FINITE VOLUME RELAXATION SCHEMES FOR MULTIDIMENSIONAL CONSERVATION LAWS

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Abstract. We consider semidiscrete and fully discrete finite volume relaxation schemes for multidimensional scalar conservation laws. These schemes are constructed by appropriate discretization of a relaxation system and it is shown to converge to the entropy solution of the conservation law with a rate of $h^{1/4}$ in $L^\infty([0,T],L^1_{loc}(\mathbb{R}^d))$.

1. Introduction

In this paper we consider a class of finite volume schemes approximating the scalar multidimensional conservation law, whose construction is motivated by discretizing the relaxation system

$$\partial_t w^\varepsilon + \mathrm{div} A w^\varepsilon = \frac{1}{\varepsilon}\sum_{i=1}^{d} G_i(w^\varepsilon, z_i^\varepsilon),\ x\in\mathbb{R}^d, \tag{1.1}$$

$$\partial_t z_i^\varepsilon + \mathrm{div} B_i z_i^\varepsilon = \frac{1}{\varepsilon} G_i(w^\varepsilon, z_i^\varepsilon),\ i=1,\dots,d,\ x\in\mathbb{R}^d, \tag{1.2}$$

in variables (w,Z) with $Z=(z_1,\dots,z_d)$. The constant vectors $A, B_i, i=1,\dots,d$ and the smooth functions $G_i:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfy certain structural assumptions, cf. Section 2. The system (1.1-2) is considered with initial data $w^\varepsilon(x,0)=w_0^\varepsilon(x),\ Z^\varepsilon(x,0)=Z_0^\varepsilon(x),\ x\in\mathbb{R}^d$. Contractive relaxation systems of the form (1.1-2) were introduced and analyzed in Katsoulakis and Tzavaras [KT1], and it was shown under certain assumptions that as $\varepsilon\to 0$ their solution is associated to the unique entropy solution of the conservation law,

$$\partial_t u + \mathrm{div} F(u) = 0,\quad x\in\mathbb{R}^d, t>0,\quad u(x,0)=u_0(x)\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\,. \tag{1.3}$$

Here, for a given conservation law (1.3), we appropriately select $A, B_i, i=1,\dots,d$ and the functions G_i, and we discretize (1.1-2) by semidisctere and fully discrete finite volume schemes. The approximations emanating from these schemes are shown to converge to the entropy solution of (1.3) with a rate of $h^{1/4}$ in $L^\infty([0,T],L^1_{loc}(\mathbb{R}^d))$.

2. Preliminaries – Relaxation schemes

We assume that for a given conservation law (1.3) we select the vectors $A, B_i, i=1,\dots,d$, and the functions G_i such that,

$$\begin{aligned} &G_i(\cdot, z_i) \text{ is strictly decreasing in } w \text{ for fixed } z_i,\\ &G_i(w,\cdot) \text{ is strictly decreasing in } z_i \text{ for fixed } w, \end{aligned} \tag{A.1}$$

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and that there exist functions $h_i : \mathbb{R} \to \mathbb{R}$,

$$\text{(A.2)} \qquad \begin{aligned} &h_i \text{ strictly decreasing, } h_i(0) = 0, \quad \lim_{w\to\pm\infty} h_i(w) = \mp\infty, \\ &G_i(w, h_i(w)) = 0, \ G_i(0,0) = 0, \quad w \in \mathbb{R}. \end{aligned}$$

Given $\mathcal{R}^{a,b} = [a,b] \times \prod_{i=1}^{d} [h_i(b), h_i(a)]$, there exists a $\sigma = \sigma(a,b) > 0$ such that

$$\text{(A.3)} \qquad |G_i(w, z_i)| \geq \sigma |h_i(w) - z_i| \quad \text{for } (w, Z) \in \mathcal{R}^{a,b},$$

and finally, if F is the flux of the conservation law (1.3), h_i should satisfy

$$\text{(A.4)} \qquad \begin{aligned} &F(\eta) = A(v) - \sum_{i=1}^{d} B_i(h_i(v)), \\ &\text{if } \eta = v - \sum_{i=1}^{d} h_i(v), \quad v \in \mathbb{R}. \end{aligned}$$

Note that as a consequence of (A.1-3) there hold

$$\text{(A.3}') \qquad \begin{aligned} &(h_i(w) - z_i) G_i(w, z_i) > 0, \\ &|G_i(w, z_i)| \leq \sigma' |h_i(w) - z_i| \quad \text{for } (w, Z) \in \mathcal{R}^{a,b}, \ \text{where } \sigma' = \sigma'(a,b) > 0. \end{aligned}$$

Lemma 4.1 of [KT1] shows that it is indeed possible to construct such functions, e.g. when $A = (\omega_1 V_1, \ldots, \omega_d V_d)$, $V_i > 0, \omega_i > 0$ $B_i = (0, \ldots, -V_i, \ldots, 0)$, and $G_i(w, z_i) = h_i(w) - z_i$, and V_i, ω_i are chosen to satisfy certain sub-characteristic conditions, cf., [KT1], [CLL] and [JX]. In this case and for $d = 1$ the relaxation system (1.1) is equivalent to the one proposed by Jin and Xin [JX] and analyzed by Natalini [N1]. The convergence properties of (1.1) for $d \geq 1$ were investigated in [KT1]. In [N2] an alternative relaxation system was proposed and analyzed.

Assumptions (A.2) and (A.4) provide a (formal) reasoning on the relationship of (1.1-2) and (1.3). Indeed (1.1-2) imply that

$$\text{(2.1)} \qquad \partial_t (w^\varepsilon - \sum_{i=1}^{d} z_i^\varepsilon) + \operatorname{div}\Big(A w^\varepsilon - \sum_{i=1}^{d} B_i z_i^\varepsilon\Big) = 0\,.$$

As $\varepsilon \to 0$ we expect that the local equilibrium, $z_i = h_i(w)$, $i = 1, \ldots, d$, will be enforced and therefore, in view of (A.4), the limiting dynamics of the relaxation system will be described by the weak solutions of (1.3), cf. [KT1]. For small ε, $w^\varepsilon - \sum_{i=1}^{d} z_i^\varepsilon$ will provide an approximation to the solution u of (1.3). Based on this observation one can construct approximating schemes to (1.3) by discretizing the relaxation system. The corresponding schemes are then called relaxation schemes.

Finite difference relaxation schemes were presented in a systematic way by Jin and Xin [JX]. Finite difference relaxation schemes based on the system (1.1) were proposed and analyzed in [KKM]. It was shown that these schemes converge to the entropy solution of the multidimensional conservation law with a rate of $h^{1/2}$ in $L^\infty([0,T], L^1(\mathbb{R}^d))$. Error estimates of difference schemes to relaxation models arising in chromatography were proved in [ScTW], [ShTW]. The convergence of finite volume schemes approximating the entropy solution of (1.3) was analyzed, e.g., in [CCL1,2], [KR], [V]. In a recent paper Rohde [R], using an appropriate extension of DiPerna's theory, has proved convergence of finite volume schemes to weakly coupled hyperbolic systems.

Space discretization. Let $\mathcal{T}_h$ be a decomposition of $\mathbb{R}^d$ into non-overlapping, nonempty and open polyhedra such that $\bigcup_{K\in\mathcal{T}_h}\bar{K}=\mathbb{R}^d$. The set of faces of K is denoted by ∂K and, on each face e on K, $\nu_{e,K}\in\mathbb{R}^d$ represents the outward unit normal to the face e. Γ_h will denote the set of all faces of the decomposition $\mathcal{T}_h$. Given a face e of K, K_e denotes the unique polyhedron that shares the face e with K. The volume of K is denoted by $|K|$ and the $(d-1)$-measure of e by $|e|$. Let h_K be the diameter of the polyhedron K and let $h=\sup_{K\in\mathcal{T}_h}h_K<1$. We shall assume that our decomposition is regular, i.e., if ρ_K is the diameter of the largest ball B, $B\subset K$,

$$h_K\le\gamma\rho_K,\quad K\in\mathcal{T}_h,$$

with a constant γ independent of h. In particular this implies that if e is a face of K, then $|e|$ and h_K are comparable. We define the finite volume scheme approximating (1.1), (1.2) as follows: We seek a piecewise constant function (w_h,Z_h), $w_h|_K=w_K$, $Z_h=(z_{1,h},\dots,z_{d,h})$, $z_{i,h}|_K=z_{i,K}$, such that

$$\begin{aligned}\partial_t w_K+\sum_{e\in\partial K}\frac{|e|}{|K|}g^K(w_K,w_{K_e})&=\frac{1}{\varepsilon}\sum_{i=1}^d G_i(w_K,z_{i,K}),\\ \partial_t z_{i,K}+\sum_{e\in\partial K}\frac{|e|}{|K|}g_i^K(z_{i,K},z_{i,K_e})&=\frac{1}{\varepsilon}G_i(w_K,z_{i,K}),\ i=1,\dots,d,\ K\in\mathcal{T}_h,\end{aligned}\tag{2.2}$$

where $g,g_i:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $i=1,\dots,d$ are discrete monotone fluxes. For initial approximations we take $w_K(0)=\frac{1}{|K|}\int_K w^0dx$, and $z_{i,K}(0)=\frac{1}{|K|}\int_K z_i^0dx$. Although g^K,g_i^K correspond to linear fluxes, it is convenient in the analysis to list their properties as in the general (nonlinear) case. We explicitly use, when it is needed, the linearity, cf. (2.6).

The discrete fluxes are assumed to satisfy:

$$g^K(u,v)=-g^{K_e}(v,u)\ ,\ g_i^K(u,v)=-g_i^{K_e}(v,u)\qquad\text{Conservation Property,}\tag{2.3}$$

$$g^K(u,u)=A(u)\cdot\nu_{e,K}\ ,\ g_i^K(u,u)=B_i(u)\cdot\nu_{e,K}\qquad\text{Consistency Property,}\tag{2.4}$$

$$\frac{\partial g^K}{\partial u},\frac{\partial g_i^K}{\partial u}\ge 0\ ,\ \frac{\partial g^K}{\partial v},\frac{\partial g_i^K}{\partial v}\le 0\qquad\text{Monotonicity Property,}\tag{2.5}$$

$$g^K(u,v),\ g_i^K(u,v)\quad\text{are linear functions of } u,v.\tag{2.6}$$

Time discretization. Let δ be the time step and $t^n=n\delta$. Then we shall consider semi-explicit fully discrete schemes: We seek a piecewise constant function $(w_{h,\delta},Z_{h,\delta})$, $w_{h,\delta}|_{K\times[t^n,t^{n+1})}=w_K^n$, $Z_{h,\delta}=(z_{1,h,\delta},\dots,z_{d,h,\delta})$, $z_{i,h,\delta}|_{K\times[t^n,t^{n+1})}=z_{i,K}^n$, such that,

$$\begin{aligned}w_K^{n+1}&=w_K^n-\frac{\delta}{|K|}\sum_{e\in\partial K}|e|g^K(w_K^n,w_{K_e}^n)+\frac{\delta}{\varepsilon}\sum_{i=1}^d G_i(w_K^{n+1},z_{i,K}^{n+1}),\\ z_{i,K}^{n+1}&=z_{i,K}^n-\frac{\delta}{|K|}\sum_{e\in\partial K}|e|g_i^K(z_{i,K}^n,z_{i,K_e}^n)+\frac{\delta}{\varepsilon}G_i(w_K^{n+1},z_{i,K}^{n+1}),\quad i=1,\dots,d,\end{aligned}\tag{2.7}$$

with initial approximations $w_K^0=\frac{1}{|K|}\int_K w_0(x)\,dx$, and $z_{i,K}^0=\frac{1}{|K|}\int_K z_{i0}(x)\,dx, i=1,\dots,d.$

The stability and convergence properties of these schemes are investigated in the next sections. For the semidiscrete case we prove that under standard assumptions on the initial data, for any $R>0$, $T>0$, there is a constant $C=C(R,T)$ such that

$$\|u(\cdot,t)-U_h(\cdot,t)\|_{L^1(B(0,R))}\le Ch^{1/4}\,,\quad t\le T,$$

where $U_h = w_h - \sum_{i=1}^{d} h_i(w_h)$, cf. Theorem 4.1. Here $B(0,R)$ is the ball with center 0 and radius R. In the case of fully discrete approximations a similar estimate holds true, provided that appropriate CFL conditions are valid, cf. Theorem 6.1; cf. also Remark 5.1.

A main advantage of relaxation schemes, is the simplicity of their construction coming from the fact that the principal part of (1.1-2) is linear, and therefore there is no need to solve local Riemann problems. Thus high order and adaptive schemes can be easily formulated. Issues related to the numerical implementation and the performance of finite volume relaxation schemes are addressed in [KZ].

Error estimates of order $O(h^{1/4})$ for finite volume approximations to (1.3) were previously obtained in [CCL1], [V], and for finite elements in [CG1]. For finite difference approximations the order of convergence $O(h^{1/2})$ was established, e.g. in [Kz], [S]. The main reason for the reduced order of convergence in the finite volume case is the lack of BV bounds for the approximate schemes in unstructured meshes. To compensate this, an estimate for the discrete gradients in L^2 was proved in [CCL1], [V], which led to the $O(h^{1/4})$ estimate. In the case of relaxation schemes considered here we are able to prove an analogous bound, cf. Lemma 3.3, Lemma 5.3. In addition for the relaxation schemes, again due to the lack of BV bounds, an estimate for the distance from the equilibrium in L^2 turns out to be crucial, cf. Lemma 3.4, Lemma 5.4. (Note that the corresponding result is new for the continuous relaxation model).

Our analysis is based on an approximation lemma for deriving error estimates for numerical approximations to conservation law (1.3), cf. Lemma 4.1. This is a result obtained in [KKM] and extends a result of Bouchut and Perthame [BP] to the case of numerical schemes. The use of this Lemma in the (complicated) case of finite volume approximations considered in this paper, avoids much of the technical work needed if one applies the original approach of doubling the variables, [Kr], [Kz], as in [CCL1], [V]. Indeed, the analysis in [CCL1], [V] is considerable simplified if one uses Lemma 4.1 along the lines of the analysis presented in Section 6. This is of some importance because the difficulties of applying Kruzkov's estimates to numerical schemes are highlighted. As noted first in [CG2], the classical approach of Kuznetsov is an "a posteriori" approach. This can be seen directly in the framework considered in this paper, simply by observing that the $E-$terms in the bound (4.5) depend only on the approximate solution u_h. Indeed, by applying a more refined analysis, explicit a posteriori error bounds suitable for adaptive mesh refinement based on Lemma 4.1 are proposed in [GM] for finite difference and finite volume approximations to (1.3), cf. also [CGa].

