ELLIPTIC RECONSTRUCTION AND A POSTERIORI ERROR ESTIMATES FOR PARABOLIC PROBLEMS

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Abstract. It is known that the energy technique for a posteriori error analysis of finite element discretizations of parabolic problems yields suboptimal rates in the norm $L^\infty(0,T;L^2(\Omega))$. In this paper we combine energy techniques with an appropriate pointwise representation of the error based on an elliptic reconstruction operator which restores the optimal order (and regularity for piecewise polynomials of degree higher than one). This technique may be regarded as the "dual a posteriori" counterpart of Wheeler's elliptic projection method in the a priori error analysis.

1. Introduction

A posteriori error estimation and adaptivity are in many cases very successful tools for efficient numerical computations of linear as well as nonlinear PDEs. In particular, a posteriori error control provides a practical, as well as mathematically sound, means of detecting multiscale phenomena and doing reliable computations. Although the a posteriori error analysis of elliptic problems is now mature [2, 3, 6, 7, 18, 23], the time dependent case is still under development. Many papers have appeared for the discontinuous Galerkin method [9, 10, 11, 13, 14, 15, 20, 19], and other schemes [1, 4, 17, 21, 24, 25], mainly for linear parabolic problems.

One of the outstanding issues related to a posteriori estimation of (linear) time dependent problems is the known fact that the energy technique for a posteriori error analysis of finite element discretizations of parabolic problems yields suboptimal rates in the norm $L^\infty(0,T;L^2(\Omega))$. Since the energy method is the most elementary technique for estimating the error in the a priori analysis, the question whether or not this method can be successfully applied in the a posteriori error analysis is very natural. In addition, we hope that examining this and related issues will enable us to increase our understanding on the important subject of error control for time dependent problems in general.

We will work with the following linear parabolic equation as a model:

$$\begin{aligned} &u_t + Au = f \quad \text{in } \Omega\times[0,T],\\ &u(\cdot,0) = u_0(\cdot) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega\times[0,T]. \end{aligned} \tag{1.1}$$

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Here A is a linear, symmetric, second order positive definite elliptic operator and Ω a bounded domain of $\mathbb{R}^d$ ($d \geq 1$) with sufficiently smooth boundary for our purposes. Let $H := L^2(\Omega)$, $V := H_0^1(\Omega)$ and $V^\star := H^{-1}(\Omega)$ be the dual of V. If $a(\cdot,\cdot)$ is the bilinear form that corresponds to A, our assumptions on A imply that

$$\|v\|_V := a(v,v)^{1/2}$$

defines a norm on V. We denote the norms on H and $V^\star$ by $\|\cdot\|_{V^\star}$ and $\|\cdot\|_H$, respectively, and we indicate with $\langle\cdot,\cdot\rangle$ the duality pairing either in H or $V^* - V$.

We assume that $f \in L^2(0,T;V^\star)$ and $u_0 \in H$, so that (1.1) admits a unique weak solution satisfying

$$\langle u_t(t), v\rangle + a(u(t),v) = \langle f, v\rangle \quad \text{for all } v \in V, \text{ a.e. } t \in [0,T].$$

In this paper we consider semidiscrete finite element discretizations of *arbitrary* degree. We combine energy techniques with an appropriate pointwise representation of the error based on a novel *elliptic reconstruction* operator which restores the optimal order in $L^\infty(0,T;L^2(\Omega))$. This technique may be regarded as the dual counterpart of Wheeler's elliptic projection method in the a priori error analysis [27]. In particular, for u_h being the finite element approximation, our estimates exhibit the following properties:

- the estimator is a computable quantity in terms of the approximate solution u_h and the data f, u_0 and Ω, but its actual form and quality depends only on the elliptic estimator at our disposal;
- the order is optimal in $L^\infty(0,T;L^2(\Omega))$ for any polynomial degree ≥ 1, and the regularity is the lowest compatible with (1.1) for polynomial degree > 1;
- the a posteriori estimates mimic completely the corresponding a priori estimates.

