Optimal rate of convergence for anisotropic vanishing viscosity limit of a scalar balance law

Charalambos Makridakis * and Benoît Perthame †

Abstract

An open question in numerical analysis of multidimensional scalar conservation laws discretized on non-structured grids is the optimal rate of convergence. The main difficulty relies on a priori BV bounds which cannot be derived by opposition to the case of structured (cartesian) grids. In this paper we consider a related question for a corresponding continuous model, namely the vanishing viscosity method for a multidimensional scalar conservation law with a general diffusion matrix which is only bounded. Then, BV estimates are not available here and we prove the $h^{1/2}$ convergence rate. Our strategy of proof differs from the classical method of Kuznetzov. It consists in using in an accurate way the entropy dissipation due to the parabolic terms. The dissipation of the conservation law is not strong enough and we thus consider an auxiliary parabolic problem to compensate that. Using the kinetic formulation and the related uniqueness method also helps to avoid unessential technicalities.

Key words. rate of convergence, vanishing viscosity method, kinetic formulation, scalar conservation laws.
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*Department of Applied Mathematics, University of Crete, 71409 Heraklion-Crete, Greece and Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion - Crete, Greece, makr@math.uoc.gr

†Département de Mathématiques et Applications, UMR 855, Ecole Normale Supérieure, 45, rue d'Ulm, 75230 Paris Cedex 05, France, perthame@dma.ens.fr

1 Introduction

We consider the entropy solution $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^+; BV(\mathbb{R}^d))$ to a multidimensional scalar conservation law

$$\begin{aligned} &\frac{\partial}{\partial t} u(t,x) + \operatorname{div} A(u) = 0, \qquad t > 0,\ x \in \mathbb{R}^d, \\ &\frac{\partial}{\partial t} S(u(t,x)) + \operatorname{div} \eta^S(u) \le 0, \qquad \text{for all } S \text{ convex}, \\ &u(t=0,x) = u^0(x) \in BV \cap L^\infty(\mathbb{R}^d), \end{aligned} \tag{1.1}$$

with the notations $\eta^S(u) = \int_0^u S'(\cdot) a(\cdot)$ and $a = A' : \mathbb{R} \to \mathbb{R}^d$.

A classical open question in the numerical analysis of this equation discretized on non-structured grids is the optimal rate of convergence. Indeed, in such situations BV bounds on the numerical approximation are not available and thus Kuznetsov's [13] classical approach does not apply and only a reduced convergence rate in $h^{1/4}$ can be established ([5], [19] see also [7] , [11], [10], [15]). This multidimensional situation is in opposition to the one dimensional case where such BV bounds are derived [18] and optimal rate of convergence $h^{1/2}$ follows. The result of Sanders [18] can be generalized in more than one dimension only for cartesian grids. Recently, Cockburn and Gremaud [6] towards proving the optimal rate, proposed a variant of Kuznetsov's approach aiming to show the expected rates by bypassing the stability estimates of the approximate problem. This approach was however restricted to strong conditions on the mesh and the discrete fluxes.
It is usual to relate numerical methods to the vanishing viscosity method (below we always use the convention of summation upon the repeated index),

$$\begin{aligned} &\frac{\partial}{\partial t} v(t,x) + \operatorname{div} A(v) = \frac{\partial}{\partial x_i}\big(a_{ij}(x) \frac{\partial}{\partial x_j} v\big), \qquad t > 0,\ x \in \mathbb{R}^d, \\ &v(t=0,x) = v^0(x) \in L^1 \cap L^\infty(\mathbb{R}^d), \end{aligned} \tag{1.2}$$

where the anisotropic matrix a_{ij} *reflects the unstructured character of the grid* and thus it is natural to only assume that for some constant $K > 0$,

$$a_{ij} \text{ is a positive definite symmetric matrix}, \quad \| a_{ij} \|_{L^\infty(\mathbb{R}^d)} = K. \tag{1.3}$$

Then the same difficulty appears that the standard method for error estimates does not apply.