An alternative "a priori" approach for deriving error estimates, which does not rely on the regularity properties of the schemes, was proposed and extensively analyzed in [CG2,3] for finite difference and in [CGY] for finite volume schemes. To carry out the program proposed in [CG1] one has to show an appropriate "discrete" stability for the scheme considered. A task considerable more complicated than the "continuous" stability used in the proof of the Lemma 4.1. Cockburn, Gremaud and Yang in [CGY] were able to prove $h^{1/2}$ estimates by using this approach for a special class of monotone finite volume schemes in symmetric (or nearly symmetric) non Cartesian meshes, cf. [CGY, Sections 2.a, b] for explicit assumptions. The development of ideas in [CG2,3], [CGY] and their application to relaxation schemes in unstructured meshes will be the subject of forthcoming work.

The paper is organized as follows. In Sections 3 and 5 we prove the necessary stability properties for the semidiscrete and fully discrete schemes respectively. We then use these properties in Sections 4 and 6 to prove convergence. In particular the relaxation schemes satisfy a basic comparison principle (Lemma 3.1 and 5.1) which then implies the L^1 contraction property (Lemma 3.2 and 5.2), the fact that $\mathcal{R}^{a,b}$ is a positively invariant region for the schemes and as consequence that the approximations are uniformly bounded in L^∞ (Lemma 3.2 and 5.2), and the discrete entropy inequalities ((3.8) and Lemma 5.2). Using the invariance of $\mathcal{R}^{a,b}$ we are then able to show the weak dissipation estimates (Lemma 3.3 and 5.3) and the estimate for the distance from equilibrium (Lemma 3.4 and 5.4) mentioned above. In the convergence proof of Section 4 we first use the discrete entropy to prove the basic error inequality (4.15) which allows us to apply then Lemma 4.1. To estimate then the E-terms of (4.5) we use Lemmata 3.3 and 3.4. The proof in Section 6 follows similar plan.

Remark 2.1. One can prove similar convergence results for the *fully-explicit* scheme,

$$w_K^{n+1} = w_K^n - \frac{\delta}{|K|}\sum_{e\in\partial K}|e|g^K(w_K^n, w_{K_e}^n) + \frac{\delta}{\varepsilon}\sum_{i=1}^d G_i(w_K^n, z_{i,K}^n),$$

$$z_{i,K}^{n+1} = z_{i,K}^n - \frac{\delta}{|K|}\sum_{e\in\partial K}|e|g_i^K(z_{i,K}^n, z_{i,K_e}^n) + \frac{\delta}{\varepsilon}G_i(w_K^n, z_{i,K}^n), \quad i = 1,\dots,d.$$

In this case however we have to assume a CFL condition of the type $\delta \le C\varepsilon$. Compare with [LV].

3. Stability estimates

We first prove a *Comparison Principle* which implies several useful properties of the scheme. We start by introducing some notation. For $a, b \in \mathbb{R}$ we set $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$. Further, for a given function f we denote by f^+, f^- the positive and negative parts of f, respectively and $\chi_{f>0}$ stands for the characteristic function of the set $\{(x,t) : f(x,t) > 0\}$, that is $\chi_{f>0} = 1$ if $f > 0$ and zero if $f \le 0$.

Lemma 3.1. *Assume that $G_i(\cdot,\cdot), i = 1,\dots,d$, satisfy assumptions $(A.1-3)$. Let (w_h, Z_h) and $(\overline{w}_h, \overline{Z}_h)$ be two solutions of (2.2) that vanish outside a ball B_M of radius M. Then we have*

(3.1)

$$\partial_t\Big\{(w_K - \bar w_K)^+ + \sum_{i=1}^d (z_{i,K} - \bar z_{i,K})^-\Big\}$$
$$+ \sum_{e\in\partial K}\frac{|e|}{|K|}\chi_{w_K-\bar w_K>0}\Big\{g^K(w_K\vee \bar w_K, w_{K_e}\vee \bar w_{K_e}) - g^K(w_K\wedge \bar w_K, w_{K_e}\wedge \bar w_{K_e})\Big\}$$
$$+ \sum_{i=1}^d\sum_{e\in\partial K}\frac{|e|}{|K|}\chi_{z_{i,K}-\bar z_{i,K}<0}\Big\{g_i^K(z_{i,K}\vee \bar z_{i,K}, z_{i,K_e}\vee \bar z_{i,K_e}) - g_i^K(z_{i,K}\wedge \bar z_{i,K}, z_{i,K_e}\wedge \bar z_{i,K_e})\Big\} \le 0.$$

Proof. Let (w_h, Z_h) and $(\overline{w}_h, \overline{Z}_h)$ be two solutions of (2.2), then we have

$$\partial_t(w_K - \bar w_K) + \sum_{e\in\partial K}\frac{|e|}{|K|}\Big\{g^K(w_K, w_{K_e}) - g^K(\bar w_K, \bar w_{K_e})\Big\} = \frac{1}{\varepsilon}\sum_{i=1}^d\Big\{G_i(w_K, z_{i,K}) - G_i(\bar w_K, \bar z_{i,K})\Big\}$$

$$\partial_t(z_{i,K} - \bar z_{i,K}) + \sum_{e\in\partial K}\frac{|e|}{|K|}\Big\{g_i^K(z_{i,K}, z_{i,K_e}) - g_i^K(\bar z_{i,K}, \bar z_{i,K_e})\Big\} = \frac{1}{\varepsilon}\Big\{G_i(w_K, z_{i,K}) - G_i(\bar w_K, \bar z_{i,K})\Big\}.$$

Using the fact that $f^+ = \chi_{f>0}f, f^- = -\chi_{f<0}f$ multiplying the first equation by $\chi_{w_K-\bar w_K>0}$, and the second by $-\chi_{z_{i,K}-\bar z_{i,K}<0}$ summing over i and adding the resulting equations, we get by using the monotonicity assumptions on G_i

(3.2)

$$\partial_t\Big\{(w_K - \bar w_K)^+ Mhcr\sum_{i=1}^d (z_{i,K} - \bar z_{i,K})^-\Big\} + \sum_{e\in\partial K}\frac{|e|}{|K|}\chi_{w_K-\bar w_K>0}\Big[g^K(w_K, w_{K_e}) - g^K(\bar w_K, \bar w_{K_e})\Big]$$
$$- \sum_{i=1}^d\sum_{e\in\partial K}\frac{|e|}{|K|}\chi_{z_{i,K}-\bar z_{i,K}<0}\Big[g_i^K(z_{i,K}, z_{i,K_e}) - g_i^K(\bar z_{i,K}, \bar z_{i,K_e})\Big] \le 0.$$

Let $\mathcal{T}_w = -\chi_{w_K-\bar{w}_K>0}[g^K(w_K, w_{K_e}) - g^K(\bar{w}_K, \bar{w}_{K_e})]$, and $\mathcal{T}_z = \chi_{z_{i,K}-\bar{z}_{i,K}<0}[g_i^K(z_{i,K}, z_{i,K_e}) - g_i^K(\bar{z}_{i,K}, \bar{z}_{i,K_e})]$. Then we have

(a) For $w_K - \bar{w}_K > 0$, we have that $w_K = w_K \vee \bar{w}_K$ and $\bar{w}_K = w_K \wedge \bar{w}_K$; otherwise $\mathcal{T}_w = 0$. Then, using (2.5) we have

$$\mathcal{T}_w \leq -\chi_{w_K-\bar{w}_K>0}\{g^K(w_K \vee \bar{w}_K, w_{K_e} \vee \bar{w}_{K_e}) - g^K(w_K \wedge \bar{w}_K, w_{K_e} \wedge \bar{w}_{K_e})\}. \tag{3.3}$$

(b) Similarly, for $z_{i,K} - \bar{z}_{i,K} < 0$ (otherwise $\mathcal{T}_z = 0$) we have that $z_{i,K} = z_{i,K} \wedge \bar{z}_{i,K}$ and $\bar{z}_{i,K} = z_{i,K} \vee \bar{z}_{i,K}$. Now, (2.5) implies

$$\mathcal{T}_z \leq -\chi_{z_{i,K}-\bar{z}_{i,K}<0}\{g_i^K(z_{i,K} \vee \bar{z}_{i,K}, z_{i,K_e} \vee \bar{z}_{i,K_e}) - g_i^K(z_{i,K} \wedge \bar{z}_{i,K}, z_{i,K_e} \wedge \bar{z}_{i,K_e})\}. \tag{3.4}$$

Therefore, (3.2), (3.3) and (3.4) yield (3.1). $\square$

Next, we show that the scheme is L^1 contractive and bounded in L^∞.

Lemma 3.2. *Under the assumptions of Lemma 3.1 we have*

$$\begin{aligned} &\|w_h(t) - \overline{w}_h(t)\|_{L^1} + \sum_{i=1}^d \|z_{i,h}(t) - \overline{z}_{i,h}(x,t)\|_{L^1} \\ &\qquad \leq \|w_h(\tau) - \overline{w}_h(\tau)\|_{L^1} + \sum_{i=1}^d \|z_{i,h}(\tau) - \overline{z}_{i,h}(\tau)\|_{L^1}, \quad 0 \leq \tau < t. \end{aligned} \tag{i}$$

(ii) *If, for some $a < b$, we have $a \leq w_K(0) \leq b$, $h_i(b) \leq z_{i,K}(0) \leq h_i(a)$, $i = 1, \dots, d, K \in \mathcal{T}_h$, then*

$$a \leq w_K(t) \leq b,\ h_i(b) \leq z_{i,K}(t) \leq h_i(a),\ K \in \mathcal{T}_h,\ i = 1, \dots, d,$$

i.e., the region $\mathcal{R}^{a,b} = [a,b] \times \prod_{i=1}^d [h_i(b), h_i(a)]$ is positively invariant.

Proof. (i) Relation (3.1) implies

$$\begin{aligned} &\partial_t\Big\{|w_K - \bar{w}_K| + \sum_{i=1}^d |z_{i,K} - \bar{z}_{i,K}|\Big\} + \sum_{e\in\partial K} \frac{|e|}{|K|} g^K(|w_K - \bar{w}_K|, |w_{K_e} - \bar{w}_{K_e}|) \\ &+ \sum_{i=1}^d \sum_{e\in\partial K} \frac{|e|}{|K|} g_i^K(|z_{i,K} - \bar{z}_{i,K}|, |z_{i,K_e} - \bar{z}_{i,K_e}|) \leq 0. \end{aligned} \tag{3.5}$$

Multiplying by $|K|$ and then summing w.r.t. to $K \in \mathcal{T}_h$ we get (i) by noticing that in each edge of our partition, $g^K(|w_K - \bar{w}_K|, |w_{K_e} - \bar{w}_{K_e}|) + g^{K_e}(|w_{K_e} - \bar{w}_{K_e}|, |w_K - \bar{w}_K|) = 0$, and a similar relation for the $z_{i,K}$ terms.

For the proof of (ii), we first observe that

$$\begin{aligned} \chi_{w_K-\bar{w}_K>0} g^K(w_K - \bar{w}_K, w_{K_e} - \bar{w}_{K_e}) &\geq g^K((w_K - \bar{w}_K)^+, (w_{K_e} - \bar{w}_{K_e})^+), \\ -\chi_{z_{i,K}-\bar{z}_{i,K}<0} g_i^K(z_{i,K} - \bar{z}_{i,K}, z_{i,K_e} - \bar{z}_{i,K_e}) &\geq -g_i^K(\,(z_{i,K} - \bar{z}_{i,K})^-, (z_{i,K_e} - \bar{z}_{i,K_e})^-). \end{aligned}$$

Indeed, by the monotonicity properties of g^K, if $\chi_{w_K-\bar{w}_K>0} = 1$,

$$\begin{aligned} \chi_{w_K-\bar{w}_K>0} g^K(w_K - \bar{w}_K, w_{K_e} - \bar{w}_{K_e}) &= g^K((w_K - \bar{w}_K)^+, (w_{K_e} - \bar{w}_{K_e})) \\ &\geq g^K((w_K - \bar{w}_K)^+, (w_{K_e} - \bar{w}_{K_e})^+), \end{aligned}$$

in the case where $\chi_{w_K-\bar{w}_K>0}=0$ it suffices to show $g^K(0,(w_{K_e}-\bar{w}_{K_e})^+)\leq 0$. But this is a consequence of (2.3), (2.4) and the fact that $(w_{K_e}-\bar{w}_{K_e})^+\geq 0$. The corresponding relation for $z_{i,K}$ is proved similarly.

