_i \quad &\iff \quad (x'' + \omega_i)^2 = (x - p_i)^2 \\ &\iff \quad 2\,(p''_i + \omega_i)x'' = (x' - p'_i)^2 + p''^2_i - \omega_i^2, \end{aligned}$$

where $p_i = (p'_i, p''_i)$, $p'_i \in \mathbb{R}^{d-1}$, $p''_i \in \mathbb{R}$. Note that our assumption $V_+(P_i) \neq \emptyset$ implies $p''_i + \omega_i > 0$. Suppose that $V_+(P_0) \cap V_+(P_i) \neq \emptyset$. The points x that are at equal distance from P_0, P_i, $i > 0$, must verify, for any $j > 0$:

$$\begin{aligned} 2\,(p''_i + \omega_i)x'' &= (x' - p'_i)^2 + p''^2_i - \omega_i^2, \\ 2\,(p''_j + \omega_j)x'' &\leq (x' - p'_j)^2 + p''^2_j - \omega_j^2. \end{aligned}$$

Eliminating x'' we get

$$\frac{1}{p''_i + \omega_i}\,(x' - p'_i)^2 + p''_i - \omega_i \leq \frac{1}{p''_j + \omega_j}\,(x' - p'_j)^2 + p''_j - \omega_j.$$

This shows that the vertical projection onto P_0 of the boundary of the cell $V_+(P_0)$ is the MW-power diagram of the AM-weighted points $Q_i = (p'_i, \frac{1}{p''_i + \omega_i}, \omega_i - p''_i)$. In particular, we have an 1–1 correspondence between the k-dimensional faces of $V_+(P_0)$ and the k-dimensional faces of the MW-power diagram in $\mathbb{R}^{d-1}$ of the Q_i's, $k = 0, \ldots, d-1$. Suppose that $p''_i = \omega_i$, $i > 0$. Then the MW-power diagram of the Q_i's is actually a multiplicatively weighted Voronoi diagram of the weighted points $M_i = (p'_i, (2\omega_i)^{-1/2})$. Since the worst case complexity of multiplicatively weighted Voronoi diagrams is $\Omega(n^{\lceil \frac{d}{2} \rceil})$, we conclude that the worst case complexity of $V_+(P_0)$ is $\Omega(n^{\lceil \frac{d}{2} \rceil})$ in this special case. Our argumentation can be applied to the general case by taking w_0 sufficiently large instead of infinite. Hence,

Theorem 4 *Let $\mathcal{E}$ be a set of n weighted points in $\mathbb{R}^d$. The worst case complexity of a single additively weighted Voronoi cell in the additively weighted Voronoi diagram $V_+(\mathcal{E})$ of $\mathcal{E}$ is $\Theta(n^{\lceil \frac{d}{2} \rceil})$.*

The construction above provides a Euclidean model for multiplicatively weighted power diagrams. A special case of this construction has been recently used in [AB02].

3.2 Correspondence with multiplicatively weighted power diagrams

Let now $\mathcal{E} = \{P_0, \dots, P_n\}$ be our set of spheres, where $P_i = (p_i, \omega_i)$, $i \geq 0$. We can assume without loss of generality that p_0 coincides with the origin. Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ centered at the origin. Let x be a point on the boundary of the additively weighted Voronoi cell $V_+(P_0)$ of P_0. Let P_i be a sphere, such that x lies on the bisector of P_0 and P_i. We denote by $x_s = \psi(x)$, the radial projection of x onto $\mathbb{S}^{d-1}$. It can easily be shown that (cf. [Wil99, Proposition 4]) :

$$x = \psi^{-1}(x_s) = \delta_+(x, P_i)\, x_s = \frac{\alpha_i}{2(\omega_i^* + x_s \cdot p_i)}\, x_s, \qquad \alpha_i = p_i^2 - (\omega_i^*)^2, \qquad \omega_i^* = \omega_i - \omega_0.$$

Note that $\alpha_i > 0$, since otherwise P_0 would be contained inside P_i and thus $V_+(P_0) = \emptyset$. It can also easily be shown that $\omega_i^* + x_s \cdot p_i > 0$ (cf. [Wil99, Proposition 4]).