Indeed, we recall that, as stated in a compact form in [1], Kuznetzov's result requires to control entropies in a weak form. Namely, error terms E^S in the hyperbolic entropy inequalities, for convex S,

$$\frac{\partial}{\partial t} S(v) + \mathrm{div} \eta^S(v) \leq \mathrm{div} E^S(t,x), \tag{1.4}$$

imply error estimates

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^d)} \leq \|u^0 - v^0\|_{L^1(\mathbb{R}^d)} + C(t)\big(\|u^0\|_{TV(\mathbb{R}^d)}\big)^{1/2} \||E|\|^{1/2}, \tag{1.5}$$

with

$$\||E|\| = \int_0^t \int_{\mathbb{R}^d} \sup_{|S'|\leq 1, S''\geq 0} |E^S(s,x)| dx \; ds.$$

For the vanishing viscosity method (1.2), we have, for S convex,

$$\frac{\partial}{\partial t} S(v) + \mathrm{div} \eta^S(v) \leq \frac{\partial}{\partial x_i}\big(a_{ij}(x) \frac{\partial}{\partial x_j} S(v)\big). \tag{1.6}$$

Therefore the inequality (1.5), applies with

$$E^S(t,x) = \nabla S(v(t,x)) = S'(v) \nabla v,$$

and we directly deduce the standard result

$$\begin{aligned} \|u(t) - v(t)\|_{L^1(\mathbb{R}^d)} & \leq \|u^0 - v^0\|_{L^1(\mathbb{R}^d)} \\ & + C(t)\big(\|u^0\|_{TV(\mathbb{R}^d)} \; \|v\|_{L^\infty((0,t);TV(\mathbb{R}^d))}\big)^{1/2} \big(\|a_{ij}\|_{L^\infty(\mathbb{R}^d)}\big)^{1/2}. \end{aligned}$$

With the only L^∞ assumption (1.3), we do not have a priori BV bound for the function v (except in one dimension). Therefore the general estimate (1.5) does not apply here.

The present paper develops new ideas to prove the

Theorem 1.1 *For a smooth matrix a_{ij} satisfying (1.3) and the smooth bounded solution $v \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ to (1.2), we have*

$$\begin{aligned} \|u(t) - v(t)\|_{L^1(\mathbb{R}^d)} & \leq \|u^0 - v^0\|_{L^1(\mathbb{R}^d)} \\ & + C(d) \|u^0\|_{TV(\mathbb{R}^d)} \; \big(t \, \| \, a_{ij} \, \|_{L^\infty(\mathbb{R}^d)}\big)^{1/2}. \end{aligned} \tag{1.7}$$

One of the ingredients of the proof relies on the precise entropy equality for (1.2), namely

$$\frac{\partial}{\partial t}S(v) + \mathrm{div}\eta^S(v) = \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial}{\partial x_j}S(v)\big) - S''(v)a_{ij}(x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}. \qquad (1.8)$$

Especially, we have included a precise parabolic entropy dissipation term $S''(v)a_{ij}(x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}$, which is essential in our analysis. This term has already been used in the proof of uniqueness for various hyperbolic/parabolic problems with the anisotropic nonlinear diffusions [4], and also by Chen and DiBenedetto [3] (but it can be recovered from a weaker entropy inequality for isotropic diffusions [2], [8]). Another idea developed here is that this entropy dissipation is not enough and a direct comparison with the hyperbolic solution u is not possible. In order to obtain such entropy dissipation we can only compare v with a solution to a parabolic equation with a constant diffusion term.
The proof covers also the case

$$\frac{\partial}{\partial t}v(t,x) + \mathrm{div}A(v) = \frac{\partial}{\partial x_i}\big(a_{ij}(x,v)\frac{\partial v}{\partial x_j}\big), \qquad t>0,\ x\in\mathbb{R}^d,$$