Using now these relations, the fact that $g^K((w_K-\bar{w}_K)^+,(w_{K_e}-\bar{w}_{K_e})^+)+g^{K_e}((w_{K_e}-\bar{w}_{K_e})^+,(w_K-\bar{w}_K)^+)=0$, and the corresponding relation for $z_{i,K}$ we obtain in view of (3.2),

$$\sum_{K\in\mathcal{T}_h}|K|\Big[(w_K(t)-\bar{w}_K(t))^+ + \sum_i (z_{i,K}(t)-\bar{z}_{i,K}(t))^-\Big] \tag{3.6}$$
$$\leq \sum_{K\in\mathcal{T}_h}|K|\Big[(w_K(0)-\bar{w}_K(0))^+ + \sum_i (z_{i,K}(0)-\bar{z}_{i,K}(0))^-\Big].$$

Then, (ii) follows by noticing that $\bar{w}_K=b, \bar{z}_{i,K}=h_i(b)$ is a solution of the semidiscrete scheme, since $\sum_{e\in\partial K}|e|g^K(a,a)=0$ and $\sum_{e\in\partial K}|e|g_i^K(h_i(b),h_i(b))=0$. □

Discrete entropy inequality. Lemma 3.1 implies a discrete entropy inequality. Indeed (3.1) is still valid if we interchange $+$ and $-$. For any $\xi\in\mathbb{R}$, we let $\bar{w}_K=\xi, \bar{z}_{i,K}=h_i(\xi),\ i=1,\dots,d$, and setting, for $u,v\in\mathbb{R}$

$$\begin{aligned} D_\xi^K(u,v) &= g^K(u\vee\xi,v\vee\xi)-g^K(u\wedge\xi,v\wedge\xi)=g^K(|u-\xi|,|v-\xi|)\ ,\\ D_\xi^{i,K}(u,v) &= g_i^K(u\vee h_i(\xi),v\vee h_i(\xi))-g_i^K(u\wedge h_i(\xi),v\wedge h_i(\xi))\\ &= g_i^K(|u-h_i(\xi)|,|v-h_i(\xi)|)\end{aligned} \tag{3.7}$$

we get after summation using (3.1), the following *Discrete Entropy Inequality*

$$\partial_t\Big\{|w_K-\xi|+\sum_{i=1}^d|z_{i,K}-h_i(\xi)|\Big\}+\sum_{e\in\partial K}\frac{|e|}{|K|}\Big\{D_\xi^K(w_K,w_{K_e})+\sum_{i=1}^d D_\xi^{i,K}(z_{i,K},z_{i,K_e})\Big\}\leq 0\ . \tag{3.8}$$

Remark 3.1. Notice that for D_ξ^K we have, for $u\in\mathbb{R}$,

$$D_\xi^K(u,u)=|u-\xi|A\cdot\nu_{e,K}\ \text{ and }\ D_\xi^{i,K}(u,u)=|u-h_i(\xi)|B_i\cdot\nu_{e,K}\ .$$

Dissipation estimate. The next lemma provides an estimate for the distance from the equilibrium $z_i=h_i(w)$ for our approximating scheme and a weak dissipation estimate for w_K and $z_{i,K}$. A stronger estimate for the distance from the equilibrium is proved in Lemma 3.4 This result compensates the lack of BV estimates for our schemes, compare with [CCL1], [V], in the proof of the convergence result in Section 4. We need some more notation: Let h_i^{-1} denote the inverse of h_i, and

$$\Psi_i(z)=-\int_0^z h_i^{-1}(\xi)\,d\xi\ ,$$

cf., [KT1]. The functions $\Psi_i,\ i=1,\dots,d$, are positive and strictly convex according to our assumptions on h_i, cf. Section 2. In particular (A.2) implies that there exists $\mu=\mu(a,b)>0$ such that

$$\Psi_i''(z)\geq\mu>0\,,\qquad z\in[h_i(b),h_i(a)]. \tag{3.9}$$

Our assumptions on the fluxes imply that

$$
\begin{aligned}
g^K(u,v) &= \frac{A\cdot\nu_{e,K}}{2}(u+v) + a_{\nu_{e,K}}(u-v),\\
g_i^K(u,v) &= \frac{B_i\cdot\nu_{e,K}}{2}(u+v) + b^i_{\nu_{e,K}}(u-v),
\end{aligned}
\tag{3.10}
$$

where $a_e := a_{\nu_{e,K}} = a_{\nu_{e,K_e}} \ge 0$ and $b^i_e := b^i_{\nu_{e,K}} = b^i_{\nu_{e,K_e}} \ge 0$. (2.5) implies $\frac{1}{2}\,|A\cdot\nu_{e,K}| \le a_e$ and $\frac{1}{2}\,|B_i\cdot\nu_{e,K}| \le b^i_e$.

Remark 3.2. Most of the well known monotone fluxes are reduced to the linear case, e.g., for $g^K(u,v)$ to

$$
g^K(u,v) = \frac{A\cdot\nu_{e,K}}{2}(u+v) + \frac{|A\cdot\nu_{e,K}|}{2}(u-v).
$$

We have now

Lemma 3.3. *Assume that the initial conditions satisfy $(w_h^0, Z_h^0)\in\mathcal{R}^{a,b}$, for some $a,b\in\mathbb{R}$. Then if $\sigma=\sigma(a,b)$ and $\mu=\mu(a,b)$ are the constants of (A.3) and (3.9), respectively, there holds*

$$
\begin{aligned}
\frac{\sigma}{\varepsilon}\int_0^t \sum_{K\in\mathcal{T}_h}|K|\sum_{i=1}^d (h_i(w_K)-z_{i,K})^2 + \int_0^t \sum_{e\in\Gamma_h}|e|\Big\{\, a_e(w_K-w_{K_e})^2 + \mu\sum_{i=1}^d b^i_e(z_{i,K}-z_{i,K_e})^2\Big\}\\
\le \sum_{K\in\mathcal{T}_h}|K|\Big\{\frac{1}{2}(w_K^0)^2 + \sum_{i=1}^d \Psi_i(z^0_{i,K})\Big\} \le C\,,
\end{aligned}
$$

where a_e, b^i_e are defined in (3.10).

Proof. First we notice that (2.4) implies $\sum_{e\in\partial K}|e|g^K(w_K,w_K)=0$, $\sum_{e\in\partial K}|e|g_i^K(z_{i,K},z_{i,K})=0$. We then multiply (2.2a) by w_K and (2.2b) by $h_i^{-1}(z_{i,K})$, sum over i and subtract the resulting equations. Next if we multiply by $|K|$ and sum we finally obtain

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}|K|\partial_t\Big\{\frac{1}{2}w_K^2 + \sum_{i=1}^d \Psi_i(z_{i,K})\Big\} + \sigma\frac{1}{\varepsilon}\sum_{K\in\mathcal{T}_h}|K|\sum_{i=1}^d (h_i(w_K)-z_{i,K})^2\\
&+\sum_{e\in\Gamma_h}|e|\Big\{w_K g^K(w_K,w_{K_e}) + w_{K_e}g^{K_e}(w_{K_e},w_K)\Big\}\\
&-\sum_{i=1}^d\sum_{e\in\Gamma_h}|e|\Big\{h_i^{-1}(z_{i,K})\big[g_i^K(z_{i,K},z_{i,K_e}) - g_i^K(z_{i,K},z_{i,K})\big]\\
&\quad + h_i^{-1}(z_{i,K_e})\big[g_i^{K_e}(z_{i,K_e},z_{i,K}) - g_i^{K_e}(z_{i,K_e},z_{i,K_e})\big]\Big\} \le 0\,,
\end{aligned}
\tag{3.11}
$$

Where we have used that $-(w-h_i^{-1}(z))G_i(w,z_i)\ge\sigma(h_i(w)-z_i)^2$, cf. (A1-3), (A.3′). We will first estimate the terms corresponding to w-fluxes. Using (3.10) we get

$$
\sum_{e\in\Gamma_h}|e|\Big\{w_K g^K(w_K,w_{K_e}) + w_{K_e}g^{K_e}(w_{K_e},w_K)\Big\} = \sum_{e\in\Gamma_h}|e|a_e(w_K-w_{K_e})^2\,,
$$

since

$$\sum_{e\in\Gamma_h} |e|\frac{1}{2}\ A\cdot\nu_{e,K}(w_K^2 - w_{K_e}^2) = \sum_{e\in\Gamma_h} |e|\Big\{\frac{1}{2}\ A\cdot\nu_{e,K}w_K^2 + \frac{1}{2}\ A\cdot\nu_{e,K_e}w_{K_e}^2\Big\}$$
$$= \sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K} |e|\frac{1}{2}\ A\cdot\nu_{e,K}w_K^2 = 0\,.$$

For the z fluxes of (3.11), using (3.10) for $g_i^K(z_{i,K}, z_{i,K_e})$, we first write,

$$-h_i^{-1}(z_{i,K})\big[g_i^K(z_{i,K}, z_{i,K_e}) - g_i^K(z_{i,K}, z_{i,K})\big] = \Psi'(z_{i,K})\big[\frac{1}{2}\ B_i\cdot\nu_{e,K} - b_e^i\big](z_{i,K_e} - z_{i,K}),$$

where, as before, $b_e^i = b^i_{\nu_{e,K}} = b^i_{\nu_{e,K_e}}$. By (2.5) $\frac{1}{2}\ B_i\cdot\nu_{e,K} - b_e^i \le 0$, and hence using Taylor's formula and (3.9), $\Psi_i'(c_1)(c_2 - c_1) \le \Psi(c_2) - \Psi(c_1) - \frac{\mu}{2}(c_1 - c_2)^2$, we get

$$-h_i^{-1}(z_{i,K})\big[g_i^K(z_{i,K}, z_{i,K_e}) - g_i^K(z_{i,K}, z_{i,K})\big] \ge \big[\frac{1}{2}\ B_i\cdot\nu_{e,K} - b_e^i\big](\Psi_i(z_{i,K_e}) - \Psi_i(z_{i,K}))$$
$$+ \frac{1}{2}\ \mu\big[b_e^i - \frac{1}{2}\ B_i\cdot\nu_{e,K}\big](z_{i,K} - z_{i,K_e})^2\,.$$

Similarly,

$$-h_i^{-1}(z_{i,K_e})\big[g_i^{K_e}(z_{i,K_e}, z_{i,K}) - g_i^{K_e}(z_{i,K_e}, z_{i,K_e})\big] \ge \big[\frac{1}{2}\ B_i\cdot\nu_{e,K_e} - b_e^i\big](\Psi_i(z_{i,K}) - \Psi_i(z_{i,K_e}))$$
$$+ \frac{1}{2}\ \mu\big[b_e^i - \frac{1}{2}\ B_i\cdot\nu_{e,K_e}\big](z_{i,K} - z_{i,K_e})^2\,.$$

But then,

$$\sum_{e\in\Gamma_h} |e|\Big\{\big[\frac{1}{2}\ B_i\cdot\nu_{e,K} - b_e^i\big](\Psi_i(z_{i,K_e}) - \Psi_i(z_{i,K})) + \big[\frac{1}{2}\ B_i\cdot\nu_{e,K_e} - b_e^i\big](\Psi_i(z_{i,K}) - \Psi_i(z_{i,K_e}))\Big\}$$
$$= \sum_{e\in\Gamma_h} |e|B_i\cdot\nu_{e,K}(\Psi_i(z_{i,K_e}) - \Psi_i(z_{i,K})) = -\sum_{e\in\Gamma_h} |e|\Big\{B_i\cdot\nu_{e,K}\Psi_i(z_{i,K}) + B_i\cdot\nu_{e,K_e}\Psi_i(z_{i,K_e})\Big\}$$
$$= -\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K} |e|B_i\cdot\nu_{e,K}\Psi_i(z_{i,K}) = 0.$$

Therefore,

$$-\sum_{e\in\Gamma_h} |e|\Big\{h_i^{-1}(z_{i,K})\big[g_i^K(z_{i,K}, z_{i,K_e}) - g_i^K(z_{i,K}, z_{i,K})\big]$$
$$+ h_i^{-1}(z_{i,K_e})\big[g_i^{K_e}(z_{i,K_e}, z_{i,K}) - g_i^{K_e}(z_{i,K_e}, z_{i,K_e})\big]\Big\} \ge \sum_{e\in\Gamma_h} |e|\mu b_e^i(z_{i,K} - z_{i,K_e})^2\,.$$

In view of these estimates (3.11) implies

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} |K|\partial_t\Big\{\frac{1}{2}w_K^2 + \sum_{i=1}^d \Psi_i(z_{i,K})\Big\} + \sigma\frac{1}{\varepsilon}\sum_{K\in\mathcal{T}_h} |K|\sum_{i=1}^d (h_i(w_K) - z_{i,K})^2 \\ + \sum_{e\in\Gamma_h} |e|\, a_e(w_K - w_{K_e})^2 + \sum_{i=1}^d\sum_{e\in\Gamma_h} |e|\mu b_e^i(z_{i,K} - z_{i,K_e})^2 \le 0\ ,\end{aligned} \tag{3.12}$$

and the proof is complete. ☐

Distance from equilibrium. Next, we estimate $\sup_t \sum_{K\in\mathcal{T}_h} |K|\sum_i |G_i(w_K(t), z_{i,K}(t))|^2$.

Lemma 3.4. *Let (w_h, Z_h) be a solution of the semidiscrete scheme emanating from data with finite total variation and lying in an (invariant) region $\mathcal{R}^{a,b}$. Assume further that*

$$\sum_{K\in\mathcal{T}_h} |K| \sum_{i=1}^{d} |G_i(w_K(0), z_{i,K}(0))|^2 \le C\varepsilon. \tag{3.13}$$

Then, for any $1 > \eta > 0$ there exists a constant $C_\eta = C(\eta, a, b)$ such that

$$\sup_t \sum_{K\in\mathcal{T}_h} |K| \sum_i |G_i(w_K(t), z_{i,K}(t))|^2 \le C_\eta\, \varepsilon', \quad \textit{with } \varepsilon' = \varepsilon^{1-\eta}\,.$$

Proof. Using the definition of the scheme, we have

$$\begin{aligned}\partial_t G_i(w_K(t), z_{i,K}(t)) =& \frac{\partial G_i}{\partial w}\Big\{ -\sum_{e\in\partial K} \frac{|e|}{|K|} g^K(w_K, w_{K_e}) + \frac{1}{\varepsilon}\sum_{j=1}^{d} G_j(w_K(t), z_{j,K}(t))\Big\}\\ &+\frac{\partial G_i}{\partial z}\Big\{ -\sum_{e\in\partial K} \frac{|e|}{|K|} g_i^K(z_{i,K}, z_{i,K_e}) + \frac{1}{\varepsilon} G_i(w_K(t), z_{i,K}(t))\Big\}\,.\end{aligned}$$

Multiplying by $G_i(w_K, z_{i,K}) = G_i$ and adding, we obtain

$$\begin{aligned}\frac{1}{2}\,\partial_t \sum_i |G_i(w_K, z_{i,K})|^2 + \frac{1}{\varepsilon}\sum_i (-\frac{\partial G_i}{\partial z})|G_i(w_K, z_{i,K})|^2 =& \frac{1}{\varepsilon}\sum_{i=1}^{d} \frac{\partial G_i}{\partial w} G_i(w_K, z_{i,K}) \sum_{j=1}^{d} G_j(w_K, z_{j,K})\\ &+\sum_{i=1}^{d} G_i\Big(\frac{\partial G_i}{\partial w}\Big[-\sum_{e\in\partial K} \frac{|e|}{|K|}(g^K(w_K, w_{K_e}) - g^K(w_K, w_K))\Big]\\ &+\frac{\partial G_i}{\partial z}\Big[-\sum_{e\in\partial K} \frac{|e|}{|K|}(g_i^K(z_{i,K}, z_{i,K_e}) - g_i^K(z_{i,K}, z_{i,K}))\Big]\Big)\,.\end{aligned}$$

