A POSTERIORI ESTIMATES FOR APPROXIMATIONS OF TIME DEPENDENT STOKES EQUATIONS

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ABSTRACT. In this paper we derive a posteriori error estimates for space discrete approximations of the time dependent Stokes equations. By using an appropriate Stokes reconstruction operator we are able to write an auxiliary error equation in pointwise form that satisfies the exact divergence free condition. Thus standard energy estimates from pde theory can be applied directly to yield a posteriori estimates that rely on available corresponding estimates of the stationary Stokes equation. Estimates of optimal order in $L^\infty(L^2)$ and in $L^\infty(H^1)$ for the velocity are derived for finite element and finite volume approximations.

1. INTRODUCTION

We consider the nonstationary Stokes problem for incompressible flow:

$$
\begin{aligned}
\mathbf{u}_t - \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \times [0,T],\\
\operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega \times [0,T],\\
\mathbf{u} &= 0 \quad \text{on } \partial\Omega \times [0,T],\\
\mathbf{u}(\cdot,0) &= \mathbf{u}_0(\cdot) \quad \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded domain of $\mathbb{R}^d$ $(d = 2,3)$ with sufficiently smooth boundary for our purposes. The above equation is discretized in space by *finite elements* or *finite volumes.* We are interested in proving a posteriori estimates for these approximations. The problem of proving a posteriori estimates for time dependent Stokes equations is rather open, although it is directly related to the important problem of error control for the time dependent Navier–Stokes equations. This is partly due to the fact that even in the case of linear parabolic problems the development of the theory of a posteriori error control is still in progress, see e.g. [14, 23] and [22, 2, 9, 21]. Additional technical difficulties appear in the case of the Stokes problem (1.1). A main problem arises from the fact that the space discrete approximations are rarely divergence–free functions. Also, the finite dimensional

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spaces are very often nonconforming. Very recently the ideas of [2] were appropriately extended to the Stokes case by Bernardi and Verfürth [4].

In the sequel we show that one can bypass the above technical problems and derive a posteriori error estimates for finite element or finite volume approximations to the Stokes system. In particular we address in a very natural way the problem that arises from the fact that although $\mathbf{u}$ is divergence free its approximation $\mathbf{u}_h$ is in general not divergence free. Thus, the error $\mathbf{u} - \mathbf{u}_h$ is not divergence free. Our approach allows the treatment of nonconforming elements, the freedom of choosing various stationary ("elliptic") estimators, and leads to estimates of optimal order in various norms by energy as well as duality techniques.

The main tool in our analysis is a *Stokes reconstruction* operator defined below, Definition 1.1., as a solution operator of an appropriate *stationary* Stokes problem. This definition is an appropriate extension of the *elliptic reconstruction* operator in the Stokes case introduced by Makridakis and Nochetto [22] for the a posteriori analysis of parabolic problems. Note that a similar operator was used in the a priori error analysis of finite element approximations of Navier–Stokes equation by Heywood and Rannacher [16, Corollary 4.3] for different purposes. In Lemma 1.1 we show that the stationary approximation of the solution $(\mathbf{U}, P)$ of the Stokes reconstruction problem is $(\mathbf{u}_h, p_h)$, i.e. the approximations of the time dependent problem. Then the derivation of the error estimates is reduced to deriving estimates for $\mathbf{e} = \mathbf{U} - \mathbf{u}$. Theorem 1.1 shows that $\mathbf{e}$ satisfies a continuous time dependent Stokes problem, and in fact it is *divergence free.* Thus, pde techniques can be applied to derive the final estimates in various norms. To focus on the main ideas, in this paper we have chosen to consider the space discrete case. The same techniques can be extended to fully discrete schemes with backward Euler time discretization along the lines of [21]. The problem of showing a posteriori bounds when considering more appropriate time discretization schemes for Stokes and Navier-Stokes equations requires new ideas.

In the rest of this section we introduce the necessary notation and the class of approximations that we will use in the sequel. Next we introduce the Stokes reconstruction and discuss its main properties. In Section 2 we show a posteriori error estimates based on the abstract setting introduced in Section 1. We derive estimates by energy as well as duality techniques. In Section 3 we apply the abstract theory to the classical nonconforming Crouzeix–Raviart pair of lowest order. We show a posteriori estimates of optimal order in $L^\infty(L^2)$ and $L^\infty(H^1)$ for the velocity error. In Section 4 we consider finite volume schemes for discretizing (1.1). Still the approximations belong to the Crouzeix–Raviart spaces. We appropriately modify the definition of the Stokes reconstruction for the finite volume case and we show estimates of optimal order in $L^\infty(L^2)$ and $L^\infty(H^1)$. Note the interesting fact that the stationary finite volume approximation of the solution $(\mathbf{U}, P)$ of the

Stokes reconstruction problem is still $(\mathbf{u}_h, p_h)$ (the approximations of the time dependent finite volume problem) in analogy to the finite element case.

In [4] optimal a posteriori error bounds in $L^2(H^1)$ are derived for fully discrete approximations of (1.1) based on backward Euler time discretization combined with conforming finite elements for the space discretization. To deal with the problem comming in the analysis from the fact that the finite element spaces are not divergence free, a special stationary problem with non zero divergence for the velocity is introduced, and, in addition, certain mesh conditions are required.