Suppose that $V_+(P_0) \cap V_+(P_i) \neq \emptyset$. Let $x \in \mathbb{R}^d$ be a point on the bisector of P_0, P_i and let x_s be its radial projection on $\mathbb{S}^{d-1}$. Since x is closer to P_i (and P_0) than to any other sphere P_j, we have, for any $j > 0$:

$$\begin{aligned}
& \delta_+(x, P_i) \leq \delta_+(x, P_j) \\
\iff\ & \frac{p_i}{\alpha_i} \cdot x_s + \frac{\omega_i^*}{\alpha_i} \geq \frac{p_j}{\alpha_j} \cdot x_s + \frac{\omega_j^*}{\alpha_j} \\
\iff\ & x_s^2 - 2\,\frac{p_i}{\alpha_i} \cdot x_s - \frac{2\omega_i^*}{\alpha_i} \leq x_s^2 - 2\,\frac{p_j}{\alpha_j} \cdot x_s - \frac{2\omega_j^*}{\alpha_j} \\
\iff\ & (x_s - \frac{p_i}{\alpha_i})^2 - \frac{2\omega_i^* \alpha_i + p_i^2}{\alpha_i^2} \leq (x_s - \frac{p_j}{\alpha_j})^2 - \frac{2\omega_j^* \alpha_j + p_j^2}{\alpha_j^2}
\end{aligned}$$

Hence x belongs to the bisector of P_0, P_i if and only if x_s belongs to the power cell of the sphere Σ_i centered at $q_i = \frac{p_i}{\alpha_i}$ of squared radius $\mu_i = \frac{2\omega_i^* \alpha_i + p_i^2}{\alpha_i^2}$. Therefore, the projection of the bisector of P_0, P_i on $\mathbb{S}^{d-1}$ coincides with the intersection of $\mathbb{S}^{d-1}$ with the power cell of Σ_i. Let $\mathcal{S} = \{\Sigma_1, \dots, \Sigma_n\}$. Let x_0 be a point of $\mathbb{S}^{d-1}$ that is in the interior of a cell of the power diagram $V_P(\mathcal{S})$ of $\mathcal{S}$ in $\mathbb{R}^d$. The sphere $\mathbb{S}^{d-1}$ is mapped by $f(\cdot; x_0)$ onto a hyperplane Π. Without loss of generality, we can assume that $x_0 = (0, \dots, 0, 1)$. Hence Π is the hyperplane $x_d = \frac{1}{2}$. By Theorem 2, the power diagram of $\mathcal{S}$ is mapped by $f(\cdot; x_0)$ to the MW-power diagram $V_{*P}(\mathcal{S}')$ of another set $\mathcal{S}' \in \mathbb{R}^d$. More precisely, the cell of Σ_i in $V_P(\mathcal{S})$ is mapped to the cell of the AM-weighted point $\Sigma_i' = (q_i', \lambda_i', \mu_i')$ in $V_{*P}(\mathcal{S}')$, where $\mathcal{S}' = \{\Sigma_1', \dots, \Sigma_n'\}$, $q_i' = \frac{q_i - x_0}{\lambda_i'} + x_0$, $\lambda_i' = (q_i - x_0)^2 - \mu_i$ and $\mu_i' = \mu_i / \lambda_i'$. By Theorem 3, the intersection of Π with $V_{*P}(\mathcal{S}')$ is a $(d-1)$-dimensional MW-power diagram $V_{*P}(\mathcal{S}'')$ of a set $\mathcal{S}''$, the centers of which lie on Π. More precisely, a point $x_s \in \mathbb{S}^{d-1}$ lies in the power cell of some Σ_i if and only if the image by $f(\cdot; x_0)$ of x_s, which lies on Π, lies in the cell of the AM-weighted point $\Sigma_i'' = (q_i'', \lambda_i', \mu_i' - \lambda_i' h_i^2)$ in $V_{*P}(\mathcal{S}'')$, where $\mathcal{S}'' = \{\Sigma_1'', \dots, \Sigma_n''\}$, q_i'' is the projection of q_i' onto Π, and $h_i = \|q_i' - q_i''\| + \frac{1}{2}$. This shows :

Lemma 2 *Let $\mathcal{E}$ be a set of n spheres in $\mathbb{R}^d$, and let $\mathcal{S}''$ be the set of additively/multiplicatively weighted points in $\mathbb{R}^{d-1}$ that we get by the transformation described above. Then the k-dimensional faces of $V_+(P_0)$ are in 1–1 correspondence with the k-dimensional faces of the multiplicatively weighted power diagram $V_{*P}(\mathcal{S}'')$ of $\mathcal{S}''$, $k = 0, \dots, d-1$.*