Observe now that (A.2) implies that $-\frac{\partial G_i}{\partial z} > c_1 = c_1(a,b) > 0$ in $\mathcal{R}^{a,b}$. Also, $|g^K(w_K, w_{K_e}) - g^K(w_K, w_K)| \le a_e|w_K - w_{K_e}|$. Therefore, if $\varepsilon \le Ch_K$, $K \in \mathcal{T}_h$, there exists a constant $c_0 = c_0(a,b) > 0$, such that

$$\partial_t \sum_{K\in\mathcal{T}_h} |K| \sum_i |G_i(w_K, z_{i,K})|^2 + \frac{c_0}{\varepsilon} \sum_{K\in\mathcal{T}_h} |K| \sum_i |G_i(w_K, z_{i,K})|^2 \le C\mathcal{A}\,, \tag{3.14}$$

where
(3.14a)

$$\mathcal{A} = \sigma\frac{1}{\varepsilon} \sum_{K\in\mathcal{T}_h} |K| \sum_{i=1}^{d} (h_i(w_K) - z_{i,K})^2 + \sum_{e\in\Gamma_h} |e|\, a_e(w_K - w_{K_e})^2 + \sum_{i=1}^{d}\sum_{e\in\Gamma_h} |e|\mu b_e^i(z_{i,K} - z_{i,K_e})^2\,,$$

and Lemma 3.3 implies that, for any $t > 0$, $\int_0^t \mathcal{A}(s)ds \le C$. By (3.14) and our assumption on the initial data, we have

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} |K| \sum_i |G_i(w_K(t), z_{i,K}(t))|^2 \le& e^{-\frac{c_0}{\varepsilon}t} \sum_{K\in\mathcal{T}_h} |K| \sum_i |G_i(w_K(0), z_{i,K}(0))|^2\\ &+ C\int_0^t e^{-\frac{c_0}{\varepsilon}(t-s)}\mathcal{A}(s)ds \le C\varepsilon + C\int_0^t e^{-\frac{c_0}{\varepsilon}(t-s)}\mathcal{A}(s)ds\,.\end{aligned}$$

Let $1 > \eta > 0$, be an arbitrarily small number and $\varepsilon' = \varepsilon^{1-\eta}$. The proof of the Lemma will be complete if we show

$$\int_0^t e^{-\frac{c_0}{\varepsilon}(t-s)} \mathcal{A}(s) ds \le C_\eta \varepsilon' . \tag{3.15}$$

Since $\frac{1}{\varepsilon} e^{-\frac{c_0}{\varepsilon^\eta}}$ is bounded for $\varepsilon \to 0$, we have, in view of Lemma 3.3,

$$\int_0^{t-\varepsilon'} e^{-\frac{c_0}{\varepsilon}(t-s)} \mathcal{A}(s) ds \le \varepsilon \int_0^{t-\varepsilon'} \frac{1}{\varepsilon} e^{-\frac{c_0}{\varepsilon^\eta}} \mathcal{A}(s) ds \le c\,\varepsilon \int_0^{t-\varepsilon'} \mathcal{A}(s) ds \le c\varepsilon .$$

On the other hand, for $N_\varepsilon = [\varepsilon^{-\eta}] + 1$,

$$\int_{t-\varepsilon'}^t e^{-\frac{c_0}{\varepsilon}(t-s)} \mathcal{A}(s) ds \le \int_{t-N_\varepsilon \varepsilon}^t \mathcal{A}(s) ds \le \sum_{m=1}^{N_\varepsilon} \int_{t-m\,\varepsilon}^{t-(m-1)\varepsilon} \mathcal{A}(s) ds . \tag{3.16}$$

Then using (3.12), we obtain

$$\begin{aligned}
&\int_{t-m\,\varepsilon}^{t-(m-1)\varepsilon} \mathcal{A}(s) ds \le C \sum_{K\in\mathcal{T}_h} |K| \Big\{ |w_K^2(t-m\,\varepsilon) - w_K^2(t-(m-1)\varepsilon)| \\
&\qquad + \sum_{i=1}^d |\Psi_i(z_{i,K}(t-m\,\varepsilon)) - \Psi_i(z_{i,K}(t-(m-1)\varepsilon))| \Big\} \\
&\le C'(a,b) \sum_{K\in\mathcal{T}_h} |K| \Big\{ |w_K(t-m\,\varepsilon) - w_K(t-(m-1)\varepsilon)| + \sum_{i=1}^d |z_{i,K}(t-m\,\varepsilon) - z_{i,K}(t-(m-1)\varepsilon)| \Big\} \\
&\le C'(a,b) \sum_{K\in\mathcal{T}_h} |K| \Big\{ |w_K(\varepsilon) - w_K(0)| + \sum_{i=1}^d |z_{i,K}(\varepsilon) - z_{i,K}(0)| \Big\} .
\end{aligned} \tag{3.17}$$

Here we also used that $(w_h, Z_h) \in \mathcal{R}^{a,b}$, and the L^1 contraction property (Lemma 3.2 (i)) for $(\overline{w}_h(\cdot), \overline{Z}_h(\cdot)) = (w_h(\cdot + \varepsilon), Z_h(\cdot + \varepsilon))$. Let us assume that

$$\sum_{K\in\mathcal{T}_h} |K| \Big\{ |w_K(\varepsilon) - w_K(0)| + \sum_{i=1}^d |z_{i,K}(\varepsilon) - z_{i,K}(0)| \Big\} \le C\varepsilon . \tag{3.18}$$

Then, (3.16), (3.17) and (3.18) imply

$$\int_{t-\varepsilon'}^t e^{-\frac{c_0}{\varepsilon}(t-s)} \mathcal{A}(s) ds \le C N_\varepsilon \,\varepsilon = C([\varepsilon^{-\eta}] + 1)\varepsilon \le C\,\varepsilon' ,$$

and the proof of (3.15) (and therefore of Lemma 3.4) will be complete. Hence it remains to verify (3.18). To this end let $0 < \tau \le \varepsilon$; then, by (2.2), we see that

$$\begin{aligned}
&|K| \Big\{ |w_K(\tau) - w_K(0)| + \sum_{i=1}^d |z_{i,K}(\tau) - z_{i,K}(0)| \Big\} \\
&\qquad \le C \int_0^\tau \sum_{e\in\partial K} |e| \Big(|w_K - w_{K_e}| + \sum_{i=1}^d |z_{i,K} - z_{i,K_e}| + \frac{1}{\varepsilon} \sum_{i=1}^d |K| |G_i(w_K, z_{i,K})| \Big) ds .
\end{aligned}$$

We estimate the terms of the right-hand side as follows

$$|G_i(w_K(s), z_{i,K}(s))| \le |G_i(w_K(0), z_{i,K}(0))| + C(a,b)(|w_K(s) - w_K(0)| + |z_{i,K}(s) - z_{i,K}(0)|)$$

and

$$|w_K(s) - w_{K_e}(s)| \le |w_K(0) - w_{K_e}(0)| + |w_K(s) - w_K(0)| + |w_{K_e}(s) - w_{K_e}(0)| .$$

Therefore in view of the stability of the local L^2 projection in BV, cf. [C], our assumptions on the initial data, we have upon summing over K and using again the fact that $\varepsilon \le Ch_K$, $K \in \mathcal{T}_h$,

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} |K| \Big\{ |w_K(\tau) - w_K(0)| + \sum_{i=1}^{d} |z_{i,K}(\tau) - z_{i,K}(0)| \Big\} \\ \le C\tau + \frac{1}{\varepsilon}\int_0^\tau \sum_{K\in\mathcal{T}_h} |K| \Big\{ |w_K(s) - w_K(0)| + \sum_{i=1}^{d} |z_{i,K}(s) - z_{i,K}(0)| \Big\}\, ds .\end{aligned}$$

Then, since $\tau \le \varepsilon$, Gronwall's lemma implies

$$\sum_{K\in\mathcal{T}_h} |K| \Big\{ |w_K(\tau) - w_K(0)| + \sum_{i=1}^{d} |z_{i,K}(\tau) - z_{i,K}(0)| \Big\} \le C e^{C\frac{\tau}{\varepsilon}}\, \tau \le C\, \varepsilon .$$

The proof is thus complete. $\square$

4. Convergence of the semidiscrete scheme

Our convergence results will be based on the following approximation lemma, [KKM], which provides a compact form for deriving error estimates for numerical approximations to conservation law (1.3). Lemma 4.1 is an extension of a result of Bouchut and Perthame [BP], and allows the explicit treatment of terms that typically arise in numerical schemes.

Lemma 4.1. ([KKM]) *Let $u_h, u \in L^\infty_{loc}([0,\infty), L^1_{loc}(\mathbb{R}^d))$ be right continuous in t, with values in $L^1_{loc}(\mathbb{R}^d)$. Assume that u is the entropy solution of a given conservation law, i.e., it satisfies (1.3) and*

$$\partial_t |u-k| + \sum_{i=1}^{d} \partial_{x_i} \big[\, (F_i(u) - F_i(k))\mathrm{sgn}(u-k) \,\big] \le 0 \ , \qquad \text{in } \mathcal{D}', \text{ for all } k \in \mathbb{R}, \tag{4.1}$$

with initial value $u^0 \in BV(\mathbb{R}^d)$. Let Ψ be a nonnegative test function, $\Psi \in C_c^\infty((0,\infty)\times\mathbb{R}^d)$, and assume that u_h with initial value u_h^0 satisfies

$$\begin{aligned}&-\iint_{(0,\infty)\times\mathbb{R}^d} \big(|u_h - k|\partial_t\Psi + \mathrm{sgn}(u_h - k)[F(u_h) - F(k)]\cdot\nabla_x\Psi\big) dtdx \\ &\le \iint_{(0,\infty)\times\mathbb{R}^d} \Big(\alpha_G|\partial_t\Psi| + \sum_j \alpha_H^j \big|\frac{\partial\Psi}{\partial x_j}\big| + \beta_G B_G(\partial_t\Psi) + \sum_j \beta_H^j B_H^j\big(\frac{\partial\Psi}{\partial x_j}\big)\Big) dxdt \\ &\qquad \text{for all } k \in \mathbb{R},\end{aligned} \tag{4.2}$$

where $F = (F_1, \dots, F_d)$ and α_G, α_H^j, β_G, β_H^j, are nonnegative k-independent functions in $L^1_{loc}([0,\infty)\times\mathbb{R}^d)$ and $\alpha_G,\ \beta_G \in L^\infty_{loc}([0,\infty), L^1_{loc}(\mathbb{R}^d))$.

Let $\Delta, \Delta' > 0$, and $\mathcal{S}_h = \{S\}$ be a given decomposition of $[0,\infty)\times\mathbb{R}^d$, into elements S, such that

$$
\begin{aligned}
&\operatorname{diam}(S_t) \le \Delta\,, \qquad \textit{if } \beta_H^j, \textit{ is not identically zero for some } j,\ j=1,\dots,d, \textit{ and} \\
&|S_x| \le \Delta'\,, \qquad \textit{if } \beta_G \textit{ is not identically zero,}
\end{aligned} \tag{4.3}
$$

where $S_x = \{t : (t,x) \in S\}$ and $S_t = \{x : (t,x)\in S\}$.

In addition, the k-independent operators B_G, B_H^j satisfy: for all $(t,x)\in S$, $1\le i,j\le d$, there holds

$$
\begin{aligned}
|B_H^j(\frac{\partial \Psi}{\partial x_j})(t,x)| &\le C \sup_{x'\in S_t} |\frac{\partial\Psi}{\partial x_j}(t,x')|\ , \\
|B_G(\partial_t\Psi)(t,x)| &\le C \sup_{t'\in K_x} |\partial_t v(t',x)| \sup_{t'\in K_x} |w(t',x)| \\
&\quad + C \sup_{t'\in K_x} |\partial_t w(t',x)| \sup_{t'\in K_x} |v(t',x)|\, \chi_{\operatorname{supp}\partial_t w}
\end{aligned} \tag{4.4}
$$

where in the second relation $\Psi = v\,w$, and χ_Ω denotes the characteristic function of Ω. Here C is a uniform constant independent of Ψ and the element decomposition $\mathcal{S}_h$.
Then the following estimate holds: for any $T\ge 0$, $x_0\in\mathbb{R}^d$, $R>0$, $\nu\ge 0$, with $M = Lip(F)$, we have:

$$
\begin{aligned}
\int_{|x-x_0|<R} |u_h(T,x)-u(T,x)|dx \le \int_{B_0} |u_h(0,x)-u(0,x)|dx + (M\,\Delta' + \Delta)TV(u^0) \\
+ C(E^G + E^H + \tilde{E}_G + \tilde{E}_H)\,.
\end{aligned} \tag{4.5}
$$

Here

$$
\begin{aligned}
E^H &= \frac{1}{\Delta}\sum_{j=1}^d \iint_{0\le t\le T\, x\in B_t} \alpha_H^j(t,x)dxdt, \quad \tilde{E}^H = \frac{1}{\Delta}\sum_{j=1}^d \iint_{0\le t\le T\, x\in B_t^\Delta} \beta_H^j(t,x)dxdt, \\
E^G &= (1+\frac{T}{\Delta'}+\frac{MT}{\Delta+\nu}) \sup_{0\le t\le 2T}\int_{B_t} \alpha_G(t,x)dx, \quad E^G = (1+\frac{T}{\Delta'}+\frac{MT}{\Delta+\nu}) \sup_{0\le t\le 2T}\int_{B_t^{\Delta'}} \beta_G(t,x)dx
\end{aligned}
$$

and $B_t = B(x_0, R+M(T-t)+\Delta+\nu)$, $B_t^\Delta = B(x_0, R+M(T-t)+2\Delta+\nu)$ and $B_t^{\Delta'} = B(x_0, R+M(T-t)+\Delta+\Delta'+\nu)$.

Remark 4.1. The terms B_H^j, β_H^j, $j=1,\dots,d$, in (4.2), (4.4) can be replaced by one term B_H, β_H. In this case (4.4) will be

$$
|B_H(\nabla_x\Psi(t,x))| \le C \sup_{x\in S_t} |\nabla_x\Psi(t,x)|\,,
$$

and $\tilde{E}^H$ will be modified accordingly.