1.1. Preliminaries - Main Definitions. Let $\mathbf{H} := (L^2(\Omega))^d$, be the usual Lebesque space equipped with the inner product

$$\langle f, g\rangle = \int_\Omega f(x)g(x)dx,$$

and $\mathbf{V} := (H^1_0(\Omega))^d$, $\Pi := \{\phi \in L^2(\Omega) : \int_\Omega \phi(x)dx = 0\}$, and $\mathbf{V}^\star := (H^{-1}(\Omega))^d$ be the dual of $\mathbf{V}$. We denote the norms on $\mathbf{H}$, Π, $\mathbf{V}$ and $\mathbf{V}^*$ by $\|\cdot\|_{\mathbf{H}}$, $\|\cdot\|_{\Pi}$, $\|\cdot\|_{\mathbf{V}}$ and $\|\cdot\|_{\mathbf{V}^*}$, respectively. Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be the bilinear forms defined as

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \sum_{i=1}^d \nabla u_i \nabla v_i \, dx \quad \mathbf{u},\mathbf{v}\in\mathbf{V}\,, \tag{1.2}$$

and

$$b(\mathbf{u},q) = -\int_\Omega (\operatorname{div}\mathbf{u})q\,dx\,, \quad \mathbf{u}\in\mathbf{V},\ q\in\Pi\,. \tag{1.3}$$

We assume that $\mathbf{f}\in L^2(0,T;\mathbf{V}^\star)$ and $\mathbf{u}_0\in\mathbf{H}$, so that (1.1) admits a unique weak solution $(\mathbf{u},p)$ satisfying

(1.4)
$$\begin{aligned}\langle \mathbf{u}_t(t),\mathbf{v}\rangle + a(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{v},p) &= \langle \mathbf{f},\mathbf{v}\rangle \quad \text{for all } \mathbf{v}\in\mathbf{V}, \text{ a.e. } t\in[0,T]\\ b(\mathbf{u},q) &= 0 \quad \text{for all } q\in\Pi\,.\end{aligned}$$

In the sequel we will assume that the the data of the problem will have sufficient (additional) reqularity for our results to hold. For detailed regularity requirements on the data see [26].

Define the closed subspace $\mathbf{Z}$ of $\mathbf{V}$ via

$$\mathbf{Z} = \{\mathbf{v}\in\mathbf{V} : b(\mathbf{v},q) = 0 \quad \text{for all } q\in\Pi\}. \tag{1.5}$$

and the closed subspace $\mathbf{J}$ of $\mathbf{H}$ via

$$\mathbf{J} = \{\mathbf{v}\in\mathbf{H} : b(\mathbf{v},q) = 0 \quad \text{for all } q\in\Pi\}. \tag{1.6}$$

Then, as usual, problem (1.4) is reduced to the following two problems: find $\mathbf{u}\in\mathbf{Z}$ such that

$$\langle \mathbf{u}_t(t),\mathbf{v}\rangle + a(\mathbf{u}(t),\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle \quad \text{for all } \mathbf{v}\in\mathbf{Z} \tag{1.7}$$

and then find $p\in\Pi$ such that

$$b(\mathbf{v},p) = -\langle \mathbf{u}_t(t),\mathbf{v}\rangle - a(\mathbf{u}(t),\mathbf{v}) + \langle \mathbf{f},\mathbf{v}\rangle \quad \text{for all } \mathbf{v}\in\mathbf{V}. \tag{1.8}$$

It is known that the well–posedness of problem (1.7) follows from the coercivity of the bilinear form $a(\cdot,\cdot)$, namely

$$\alpha\|\mathbf{v}\|_{\mathbf{V}}^2 \le a(\mathbf{v},\mathbf{v}) \quad \text{for all } \mathbf{v}\in\mathbf{V}, \tag{1.9}$$

where $\alpha > 0$, and the well–posedness of problem (1.8) follows from the continuous inf–sup condition

$$\beta\|p\|_\Pi \le \sup_{\mathbf{w}\in\mathbf{V}} \frac{b(\mathbf{w},p)}{\|\mathbf{w}\|_{\mathbf{V}}} \quad \text{for all } p\in\Pi, \tag{1.10}$$

where $\beta > 0$.

Now let $(\mathbf{V}_h,\Pi_h)$ be an appropriate pair of finite dimensional spaces that is chosen for the discretization of the stationary Stokes problem. Then the space–discrete time dependent counterpart of (1.4) is: find $(\mathbf{u}_h,p_h) : [0,T]\to\mathbf{V}_h\times\Pi_h$ such that

$$\begin{aligned} \langle\mathbf{u}_{h,t}(t),\boldsymbol{\varphi}\rangle + a(\mathbf{u}_h(t),\boldsymbol{\varphi}) + b(\boldsymbol{\varphi},p_h) = \langle\mathbf{f},\boldsymbol{\varphi}\rangle \quad &\text{for all } \boldsymbol{\varphi}\in\mathbf{V}_h,\\ b(\mathbf{u}_h,q_h) = 0 \quad &\text{for all } q_h\in\Pi_h. \end{aligned} \tag{1.11}$$

Note that we do not assume that $\mathbf{V}_h\subset\mathbf{V}$, but just that $a(\cdot,\cdot)$ can be extended to $(\mathbf{V}_h+\mathbf{V})\times(\mathbf{V}_h+\mathbf{V})$, and $b(\cdot,\cdot)$ can be extended to $(\mathbf{V}_h+\mathbf{V})\times\Pi$. In addition for $\mathbf{v}\in\mathbf{V}_h$, with a slight abuse of notation, we still denote its norm in $\mathbf{V}_h$ by $\|v\|_{\mathbf{V}}$ (understood in the elementwise sense).