Hereafter, the use of "optimal order" and "optimal regularity" is consistent with classical terminology in approximation theory. Consequently, "optimal order" corresponds to the largest exponent r for which the error is $O(h^r)$, where h is the biggest element diameter of the partition. Likewise, "optimal regularity" refers to the lowest regularity which is compatible with (1.1) and an error of $O(h^r)$.

Finite Element Approximation. For $\mathcal{T}_h$ being a shape-regular partition of Ω consider the finite element space

$$V_h = \{\chi \in H_0^1(\Omega) : \chi|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}_h\},$$

where $\mathbb{P}_k(K)$ is the space of polynomials of degree $\leq k$ over K. The finite element approximation $u_h : [0,T] \to V_h$ of u is defined to satisfy the following linear ODE

$$\begin{aligned} \langle u_{h,t}, \chi\rangle + a(u_h,\chi) &= \langle f, \chi\rangle \quad \text{for all } \chi \in V_h, \text{ a.e. } t \in [0,T], \\ u_h(\cdot,0) &= u_h^0 \in V_h\,. \end{aligned} \tag{1.2}$$

A Posteriori Error Estimation. Residual based a posteriori estimates are usually proved by estimating the linear functional $R \in V^\star$, so-called *residual*,

$$\begin{aligned} -\langle R, v\rangle &= \int_0^T \Big(\langle u_{h,t}, v\rangle + a(u_h, v) - \langle f, v\rangle\Big)dt \\ &= \int_0^T \Big(\langle u_{h,t}, v - I_h v\rangle + a(u_h, v - I_h v) - \langle f, v - I_h v\rangle\Big)dt, \end{aligned} \tag{1.3}$$

in appropriate norms. Here in the second equality we have used the definition of the semidiscrete scheme (1.2), and an interpolation operator $I_h : V \to V_h$ stable in V (e.g., Clement's interpolant). Then, for $e = u - u_h$ being the error to be estimated, we have

$$\frac{1}{2}\|e(T)\|_H^2 + \int_0^T a(e,e)dt = \frac{1}{2}\|e(0)\|_H^2 + \langle R, e\rangle. \tag{1.4}$$

Due to the presence of $\int_0^T a(u_h, e - I_h e)dt$, which gives rise to the integral of an H^1 elliptic residual, the ensuing a posteriori estimate is of optimal order in $L^2(0,T;H_0^1(\Omega))$, as corresponds to an estimate of $\int_0^T a(e,e)dt$, but *suboptimal* in $L^\infty(0,T;L^2(\Omega))$. It is well known that an analogous phenomenon occurs in the a priori analysis, and that the use of an elliptic projection operator overcomes the difficulty [27]. This is now a standard tool in the finite element analysis.

In this paper we introduce an *elliptic reconstruction* operator which restores the optimal order in the a posteriori error estimation in $L^\infty(0,T;L^2(\Omega))$. The key properties of the elliptic reconstruction U, cf. Definition 2.1, are (i) $u - U$ satisfies an appropriate pointwise equation, cf. (3.2), that can be used to derive estimates in terms of $u_{h,t} - U_t$ and (ii) u_h is the finite element solution of an elliptic problem whose exact solution is U, and therefore $u_h - U$ (as well as $u_{h,t} - U_t$) can be estimated in various norms by any given a posteriori elliptic estimator. Note that a similar function U was introduced in [12] for a different purpose.

For clarity of exposition we present the method in the simplest framework. The ideas of the present paper might be useful for linear problems of non-dissipative character, as well as for nonlinear dissipative problems. In this direction they should be explored together with the recent a posteriori results of time discretization of nonlinear problems [17, 19]. The a posteriori analysis of [17, 19] is based on the same principles as in the present paper, namely an appropriate pointwise representation of the error and energy arguments.