under appropriate smoothness assumptions on v and provided that matrix a_{ij} still is bounded, (1.3). Note that estimates of the viscosity approximation

$$\frac{\partial}{\partial t}v(t,x) + \mathrm{div}A(v) = \frac{\partial}{\partial x_i}\big(B_{ij}(v)\frac{\partial v}{\partial x_j}\big), \qquad t>0,\ x\in\mathbb{R}^d,$$

without using the TV stability of v were first proved in [6] in one dimension and extended in many dimensions in [1]. These proofs do not cover our case (1.3) since when applied to (1.2) require a_{ij} to be differentiable.

Remark 1.1 *The method does not use the smoothness of the function v, neither the positivity and smoothness of the matrix a_{ij}, and it could be extended to a purely $C(\mathbb{R}^+;L^1(\mathbb{R}^d))$ setting using kinetic solutions along the lines [17]. We have chosen, for simplicity, to use this framework on v in order to avoid unessential technicalities.*

2 Proof of Theorem 1.1

2.1 More entropy dissipation

In fact we are going to prove a variant of Theorem 1.1, comparing v with the solution $w \in C(\mathbb{R}^+; L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^+; BV(\mathbb{R}^d))$ to the parabolic equation (recall the definition of K in (1.3))

$$\begin{aligned} \frac{\partial}{\partial t} w + \operatorname{div} A(w) = K \Delta w, \\ w(t=0,x) = u^0(x). \end{aligned} \tag{2.1}$$

Theorem 2.1 *With the assumptions of Theorem 1.1, we have*

$$\begin{aligned} \|w(t) - v(t)\|_{L^1(\mathbb{R}^d)} \;&\le \|u^0 - v^0\|_{L^1(\mathbb{R}^d)} \\ &+ C(d)\ \|u^0\|_{TV(\mathbb{R}^d)}\ (K\,t)^{1/2}. \end{aligned} \tag{2.2}$$

Theorem 1.1 follows directly from this because we can apply (1.5) to compare u and w. Since we have, for all $t \ge 0$,

$$\|w(t)\|_{TV(\mathbb{R}^d)} \le \|u^0\|_{TV(\mathbb{R}^d)},$$

we indeed deduce from (1.5) that

$$\|w(t) - u(t)\|_{L^1(\mathbb{R}^d)} \le C(t) \|u^0\|_{TV(\mathbb{R}^d)}\ K^{1/2}.$$

The proof is therefore reduced to prove Theorem 2.1. This will be shown in the sequel; a main point here is the fact that (2.1) contains more entropy dissipation than (1.1).

2.2 Kinetic formulations

We use the kinetic framework [14], [16], [17] which simplifies very much uniqueness arguments compared to the initial Kruzkov approach [12]. This needs some notations. We define after [14] the 'equilibrium' function of density w by $\boldsymbol{\chi}(t,x,\xi) := \boldsymbol{\chi}(\xi; w(t,x))$ by

$$\boldsymbol{\chi}(\xi; w) = \begin{cases} +1, & \text{for } \ 0 < \xi < u(t,x), \\ -1, & \text{for } \ u(t,x) < \xi < 0, \\ \ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

The theory of kinetic formulations states that (2.1) is equivalent to write the kinetic equation on $\boldsymbol{\chi}$

$$\partial_t \boldsymbol{\chi} + a(\xi)\cdot\nabla_x \boldsymbol{\chi} = K\ \Delta \boldsymbol{\chi} + \frac{\partial}{\partial_\xi} m(t,x,\xi), \tag{2.4}$$

for some the nonnegative bounded measure m given by

$$m(t,x,\xi) = K\ \delta(\xi - w(t,x))\ |\nabla w|^2. \tag{2.5}$$

The derivation of this equation from (2.1) shows that the measure m expresses the entropy dissipation. Indeed, after multiplying (2.4) by $S'(\xi)$ and ξ integration, we obtain