Then, indeed, as in the continuous case, (1.11) is reduced to: find $\mathbf{u}_h\in\mathbf{Z}_h$ such that

$$\langle\mathbf{u}_{h,t}(t),\boldsymbol{\varphi}\rangle + a(\mathbf{u}_h(t),\boldsymbol{\varphi}) = \langle\mathbf{f},\boldsymbol{\varphi}\rangle \quad \text{for all } \boldsymbol{\varphi}\in\mathbf{Z}_h, \tag{1.12}$$

and then find $p_h\in\Pi_h$ such that

$$b(\mathbf{u}_h,p_h) = -\langle\mathbf{u}_{h,t}(t),\boldsymbol{\varphi}\rangle - a(\mathbf{u}_h(t),\boldsymbol{\varphi}) + \langle\mathbf{f},\boldsymbol{\varphi}\rangle \quad \text{for all } \boldsymbol{\varphi}\in\mathbf{V}_h \tag{1.13}$$

where $\mathbf{Z}_h\subset\mathbf{V}_h$ is the "discrete divergence free" subspace of $\mathbf{V}_h$

$$\mathbf{Z}_h = \{\boldsymbol{\varphi}\in\mathbf{V}_h : b(\boldsymbol{\varphi},q) = 0 \quad \text{for all } q\in\Pi_h\}.$$

The uniqueness of the $(\mathbf{u}_h,p_h)$ is well–known [5, p. 248] and [15, p. 59] under the following conditions:

$$\alpha^*\|\mathbf{v}\|_{\mathbf{V}}^2 \le a(\mathbf{v},\mathbf{v}) \quad \text{for all } \mathbf{v}\in\mathbf{V}_h, \tag{1.14}$$

where $\alpha^* > 0$, and

$$0 < \beta^* := \inf_{q\in\Pi_h}\sup_{\mathbf{v}\in\mathbf{V}_h} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{\mathbf{V}}\|q\|_\Pi}. \tag{1.15}$$

We proceed now to define the *a posteriori estimator functions* for the stationary problem. For $\mathbf{g}\in\mathbf{V}^\star$, let $(\mathbf{w},q)\in\mathbf{V}\times\Pi$ be the unique solution of the stationary Stokes equation

$$\begin{aligned} -\Delta\mathbf{w} + \nabla q = \mathbf{g} \quad &\text{in } \Omega,\\ \operatorname{div}\mathbf{w} = 0 \quad &\text{in } \Omega, \end{aligned} \tag{1.16}$$

or in weak form

$$
\begin{aligned}
a(\mathbf{w},\mathbf{v}) + b(\mathbf{v},q) &= \langle \mathbf{g},\mathbf{v}\rangle \quad \text{for all } \mathbf{v}\in\mathbf{V},\\
b(\mathbf{w},\tilde q) &= 0 \quad \text{for all } \tilde q \in \Pi.
\end{aligned} \tag{1.17}
$$

Let $(\mathbf{w}_h, q_h) \in \mathbf{V}_h \times \Pi_h$ be the corresponding finite element solution, i.e.

$$
\begin{aligned}
a(\mathbf{w}_h,\boldsymbol{\varphi}) + b(\boldsymbol{\varphi},q_h) &= \langle \mathbf{g},\boldsymbol{\varphi}\rangle \quad \text{for all } \boldsymbol{\varphi}\in\mathbf{V}_h,\\
b(\mathbf{w}_h,\tilde q) &= 0 \quad \text{for all } \tilde q \in \Pi_h.
\end{aligned} \tag{1.18}
$$

We thus assume the availability of a posteriori estimators for this problem:

Assumption 1.1. *Let $(\mathbf{w}, q) \in \mathbf{Z}\times\Pi$ and $(\mathbf{w}_h, q_h) \in \mathbf{Z}_h \times \Pi_h$ be the exact solution and its finite element approximation given in (1.17) and (1.18) above. We assume that there exist a posteriori estimator functions $\mathcal{E} = \mathcal{E}((\mathbf{w}_h, q_h), \mathbf{g}; X)$, $X = \mathbf{H}, \mathbf{V}, \mathbf{V}^*$, $\mathcal{E}_{pres} = \mathcal{E}_{pres}((\mathbf{w}_h, q_h), \mathbf{g}; \Pi)$, which depend on $(\mathbf{w}_h, q_h)$, $\mathbf{g}$ and the corresponding norm, such that*

$$
\|\mathbf{w} - \mathbf{w}_h\|_X \le \mathcal{E}((\mathbf{w}_h, q_h), \mathbf{g}; X), \quad X = \mathbf{H},\, \mathbf{V},\, \mathbf{V}^*\,, \tag{1.19}
$$

and

$$
\|q - q_h\|_\Pi \le \mathcal{E}_{pres}((\mathbf{w}_h, q_h), \mathbf{g}; \Pi)\,. \tag{1.20}
$$

Next we will define the *Stokes reconstruction.* Let $\tilde\Delta : \mathbf{H}^2 \cap \mathbf{Z} \subset \mathbf{J} \to \mathbf{J}$ be the *Stokes operator*, namely the L^2-projection of the Laplace operator onto $\mathbf{J}$ [16]. We then introduce a discrete version of the Stokes operator $\tilde\Delta_h : \mathbf{Z}_h \to \mathbf{Z}_h$ by

$$
\langle \tilde\Delta_h \mathbf{v}, \boldsymbol{\chi}\rangle = -a(\mathbf{v}, \boldsymbol{\chi}) \quad \text{for all } \boldsymbol{\chi} \in \mathbf{Z}_h. \tag{1.21}
$$

We denote by $\mathbf{f}_h$ the L^2-projection of $\mathbf{f}$ onto $\mathbf{Z}_h$, i.e.