Although it is possible to derive quasi-optimal order-regularity estimators in $L^\infty(0,T;L^2(\Omega))$ via *parabolic duality* [9, 22], this technique hinges on the parabolic regularizing effect which is not valid for estimates in $L^2(0,T;H_0^1(\Omega))$. For the latter, duality leads invariably to estimators similar to those obtained with the energy approach, and which also bound the error in $L^\infty(0,T;L^2(\Omega))$ but with suboptimal order. In contrast, several contributions over the last few years are devoted to estimates that are based on the (forward) energy approach. Picasso [21] derives a posteriori error estimates of residual type that are optimal in $L^2(0,T;H_0^1(\Omega))$ for piecewise linear elements for space discretization and backward Euler for time discretization. Towards overcoming the barrier described above, Babuška, Feistauer and Šolín [4] derive estimates in $L^2(0,T;L^2(\Omega))$ for (1.2) by a double integration in time; see also [1, 5]. In [24, 25] Verfürth proves a posteriori estimates in $L^r(0,T;L^\rho(\Omega))$, with $1 < r, \rho < \infty$, for fully discrete approximations of quasilinear parabolic equations.

The paper is organized as follows. We introduce the elliptic reconstruction operator in section 2, and we derive abstract a posteriori error estimates in section 3. In particular our estimator of Theorem 3.1 depends on an abstract *elliptic estimator function* for elliptic problems; any such estimator can be used. In section 4 we specify the form of the estimates for the classical residual type elliptic estimators.

2. Elliptic Reconstruction

We now introduce the elliptic reconstruction operator $\mathcal{R} : V_h \to V$. To this end, let $P_h^1 : V \to V_h$ be the elliptic projection operator, i.e.,

$$a(P_h^1 w, \chi) = a(w, \chi) \quad \text{for all } \chi \in V_h, \tag{2.1}$$

and let $P_h^0 : H \to V_h$ be the L^2-projection operator, i.e.,

$$(P_h^0 w, \chi) = \langle w, \chi \rangle \quad \text{for all } \chi \in V_h. \tag{2.2}$$

Let $w \in V$ satisfy the elliptic problem $Aw = g \in V^*$, or in weak form,

$$w \in V : \qquad a(w, v) = \langle g, v \rangle \quad \text{for all } v \in V. \tag{2.3}$$

Let $w_h \in V_h$ be the corresponding finite element solution

$$w_h \in V_h : \qquad a(w_h, \chi) = \langle g, \chi \rangle \quad \text{for all } \chi \in V_h; \tag{2.4}$$

hence $w_h = P_h^1 w$. We assume that we have at our disposal a posteriori estimators that control the error $\|w - w_h\|_X$ in the spaces $X = H, V$, or $V^\star$.

Assumption 2.1. *Let w and w_h be the exact solution and its finite element approximation given in (2.3) and (2.4) above. We assume that there exists an a posteriori estimator function $\mathcal{E} = \mathcal{E}(w_h, g; X)$, which depends on w_h, g and the space $X = H, V$, or $V^\star$ such that*

$$\|w - w_h\|_X \leq \mathcal{E}(w_h, g; X). \tag{2.5}$$

Let $A_h : V_h \to V_h$ be the following discrete version of A:

$$\langle A_h v, \chi \rangle = a(v, \chi) \quad \text{for all } \chi \in V_h. \tag{2.6}$$

Then we have:

Definition 2.1. *Let u_h be the finite element solution of (1.2) and $f_h := P_h^0 f$. We define the elliptic reconstruction $U = \mathcal{R} u_h \in H_0^1(\Omega)$ of u_h to be the solution of the elliptic problem in weak form*

$$a(U(t), v) = \langle g_h(t), v \rangle \quad \textit{for all } v \in H_0^1(\Omega), \textit{ a.e. } t \in [0, T], \tag{2.7}$$

where

$$g_h := A_h u_h - f_h + f. \tag{2.8}$$

We note that a similar function U was defined at the final time T in [12] in a different context, i.e., in post-processing the Galerkin method at T with the aim of improving the order of convergence. We observe that U satisfies the strong form

$$AU = A_h u_h - f_h + f, \tag{2.9}$$

as well as

$$a(U, \varphi) = a(u_h, \varphi) - \langle f_h - f, \varphi \rangle = a(u_h, \varphi) \quad \text{for all } \varphi \in V_h, \tag{2.10}$$

because $f_h = P_h^0 f$. This relation implies that *u_h is the finite element solution of the elliptic problem whose exact solution is the elliptic reconstruction U*, namely,

$$u_h = P_h^1 U. \tag{2.11}$$

Assume that $f \in H^1(0, T; V^*)$. Since $a(\cdot, \cdot)$ is independent of t there holds $a(U_t, \varphi) = a(u_{h,t}, \varphi)$ for all $\varphi \in V_h$, or

$$u_{h,t} = P_h^1 U_t. \tag{2.12}$$

In addition

$$a(U_t, v) = \langle g_{h,t}, v\rangle \quad \text{for all } v \in V\,. \tag{2.13}$$

3. Abstract A Posteriori Error Analysis

In this section we establish the improved a posteriori error estimate in H, and make several comments about its optimality regarding both order and regularity.

Theorem 3.1. *Assume that u is the solution of (1.1) and u_h is its finite element approximation (1.2). Let U be the elliptic reconstruction of u_h and $\mathcal{E}$ be as defined in Assumption 2.1. Then the following a posteriori error bounds hold for* $0 < t \le T$

$$\|(u-U)(t)\|_H^2 + \int_0^t \|u-U\|_V^2 ds \le \|u(0)-U(0)\|_H^2 + \int_0^t \mathcal{E}(u_{h,t}, g_{h,t}; V^\star)^2 ds\,,$$

and

$$\begin{aligned}\|(u-u_h)(t)\|_H \le \|u_0 - u_h^0\|_H &+ \Big(\int_0^t \mathcal{E}(u_{h,t}, g_{h,t}; V^\star)^2 ds\Big)^{1/2} \\ &+ \mathcal{E}(u_h(0), g_h(0); H) + \mathcal{E}(u_h(t), g_h(t); H).\end{aligned}$$

Proof. By virtue of definitions (1.2) and (2.9) of u_h and U, we have

$$u_{h,t} + AU = f\,,$$

whence U satisfies the following pointwise equation

$$U_t + AU = f + (U - u_h)_t\,. \tag{3.1}$$

Thus the error equation for $u - U$ reads

$$(u-U)_t + A(u-U) = (u_h - U)_t\,. \tag{3.2}$$

Multiplying by $u - U$, and using standard energy arguments, yields

$$\begin{aligned}\|(u-U)(t)\|_H^2 + \int_0^t \|(u-U)(s)\|_V^2 ds \le{}& \|u(0)-U(0)\|_H^2 \\ &+ \int_0^t \|(u_{h,t} - U_t)(s)\|_{V^\star}^2 ds\,.\end{aligned} \tag{3.3}$$

Relations (2.12) and (2.13), in conjunction with Assumption 2.1, imply

$$\|u_{ht} - U_t\|_{V^\star} \le \mathcal{E}(u_{h,t}, g_{h,t}; V^\star)\,,$$

which in turn leads to the first assertion of Theorem 3.1. To show the second one it suffices to note that (2.11) and Assumption 2.1 yield

$$\|(u_h - U)(t)\|_H \le \mathcal{E}(u_h(t), g_h(t); H) \quad \text{for all } 0 \le t \le T\,, \tag{3.4}$$

which, together with

$$\begin{aligned}\|u(0)-U(0)\|_H &\le \|u(0) - u_h(0)\|_H + \|P_h^1 U(0) - U(0)\|_H \\ &\le \|u_0 - u_h^0\|_H + \mathcal{E}(u_h(0), g_h(0); H),\end{aligned}$$