$$\frac{\partial}{\partial t} S(w) + \text{div}\eta^S(w) = K\Delta S(w) - S''(w)\ K\ |\nabla w|^2, \tag{2.6}$$

which is the entropy equality for (2.1). Indeed, the function $\boldsymbol{\chi}$ is chosen because it provides the equalities

$$S(w) = \int_{\mathbb{R}} S'(\xi)\, \boldsymbol{\chi}(t,x,\xi)\, d\xi, \quad \eta^S(w) = \int_{\mathbb{R}} S'(\xi)\, a(\xi)\, \boldsymbol{\chi}(t,x,\xi)\, d\xi.$$

Similarly, we can perform the same construction for the function v and define, using still the notation (2.3), $\bar{\boldsymbol{\chi}}(t,x,\xi) := \boldsymbol{\chi}(\xi; v(t,x))$. It solves

$$\partial_t \bar{\boldsymbol{\chi}} + a(\xi)\cdot\nabla_x \bar{\boldsymbol{\chi}} = \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial}{\partial x_j}\bar{\boldsymbol{\chi}}\big) + \frac{\partial}{\partial_\xi}\bar{m}(t,x,\xi), \tag{2.7}$$

$$\bar{m}(t,x,\xi) = \delta(\xi - v(t,x))\ a_{ij}\frac{\partial}{\partial x_i} v \frac{\partial}{\partial x_j} v. \tag{2.8}$$

2.3 Regularization

We shall need more regularity than available on the function $\boldsymbol{\chi}(\xi; w(t,x))$. We set $\varepsilon = (\varepsilon_1, \varepsilon_2)$, ε_1 for the forward time regularization and ε_2 for the space regularization, and we define

$$\varphi_\varepsilon(t,x) = \frac{1}{\varepsilon_1}\varphi_1(\frac{t}{\varepsilon_1})\ \frac{1}{\varepsilon_2^d}\varphi_2(\frac{x}{\varepsilon_2}),$$

where $\varphi_j \geq 0, j = 1, 2$, denote the normalized regularizing kernels with $\int \varphi_j = 1$, $supp(\varphi_1) \subset (-1, 0)$ in order to allow the time regularization. Next we set

$$\boldsymbol{\chi}_\varepsilon(t, x, \xi) = \boldsymbol{\chi}(\xi; w(t, x)) \star_{(t,x)} \varphi_\varepsilon. \tag{2.9}$$

The regularity of the kinetic formulation leads to an equation on $\boldsymbol{\chi}_\varepsilon$,

$$\partial_t \boldsymbol{\chi}_\varepsilon + a(\xi) \cdot \nabla_x \boldsymbol{\chi}_\varepsilon = K \Delta \boldsymbol{\chi}_\varepsilon + \frac{\partial}{\partial_\xi} m_\varepsilon(t, x, \xi), \tag{2.10}$$

$$m_\varepsilon(t, x, \xi) = m(t, x, \xi) \star_{(t,x)} \varphi_\varepsilon. \tag{2.11}$$

2.4 Decay functional

Following [16], we introduce the decay functional

$$Q_\varepsilon(t) = \int_{\mathbb{R} \times \mathbb{R}^d} [|\boldsymbol{\chi}_\varepsilon(t, x, \xi)| + |\bar{\boldsymbol{\chi}}(t, x, \xi)| - 2\boldsymbol{\chi}_\varepsilon(t, x, \xi)\ \bar{\boldsymbol{\chi}}(t, x, \xi)]\, d\xi\, dx \geq 0. \tag{2.12}$$