$$
\langle \mathbf{f}, \boldsymbol{\chi}\rangle = \langle \mathbf{f}_h, \boldsymbol{\chi}\rangle \quad \text{for all } \boldsymbol{\chi} \in \mathbf{Z}_h. \tag{1.22}
$$

Then we have, compare to [22]:

Definition 1.1. (Stokes Reconstruction) *For fixed $t \in [0, T]$ let $(\mathbf{U}, P) \in \mathbf{V}\times\Pi$ be the solution of the stationary Stokes problem*

$$
\begin{aligned}
a(\mathbf{U},\mathbf{v}) + b(\mathbf{v},P) &= \langle \mathbf{g}_h(t),\mathbf{v}\rangle \quad \text{for all } \mathbf{v}\in\mathbf{V},\\
b(\mathbf{U},q) &= 0 \quad \text{for all } q \in \Pi,
\end{aligned} \tag{1.23}
$$

where

$$
\mathbf{g}_h := -\tilde\Delta_h \mathbf{u}_h - \mathbf{f}_h + \mathbf{f}. \tag{1.24}
$$

We call $(\mathbf{U}, P) = (\mathbf{U}(t), P(t))$ the Stokes Reconstruction of $(\mathbf{u}_h(t), p_h(t))$.

As before, $\mathbf{U} \in \mathbf{Z}$ and $P \in \Pi$ are, respectively, the solutions of the following problems

$$
a(\mathbf{U}, \mathbf{v}) = \langle \mathbf{g}_h, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{Z}, \tag{1.25}
$$

and

$$
b(\mathbf{v}, P) = -a(\mathbf{U}, \mathbf{v}) + \langle \mathbf{g}_h, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{V}. \tag{1.26}
$$

Lemma 1.1. *Assume that* $(\mathbf{U}, P)$ *is the unique solution of the stationary Stokes problem* (1.23)*. Let* $(\mathbf{U}_h, P_h)$ *be the finite element solution of* (1.23)*, namely*

$$a(\mathbf{U}_h, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, P_h) = \langle \mathbf{g}_h, \boldsymbol{\varphi}\rangle \quad \textit{for all } \boldsymbol{\varphi} \in \mathbf{V}_h, \qquad b(\mathbf{U}_h, q) = 0 \quad \textit{for all } q \in \Pi_h. \tag{1.27}$$

Then

$$\mathbf{U}_h(t) = \mathbf{u}_h(t) \quad \textit{and} \quad P_h(t) = p_h(t)\,, \tag{1.28}$$

where $(\mathbf{u}_h, p_h)$ *is the solution of* (1.11).

Proof. Let $\mathbf{v}_h \in \mathbf{Z}_h$; then $b(\mathbf{v}_h, P_h) = 0$ and $a(\mathbf{U}_h, \mathbf{v}_h) = \langle \mathbf{g}_h, \mathbf{v}_h\rangle$. Now

$$\begin{aligned}\langle \mathbf{g}_h, \mathbf{v}_h\rangle &= -\langle \tilde{\Delta}_h \mathbf{u}_h, \mathbf{v}_h\rangle - \langle \mathbf{f}_h, \mathbf{v}_h\rangle + \langle \mathbf{f}, \mathbf{v}_h\rangle \\ &= a(\mathbf{u}_h, \mathbf{v}_h),\end{aligned} \tag{1.29}$$

i.e.,

$$a(\mathbf{U}_h - \mathbf{u}_h, \mathbf{v}_h) = 0 \quad \text{for all } \mathbf{v}_h \in \mathbf{Z}_h. \tag{1.30}$$

Since $\mathbf{U}_h,\ \mathbf{u}_h \in \mathbf{Z}_h$ we get $\mathbf{U}_h(t) = \mathbf{u}_h(t)$. Also, according to (1.12) we obtain

$$\langle \mathbf{u}_{h,t} - \tilde{\Delta}_h \mathbf{u}_h - \mathbf{f}_h, \boldsymbol{\varphi}\rangle = 0 \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{Z}_h$$

so $\mathbf{u}_h$ satisfies the relation in L^2

$$\mathbf{u}_{h,t} - \tilde{\Delta}_h \mathbf{u}_h - \mathbf{f}_h = 0. \tag{1.31}$$

Further,

(1.32)

$$\begin{aligned}b(\boldsymbol{\varphi}, p_h) - b(\boldsymbol{\varphi}, P_h) &= -a(\mathbf{u}_h, \boldsymbol{\varphi}) - \langle \mathbf{u}_{h,t}, \boldsymbol{\varphi}\rangle + \langle \mathbf{f}, \boldsymbol{\varphi}\rangle - [-a(\mathbf{U}_h, \boldsymbol{\varphi}) + \langle \mathbf{g}_h, \boldsymbol{\varphi}\rangle] \\ &= a(\mathbf{U}_h - \mathbf{u}_h, \boldsymbol{\varphi}) - \langle \mathbf{u}_{h,t} - \tilde{\Delta}_h \mathbf{u}_h - \mathbf{f}_h, \boldsymbol{\varphi}\rangle\end{aligned}$$

for all $\boldsymbol{\varphi} \in \mathbf{V}_h$. According to (1.31) and the fact, as we proved above, that $\mathbf{U}_h = \mathbf{u}_h$ we get

$$b(\boldsymbol{\varphi}, p_h - P_h) = 0 \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{V}_h.$$