4 Convex hulls of spheres

Let $\delta_\varepsilon(x,\Pi)$ denote the signed distance of a point $x \in \mathbb{R}^d$ from a hyperplane Π. We define the distance $\delta_+(P,\Pi)$ of a weighted point $P=(p,\omega)$ from a hyperplane Π to be $\delta_+(P,\Pi)=\delta_\varepsilon(p,\Pi)-\omega$. Finally we define the distance $\delta_+(P,Q)$ between two weighted points $P=(p,\omega_P)$ and $Q=(q,\omega_Q)$ to be

$$\delta_+(P,Q)=\|p-q\|-\omega_P-\omega_Q=\delta_+(p,Q)-\omega_P=\delta_+(q,P)-\omega_Q.$$

Observe that, if P and Q are two spheres, $\delta_+(P,Q)>0$ (resp. $=0$) if and only if the two balls bounded by P and Q do not intersect (resp. are tangent). Let again $\mathcal{E}=\{P_0,\dots,P_n\}$, $P_i=(p_i,\omega_i)$ be a set of spheres in $\mathbb{R}^d$, and suppose that $V_+(P_0)\neq\emptyset$. Let u_k be a point of a k-dimensional face of $V_+(P_0)$, $0\le k\le d$. In particular, u_0 is a Voronoi vertex of $V_+(P_0)$ and u_d is a point in the interior of $V_+(P_0)$. The co-dimension $(d-k)$ of the face of $V_+(P_0)$ containing u_k is called the *Voronoi dimension* (V-dimension) of u_k. Let $\beta_k=\delta_+(u_k,P_0)$. The distance β_k may be positive, zero or negative, since u_k may lie on the exterior, boundary or interior of P_0, respectively. We call the weighted point $U_k=(u_k,\beta_k)$ the *Voronoi weighted point* associated with u_k. We use the term *Voronoi sphere* to refer to a Voronoi weighted point with non-negative weight. We define the V-dimension of U_k to be the V-dimension of u_k.

Let us consider the convex hull $CH(\mathcal{S})$ of a set $\mathcal{S}$ of spheres. We say that a supporting hyperplane Π of $\mathcal{S}$ has *convex hull dimension* (CH-dimension) k, if it is tangent to exactly k spheres of $\mathcal{S}$. Finally, a *face of* $CH(\mathcal{S})$ *of circularity* k, $0\le k\le d-1$, is a maximal connected portion of the boundary of $CH(\mathcal{S})$, consisting of points where the supporting hyperplanes are tangent to a given set of $(d-k)$ spheres.

4.1 A special case

We assume that $\omega_0=0$. Let $\Sigma_i=(c_i,\rho_i)=f(P_i;p_0)$, $i>0$. Since $V_+(P_0)\neq\emptyset$, none of the spheres P_i pass through p_0 and thus the Σ_i are spheres with

$$c_i=\frac{p_i-p_0}{(p_i-p_0)^2-\omega_i^2},\qquad \rho_i=\frac{\omega_i}{(p_i-p_0)^2-\omega_i^2}.$$

Let u_k, $k<d$, be a point of $V_+(P_0)$ of V-dimension $(d-k)$ and let $U_k=(u_k,\beta_k)$ be the corresponding Voronoi sphere. Let $\Pi_k=f(U_k;p_0)$. Since U_k passes through p_0, Π_k is a hyperplane in $\mathbb{R}^d$. The normal of Π_k is chosen such that the points at positive distance to U_k map to points that are at positive distance to Π_k. Without loss of generality, let P_i, $i=1,\dots,d-k$, be the weighted points that define U_k along with P_0. By construction,

$$\begin{aligned}\delta_+(P_i,U_k)&=0, &\quad 0\le i\le d-k,\\ \delta_+(P_i,U_k)&>0, &\quad i>d-k.\end{aligned}$$

The above relations are equivalent to

$$\begin{aligned}\delta_+(\Sigma_i,\Pi_k)&=0, &\quad 1\le i\le d-k,\\ \delta_+(\Sigma_i,\Pi_k)&>0, &\quad i>d-k.\end{aligned}$$

Hence Π_k is a supporting hyperplane of the convex hull of the set of spheres $\mathcal{S}=\{\Sigma_1,\dots,\Sigma_n\}$ of CH-dimension $(d-k)$ and conversely, a hyperplane Π of CH-dimension $(d-k)$ maps to a point of $V_+(P_0)$ of V-dimension $(d-k)$. In particular, this implies an 1–1 correspondence between the faces of $CH(\mathcal{S})$ of circularity k and the k-dimensional faces of $V_+(P_0)$.