Since $|\boldsymbol{\chi}_\varepsilon| = sgn(\xi)\boldsymbol{\chi}_\varepsilon$, and using the L^1 assumption which allows to integrate by parts, we have

$$\begin{aligned}
\frac{d}{dt} Q_\varepsilon(t) = &-2 \int_{\mathbb{R} \times \mathbb{R}^d} [m_\varepsilon(t, x, \xi = 0) + \bar{m}(t, x, \xi = 0)]\, d\xi\, dx \\
&+2 \int_{\mathbb{R}^d} a_{ij} \frac{\partial}{\partial x_i} \bar{\boldsymbol{\chi}}\ \frac{\partial}{\partial x_j} \boldsymbol{\chi}_\varepsilon\ dx + 2 \int_{\mathbb{R} \times \mathbb{R}^d} \bar{m}(t, x, \xi)\ \frac{\partial}{\partial_\xi} \boldsymbol{\chi}_\varepsilon\, d\xi\, dx \\
&-2 \int_{\mathbb{R} \times \mathbb{R}^d} K \bar{\boldsymbol{\chi}}\ \Delta \boldsymbol{\chi}_\varepsilon\, d\xi\, dx + 2 \int_{\mathbb{R} \times \mathbb{R}^d} m_\varepsilon(t, x, \xi)\ \frac{\partial}{\partial_\xi} \bar{\boldsymbol{\chi}}\, d\xi\, dx \\
= -2 \int_{\mathbb{R}^+ \times \mathbb{R}^{2d}} &\bar{m}(t, x, \xi = w(s, y))\, \varphi_\varepsilon(t-s, x-y) ds dy dx - 2 \int_{\mathbb{R}^d} m_\varepsilon(t, x, \xi = v(t, x))\, dx \\
&+2 \int_{\mathbb{R} \times \mathbb{R}^d} a_{ij} \frac{\partial}{\partial x_i} \bar{\boldsymbol{\chi}}\ \frac{\partial}{\partial x_j} \boldsymbol{\chi}_\varepsilon\, d\xi\, dx - 2 \int_{\mathbb{R} \times \mathbb{R}^d} K \bar{\boldsymbol{\chi}}\ \Delta \boldsymbol{\chi}_\varepsilon\, d\xi\, dx.
\end{aligned}$$

We refer to [16], [17] for the justification of significance of all these terms. Here the two negative terms containing m are favorable to prove decay of Q_ε, and the two other terms have to be controlled, which we do now.

We begin with the worse, containing a_{ij} which is treated in an original way here.

$$\int_{\mathbb{R}\times\mathbb{R}^d} a_{ij}\frac{\partial}{\partial x_i}\bar{\boldsymbol{\chi}}\ \frac{\partial}{\partial x_j}\boldsymbol{\chi}_\varepsilon\, d\xi\, dx$$

$$= \int_{\mathbb{R}^+\times\mathbb{R}^{2d+1}} \delta(\xi - v(t,x))\delta(\xi - w(s,y))\, a_{ij}(x)\frac{\partial}{\partial x_i}v(t,x)\ \frac{\partial}{\partial x_j}w(s,y)\ \varphi_\varepsilon(t-s, x-y)$$

$$\leq \frac{1}{2}\int_{\mathbb{R}^+\times\mathbb{R}^{2d+1}} \delta(\xi - v(t,x))\delta(\xi - w(s,y))\, a_{ij}(x)[\frac{\partial}{\partial x_i}v(t,x)\ \frac{\partial}{\partial x_j}v(t,x) +$$

$$\frac{\partial}{\partial x_i}w(s,y)\ \frac{\partial}{\partial x_j}w(s,y)]\ \varphi_\varepsilon(t-s, x-y)\, d\xi\, dx\, dy\, ds$$

$$\leq \frac{1}{2}\int_{\mathbb{R}^+\times\mathbb{R}^{2d}} \bar{m}(t,x,\xi = w(s,y))\ \varphi_\varepsilon(t-s, x-y)$$

$$+\frac{1}{2}\int_{\mathbb{R}^d} m_\varepsilon(t,x,\xi = v(t,x))\ dx$$

where we have used the definitions of m_ε, $\bar{m}$, and the bound in (1.3). Hence we conclude