We will use Lemma 4.1 to prove our convergence results. We introduce notation that will be used in the sequel along with some preliminary results. In particular, for any $k\in\mathbb{R}$ we define $\xi\in\mathbb{R}$ such that

$$
k = \xi - \sum_{i=1}^d h_i(\xi) \quad \text{and we set} \quad U_h = w_h - \sum_{i=1}^d h_i(w_h) \ \text{ i.e., } U_K = w_K - \sum_{i=1}^d h_i(w_K),\ K\in\mathcal{T}_h\,.
$$

Then $U_K - k = [w_K - \xi] - \sum_{i=1}^d [h_i(w_K) - h_i(\xi)]$, and, since we assumed that the functions $h_i, i = 1, \dots, d$ are decreasing, we get $|U_K - k| = |w_K - \xi| + \sum_{i=1}^d |h_i(w_K) - h_i(\xi)|$, i.e.,

$$|U_K - k| = |w_K - \xi| + \sum_{i=1}^d |z_{i,K} - h_i(\xi)| + J_K \quad \text{with} \quad |J_K| \le \frac{1}{\sigma} \sum_{i=1}^d |G_i(w_K, z_{i,K})|. \tag{4.6}$$

In view of (A.4) we have $F(U_K) = A(w_K) - \sum_{i=1}^d B_i(h_i(w_K))$. Hence

$$\Big[F(U_K) - F(k)\Big]\mathrm{sgn}(U_K - k) = \Big\{[A(w_K) - A(\xi)] - \sum_{i=1}^d [B_i(h_i(w_K)) - B_i(h_i(\xi))]\Big\}\mathrm{sgn}(U_K - k). \tag{4.7}$$

Now for $w_K - \xi > 0$, we have by (A.2), $h_i(w_K) - h_i(\xi) < 0$, hence, $\mathrm{sgn}\big[(w_K - \xi) - \sum_{i=1}^d (h_i(w_K) - h_i(\xi))\big] > 0$. So, by (4.7) we get

$$\Big[F(U_K) - F(k)\Big]\mathrm{sgn}(U_K - k) = |w_K - \xi| A + \sum_{i=1}^d |h_i(w_K) - h_i(\xi)| B_i\,. \tag{4.7a}$$

Similarly, (4.7a) holds, if $w_K - \xi < 0$. Therefore,

$$\begin{aligned} \Big[F(U_K) - F(k)\Big]\mathrm{sgn}(U_K - k) = &|w_K - \xi| A + \sum_{i=1}^d |z_{i,K} - h_i(\xi)| B_i + H_K\,, \\ &\text{with } |H_K| \le \sum_{i=1}^d |h_i(w_K) - z_{i,K}||B_i|\,. \end{aligned} \tag{4.8}$$

Now we are ready to prove our convergence theorem for the semidiscrete scheme.

Theorem 4.1. *Let u be the entropy solution of the conservation law (1.3) with initial data $u_0 \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. For $U_h = w_h - \sum_{i=1}^d h_i(w_h)$, where (w_h, Z_h) is the solution of the semidiscrete finite volume scheme (2.2), assume that the assumptions of Lemma 3.4 hold and for some $1 > \eta > 0$, $\varepsilon^{1-\eta} \le Ch$. Then, for any time $t \le T$ and $R > 0$, there exists a constant $C = (R + MT)^{d/4} T^{1/2} c(a, b)$ such that the following error estimate holds*

$$\|u(\cdot, t) - U_h(\cdot, t)\|_{L^1(B(0,R))} \le Ch^{1/4} + \|u(\cdot, 0) - U_h(\cdot, 0)\|_{L^1}\,.$$

If in addition $\|u_0 - (w_0^\varepsilon - \sum_{i=1}^d h_i(w_0^\varepsilon))\|_{L^1} \le C\varepsilon$, then

$$\|u(\cdot, t) - U_h(\cdot, t)\|_{L^1(B(0,R))} \le Ch^{1/4}\,.$$

Proof. To apply Lemma 4.1 we consider a nonnegative test function Ψ with compact support, $\mathrm{supp}\,\Psi = \Omega$. We also set

$$V_K := |U_K - k| \quad \text{and} \quad V_{F,K} := [F(U_K) - F(k)]\mathrm{sgn}(U_K - k).$$

Then, we would like to estimate the following quantity

$$E := -\int \sum_{K \in \mathcal{T}_h} \int_K [V_K \Psi_t + V_{F,K} \cdot \nabla_x \Psi] dx\, dt =: -(E_1 + E_2). \tag{4.9}$$

For the first term, we have

$$
\begin{aligned}
E_1 &= \int \sum_{K\in\mathcal{T}_h} \int_K V_K \Psi_t \,dx\,dt = \int \sum_{K\in\mathcal{T}_h} V_K \int_K \Psi_t \,dx\,dt \\
&= \sum_{K\in\mathcal{T}_h} \int V_K \bar\Psi_t^K \,dt \quad \text{where } \bar\Psi_t^K = \int_K \Psi_t\,dt\,.
\end{aligned} \tag{4.10}
$$

For the second term we have,

$$
\begin{aligned}
E_2 &= \int \sum_{K\in\mathcal{T}_h} V_{F,K}\cdot \int_K \nabla_x \Psi \,dx\,dt = \int \sum_{K\in\mathcal{T}_h} \sum_{e\in\partial K} V_{F,K}\cdot \nu_{e,K} \int_e \Psi\,ds\,dt \\
&= \int \sum_{e\in\Gamma_h} (V_{F,K}\cdot\nu_{e,K} + V_{F,K_e}\cdot \nu_{e,K_e}) \bar\Psi^e \,dt \quad \text{where } \bar\Psi^e = \int_e \Psi\,ds\,.
\end{aligned} \tag{4.11}
$$

Now, (4.8) and (3.7), cf. Remark 3.1, imply

$$
V_{F,K}\cdot\nu_{e,K} = D_\xi^K(w_K,w_K) + \sum_{i=1}^d D_\xi^{i,K}(z_{i,K},z_{i,K}) + H_K\cdot\nu_{e,K}\,.
$$

Combining (4.9), (4.10) and (4.11) we have

$$
\begin{aligned}
E = &- \sum_{K\in\mathcal{T}_h} \int \Big\{ |w_K-\xi| + \sum_{i=1}^d |z_{i,K} - h_i(\xi)| \Big\} \bar\Psi_t^K\,dt - \sum_{e\in\Gamma_h} \int \Big\{ [D_\xi^K(w_K,w_K) + D_\xi^{K_e}(w_{K_e},w_{K_e})] \\
&+ \sum_{i=1}^d [D_\xi^{i,K}(z_{i,K},z_{i,K}) + D_\xi^{i,K_e}(z_{i,K_e},z_{i,K_e})] \Big\} \bar\Psi^e\,dt - \sum_{K\in\mathcal{T}_h} \Big(\int J_K \bar\Psi_t^K\,dt + \int H_K\cdot \int_K \nabla_x \Psi dx\,dt \Big)\,.
\end{aligned} \tag{4.12}
$$

There holds $\sum_{e\in\partial K} |e| D_\xi^K(w_K,w_K) = 0$ and $\sum_{e\in\partial K}|e| D_\xi^{i,K}(z_{i,K},z_{i,K}) = 0,\ i=1,\ldots,d$. Thus, if we multiply the discrete entropy inequality (3.8) by $\bar\Psi^K$ and sum for all $K\in\mathcal{T}_h$, and get

$$
\begin{aligned}
&- \sum_{K\in\mathcal{T}_h} \int \Big\{ |w_K-\xi| + \sum_{i=1}^d |z_{i,K}-h_i(\xi)| \Big\} \bar\Psi_t^K\,dt \\
&+ \sum_{e\in\Gamma_h} \int |e| \Big\{ (\mathcal{F}_w^K + \mathcal{F}_w^{K_e}) + \sum_{i=1}^d (\mathcal{F}_{z_i}^K + \mathcal{F}_{z_i}^{K_e}) \Big\} dt \le 0,
\end{aligned} \tag{4.13}
$$

where $\mathcal{F}_w^K = \frac{1}{|K|}[D_\xi^K(w_K,w_{K_e}) - D_\xi^K(w_K,w_K)]\bar\Psi^K$, $\mathcal{F}_{z_i}^K = \frac{1}{|K|}[D_\xi^{i,K}(z_{i,K},z_{i,K_e}) - D_\xi^{i,K}(z_{i,K},z_{i,K})]\bar\Psi^K$, and $\mathcal{F}_w^{K_e}$, $\mathcal{F}_{z_i}^{K_e}$ are defined by the same formulas with K and K_e interchanged. In view of (4.13), we see that (4.12) implies

$$
\begin{aligned}
E \le &- \sum_{e\in\Gamma_h} \int |e| \Big\{ (\mathcal{F}_w^K + \mathcal{F}_w^{K_e}) + \sum_{i=1}^d (\mathcal{F}_{z_i}^K + \mathcal{F}_{z_i}^{K_e}) \Big\} dt \\
&- \sum_{e\in\Gamma_h} \int \Big\{ D_\xi^K(w_K,w_K) + D_\xi^{K_e}(w_{K_e},w_{K_e}) \Big\} \bar\Psi^e\,dt \\
&- \sum_{e\in\Gamma_h} \int \Big\{ \sum_{i=1}^d [D_\xi^{i,K}(z_{i,K},z_{i,K}) + D_\xi^{i,K_e}(z_{i,K_e},z_{i,K_e})] \Big\} \bar\Psi^e\,dt \\
&- \sum_{K\in\mathcal{T}_h} \int \Big(J_K\bar\Psi_t^K + H_K\cdot \int_K \nabla_x\Psi dx \Big)\,dt.
\end{aligned} \tag{4.14}
$$

Now for the w-terms in (4.14) we have using the properties of the discrete fluxes,

$$
\begin{aligned}
&-\frac{|e|}{|K|}\Big\{D_\xi^K(w_K,w_{K_e})-D_\xi^K(w_K,w_K)\Big\}\bar\Psi^K-\frac{|e|}{|K_e|}\Big\{D_\xi^{K_e}(w_{K_e},w_K)-D_\xi^{K_e}(w_{K_e},w_{K_e})\Big\}\bar\Psi^{K_e}\\
&-\Big\{D_\xi^K(w_K,w_K)+D_\xi^{K_e}(w_{K_e},w_{K_e})\Big\}\bar\Psi^e+\Big\{D_\xi^K(w_K,w_{K_e})+D_\xi^{K_e}(w_{K_e},w_K)\Big\}\bar\Psi^e\\
&=\Big\{D_\xi^K(w_K,w_{K_e})-D_\xi^K(w_K,w_K)\Big\}\Big\{\bar\Psi^e-\frac{|e|}{|K|}\bar\Psi^K\Big\}\\
&\quad+\Big\{D_\xi^{K_e}(w_{K_e},w_K)-D_\xi^{K_e}(w_{K_e},w_{K_e})\Big\}\Big\{\bar\Psi^e-\frac{|e|}{|K_e|}\bar\Psi^{K_e}\Big\}\\
&\le C\,a_e|w_K-w_{K_e}|\Big|\bar\Psi^e-\frac{|e|}{|K|}\bar\Psi^K\Big|+C\,a_e|w_K-w_{K_e}|\Big|\bar\Psi^e-\frac{|e|}{|K_e|}\bar\Psi^{K_e}\Big|.
\end{aligned}
$$

A similar inequality holds true for the z-terms in (4.14). Hence summing back to the elements K,

$$
\begin{aligned}
E\le&\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}\int\Big\{a_e|w_K-w_{K_e}|+\sum_{i=1}^d b_e^i|z_{i,K}-z_{i,K_e}|\Big\}\Big|\bar\Psi^e-\frac{|e|}{|K|}\bar\Psi^K\Big|dt\\
&-\sum_{K\in\mathcal{T}_h}\int\Big(J_K\bar\Psi_t^K+H_K\cdot\int_K\nabla_x\Psi dx\Big)\,dt.
\end{aligned}
\tag{4.15}
$$

To adjust to the notation of Lemma 4.1 let $\mathcal{S}_h=\{S^K\}$, $S^K=([0,+\infty)\times K)$, $K\in\mathcal{T}_h$ be a partition of $[0,+\infty)\times\mathbb{R}^d$. Then, for any $t>0$, $(S^K)_t=K$.

Further, we set

$$
B_H(\nabla_x\Psi)\Big|_{S^K}(x,t)=\frac{1}{|K|}\Big||e|\Psi(x,t)-\bar\Psi^e(t)\Big|.
$$

Then, since $x\in K$, we have

$$
\begin{aligned}
\frac{1}{|K|}\Big||e|\Psi(x,t)-\bar\Psi^e(t)\Big|&=\frac{1}{|K|}\Big||e|\Psi(x,t)-\int_e\Psi(S,t)\,dS\Big|\\
&\le\frac{1}{|K|}Ch_K|e|\sup_{x'\in K}|\nabla\Psi(x',t)|\le C\sup_{x'\in K}|\nabla\Psi(x',t)|\,,
\end{aligned}
\tag{4.16}
$$

i.e., (4.4) is satisfied.

In view of (4.15), U_h satisfies (4.2) with $\mathcal{S}_h=\{S^K\}$, $K\in\mathcal{T}_h$ as above, and

$$
\begin{aligned}
&\alpha_H\Big|_{S^K}=|H_K|\,,\quad\alpha_G\Big|_{S^K}=|J_K|\,,\\
&\beta_H\Big|_{S^K}=C\sum_{e\in\partial K}\Big\{a_e|w_K-w_{K_e}|+\sum_{i=1}^d b_e^i|z_{i,K}-z_{i,K_e}|\Big\}\,.
\end{aligned}
\tag{4.17}
$$

Next, we will estimate the terms on the right hand side of (4.5) in our case for $\nu=0,\Delta=\Delta'$ and $u_h=U_h$. The only nonzero $E-$terms are E^H, E^G and $\tilde E^H$. By (4.17), (4.8) and Lemma 3.3, we obtain for R,T fixed,

$$
E^H\le\frac{1}{\Delta}\Big\{\iint_{0\le t\le T\,x\in B_t}\Big(\alpha_H(t,x)\Big)^2dxdt\Big\}^{1/2}\Big\{\int_{0\le t\le T}|B_t|dt\Big\}^{1/2}\le\frac{1}{\Delta}(R+MT)^{d/2}T^{1/2}C\varepsilon^{1/2}\,.
$$

Similarly, (4.17), (4.6) and Lemma 3.4 imply

$$E^G = (1 + \frac{(1+M)T}{\Delta}) \sup_{0\le t\le 2T} \int_{B_t} \alpha_G(t,x)dx \le (R+MT)^{d/2} C_\eta (1 + \frac{MT}{\Delta}) \varepsilon^{1/2-\eta/2} \ .$$

Finally, (4.17) and Lemma 3.3 yields

$$\begin{aligned}
\tilde{E}^H &\le \frac{1}{\Delta} C \int_{0\le t\le T} \sum_{K\cap B_t^\Delta} |K| \sum_{e\in\partial K} \Big\{ a_e |w_K - w_{K_e}| + \sum_{i=1}^d b_e^i |z_{i,K} - z_{i,K_e}| \Big\} \\
&\le \frac{1}{\Delta} (R+MT)^{d/2} T^{1/2} C \Big\{ \int_{0\le t\le T} h \sum_{e\in\Gamma_h} |e| \Big\{ a_e |w_K - w_{K_e}|^2 + \sum_{i=1}^d b_e^i |z_{i,K} - z_{i,K_e}|^2 \Big\} \Big\}^{1/2} \\
&\le \frac{Ch^{1/2}}{\Delta} (R+MT)^{d/2} T^{1/2} \ .
\end{aligned}$$

Using the above estimates in Lemma 4.1 we have, for $t \le T$,

$$\int_{|x|<R} |u_h(t,x) - u(t,x)| dx \le C\Big(\Delta + \frac{(R+MT)^{d/2} T h^{1/2}}{\Delta} \Big) + \int_{B_0} |U_h(0,x) - u(0,x)| dx \, ,$$

and the proof of the theorem is complete by minimizing over Δ. □

5. Stability estimates for the fully discrete schemes

We consider now fully discrete finite volume schemes for (1.1-2) defined in section 2. In the proofs of the estimates in this section we shall assume that the following CFL condition is satisfied

$$\frac{\delta |\partial K|}{|K|} \max\{a_K, \frac{\bar{\mu}^2}{\mu^2} b_K\} \le \frac{1}{8} \ , \tag{5.1}$$

with $a_K = \max_{e\in\partial K} a_e, b_K = \max_{1\le i\le d} \max_{e\in\partial K} b_e^i$, and $\bar{\mu} = \bar{\mu}(a,b) = \max_{1\le i\le d} \sup_{\zeta\in[h_i(b), h_i(a)]} |\Psi_i''(\zeta)|$,

a_e, b_e^i, Ψ_i and μ are defined in Section 3, cf. (3.9). In Lemmas 5.1 and 5.2 we actually need a weaker mesh condition than (5.1), cf. (5.9).