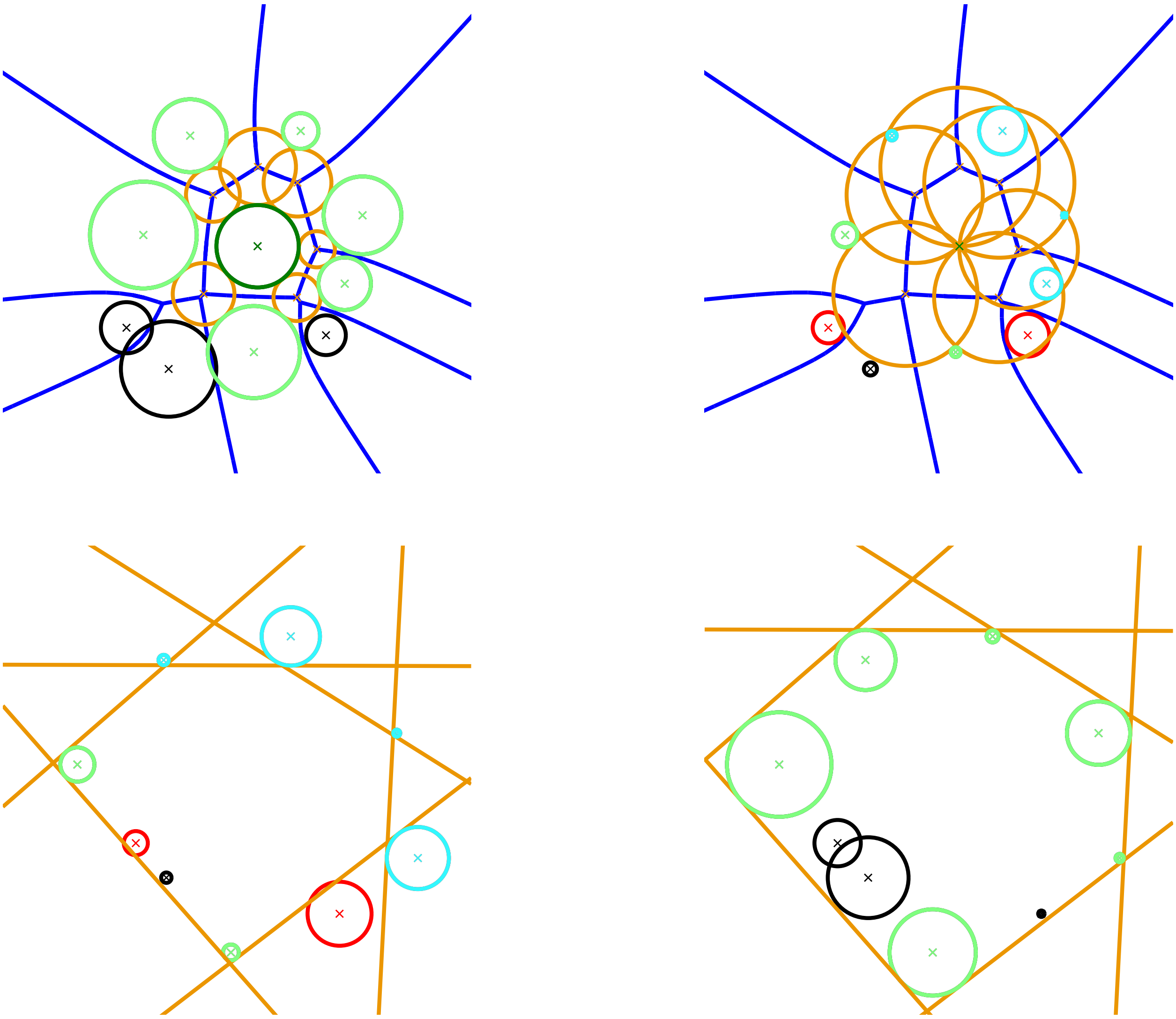

Figure 2: The equivalence relationship between additively weighted Voronoi cells and convex hulls of spheres in two dimensions. P_0 is shown in dark green. Light green and black spheres have positive weight. Light blue and red spheres have negative weight. Light green and light blue spheres correspond to neighbors of P_0 in $V_+(\mathcal{E})$. Black and red spheres do not correspond to neighbors of P_0 in $V_+(\mathcal{E})$. Top left: the set $\{P_0, \ldots, P_n\}$. The Voronoi spheres U_k of V-dimension 0 are shown in yellow. The Voronoi skeleton is shown in blue. Top right: the set $\{P'_0, \ldots, P'_n\}$. The spheres U'_k are in yellow. The Voronoi skeleton remains the same and P'_0 is a point. Bottom left: the set $\{\Sigma_1, \ldots, \Sigma_n\}$. The hyperplanes Π_k are in yellow. Bottom right: the set $\{\Sigma'_1, \ldots, \Sigma'_n\}$. The hyperplanes Π'_k are in yellow.