Due to the discrete inf–sup assumption (1.15) we have that $p_h = P_h$. So we conclude that $(\mathbf{u}_h, p_h) \in \mathbf{Z}_h \times \Pi_h$ is the finite element solution of the stationary Stokes equation whose exact solution is $(\mathbf{U}, P)$. □

We have the following

Theorem 1.1. (Error equation) *Let* $(\mathbf{U}, P)$ *be the Stokes reconstruction and* $(\mathbf{u}, p)$ *be the solution of the Stokes problem* (1.1)*. If* $\mathbf{e} := \mathbf{U} - \mathbf{u}$ *and* $\varepsilon := P - p$ *then* $(\mathbf{e}, \varepsilon)$ *is the weak solution of the problem*

$$\begin{aligned}\mathbf{e}_t - \Delta \mathbf{e} + \nabla \varepsilon &= (\mathbf{U} - \mathbf{u}_h)_t \\ \operatorname{div} \mathbf{e} &= 0\,.\end{aligned} \tag{1.33}$$

In addition $\mathbf{U} - \mathbf{u}_h$ *satisfies the estimates*

$$\|\partial_t^{(j)}(\mathbf{U} - \mathbf{u}_h)\|_X \le \mathcal{E}((\partial_t^{(j)}\mathbf{u}_h, \partial_t^{(j)} p_h), \partial_t^{(j)}\mathbf{g}_h; X), \quad j = 0,\ 1, \tag{1.34}$$

$X = \mathbf{H}, \mathbf{V}, \mathbf{V}^*$, *where* $\mathcal{E}$ *is the a posteriori estimator function defined in* Assumption 1.1.

Proof. The pair $(\mathbf{U}, P)$ is the unique solution of the stationary Stokes problem

$$\begin{aligned} -\Delta \mathbf{U} + \nabla P &= \mathbf{g}_h \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{U} &= 0 \quad \text{in } \Omega. \end{aligned}$$

According to (1.31) we have

$$\mathbf{U}_t - \Delta \mathbf{U} + \nabla P = (\mathbf{U} - \mathbf{u}_h)_t + \mathbf{f},$$

and the assertion of this theorem is now obvious. □

2. Error estimates

2.1. **Energy estimates.** Next we derive a posteriori estimates by using energy techniques to the auxiliary problem (1.33). We start with $L^\infty(\mathbf{H})$ and $L^2(\mathbf{V})$ error estimates for the velocity.

Theorem 2.1. ($L^\infty(\mathbf{H})$ and $L^2(\mathbf{V})$–norm error estimates) *Assume that* $(\mathbf{u}, p)$ *is the solution of the time dependent Stokes Problem* (1.1) *and* $(\mathbf{u}_h, p_h)$ *its finite element approximation* (1.11). *Let* $(\mathbf{U}, P)$ *be the solution of the stationary Stokes problem* (1.17) *and* $\mathcal{E}$ *be as defined in* Assumption 1.1. *Then the following a posteriori error bounds hold for* $0 < t \le T$

$$\begin{aligned} \|(\mathbf{u} - \mathbf{U})(t)\|_{\mathbf{H}}^2 + &\int_0^t \|(\mathbf{u} - \mathbf{U})(s)\|_{\mathbf{V}}^2 ds \\ &\le \|\mathbf{u}(0) - \mathbf{U}(0)\|_{\mathbf{H}}^2 + \int_0^t \mathcal{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{V}^\star)^2 ds \,. \end{aligned}$$

In addition there holds

$$\begin{aligned} \|(\mathbf{u} - \mathbf{u}_h)(t)\|_{\mathbf{H}} \le \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{H}} &+ \Big(\int_0^t \mathcal{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{V}^\star)^2 ds \Big)^{1/2} \\ &+ \mathcal{E}((\mathbf{u}_h(0), p_h(0)), \mathbf{g}_h(0); \mathbf{H}) + \mathcal{E}((\mathbf{u}_h(t), p_h(t)), \mathbf{g}_h(t); \mathbf{H}). \end{aligned}$$

and

$$\begin{aligned} \Big(\int_0^t \|(\mathbf{u} - \mathbf{u}_h)(s)\|_{\mathbf{V}}^2 ds \Big)^{1/2} &\le \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{H}} + \Big(\int_0^t \mathcal{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{V}^\star)^2 ds \Big)^{1/2} \\ &+ \mathcal{E}((\mathbf{u}_h(0), p_h(0)), \mathbf{g}_h(0); \mathbf{H}) + \Big(\int_0^t \mathcal{E}((\mathbf{u}_h, p_h), \mathbf{g}_h; \mathbf{V})^2 ds \Big)^{1/2} . \end{aligned}$$

Proof. Using again (1.31) we have

$$\langle \mathbf{u}_{h,t} - \tilde{\Delta}_h \mathbf{u}_h - \mathbf{f}_h, \mathbf{v} \rangle = 0 \quad \text{for all } \mathbf{v} \in \mathbf{V}.$$

According to (1.23) and the above relation we get

$$\langle \mathbf{u}_{h,t}, \mathbf{v} \rangle + a(\mathbf{U}, \mathbf{v}) + b(\mathbf{v}, P) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in \mathbf{V}. \tag{2.1}$$