We prove first a comparison principle analogous to Lemma 3.1.

Lemma 5.1. *Let $(w_{h,\delta}, Z_{h,\delta})$ and $(\overline{w}_{h,\delta}, \overline{Z}_{h,\delta})$ be two solutions of (2.7) that vanish outside a ball B_M of radius M. Let also $\mathcal{W}^n = w_K^n - \bar{w}_K^n$ and $\mathcal{Z}_i^n = z_{i,K}^n - \bar{z}_{i,K}^n, i = 1,\dots,d$, $K \in \mathcal{T}_h$. Then we have*

$$\begin{aligned}
&(\mathcal{W}^{n+1})^+ + \sum_{i=1}^d (\mathcal{Z}_i^{n+1})^- \\
&+ \frac{\delta}{|K|} \sum_{e\in\partial K} |e| \chi_{\mathcal{W}^{n+1}>0} \big[g^K(w_K^n \vee \bar{w}_K^n, w_{K_e}^n \vee \bar{w}_{K_e}^n) - g^K(w_K^n \wedge \bar{w}_K^n, w_{K_e}^n \wedge \bar{w}_{K_e}^n) \big] \\
&+ \frac{\delta}{|K|} \sum_{i=1}^d \chi_{\mathcal{Z}_i^{n+1}<0} \sum_{e\in\partial K} |e| \big[g_i^K(z_{i,K}^n \vee \bar{z}_{i,K}^n, z_{i,K_e}^n \vee \bar{z}_{i,K_e}^n) - g_i^K(z_{i,K}^n \wedge \bar{z}_{i,K}^n, z_{i,K_e}^n \wedge \bar{z}_{i,K_e}^n) \big] \\
&\le (\mathcal{W}^n)^+ + \sum_{i=1}^d (\mathcal{Z}_i^n)^- \ ,
\end{aligned} \tag{5.2}$$

provided the CFL- condition (5.1) holds.

Proof. Multiplying the equations that $\mathcal{W}^{n+1}$, $\mathcal{Z}_i^{n+1}$, satisfy cf. (2.7), by $\chi_{\mathcal{W}^{n+1}>0}$ and $-\chi_{\mathcal{Z}_i^{n+1}<0}$, respectively, sum over i and and using the monotonicity properties of $G_i,\ i=1,\dots,d$, we obtain

$$(5.3)\qquad (\mathcal{W}^{n+1})^+ + \sum_{i=1}^{d}(\mathcal{Z}_i^{n+1})^- - \frac{\delta}{|K|}\sum_{e\in\partial K}|e|\big(T_w^e + \sum_{i=1}^{d}T_{z_i}^e\big) \le \chi_{\mathcal{W}^{n+1}>0}\mathcal{W}^n - \sum_{i=1}^{d}\chi_{\mathcal{Z}_i^{n+1}<0}\mathcal{Z}_i^n,$$

where $T_w^e := -\chi_{\mathcal{W}^{n+1}>0}\big[g^K(w_K^n,w_{K_e}^n) - g^K(\bar w_K^n,\bar w_{K_e}^n)\big]$, and $T_{z_i}^e := \chi_{\mathcal{Z}_i^{n+1}<0}\big[g_i^K(z_{i,K}^n,z_{i,K_e}^n) - g_i^K(\bar z_{i,K}^n,\bar z_{i,K_e}^n)\big]$. Next, we estimate $T_w = T_w^e$ and $T_{z_i} = T_{z_i}^e$. We distinguish two cases :
(i). First we assume that $\chi_{\mathcal{W}^{n+1}>0} = \chi_{\mathcal{W}^n>0}$ and $\chi_{\mathcal{Z}_i^{n+1}<0} = \chi_{\mathcal{Z}_i^n<0}$. Then, as in Lemma 3.1, we have

$$T_w \le -\chi_{\mathcal{W}^{n+1}>0}\big[g^K(w_K^n\vee\bar w_K^n, w_{K_e}^n\vee\bar w_{K_e}^n) - g^K(w_K^n\wedge\bar w_K^n, w_{K_e}^n\wedge\bar w_{K_e}^n)\big],$$
$$T_{z_i} \le -\chi_{\mathcal{Z}_i^{n+1}<0}\big[g_i^K(z_{i,K}^n\vee\bar z_{i,K}^n, z_{i,K_e}^n\vee\bar z_{i,K_e}^n) - g^K(z_{i,K}^n\wedge\bar z_{i,K}^n, z_{i,K_e}^n\wedge\bar z_{i,K_e}^n)\big].$$

Hence (5.2) follows.
(ii). Now suppose that $\chi_{\mathcal{W}^{n+1}>0} = \chi_{\mathcal{W}^n<0}$ and $\chi_{\mathcal{Z}_i^{n+1}<0} = \chi_{\mathcal{Z}_i^n>0}$.

For the first term we have

$$\begin{aligned}
T_w \le &- \chi_{\mathcal{W}^n<0}\big[g^K(w_K^n\vee\bar w_K^n, w_{K_e}^n\vee\bar w_{K_e}^n) - g^K(w_K^n\wedge\bar w_K^n, w_{K_e}^n\wedge\bar w_{K_e}^n)\big]\\
&+ \chi_{\mathcal{W}^n<0}\big[g^K(w_K^n\vee\bar w_K^n, w_{K_e}^n\vee\bar w_{K_e}^n) - g^K(w_K^n\wedge\bar w_K^n, w_{K_e}^n\vee\bar w_{K_e}^n)\big]\\
&- \chi_{\mathcal{W}^n<0}\big[g^K(w_K^n\wedge\bar w_K^n, w_{K_e}^n\wedge\bar w_{K_e}^n) - g^K(w_K^n\vee\bar w_K^n, w_{K_e}^n\wedge\bar w_{K_e}^n)\big]\\
= &- \chi_{\mathcal{W}^n<0}\big[g^K(w_K^n\vee\bar w_K^n, w_{K_e}^n\vee\bar w_{K_e}^n) - g^K(w_K^n\wedge\bar w_K^n, w_{K_e}^n\wedge\bar w_{K_e}^n)\big]\\
&+ 2a\chi_{\mathcal{W}^n<0}(w_K^n\vee\bar w_K^n - w_K^n\wedge\bar w_K^n),
\end{aligned}$$

with $a = \frac{\partial g^K(u,v)}{\partial u} = a_e + A\cdot\nu/2 \ge 0$. If $\chi_{\mathcal{W}^n<0} = 1$ then

$$(5.4)\qquad T_w \le -\chi_{\mathcal{W}^n<0}\big[g^K(w_K^n\vee\bar w_K^n, w_{K_e}^n\vee\bar w_{K_e}^n) - g^K(w_K^n\wedge\bar w_K^n, w_{K_e}^n\wedge\bar w_{K_e}^n)\big] + 2a(\mathcal{W}^n)^-,$$

and if $\chi_{\mathcal{W}^n<0} = 0$ then (5.4) holds trivially. A similar argument applies for the $\mathcal{Z}_i^n$–term as well, and therefore, if $b_i = b_e^i + B_i\cdot\nu/2 \ge 0$,

$$(5.5)\quad T_{z_i} \le -\chi_{\mathcal{Z}_i^{n+1}<0}\big[g_i^K(z_{i,K}^n\vee\bar z_{i,K}^n, z_{i,K_e}^n\vee\bar z_{i,K_e}^n) - g^K(z_{i,K}^n\wedge\bar z_{i,K}^n, z_{i,K_e}^n\wedge\bar z_{i,K_e}^n)\big] + 2b_i(\mathcal{Z}_i^n)^+.$$

Now if $\chi_{\mathcal{W}^{n+1}>0} = \chi_{\mathcal{W}^n<0} = 1$ (otherwise the inequality reduces to a trivial one) we have $\chi_{\mathcal{W}^n>0} = 0$ and similarly $\chi_{\mathcal{Z}_i^n<0} = 0$; hence, from (5.4), (5.5) and (5.3) we get relation (5.2) with right-hand side

$$(\mathcal{W}^n)^+ + \sum_{i=1}^{d}(\mathcal{Z}_i^n)^- + \Big\{-1 + \frac{\delta|\partial K|}{|K|}4a_K\Big\}(\mathcal{W}^n)^- + \sum_{i=1}^{d}\Big\{-1 + \frac{\delta|\partial K|}{|K|}4b_K\Big\}(\mathcal{Z}_i^n)^+ .$$

But then the CFL condition (5.1) implies that the last two terms are non positive and (5.2) follows. The other cases are treated similarly and the proof of the lemma is complete. □

The comparison principle (5.2) now gives the L^1–contraction property of the fully discrete scheme as well as a discrete entropy inequality.

Lemma 5.2. (i) *Under the assumptions of Lemma 5.1 we have*

$$\begin{aligned}\|w_{h,\delta}(t^{n+1}) - \bar w_{h,\delta}(t^{n+1})\|_{L^1} + \sum_{i=1}^{d} \|z_{i,h,\delta}(t^{n+1}) - \bar z_{i,h,\delta}(t^{n+1})\|_{L^1} \\ \le \|w_{h,\delta}(t^{n}) - \bar w_{h,\delta}(t^{n})\|_{L^1} + \sum_{i=1}^{d} \|z_{i,h,\delta}(t^{n}) - \bar z_{i,h,\delta}(t^{n})\|_{L^1}.\end{aligned} \tag{5.6}$$

(ii) *(Entropy Inequality). For any $\xi \in \mathbb{R}$ we have*

$$\begin{aligned}|w_K^{n+1} - \xi| + \sum_{i=1}^{d} |z_{i,K}^{n+1} - h_i(\xi)| + \frac{\delta}{|K|}\sum_{e\in\partial K} |e| \Big\{ D_\xi^K(w_K^n, w_{K_e}^n) + \sum_{i=1}^{d} D_\xi^{i,K}(z_{i,K}^n, z_{i,K_e}^n)\Big\} \\ \le |w_K^n - \xi| + \sum_{i=1}^{d} |z_{i,K}^n - h_i(\xi)|.\end{aligned} \tag{5.7}$$

(iii) *If, for some $a < b$, we have $a \le w_K^0 \le b$, $h_i(b) \le z_{i,K}^0 \le h_i(a)$, $i = 1, \dots, N, K \in \mathcal{T}_h$, then*

$$a \le w_K^n \le b,\ h_i(b) \le z_{i,K}^n \le h_i(a),\ K \in \mathcal{T}_h,\ i = 1, \dots, N,$$

i.e., the region $\mathcal{R}^{a,b} = [a,b] \times \prod_{i=1}^{d}[h_i(b), h_i(a)]$ is positively invariant.

Proof. (i) The L^1 contraction property is a direct application comparison principle and of the fact that $g^K(|w_K^n - \bar w_K^n|, |w_{K_e}^n - \bar w_{K_e}^n|)] + g^{K_e}(|w_{K_e}^n - \bar w_{K_e}^n|, |w_K^n - \bar w_K^n|)] = 0$, cf. Lemma 3.2.

(ii) Letting $\bar w_K^n = \xi$ and $\bar z_{i,K}^n = h_i(\xi)$, $i = 1, \dots, d$, for any $\xi \in \mathbb{R}$, (5.2) yields (5.7).

(iii) To prove this part it suffices to combine arguments similar to those used in the proofs of Lemma 3.2(ii) and Lemma 5.1. Let us only notice that in the case $\chi_{\mathcal{W}^{n+1}>0} = \chi_{\mathcal{W}^n<0}$ and $\chi_{\mathcal{Z}_i^{n+1}<0} = \chi_{\mathcal{Z}_i^n>0}$, one can see that

$$\begin{aligned}\chi_{\mathcal{W}^{n+1}>0} g^K(w_K^n - \bar w_K^n, w_{K_e}^n - \bar w_{K_e}^n) &= \chi_{\mathcal{W}^n<0} g^K(w_K^n - \bar w_K^n, w_{K_e}^n - \bar w_{K_e}^n) \\ &\ge g^K((w_K^n - \bar w_K^n)^+, (w_{K_e}^n - \bar w_{K_e}^n)^+) - 2a_e(w_K^n - \bar w_K^n)^- \\ -\chi_{\mathcal{Z}_i^{n+1}<0} g_i^K(z_{i,K}^n - \bar z_{i,K}^n, z_{i,K_e}^n - \bar z_{i,K_e}^n) &= -\chi_{\mathcal{Z}_i^n>0} g_i^K(z_{i,K}^n - \bar z_{i,K}^n, z_{i,K_e}^n - \bar z_{i,K_e}^n) \\ &\ge -g^K((z_{i,K}^n - \bar z_{i,K}^n)^-, (z_{i,K_e}^n - \bar z_{i,K_e}^n)^-) - 2b_e^i(z_{i,K}^n - \bar z_{i,K}^n)^+,\end{aligned}$$

and the CFL condition implies

$$\sum_{K\in\mathcal{T}_h} |K| \Big\{ (\mathcal{W}^{n+1})^+ + \sum_{i=1}^{d} (\mathcal{Z}_i^{n+1})^- \Big\} \le \sum_{K\in\mathcal{T}_h} |K| \Big\{ (\mathcal{W}^{n})^+ + \sum_{i=1}^{d} (\mathcal{Z}_i^{n})^- \Big\}.$$

Hence (iii) follows. □

Next we show the analogous of Lemma 3.3 for the fully discrete scheme.