4.2 The general case

In this subsection we want to show the equivalence of the previous subsection when $\omega_0 \geq 0$. In particular, we want to find a set of spheres the convex hull of which is combinatorially equivalent to the additively weighted Voronoi cell $V_+(P_0)$ of P_0.

Let $P_i' = (p_i, \omega_i - \omega_0)$, $i = 0, \ldots, n$ (see Fig. 2(top right)), and let $\Sigma_i = (c_i, \rho_i) = f(P_i'; p_0)$ (see Fig. 2(bottom left)). In this case :

$$c_i = \frac{p_i - p_0}{(p_i - p_0)^2 - (\omega_i - \omega_0)^2}, \qquad \rho_i = \frac{\omega_i - \omega_0}{(p_i - p_0)^2 - (\omega_i - \omega_0)^2}.$$

Note that the additively weighted Voronoi diagram does not change combinatorially, as well as geometrically, if we translate the weights by the same quantity, which implies that the Voronoi cells $V_+(P_0)$ and $V_+(P_0')$ are exactly the same. Let u_k, $k < d$ be a point of $V_+(P_0)$ of V-dimension $(d-k)$ and let $U_k = (u_k, \beta_k)$ be the corresponding Voronoi weighted point. In this case β_k may be positive as well as zero or negative. Let $U_k' = (u_k, \beta_k + \omega_0)$. Then :

$$\delta_+(P_0, U_k) = 0 \iff \|p_0 - u_k\| - (\omega_0 + \beta_k) = 0 \iff \delta_+(p_0, U_k') = 0.$$

Trivially, $\omega_0 + \beta_k = \|p_0 - u_k\| \geq 0$. Hence U_k' is a Voronoi sphere that passes through p_0, and corresponds to u_k in $V_+(P_0')$. Let $\Pi_k = f(U_k'; p_0)$. The orientation of Π_k is as in the previous subsection. Clearly,

$$\begin{aligned} \delta_+(P_i', U_k') &= 0, & 0 \leq i \leq d-k, \\ \delta_+(P_i', U_k') &> 0, & i > d-k, \end{aligned}$$

which in turns implies that :

$$\begin{aligned} \delta_+(\Sigma_i, \Pi_k) &= 0, & 1 \leq i \leq d-k, \\ \delta_+(\Sigma_i, \Pi_k) &> 0, & i > d-k. \end{aligned}$$

Let $R \in \mathbb{R}$ be a sufficiently large number such that $\rho_i + R \geq 0$, and let $\Sigma_i' = (c_i, \rho_i + R)$, $i > 0$. Finally, let Π_k' be the translation of Π_k by R in the opposite direction of its normal (see Fig. 2(bottom right)). Obviously :

$$\begin{aligned} \delta_+(\Sigma_i', \Pi_k') &= 0, & 1 \leq i \leq d-k, \\ \delta_+(\Sigma_i', \Pi_k') &> 0, & i > d-k, \end{aligned}$$

i.e., Π_k' is a supporting hyperplane of the set of spheres $\mathcal{S} = \{\Sigma_1', \ldots, \Sigma_n'\}$ of CH-dimension $(d-k)$. As in the preceding subsection, we can show that, by means of the inverse transformation, a supporting hyperplane of $\mathcal{S}$ of CH-dimension $(d-k)$, maps to a point of $V_+(P_0)$ of V-dimension $(d-k)$. Hence,

Lemma 3 *Let $\mathcal{E} = \{P_0, \ldots, P_n\}$ be a set of $n+1$ spheres in $\mathbb{R}^d$, and let $\mathcal{S}$ be the set of n spheres that we get by the transformation described above. Then the k-dimensional faces of $V_+(P_0)$ are in 1–1 correspondence with the faces of $CH(\mathcal{S})$ of circularity k, $k = 0, \ldots, d-1$.*

An immediate consequence of the above lemma is that the worst case complexity of the convex hull of a set of spheres in dimension d is the same with the worst case complexity of an additively weighted Voronoi cell in dimension d, i.e.,