Let $\mathbf{e} := \mathbf{U} - \mathbf{u}$ and $\varepsilon := P - p$. By combining (1.4) and (2.1) and using the definitions of $\mathbf{u}$ and $\mathbf{U}$ we have that $(\mathbf{e}, \varepsilon) \in \mathbf{V} \times \Pi$ is the solution of the next nonstationary Stokes equation

$$\begin{aligned} \langle \mathbf{e}_t, \mathbf{v}\rangle + a(\mathbf{e}, \mathbf{v}) + b(\mathbf{v}, \varepsilon) =& \langle (\mathbf{U} - \mathbf{u}_h)_t, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{V}, \\ \langle \operatorname{div} \mathbf{e}, q\rangle =& 0 \quad \text{for all } q \in \Pi, \end{aligned} \tag{2.2}$$

or, $\mathbf{e} \in \mathbf{Z}$ is the solution of

$$\langle \mathbf{e}_t, \mathbf{v}\rangle + a(\mathbf{e}, \mathbf{v}) = \langle (\mathbf{U} - \mathbf{u}_h)_t, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{Z}, \tag{2.3}$$

and then $\varepsilon \in \Pi$ is the solution of

$$b(\mathbf{v}, \varepsilon) = -\langle \mathbf{e}_t, \mathbf{v}\rangle - a(\mathbf{e}, \mathbf{v}) + \langle (\mathbf{U} - \mathbf{u}_h)_t, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{V}. \tag{2.4}$$

Now, since $\mathbf{e} \in \mathbf{Z}$, we can choose $\mathbf{v} = \mathbf{e}$ in (2.3) to get

$$\begin{aligned} \|(\mathbf{u} - \mathbf{U})(t)\|^2_{\mathbf{H}} + \int_0^t \|(\mathbf{u} - \mathbf{U})(s)\|^2_{\mathbf{V}} ds \\ \leq \|\mathbf{u}(0) - \mathbf{U}(0)\|^2_{\mathbf{H}} + \int_0^t \|(\mathbf{u}_{h,t} - \mathbf{U}_t)(s)\|^2_{\mathbf{V}^*} ds\,. \end{aligned}$$

Assumption 1.1 implies that

$$\|\mathbf{u}_{h,t} - \mathbf{U}_t\|_{\mathbf{V}^\star} \leq \mathcal{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{V}^\star)\,, \tag{2.5}$$

which in turn leads to the first assertion of Theorem 2.1. To show the second one it suffices to note that Assumption 1.1 yields

$$\|(\mathbf{u}_h - \mathbf{U})(t)\|_{\mathbf{H}} \leq \mathcal{E}((\mathbf{u}_h(t), p_h(t)), \mathbf{g}_h(t); \mathbf{H}) \quad \text{for all } 0 \leq t \leq T\,, \tag{2.6}$$

which, together with

$$\begin{aligned} \|\mathbf{u}(0) - \mathbf{U}(0)\|_{\mathbf{H}} \leq& \|\mathbf{u}(0) - \mathbf{u}_h(0)\|_{\mathbf{H}} + \|\mathbf{u}_h(0) - \mathbf{U}(0)\|_{\mathbf{H}} \\ \leq& \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{H}} + \mathcal{E}((\mathbf{u}_h(0), p_h(0)), \mathbf{g}_h(0); \mathbf{H}), \end{aligned} \tag{2.7}$$

concludes the proof. □

Next we show estimates in $L^\infty(\mathbf{V})$ for the velocity error.

Theorem 2.2. ($L^\infty(\mathbf{V})$–norm error estimates) *Under the assumptions of* Theorem 2.1, *the following a posteriori error bounds hold for* $0 < t \leq T$

$$\begin{aligned} \int_0^t \|(\mathbf{u} - \mathbf{U})_t(s)\|^2_{\mathbf{H}}\, ds + \|\nabla(\mathbf{u} - \mathbf{U})(t)\|^2_{\mathbf{H}} \\ \leq \|\nabla(\mathbf{u} - \mathbf{U})(0)\|^2_{\mathbf{H}} + \int_0^t \mathcal{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{H})^2 ds \end{aligned}$$

and

$$\begin{aligned} \|\nabla(\mathbf{u} - \mathbf{u}_h)(t)\|_{\mathbf{H}} \leq \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{V}} + \Big(\int_0^t \mathcal{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{H})^2 ds\Big)^{1/2} \\ + \mathcal{E}((\mathbf{u}_h(0), p_h(0)), \mathbf{g}_h(0); \mathbf{V}) + \mathcal{E}((\mathbf{u}_h(t), p_h(t)), \mathbf{g}_h(t); \mathbf{V}). \end{aligned}$$

Proof. The first assertion of theorem follows by selecting $\mathbf{v} = \mathbf{e}_t$ in (2.3). Also, in view of Assumption 1.1 we get

$$\|(\mathbf{u}_h - \mathbf{U})(t)\|_{\mathbf{V}} \leq \mathcal{E}((\mathbf{u}_h(t), p_h(t)), \mathbf{g}_h(t); \mathbf{V}) \quad \text{for all } 0 \leq t \leq T\,, \tag{2.8}$$

which, together with

$$\begin{aligned}\|\nabla(\mathbf{u} - \mathbf{U})(0)\|_{\mathbf{H}} &\leq \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{V}} + \|(\mathbf{u}_h - \mathbf{U})(0)\|_{\mathbf{V}} \\ &\leq \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{V}} + \mathcal{E}((\mathbf{u}_h(0), p_h(0)), \mathbf{g}_h(0); \mathbf{V}).\end{aligned}$$

prove the second assertion of this theorem. □

Remark 2.1. *Another way to prove the* $\mathbf{H}^1$*–norm error estimate is as follows: Obviously* $\tilde{\Delta}\mathbf{e} \notin Z$*, but after integrating by parts in* (2.3)*, one can justify setting* $\mathbf{v} = \tilde{\Delta}\mathbf{e}$*. Then, noting that* $\langle \mathbf{e}_t, \tilde{\Delta}\mathbf{e}\rangle = \langle \mathbf{e}_t, \Delta\mathbf{e}\rangle = -\langle \nabla\mathbf{e}_t, \nabla\mathbf{e}\rangle$*,* [16]*, we find*

$$\begin{aligned}\max_{0\leq t\leq T} \|\nabla(\mathbf{u} - \mathbf{U})(t)\|_{\mathbf{H}}^2 + \int_0^T \|\tilde{\Delta}(\mathbf{u} - \mathbf{U})(s)\|_{\mathbf{H}}^2 ds \\ \leq \|\nabla(\mathbf{u} - \mathbf{U})(0)\|_{\mathbf{H}}^2 + \int_0^T \mathcal{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{H})^2 ds\,.\end{aligned}$$