concludes the proof. ☐

Remark 3.1. (L^2-based estimate). An alternative estimate that follows from the proof of Theorem 3.1 is

$$\max_{0\le t\le T}\|u-U\|_H^2 \le \|u(0)-U(0)\|_H^2 + \max_{0\le t\le T}\|u-U\|_H\int_0^T \|u_{h,t}-U_t\|_H dt$$
$$\le \max_{0\le t\le T}\|u-U\|_H\Big(\|u(0)-U(0)\|_H + \int_0^T \|u_{h,t}-U_t\|_H dt\Big).$$

Therefore, (2.5) and (3.4) imply

$$\max_{0\le t\le T}\|u-U\|_H \le \|u(0)-U(0)\|_H + \int_0^T \mathcal{E}(u_{h,t},g_{h,t};H)dt\,,$$

along with the corresponding a posteriori error bound

$$\max_{0\le t\le T}\|u-u_h\|_H \le \|u_0-u_h^0\|_H + \mathcal{E}(u_h(0),g_h(0);H)\ + 2\int_0^T \mathcal{E}(u_{h,t},g_{h,t};H)dt.$$

Remark 3.2. (A priori vs a posteriori bounds). Note that the elliptic reconstruction is an "a posteriori dual" to Wheeler's elliptic projection [22, 27]. Furthermore the two results in Theorem 3.1 are indeed an *a posteriori dual* to the classical a priori estimate for semidiscrete linear parabolic problems [22, 27]

$$\begin{aligned}\|(u_h-P_h^1u)(t)\|_H^2 &+ \int_0^t \|u_h-P_h^1u\|_V^2 ds\\ &\le \|u_h(0)-P_h^1u(0)\|_H^2 + \int_0^t \|u_t-P_h^1u_t\|_{V^\star}^2 ds\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}\|(u-u_h)(t)\|_H \le\ & \|(u-P_h^1u)(t)\|_H\\ &+\Big(\|u_h(0)-P_h^1u(0)\|_H^2 + \int_0^t \|u_t-P_h^1u_t\|_{V^\star}^2 dt\Big)^{1/2}.\end{aligned}\tag{3.6}$$

Remark 3.3. (Optimal regularity). The a priori bound in (3.5) (and therefore in (3.6)) is of optimal order. The regularity required is optimal only for polynomial degree $k\ge 2$. Indeed by exploiting standard results on superconvergence in negative norms of elliptic finite element problems we see that the following bound for the error of the elliptic projection holds [22, 26]:

$$\|v-P_h^1v\|_{V^\star} \le Ch^{(k+1)}\|v\|_k.\tag{3.7}$$

This estimate follows from the definition of the dual norm $\|w\|_{V^\star}=\sup_{\|z\|_V=1}\langle w,z\rangle$ and a standard duality argument. Using (3.7) we obtain

$$\int_0^T \|u_t-P_h^1u_t\|_{V^\star}^2 dt \le C\int_0^T h^{2(k+1)}\|u_t\|_k^2 dt \le Ch^{2(k+1)}\int_0^T \|u\|_{k+2}^2 dt\,;$$

here $\|\cdot\|_s$ denotes the Sobolev norm of $H^s(\Omega)$, and for simplicity take $A=-\Delta$ and $f=0$. For an (optimal) convergence rate of order $O(h^{k+1})$ in $L^\infty(0,T;L^2(\Omega))$, the minimal regularity required by our finite element space is $u\in L^\infty(0,T;H^{k+1}(\Omega))$. But it is a simple matter to check that for (1.1) both

$$\int_0^T \|u\|_{k+2}^2 dt \quad\text{and}\quad \max_{0\le t\le T}\|u\|_{k+1}^2$$

are bounded by the same constant depending on data. Thus the classical a priori estimate (3.6) is of optimal order and regularity for $k \geq 2$. The negative norm $\|\cdot\|_{V^*}$ appears in a complete similar fashion in the a posteriori error analysis of Theorem 3.1, and thus for polynomial degree $k \geq 2$ this indicates that the estimator is of *optimal order-regularity.*