Theorem 5 *Let $\mathcal{S}$ be a set of n spheres in $\mathbb{R}^d$. The worst case complexity of the convex hull $CH(\mathcal{S})$ of $\mathcal{S}$ is $\Theta(n^{\lceil \frac{d}{2} \rceil})$.*

It has been shown in [BCD$^+$96] that the worst case complexity of the convex hull of a set of n d-dimensional spheres is $O(n^{\lceil \frac{d}{2} \rceil})$. It has also been shown that the worst case complexity of the convex hull of n spheres is $\Omega(n^{\lfloor \frac{d}{2} \rfloor})$. Our construction provides an alternative way to prove the upper bound in [BCD$^+$96], and at the same time it gives us a tight lower bound. A corollary of Theorem 5 is that the algorithm presented in [BCD$^+$96] for the construction of the convex hull of spheres in dimension d is optimal in any dimension.

5 Computing a cell of an additively weighted Voronoi diagram

The algorithm of Aurenhammer [Aur87] for the computation of the entire additively weighted Voronoi diagram suggests also an algorithm for the computation of a single additively weighted Voronoi cell. This algorithm runs in time $O(n^{\lfloor \frac{d}{2} \rfloor + 1})$ and it is worst case optimal only for odd d.

The construction described in Subsection 4.2 provides an alternative to the above algorithm of Aurenhammer for the computation of a single additively weighted Voronoi cell in any dimension. Suppose that we are given a set $\mathcal{E} = \{P_0, \ldots, P_n\}$ of weighted points in $\mathbb{R}^d$ and suppose we want to compute the additively weighted Voronoi cell $V_+(P_0)$ of $P_0 = (p_0, \omega_0)$. The first step is to decrease the weights of all P_i by ω_0. Then we invert all P_i's, $i > 0$, using p_0 as the pole of inversion. After the inversion we get a new set of n weighted points $\mathcal{S} = \{\Sigma_1, \ldots, \Sigma_n\}$. We enlarge the weights of all Σ_i by the same quantity R, so that they become non-negative. Finally, we use the algorithm in [BCD+96] to construct the convex hull $CH(\mathcal{S})$ of $\mathcal{S}$. The additively weighted Voronoi cell $V_+(P_0)$ of P_0 can now be constructed from $CH(\mathcal{S})$ in time proportional to its complexity. By Lemma 3 and Theorem 5 we conclude that the algorithm just described is worst case optimal in any dimension, i.e.,

Theorem 6 *Let $\mathcal{E}$ be a set n of weighted points in $\mathbb{R}^d$. A single additively weighted Voronoi cell of $V_+(\mathcal{E})$ can be computed in worst case optimal time $O(n \log n + n^{\lceil \frac{d}{2} \rceil})$.*

Yet another worst case optimal algorithm is that suggested in Subsection 3.2. Assuming that P_0 is the origin, we first compute the set of spheres $\mathcal{S}$, such that the intersection of $V_P(\mathcal{S})$ with the unit sphere $\mathbb{S}^{d-1}$ coincides with the projection of $V_+(P_0)$ with $\mathbb{S}^{d-1}$. Then we invert $\mathcal{S}$ using a suitable point x_0 on $\mathbb{S}^{d-1}$, to get a set of AM-weighted points $\mathcal{S}'$. Let Π be the image of $\mathbb{S}^{d-1}$ under the inversion. The next step is to project the set $\mathcal{S}'$ on Π. This gives us another set of $(d-1)$-dimensional AM-weighted points $\mathcal{S}''$, the MW-power of which can be computed as per Lemma 1. $V_+(P_0)$ can then be constructed from $V_{*P}(\mathcal{S}'')$ in time proportional to its complexity.

6 Conclusion

In this paper we presented a equivalence relationship between additively weighted Voronoi cells in $\mathbb{R}^d$, convex hulls of spheres in $\mathbb{R}^d$, power diagrams in $\mathbb{R}^d$ and multiplicatively weighted power diagrams in $\mathbb{R}^{d-1}$. Using this equivalence, we proved tight bounds on the worst case complexity of a single Voronoi cell and the convex hull for a set of spheres in dimension d. We also presented two worst case optimal algorithms for the construction of a single additively weighted Voronoi cell in any dimension.

The worst case complexity of the whole additively weighted Voronoi diagram in even dimensions $d > 2$ is still an open problem. It is also unknown what is the complexity of a single additively weighted Voronoi cell, the whole additively weighted Voronoi diagram or the convex hull of a set of spheres, if the spheres have a constant number of different radii.

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