$$\frac{d}{dt}Q_\varepsilon(t) \leq -2\int_{\mathbb{R}\times\mathbb{R}^d} K\bar{\boldsymbol{\chi}}\ \Delta\boldsymbol{\chi}_\varepsilon\, d\xi\, dx \leq 2K\int_{\mathbb{R}\times\mathbb{R}^d} |\Delta\boldsymbol{\chi}_\varepsilon|\, d\xi\, dx. \tag{2.13}$$

To proceed further, we upper bound the right hand side of (2.13) by

$$|\Delta\boldsymbol{\chi}_\varepsilon| = |\int_{\mathbb{R}^+\times\mathbb{R}^d} \Delta\boldsymbol{\chi}(s,y,\xi)\varphi_\varepsilon(t-s, x-y)ds\, dy|$$

$$= |\int_{\mathbb{R}^+\times\mathbb{R}^d} \nabla\boldsymbol{\chi}(s,y,\xi)\cdot\nabla\varphi_\varepsilon(t-s, x-y)ds\, dy|$$

$$= |\int_{\mathbb{R}^+\times\mathbb{R}^d} \delta(\xi - w(s,y))\,\nabla w(s,y)\cdot\nabla\varphi_\varepsilon(t-s, x-y)ds\, dy|$$

and we conclude that

$$\int_{\mathbb{R}\times\mathbb{R}^d} |\Delta\boldsymbol{\chi}_\varepsilon|\, d\xi\, dx \leq \frac{C}{\varepsilon_2}\|u^0\|_{TV(\mathbb{R}^d)}.$$

$$\frac{d}{dt}Q_\varepsilon(t) \leq \frac{C\,K}{\varepsilon_2}\ \|u^0\|_{TV(\mathbb{R}^d)}. \tag{2.14}$$

2.5 Conclusion of the proof

We can now conclude the proof. We deduce from (2.14) that

$$Q_\varepsilon(t) \le Q_\varepsilon(0) + \frac{C\,K\,t}{\varepsilon_2}\,\|u^0\|_{TV(\mathbb{R}^d)}.$$

On the other hand, we can upper bound the initial error by

$$Q_\varepsilon(0) = \int_{\mathbb{R}^{2d}} [|w(s,y)| + |v^0(x)| - 2\min(|w(s,y)|, |v^0(x)|)\mathbb{1}_{\{sgn(w(s,y)v^0(x))\ge 0\}}]$$
$$\varphi_\varepsilon(-s, x-y)\,dx\;dy\;ds$$
$$= \int_{\mathbb{R}^{2d}} |u^0(s,y) - v^0(x)|\;\varphi_\varepsilon(-s,x-y)\,dx\;dy\;ds.$$

At this level we may pass to limit as ε_1 vanishes and we find, (with the obvious modification on the definition of Q_{ε_2})

$$Q_{\varepsilon_2}(t) \le \|u^0 - v^0\|_{L^1(\mathbb{R}^d)} + C\varepsilon_2\|u^0\|_{TV(\mathbb{R}^d)} + \frac{C\,K\,t}{\varepsilon_2}\,\|u^0\|_{TV(\mathbb{R}^d)}. \qquad (2.15)$$

Finally, following the above lines, we lower bound $Q_{\varepsilon_2}(t)$ by

$$Q_{\varepsilon_2}(t) = \int_{\mathbb{R}^{2d}} [|w(t,y) - v(t,x)|\varphi_{\varepsilon_2}(x-y)\,dx\;dy$$
$$\ge \|w(t) - v(t)\|_{L^1(\mathbb{R}^d)} - C\varepsilon_2\|u^0\|_{TV(\mathbb{R}^d)}.$$

Together with (2.15) we find

$$\|w(t) - v(t)\|_{L^1(\mathbb{R}^d)} \le \|u^0 - v^0\|_{L^1(\mathbb{R}^d)} + C\varepsilon_2\|u^0\|_{TV(\mathbb{R}^d)} + \frac{C\,K\,t}{\varepsilon_2}\,\|u^0\|_{TV(\mathbb{R}^d)}$$

and optimizing the parameter ε_2, we conclude the proof of Theorem 2.1.