Estimating the pressure error for the time dependent problem is a delicate issue. Unlike the stationary problem where $\|p - p_h\|_\Pi$ behaves like $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}}$, a corresponding estimate for the pressure error in the time dependent problem is not clear even in the a priori analysis. In our approach this problem is reduced to a "balanced" estimate involving $\|p - P\|_\Pi$ and $\|\mathbf{u} - \mathbf{U}\|_{\mathbf{V}}$ in terms of the right hand side of the pde (1.33). It turns out that it is not clear how this can be done with simple energy arguments. We refer to the works of Solonnikov, and Koch and Solonnikov [25, 18, 19] where fine stability issues for the time dependent Stokes problem are addressed. Note also that estimates of the pressure error in $H^{-1}(L^2)$ can be derived by adopting arguments of Bernardi and Raugel [3] in estimating (1.33). Below we estimate $\max_{t\in[0,T]} \|p(t) - P(t)\|_\Pi$ at the expense of an extra time derivative in the estimator function compared to the estimate in Theorem 2.2 for $\max_{t\in[0,T]} \|\mathbf{u} - \mathbf{U}\|_{\mathbf{V}}$. Nevertheless, the order of the estimator is the right one.

Theorem 2.3. (Pressure error estimate) *Under the assumptions of Theorem 2.2 we have*

$$\begin{aligned}\beta^2 \|p - P\|_\Pi{}^2 \leq C\big[&\|(\mathbf{u} - \mathbf{U})_t(0)\|_{\mathbf{H}}^2 + \|\nabla(\mathbf{u} - \mathbf{U})(0)\|_{\mathbf{H}}^2 \\ &+ \int_0^t \mathcal{E}((\mathbf{u}_{h,tt}, p_{h,tt}), \mathbf{g}_{h,tt}; \mathbf{H})^2 ds \\ &+ \int_0^t \mathcal{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{H})^2 ds + \mathcal{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{V}^*)^2\big].\end{aligned}$$

and

$$\begin{aligned}\beta\|p-p_h\|_\Pi \le C\big[&\|(\mathbf{u}-\mathbf{U})_t(0)\|_{\mathbf{H}}+\|\nabla(\mathbf{u}-\mathbf{U})(0)\|_{\mathbf{H}}\\&+\Big(\int_0^t \mathcal{E}((\mathbf{u}_{h,tt},p_{h,tt}),\mathbf{g}_{h,tt};\mathbf{H})^2 ds\\&+\int_0^t \mathcal{E}((\mathbf{u}_{h,t},p_{h,t}),\mathbf{g}_{h,t};\mathbf{H})^2 ds\Big)^{1/2}+\mathcal{E}((\mathbf{u}_{h,t},p_{h,t}),\mathbf{g}_{h,t};\mathbf{V}^*)\big]\\&+\beta\mathcal{E}_{pres}((\mathbf{u}_h,p_h),\mathbf{g}_h;\Pi)\,.\end{aligned}$$

Proof. According to the discrete inf–sup condition (1.15) and Poincare's inequality, one finds

$$\begin{aligned}\beta\|\varepsilon\|_\Pi &\le \sup_{\mathbf{v}\in\mathbf{V}}\frac{b(\mathbf{v},\varepsilon)}{\|\mathbf{v}\|_{\mathbf{V}}}=\sup_{\mathbf{v}\in\mathbf{V}}\frac{-\langle\mathbf{e}_t,\mathbf{v}\rangle-a(\mathbf{e},\mathbf{v})+\langle(\mathbf{U}-\mathbf{u}_h)_t,\mathbf{v}\rangle}{\|\mathbf{v}\|_{\mathbf{V}}}\\&\le c\left(\|\mathbf{e}_t\|_{\mathbf{H}}+\|\nabla\mathbf{e}\|_{\mathbf{H}}+\|(\mathbf{U}-\mathbf{u}_h)_t\|_{\mathbf{V}^*}\right)\end{aligned}\tag{2.9}$$

Differentiating (2.3) with respect to t we get

$$\langle\mathbf{e}_{tt},\mathbf{v}\rangle+a(\mathbf{e}_t,\mathbf{v})=\langle(\mathbf{U}-\mathbf{u}_h)_{tt},\mathbf{v}\rangle\quad\text{for all }\mathbf{v}\in\mathbf{Z}.\tag{2.10}$$

Now, setting $\mathbf{v}=\mathbf{e}_t$ in the last equation and integrating with respect to t we obtain

$$\|(\mathbf{u}-\mathbf{U})_t(t)\|^2_{\mathbf{H}}+\int_0^t\|\nabla(\mathbf{u}-\mathbf{U})_t(s)\|^2_{\mathbf{H}}ds\le\|(\mathbf{u}-\mathbf{U})_t(0)\|^2_{\mathbf{H}}+\int_0^t\|(\mathbf{U}-\mathbf{u}_h)_{tt}(s)\|^2_{\mathbf{H}}\,ds\,.\tag{2.11}$$

According to (2.9), (2.11), the Assumption 1.1 and the previous error estimates for velocity, we derive the result of this theorem.