4. Application: Residual-Type Error Estimators

In this section we derive the specific form of the estimates of Section 3 in case we choose the classical residual type estimators for (2.5) [6, 23]. Of course any other choice, such as solving local problems [2, 7, 18, 23] or averaging techniques [3], is possible according to Theorem 3.1. For simplicity we assume that $A = -\Delta$ and that Ω is sufficiently smooth in order for (4.2) below to be valid. However, Theorem 3.1 is general enough to allow for geometric singularities and corresponding elliptic estimators. We refer to [16] for *weighted* a posteriori estimators which account for corner singularities in both H and $V^\star$ in an optimal fashion, as well as to [8] where an error estimator is derived for an elliptic problem with curved boundaries.

We first calculate $\mathcal{E}(u_{h,t}, g_{h,t}; V^\star)$, or equivalently estimate

$$\|\rho\|_{V^\star} = \sup_{\|\phi\|_V \leq 1} \langle \rho, \phi \rangle, \quad \rho = (U - u_h)_t.$$

We accomplish this via standard duality arguments. Given $\phi \in V$, let $\psi \in V$ be defined by

$$a(\psi, v) = \langle \nabla\psi, \nabla v \rangle = \langle v, \phi \rangle \quad \forall v \in V, \tag{4.1}$$

and suppose there exists a constant $C_\Omega > 0$, depending on the domain Ω, such that

$$\|\psi\|_{H^3(\Omega)} \leq C_\Omega \|\phi\|_{H^1(\Omega)}. \tag{4.2}$$

If $\mathcal{T}_h = \{K\}$ is a shape-regular partition of Ω into finite elements K, then $\mathcal{S}_h = \{S\}$ denotes the set of internal interelement sides and $\mathcal{N}_h(E)$ stands for the union of all elements of $\mathcal{T}_h$ intersecting the *closed* set E ($= K$ or S). Then, assuming for the time being that the polynomial degree is $k \geq 2$ and recalling (2.12), we can write

$$\begin{aligned}
\langle \rho, \phi \rangle &= a(\psi, \rho) = a(\psi - I_h\psi, \rho) \\
&\leq \sum_{K \in \mathcal{T}_h} |(\psi - I_h\psi, \Delta\rho)_K| + \sum_{S \in \mathcal{S}_h} \int_S |\psi - I_h\psi| |[\partial_n \rho]| \, ds \\
&\leq C_I \sum_{K \in \mathcal{T}_h} h_K^3 |\psi|_{3,\mathcal{N}_h(K)} \|\Delta\rho\|_{L^2(K)} \\
&\quad + C_I \sum_{S \in \mathcal{S}_h} h_S^{5/2} |\psi|_{3,\mathcal{N}_h(S)} \|[\partial_n \rho]\|_{L^2(S)},
\end{aligned} \tag{4.3}$$

where $C_I > 0$ is an interpolation constant associated with the local interpolation operator I_h. If we further set

$$\eta_{-1}(u_{h,t})^2 = \sum_{K \in \mathcal{T}_h} h_K^6 \|\Delta\rho\|^2_{L^2(K)} + \sum_{S \in \mathcal{S}_h} h_S^5 \|[\partial_n u_{h,t}]\|^2_{L^2(S)},$$

and make use of (4.2), then we end up with the a posteriori error estimate

$$\mathcal{E}(u_{h,t}, g_{h,t}; V^\star) = \|\rho\|_{V^\star} \leq C_I C_\Omega \eta_{-1}(u_{h,t}),$$

where C_I now contains an additional factor to account for the h-independent overlap of sets $\mathcal{N}_h(E)$ in (4.3).