References

[1] F. Bouchut and B. Perthame, Kruzhkov's estimates for scalar conservation laws revisited, *Trans. Amer. Math. Soc.* **350** (1998), 2847–2870.

[2] J. Carrillo, Entropy solutions for nonlinear degenerate problems, *Arch. Rational Mech. Anal.* **147** (1999), 269–361.

[3] G.-Q. Chen and E. DiBenedetto, Stability of entropy solutions to the Cauchy problem for a class of nonlinear hyperbolic-parabolic equations, *SIAM J. Math. Anal.* **33** (2001), 751-762.

[4] G.-Q. Chen and B. Perthame, Kinetic Formulation and Well-Posedness for Kinetic Solutions to Degenerate Parabolic-Hyperbolic Equations. To appear in *Ann. Inst. H. Poincaré*.

[5] B. Cokburn, F. Coquel and P.G. LeFloch, Convergence of the finite volume method for multidimensional conservation laws, *SIAM J. Numer. Anal.* **32** (1995), no. 3, 687–705.

[6] B. Cokburn and P.-A. Gremaud, A priori estimates for numerical methods for scalar conservation laws. Part I: The general approach, *Math. Comp.* **65** (1996), 533–573.

[7] R. Eymard, T. Gallouët and R. Herbin, *Finite volume methods*, In Handboook of numerical analysis, **vol. VIII**, P.G. Ciarlet and J-L. Lions eds., North-Holland, Amsterdam (2001).

[8] R. Eymard, T. Gallouët, R. Herbin, and A. Michel, Convergence of a finite volume scheme for nonlinear degenerate parabolic equations, Preprint, 2001.

[9] E. Godlewski and P.-A. Raviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Appl. Math. Series 118. Springer-Verlag (1996).

[10] C. Hillairet, Finite volume schemes for a nonlinear hyperbolic equation. Convergence towards entropy solutions and error estimates, *ESAIM: Math. Mod. and Num. Anal.*, **33** (1) (1999), 129–156.

[11] D. Kröner, *Numerical schemes for conservation laws*, Wiley and Teubner (1997).

[12] S. Kruzhkov, First order quasilinear equations with several space variables, *Mat. Sbornik,* **123** (1970), 228–255; Engl. Transl. *Math. USSR Sb.* **10** (1970), 217–273.

[13] N.N. Kuznetsov, Accuracy of some approximate methods for computing the weak solutions of a first order quasilinear equation, *USSR Comp. Math. and Math. Phys.* **16** (1976) 105–119.

[14] P.L. Lions, B. Perthame and E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related questions. *J. Amer. Math. Soc.*, **7** (1994), 169–191.

[15] M. Ohlberger, A posteriori error estimates for vertex centered finite volume approximations of convection-diffusion-reaction equations, *ESAIM: Math. Mod. and Num. Anal.*, **35** (2) (2001), 355–387.

[16] B. Perthame, Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure, *J. Math. Pure Appl.* **77** (1998), 1055–1064.

[17] B. Perthame, Kinetic Formulations of conservation laws, *Oxford series in mathematics and its applications*, Oxford University Press (2002).

[18] R. Sanders, On convergence of monotone finite difference schemes with variable spatial differencing, *Math. Comp.* **40** (1983) 499–518.

[19] J.-P. Vila, Convergence and error estimates in finite volume schemes for general multidimensional scalar conservation laws, *RAIRO: M2AN Math. Model. Numer. Anal.* **28** (1994), 267–295.