Lemma 5.3. *Assume that the initial data $(w_K^0, z_{i,K}^0) \in \mathcal{R}^{a,b}, i = 1, \dots, d$, for some $a, b \in \mathbb{R}$. Then if $\sigma = \sigma(a,b)$ and $\mu = \mu(a,b)$ are the constants of (A.3) and (3.9), respectively, there holds*

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} |K| \Big\{ \frac{1}{2}(w_K^{n+1})^2 + \sum_{i=1}^{d} \Psi_i(z_{i,K}^{n+1}) \Big\} + \delta \sum_{e\in\Gamma_h} \frac{a_e}{2} |e| (w_K^{n+1} - w_{K_e}^{n+1})^2 \\ + \delta \sum_{i=1}^{d} \sum_{e\in\Gamma_h} \mu \frac{b_e^i}{2} |e| (z_{i,K}^{n+1} - z_{i,K_e}^{n+1})^2 + \frac{\sigma\delta}{\varepsilon} \sum_{i=1}^{d} \sum_{K\in\mathcal{T}_h} |K| (h_i(w_K^{n+1}) - z_{i,K}^{n+1})^2 \\ \le \sum_{K\in\mathcal{T}_h} |K| \Big\{ \frac{1}{2}(w_K^{n})^2 + \sum_{i=1}^{d} \Psi_i(z_{i,K}^{n}) \Big\},\end{aligned} \tag{5.8}$$

provided that the CFL condition (5.1) holds.

Proof. We multiply (2.7a) by w_K^{n+1} and (2.7b) by $h_i^{-1}(z_{i,K}^{n+1})$, to obtain

$$(w_K^{n+1})^2 = w_K^{n+1} w_K^n - \frac{\delta}{|K|} \sum_{e\in\partial K} |e| w_K^{n+1} g^K(w_K^n, w_{K_e}^n) + \frac{\delta}{\varepsilon} \sum_{i=1}^d w_K^{n+1} G_i(w_K^{n+1}, z_{i,K}^{n+1})$$

and

$$\begin{aligned} -\Psi_i'(z_{i,K}^{n+1}) z_{i,K}^{n+1} = &- \Psi_i'(z_{i,K}^{n+1}) z_{i,K}^n \\ &- \frac{\delta}{|K|} \sum_{e\in\partial K} |e| h_i^{-1}(z_{i,K}^{n+1}) g_i^K(z_{i,K}^n, z_{i,K_e}^n) + \frac{\delta}{\varepsilon} h_i^{-1}(z_{i,K}^{n+1}) G_i(w_K^{n+1}, z_{i,K}^{n+1}). \end{aligned}$$

Now, since Ψ_i satisfies (3.9), $\Psi_i'(y)(x-y) \le \Psi_i(x) - \Psi_i(y) - \frac{\mu}{2}(x-y)^2$, we get by summing over i and subtracting the resulting relations

$$\begin{aligned} \sum_{K\in\mathcal{T}_h} |K| \Big\{ \frac{1}{2}(w_K^{n+1})^2 + \sum_{i=1}^d \Psi_i(z_{i,K}^{n+1}) \Big\} + \sum_{K\in\mathcal{T}_h} |K| \frac{1}{2}(w_K^n - w_K^{n+1})^2 + \sum_{K\in\mathcal{T}_h} |K| \frac{\mu}{2}(z_{i,K}^n - z_{i,K}^{n+1})^2 \\ + \delta \sum_{K\in\mathcal{T}_h} \sum_{e\in\partial K} |e| w_K^{n+1} g^K(w_K^n, w_{K_e}^n) - \delta \sum_{i=1}^d \sum_{K\in\mathcal{T}_h} \sum_{e\in\partial K} |e| h_i^{-1}(z_{i,K}^{n+1}) g_i^K(z_{i,K}^n, z_{i,K_e}^n) \\ + \frac{\delta}{\varepsilon} \sum_{i=1}^d \sum_{K\in\mathcal{T}_h} |K| \phi_i(w_K^{n+1}, z_{i,K}^{n+1}) \le \sum_{K\in\mathcal{T}_h} |K| \Big\{ \frac{1}{2}(w_K^n)^2 + \sum_{i=1}^d \Psi_i(z_{i,K}^n) \Big\}, \end{aligned}$$

where $\phi_i(w, z_i) = -(w - h_i^{-1}(z)) G_i(w, z_i)$. As in Lemma 3.3, and in view of our assumptions on h_i, G_i, we have $\phi_i(w, z_i) \ge \sigma(h_i(w) - z_i)^2$.

We next estimate the terms corresponding to the w-fluxes. We have

$$\begin{aligned} \sum_{K\in\mathcal{T}_h} \sum_{e\in\partial K} |e| w_K^{n+1} g^K(w_K^n, w_{K_e}^n) = \sum_{K\in\mathcal{T}_h} \sum_{e\in\partial K} |e| w_K^{n+1} g^K(w_K^{n+1}, w_{K_e}^{n+1}) \\ + \sum_{K\in\mathcal{T}_h} \sum_{e\in\partial K} |e| \Big\{ w_K^{n+1} g^K(w_K^n, w_{K_e}^n) - w_K^{n+1} g^K(w_K^{n+1}, w_{K_e}^{n+1}) \Big\} := W_1 + W_2 \end{aligned}$$

The W_1–term is treated like in the semidiscrete case and we have,

$$W_1 = \sum_{e\in\Gamma_h} |e| a_e (w_K^{n+1} - w_{K_e}^{n+1})^2.$$

On the other hand we have for the W_2–term, using the arith. geom. mean inequality and $\frac{|A\cdot\nu_{e,K}|}{2} \le a_e$,

$$\begin{aligned} -W_2 &\le \sum_{e\in\Gamma_h} |e| 2a_e |w_{K_e}^{n+1} - w_K^{n+1}| \Big(|w_K^n - w_K^{n+1}| + |w_{K_e}^n - w_{K_e}^{n+1}| \Big) \\ &\le \sum_{e\in\Gamma_h} |e| \Big\{ \frac{a_e}{2}(w_{K_e}^{n+1} - w_K^{n+1})^2 + 4a_e(w_K^n - w_K^{n+1})^2 + 4a_e(w_{K_e}^n - w_{K_e}^{n+1})^2 \Big\}. \end{aligned}$$

For the terms corresponding to the z–fluxes we have

$$-\sum_{i=1}^{d}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}|e|h_i^{-1}(z_{i,K}^{n+1})g_i^K(z_{i,K}^n,z_{i,K_e}^n)=-\sum_{i=1}^{d}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}|e|h_i^{-1}(z_{i,K}^{n+1})g_i^K(z_{i,K}^{n+1},z_{i,K_e}^{n+1})$$
$$+\sum_{i=1}^{d}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}|e|\Big\{h_i^{-1}(z_{i,K}^{n+1})g_i^K(z_{i,K}^{n+1},z_{i,K_e}^{n+1})-h_i^{-1}(z_{i,K}^{n+1})g_i^K(z_{i,K}^n,z_{i,K_e}^n)\Big\}:=Z_1+Z_2.$$

As before the Z_1–term is treated as in the semidiscrete case, $Z_1\ge\sum_{i=1}^d\sum_{e\in\Gamma_h}|e|\,\mu\,b_e^i(z_{i,K}^{n+1}-z_{i,K_e}^{n+1})^2$. Further for the Z_2–term we have

$$-Z_2\le\sum_{i=1}^{d}\sum_{e\in\Gamma_h}|e|\,2b_e^i\left|\Psi_i'(z_{i,K}^{n+1})-\Psi_i'(z_{i,K_e}^{n+1})\right|\big(|z_{i,K}^{n+1}-z_{i,K}^n|+|z_{i,K_e}^{n+1}-z_{i,K_e}^n|\big).$$

Hence using the arith. geom. mean inequality we get

$$-Z_2\le\sum_{i=1}^{d}\sum_{e\in\Gamma_h}|e|\Big\{\mu\frac{b_e^i}{2}(z_{i,K}^{n+1}-z_{i,K_e}^{n+1})^2+4\frac{\bar\mu^2}{\mu}b_e^i\Big((z_{i,K}^{n+1}-z_{i,K}^n)^2+(z_{i,K_e}^{n+1}-z_{i,K_e}^n)^2\Big)\Big\}.$$

Notice now that

$$\sum_{e\in\Gamma_h}|e|\Big\{4a_e(w_K^n-w_K^{n+1})^2+4a_e(w_{K_e}^n-w_{K_e}^{n+1})^2\Big\}=\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}|e|4a_e(w_K^n-w_K^{n+1})^2.$$

A similar relation holds for the z-terms. Therefore

$$\sum_{K\in\mathcal{T}_h}|K|\Big\{\frac12(w_K^{n+1})^2+\sum_{i=1}^d\Psi_i(z_{i,K}^{n+1})\Big\}+\sum_{K\in\mathcal{T}_h}|K|\frac12(w_K^n-w_K^{n+1})^2+\sum_{K\in\mathcal{T}_h}|K|\frac{\mu}{2}(z_{i,K}^n-z_{i,K}^{n+1})^2$$
$$+\delta\sum_{e\in\Gamma_h}|e|\Big(\frac{a_e}{2}(w_K^{n+1}-w_{K_e}^{n+1})^2+\sum_{i=1}^d\mu\,\frac{b_e^i}{2}(z_{i,K}^{n+1}-z_{i,K_e}^{n+1})^2\Big)+\frac{\delta}{\varepsilon}\sum_{i=1}^d\sum_{K\in\mathcal{T}_h}|K|\sigma(h_i(w_K^{n+1})-z_{i,K}^{n+1})^2$$
$$\le\sum_{K\in\mathcal{T}_h}\Big(|K|\Big\{\frac12(w_K^n)^2+\sum_{i=1}^d\Psi_i(z_{i,K}^n)\Big\}+\delta|\partial K|4a_K(w_K^n-w_K^{n+1})^2+\delta|\partial K|\frac{4b_K^i\bar\mu^2}{\mu}(z_{i,K}^{n+1}-z_{i,K}^n)^2\Big)$$

and (5.8) follows by using the CFL condition (5.1). □

Lemma 5.4. *Let $(w_{h,\delta},Z_{h,\delta})$ be a solution of the scheme (2.7) emanating from data with finite total variation and lying in an (invariant) region $\mathcal{R}^{a,b}$. In addition to (A.1-3) we assume that*

$$\Big(-\frac{\partial G_i}{\partial z_i}-\sum_j\Big|\frac{\partial G_j}{\partial w}\Big|\Big)\ge c_1>0\quad\text{ in }\mathcal{R}^{a,b}.\tag{5.9}$$

Let $\varepsilon^{1-\eta}\le C\delta$, where $1>\eta>0$ is any small number. Assume further that (3.13) and (5.8) hold. Then for any $n=0,1,\dots$, there holds

$$\sum_{K\in\mathcal{T}_h}|K|\Big(|w_K^{n+1}-w_K^n|+\sum_i|z_{i,K}^{n+1}-z_{i,K}^n|\Big)\le C\delta\quad\text{and}\tag{5.10}$$

$$\sum_{K\in\mathcal{T}_h}|K|\sum_i|G_i(w_K^n,z_{i,K}^n)|^2\le C\delta.\tag{5.11}$$

Proof. By the L^1 contraction property (Lemma 5.2 (i)) it follows,

$$\sum_{K\in\mathcal{T}_h} |K|\Big(|w_K^{n+1}-w_K^n| + \sum_i |z_{i,K}^{n+1}-z_{i,K}^n|\Big) \le \sum_{K\in\mathcal{T}_h} |K|\Big(|w_K^1-w_K^0| + \sum_i |z_{i,K}^1-z_{i,K}^0|\Big).$$

To estimate $w_K^1 - w_K^0$ and $z_{i,K}^1 - z_{i,K}^0$ we have to estimate first $|G_i(w_K^1, z_{i,K}^1)|$: As in [KKM, Proposition 4.6] we have

$$\begin{aligned} G_i(w_K^1,&z_{i,K}^1) - G_i(w_K^0, z_{i,K}^0) = \\ &= \Big(\int_0^1 \frac{\partial G_i}{\partial w}(\ell(s))ds\Big)\Big[-\delta\sum_{e\in\partial K}\frac{|e|}{|K|}g^K(w_K^0, w_{K_e}^0) + \frac{\delta}{\varepsilon}\sum_j G_j(w_K^1, z_{j,q}^1)\Big] \\ &+ \Big(\int_0^1 \frac{\partial G_i}{\partial z}(\ell(s))ds\Big)\Big[-\delta\sum_{e\in\partial K}\frac{|e|}{|K|}g_i^K(z_{i,K}^0, z_{i,K_e}^0) + \frac{\delta}{\varepsilon}G_i(w_K^1, z_{i,K}^1)\Big]\,, \end{aligned} \tag{5.12}$$

where $\ell(s) = \big(w_K^0 + (w_K^1 - w_K^0)s, z_{i,K}^0 + (z_{i,K}^1 - z_{i,K}^0)s\big)$. Multiplying (5.12) by $\operatorname{sgn} G_i(w_K^1, z_{i,K}^1)$, and summing with respect to i, we finally obtain,

$$\begin{aligned} \sum_{i=1}^d |K|\,|G_i(w_K^1, z_{i,K}^1)| + \frac{\delta}{\varepsilon}\sum_{i=1}^d\int_0^1\Big(-\frac{\partial G_i}{\partial z_i}(\ell(s)) - \sum_j\Big|\frac{\partial G_j}{\partial w}(\ell(s))\Big|\Big)ds\,|K|\,|G_i(w_K^1, z_{i,K}^1)| \\ \le \sum_{i=1}^d |K|\,|G_i(w_K^0, z_{i,K}^0)| + C\delta\sum_{e\in\partial K}|e|\Big(|w_K^0 - w_{K_e}^0| + \sum_{i=1}^d |z_{i,K}^0 - z_{i,K_e}^0|\Big)\,. \end{aligned}$$

Hence using (5.9), (2.7) implies

$$\begin{aligned} \sum_{K\in\mathcal{T}_h} |K|\Big(|w_K^1 - &w_K^0| + \sum_i |z_{i,K}^1 - z_{i,K}^0|\Big) \\ &\le C\sum_{i=1}^d |K|\,|G_i(w_K^0, z_{i,K}^0)| + C\delta\sum_{e\in\partial K}|e|\Big(|w_K^0 - w_{K_e}^0| + \sum_{i=1}^d |z_{i,K}^0 - z_{i,K_e}^0|\Big)\,. \end{aligned}$$

Thus (5.10) follows in view of the BV stability of the L^2 projection, cf [C], and of (3.13).