The form of $\eta_{-1}(u_{h,t})$ can be further simplified upon using the definition of the elliptic reconstruction and the semidiscrete scheme:

$$\Delta\rho = \Delta U_t - \Delta u_{h,t} = -A_h u_{h,t} + f_{h,t} - f_t - \Delta u_{h,t}\,.$$

Since $u_{h,tt} + A_h u_{h,t} = f_{h,t}$, we have

$$\Delta\rho = -f_{h,t} + u_{h,tt} + f_{h,t} - f_t - \Delta u_{h,t} = (u_{h,t} - \Delta u_h - f)_t\,.$$

If we denote the element residuals as

$$r|_K := u_{h,t} - \Delta u_h - f \quad \forall K \in \mathcal{T}_h, \qquad j|_S := [\partial_n u_h] \quad \forall S \in \mathcal{S}_h,$$

we finally get

$$\eta_{-1}(u_{h,t})^2 = \sum_{K\in\mathcal{T}_h} h_K^6 \|r_t\|^2_{L^2(K)} + \sum_{S\in\mathcal{S}_h} h_S^5 \|j_t\|^2_{L^2(S)}, \tag{4.4}$$

and

$$\mathcal{E}(u_{h,t}, g_{h,t}; V^\star) \le C_I C_\Omega \eta_{-1}(u_{h,t}) \quad \text{if } k \ge 2.$$

Using similar arguments we can derive

$$\mathcal{E}(u_h, g_h; H) \le C_I C_\Omega \eta_0(u_h) \quad \text{if } k \ge 2,$$

where

$$\eta_0(u_h)^2 = \sum_{K\in\mathcal{T}_h} h_K^4 \|r\|^2_{L^2(K)} + \sum_{S\in\mathcal{S}_h} h_S^3 \|j\|^2_{L^2(S)}. \tag{4.5}$$

Note that the constants C_I, C_Ω may have different values now. Finally in the case $k = 1$ the use of negative norm does not give better results because of the lack of superconvergence in $V^\star$. Hence

$$\mathcal{E}(u_{h,t}, g_{h,t}; V^\star) \le \mathcal{E}(u_{h,t}, g_{h,t}; H) \le C_I C_\Omega \eta_0(u_{h,t})\,. \tag{4.6}$$

In summary, in view of Theorem 3.1, we have derived the following *explicit* error estimate.

Theorem 4.1. (A posteriori estimators of residual type). *Assume that the domain Ω is sufficiently smooth and let $t \in (0,T]$. If $k = 1$, then the following a posteriori estimate holds*

$$\begin{aligned}\|(u-u_h)(t)\|_H &\le \|u^0 - u_h^0\|_H \\ &+ C_I C_\Omega \Big\{ \eta_0(u_h(0)) + \eta_0(u_h(t)) + \Big(\int_0^t \eta_0(u_{h,t}(s))^2 ds \Big)^{1/2} \Big\}.\end{aligned}$$

In addition, for $k \ge 2$ we have

$$\begin{aligned}\|(u-u_h)(t)\|_H &\le \|u^0 - u_h^0\|_H \\ &+ C_I C_\Omega \Big\{ \eta_0(u_h(0)) + \eta_0(u_h(t)) + \Big(\int_0^t \eta_{-1}(u_{h,t}(s))^2 ds \Big)^{1/2} \Big\}.\end{aligned}$$

where the estimators η_0 and η_{-1} are given by (4.5) and (4.4) respectively.

Remark 4.1. The reasoning of Remark 3.3 applies and indicates that the estimator in Theorem 4.1 is of optimal order for polynomial degree $k \ge 1$, and of optimal regularity for $k \ge 2$. We do not actually show an a priori convergence rate for the a posteriori estimators $\eta_0(u_h)$ and $\eta_1(u_h)$, which is of interest but lies outside the scope of this paper.

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