To prove (5.11), we start from (5.12) for $n, n+1$ instead of $0, 1$, we multiply by $G_i(w_K^{n+1}, z_{i,K}^{n+1})$, and as in the semidiscrete case, cf. (3.14), we finally obtain

$$\Big(1 + \frac{c_1\delta}{\varepsilon}\Big)\sum_{K\in\mathcal{T}_h}|K|\sum_i |G_i(w_K^{n+1}, z_{i,K}^{n+1})|^2 \le \sum_{K\in\mathcal{T}_h}|K|\sum_i |G_i(w_K^n, z_{i,K}^n)|^2 + C\mathcal{A}^n\,, \tag{5.13}$$

where

$$\begin{aligned} \mathcal{A}^n =&\sigma\frac{\delta}{\varepsilon}\sum_{K\in\mathcal{T}_h}|K|\sum_{i=1}^d (h_i(w_K^n) - z_{i,K}^n)^2 \\ &+ \delta\sum_{e\in\Gamma_h}|e|\,a_e(w_K^n - w_{K_e}^n)^2 + \delta\sum_{i=1}^d\sum_{e\in\Gamma_h}|e|\mu b_e^i(z_{i,K}^n - z_{i,K_e}^n)^2\,. \end{aligned} \tag{5.14}$$

Relation (5.13) implies that for any n,
(5.15)

$$\sum_{K\in\mathcal{T}_h}|K|\sum_i|G_i(w_K^n,z_{i,K}^n)|^2\le\Big(1+\frac{c_1\delta}{\varepsilon}\Big)^{-n}\sum_{K\in\mathcal{T}_h}|K|\sum_i|G_i(w_K^0,z_{i,K}^0)|^2+C\sum_{j=0}^{n-1}\Big(1+\frac{c_1\delta}{\varepsilon}\Big)^{-(n-j)}\mathcal{A}^j\,.$$

Next we show that the last term of (5.15) is of the order $O(\delta)$. To this end, let $1>\eta>0$, an arbitrary small number. In view of our assumption $\varepsilon^{1-\eta}\le C\delta$, we see that as $\varepsilon\to 0$

$$\varepsilon^{-\eta}\Big(1+\frac{c_1\delta}{\varepsilon}\Big)^{-1}\le\varepsilon^{-\eta}\Big(1+\frac{C}{\varepsilon^\eta}\Big)^{-1}\le C_\eta.$$

Let M_η be the smallest integer such that $\eta M_\eta\ge 1$. By Lemma 5.3, $\sum_{j=0}^n\mathcal{A}^j\le C$, and therefore,

$$\begin{aligned}\sum_{j=0}^{n-1}\Big(1+\frac{c_1\delta}{\varepsilon}\Big)^{-(n-j)}\mathcal{A}^j&=\sum_{j=0}^{n-M_\eta}\Big(1+\frac{c_1\delta}{\varepsilon}\Big)^{-(n-j)}\mathcal{A}^j+\sum_{j=n-M_\eta+1}^{n-1}\Big(1+\frac{c_1\delta}{\varepsilon}\Big)^{-(n-j)}\mathcal{A}^j\\&\le\Big(1+\frac{C}{\varepsilon^\eta}\Big)^{-M_\eta}\sum_{j=0}^{n-M_\eta}\mathcal{A}^j+\Big(1+\frac{C}{\varepsilon^\eta}\Big)^{-1}\sum_{j=n-M_\eta+1}^{n-1}\mathcal{A}^j\\&\le C\varepsilon^{\eta\,M_\eta}+C\varepsilon^\eta\sum_{j=n-M_\eta+1}^{n-1}\mathcal{A}^j\le C\varepsilon+C\varepsilon^\eta\sum_{j=n-M_\eta+1}^{n-1}\mathcal{A}^j\,.\end{aligned}$$

Using (5.8) and (5.10) each $\mathcal{A}^j$ can be estimated as (compare with (3.17))

$$\mathcal{A}^j\le C(a,b)\sum_{K\in\mathcal{T}_h}|K|\Big(|w_K^j-w_K^{j-1}|+\sum_i|z_{i,K}^j-z_{i,K}^{j-1}|\Big)\le C\delta\,.$$

Hence $\sum_{j=0}^{n-1}\Big(1+\frac{c_1\delta}{\varepsilon}\Big)^{-(n-j)}\mathcal{A}^j\le C\,\varepsilon+CM_\eta\varepsilon^\eta\,\delta\le C\,\delta$, and the proof is complete . ☐

Remark 5.1. The assumption (5.9) was used only to show

(5.9′)
$$\sum_{K\in\mathcal{T}_h}|K|\Big(|w_K^1-w_K^0|+\sum_i|z_{i,K}^1-z_{i,K}^0|\Big)\le C\delta.$$

So, it can be replaced by assuming that (5.9′) holds. In any case, Lemma 4.1 of [KT1] shows that (5.9) is satisfied by choosing h_i, and G_i appropriately. Note also that this assumption was used in the proofs of the error estimates in [KT1] and [KKM].

6. Convergence of the fully discrete scheme

As in section 4 we will use Lemma 4.1. For any $k\in\mathbb{R}$ let $\xi\in\mathbb{R}$ such that $k=\xi-\sum_{i=1}^d h_i(\xi)$ and we set

$$U_{h,\delta}=w_{h,\delta}-\sum_{i=1}^d h_i(w_{h,\delta})\ \text{ and } U_K^n=w_K^n-\sum_{i=1}^d h_i(w_K^n),\ K\in\mathcal{T}_h\,,\quad n=0,1,\ldots\,.$$

Also,

(6.1)
$$|U_K^n-k|=|w_K^n-\xi|+\sum_{i=1}^d|z_{i,K}^n-h_i(\xi)|+J_K^n\ \text{ with }\ |J_K^n|\le\frac{1}{\sigma}\sum_{i=1}^d|G_i(w_K^n,z_{i,K}^n)|,$$

and

$$\Big[F(U_K^n)-F(k)\Big]\mathrm{sgn}(U_K^n-k)=|w_K^n-\xi|A+\sum_{i=1}^d |z_{i,K}^n-h_i(\xi)|B_i+H_K^n\,,\qquad \text{with } |H_K^n|\le \sum_{i=1}^d |h_i(w_K^n)-z_{i,K}^n||B_i|\,. \tag{6.2}$$

We prove the following convergence result for the fully discrete scheme

Theorem 6.1. *Let u be the entropy solution of the conservation law (1.3) with initial data $u_0\in BV(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, and let $\|u_0-(w_0^\varepsilon-\sum_{i=1}^d h_i(w_0^\varepsilon))\|_{L^1}\le C\varepsilon$. For $U_{h,\delta}=w_{h,\delta}-\sum_{i=1}^d h_i(w_{h,\delta})$, where $(w_{h,\delta},Z_{h,\delta})$ is the solution of the finite volume scheme (2.7), assume that the assumptions of Lemma 5.4 hold. Then for any time $t\le T$, and $R>0$ there is a constant $C=(R+MT)^{d/4}T^{1/2}c(a,b)$ such that the following error estimate holds*

$$\|u(\cdot,t)-U_{h,\delta}(\cdot,t)\|_{L^1(B(0,R))}\le C\left(h^{1/4}+\delta^{1/4}\right)+\|u(\cdot,0)-U_{h,\delta}(\cdot,0)\|_{L^1}\,.$$

If in addition $\|u_0-(w_0^\varepsilon-\sum_{i=1}^d h_i(w_0^\varepsilon))\|_{L^1}\le C\varepsilon$, then

$$\|u(\cdot,t)-U_{h,\delta}(\cdot,t)\|_{L^1(B(0,R))}\le C\left(h^{1/4}+\delta^{1/4}\right)$$

Proof. Let Ψ a nonnegative test function with compact support and set

$$V_K^n:=|U_K^n-k|\quad \text{and}\quad V_{F,K}^n:=[F(U_K^n)-F(k)]\mathrm{sgn}(U_K^n-k).$$

To apply Lemma 4.1 we have to estimate

$$E:=-\sum_{n=0}^\infty\int_{I_n}\sum_{K\in\mathcal{T}_h}\int_K[V_K^n\Psi_t+V_{F,K}^n\cdot\nabla_x\Psi]dx\,dt=:-(E_1+E_2), \tag{6.3}$$

where $I_n=[n\delta,(n+1)\delta)$.

For term corresponding to space discretization we have as in the semidiscrete case,

$$E_2=\sum_{n=0}^\infty\sum_{e\in\Gamma_h}(V_{F,K}^n\cdot\nu_{e,K}+V_{F,K_e}^n\cdot\nu_{e,K_e})\int_{I_n}\bar\Psi^e\quad \text{where}\quad \bar\Psi^e=\int_e\Psi\,ds\,. \tag{6.4}$$

To estimate the other term, let $W_K^n:=|w_K^n-\xi|+\sum_{i=1}^d|z_{i,K}^n-h_i(\xi)|$ then $V_K^n=W_K^n+J_K^n$, cf., (6.1). We then have summing by parts for $\bar\Psi^K=\int_K\Psi\,dt$ and $\bar\Psi_t^K=\int_K\Psi_t\,dt$,

$$\begin{aligned}E_1&=\sum_{n=0}^\infty\sum_{K\in\mathcal{T}_h}V_K^n\int_{I_n}\int_K\Psi_t\,dt\,dx=\sum_{n=0}^\infty\sum_{K\in\mathcal{T}_h}W_K^n\int_{I_n}\bar\Psi_t^K\,dt+\sum_{n=0}^\infty\sum_{K\in\mathcal{T}_h}J_K^n\int_{I_n}\bar\Psi_t^K\,dt\\&=\sum_{K\in\mathcal{T}_h}\sum_{n=0}^\infty\left(W_K^n-W_K^{n+1}\right)\frac{1}{\delta}\int_{I_n}\bar\Psi^K dt+\sum_{K\in\mathcal{T}_h}\sum_{n=0}^\infty\left(W_K^n-W_K^{n+1}\right)\Big[\bar\Psi^K(t^{n+1})-\frac{1}{\delta}\int_{I_n}\bar\Psi^K dt\Big]\\&\quad+\sum_{n=0}^\infty\sum_{K\in\mathcal{T}_h}J_K^n\int_{I_n}\bar\Psi_t^K\,dt\,.\end{aligned}$$

We multiply the discrete entropy inequality (5.7) by $\frac{1}{\delta}\int_{I_n} \bar\Psi^K dt$ and we sum with respect to K and n. As in the proof of Theorem 4.1 we finally conclude

$$
\begin{aligned}
E \leq & \sum_{n=0}^{\infty} \sum_{K\in\mathcal{T}_h} \sum_{e\in\partial K} \Big\{ a_e|w_K^n - w_{K_e}^n| + \sum_{i=1}^{d} b_e^i |z_{i,K}^n - z_{i,K_e}^n| \Big\} \int_{I_n} \Big| \bar\Psi^e - \frac{|e|}{|K|}\bar\Psi^K \Big| dt \\
& + \sum_{n=0}^{\infty} \sum_{K\in\mathcal{T}_h} \big|W_K^{n+1} - W_K^n\big| \int_K \Big| \Psi(t^{n+1},x) - \frac{1}{\delta}\int_{I_n} \Psi(t,x)dt \Big| dx \qquad (6.5) \\
& - \sum_{n=0}^{\infty} \sum_{K\in\mathcal{T}_h} \Big(J_K^n \int_{I_n} \bar\Psi_t^K dt + H_K^n \cdot \int_{I_n}\int_K \nabla_x \Psi dx \Big) dt.
\end{aligned}
$$

To adjust to the notation of Lemma 4.1 let $\mathcal{S}_h = \{S_K^n\}$, $S_K^n = I_n \times K$, $K \in \mathcal{T}_h$ be a partition of $[0,+\infty)\times\mathbb{R}^d$. Then, for any $t>0$, $(S_K^n)_t = K$. If

$$
B_H(\nabla_x\Psi)\Big|_{S^K}(x,t) = \frac{1}{|K|}\Big| |e|\Psi(x,t) - \bar\Psi^e(t) \Big|,
$$

then (4.4) is satisfied, cf. (4.16). Let also

$$
B_G(\partial_t\Psi)\Big|_{S_K^n}(x,t) = \frac{1}{\delta^2}\int_{I_n} \Big| \Psi(t^{n+1},x) - \Psi(t,x)dt \Big| .
$$

If $\Psi = vw$, then

$$
|\Psi(t^{n+1},x) - \Psi(t,x)| \leq |(v(t^{n+1},x) - v(t,x))w(t^{n+1},x)| + |(w(t^{n+1},x) - w(t,x))v(t,x)|,
$$

and (4.4) follows upon integrating. Therefore (6.5) implies that $U_{h,\delta}$ satisfies (4.2) if

$$
\begin{aligned}
&\alpha_H\Big|_{S_K^n} = |H_K^n| , \quad \alpha_G\Big|_{S_K^n} = |J_K^n| , \\
&\beta_H\Big|_{S_K^n} = C \sum_{e\in\partial K} \Big\{ a_e|w_K^n - w_{K_e}^n| + \sum_{i=1}^{d} b_e^i|z_{i,K}^n - z_{i,K_e}^n| \Big\}, \\
&\beta_G\Big|_{S_K^n} = |w_K^{n+1} - w_K^n| + \sum_i |z_{i,K}^{n+1} - z_{i,K}^n| .
\end{aligned}
$$

Then (4.5) holds for $\nu = 0, \Delta = \Delta'$ and $u_h = U_{h,\delta}$. Let R, T fixed. The terms E^H, E^G and $\tilde E^H$ are estimated as in the semidiscrete case using here the stability estimates of Lemmas 5.3 and 5.4. By Lemma 5.4,

$$
\tilde E^G = (1 + \frac{(1+M)T}{\Delta}) \sup_{0\leq t\leq 2T} \int_{B_t^\Delta} \beta_G(t,x)dx \leq C\delta .
$$

The desired result now follows by minimizing over Δ (4.5).

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