□

2.2. **Estimates by parabolic duality.** We briefly discuss now how one can apply our ideas to derive estimates using parabolic duality, [27, 14]. Thus consider the backward parabolic Stokes problem: Fix $t_\star\in(0,T]$ and let $(\mathbf{z},s)\in\mathbf{Z}\times\Pi$ be the solution of the *backward* problem

$$\begin{aligned}\mathbf{z}_t+\Delta\mathbf{z}-\nabla s&=0\quad\text{in }\Omega\times(0,t_\star),\\\operatorname{div}\mathbf{z}&=0\quad\text{in }\Omega\times(0,t_\star),\\\mathbf{z}&=0\quad\text{on }\partial\Omega\times(0,t_\star),\\\mathbf{z}(\cdot,t_\star)&=\mathbf{e}(\cdot,t_\star)\quad\text{in }\Omega,\end{aligned}\tag{2.12}$$

where $\mathbf{e}=\mathbf{U}-\mathbf{u}$. Then for any τ, $0<\tau<t_\star$, there holds

$$\begin{aligned}\max_{t\in[0,t_\star]}\|\mathbf{z}(t)\|_{\mathbf{H}}&\le\|\mathbf{e}(t_\star)\|_{\mathbf{H}}\,,\\\int_0^{t_\star-\tau}\|\mathbf{z}_t\|_{\mathbf{H}}ds&\le\frac{1}{2}L_\tau\,\|\mathbf{e}(t_\star)\|_{\mathbf{H}}\,,\end{aligned}\tag{2.13}$$

where $L_\tau = (\log(\frac{t_\star}{\tau}))^{1/2}$. This bound follows by entirely similar arguments as in [27, Lemma 12.5], working with the one field equation of the Stokes problem

$$\mathbf{z}_t + \tilde{\Delta}\mathbf{z} = 0 \quad \text{in } \Omega \times (0, t_\star) \ . \tag{2.14}$$

Theorem 2.4. *Assume that* $(\mathbf{u}, p)$ *is the solution of the time dependent Stokes problem* (1.1) *and* $(\mathbf{u}_h, p_h)$ *is its finite element approximation* (1.11). *Let* $(\mathbf{U}, P)$ *be the solution of the stationary Stokes problem* (1.17). *Then the following a posteriori error bound holds: for* $0 < t_\star \le T$ *and any* τ, $0 < \tau < t_\star$ *we have*

$$\|(\mathbf{u} - \mathbf{u}_h)(t_\star)\|_{\mathbf{H}} \le \big(1+\frac{1}{2}L_\tau\big) \max_{0\le t\le t_\star} \|(\mathbf{U} - \mathbf{u}_h)(t)\|_{\mathbf{H}} + \int_{t_\star-\tau}^{t_\star} \|(\mathbf{U} - \mathbf{u}_h)_t\|_{\mathbf{H}}\, dt + \|(\mathbf{u} - \mathbf{u}_h)(0)\|_{\mathbf{H}}\,,$$

where $L_\tau = \log(\frac{t_\star}{\tau})^{1/2}$.

Proof. The proof of this estimate is based on the formula obtained by a simple integration by parts

$$\int_0^{t_\star} \langle \mathbf{v}_t, \mathbf{z}\rangle + a(\mathbf{v}, \mathbf{z}) + \langle \nabla s, \mathbf{v}\rangle\, dt + \langle \mathbf{v}(0), \mathbf{z}(0)\rangle = \langle \mathbf{v}(t_\star), \mathbf{z}(t_\star)\rangle, \quad \text{for } \mathbf{v} \in \mathbf{V}\,. \tag{2.15}$$

In particular for $\mathbf{v} \in \mathbf{Z}$ this relation reduces to

$$\int_0^{t_\star} \langle \mathbf{v}_t, \mathbf{z}\rangle + a(\mathbf{v}, \mathbf{z})\, dt + \langle \mathbf{v}(0), \mathbf{z}(0)\rangle = \langle \mathbf{v}(t_\star), \mathbf{z}(t_\star)\rangle, \quad \text{for } \mathbf{v} \in \mathbf{Z}\,. \tag{2.16}$$

Thus since $\mathbf{e} \in \mathbf{Z}$, (1.33) yields

$$\begin{aligned}
\|\mathbf{e}(t_\star)\|_{\mathbf{H}}^2 &= \langle \mathbf{e}(t_\star), \mathbf{z}(t_\star)\rangle \\
&= \int_0^{t_\star} \langle \mathbf{e}_t, \mathbf{z}\rangle + a(\mathbf{e}, \mathbf{z})\, dt + \langle \mathbf{e}(0), \mathbf{z}(0)\rangle \\
&= \int_{t_\star-\tau}^{t_\star} \langle (\mathbf{U} - \mathbf{u}_h)_t, \mathbf{z}\rangle\, dt + \int_0^{t_\star-\tau} \langle (\mathbf{U} - \mathbf{u}_h)_t, \mathbf{z}\rangle\, dt + \langle \mathbf{e}(0), \mathbf{z}(0)\rangle \\
&= \int_{t_\star-\tau}^{t_\star} \langle (\mathbf{U} - \mathbf{u}_h)_t, \mathbf{z}\rangle\, dt + \langle (\mathbf{U} - \mathbf{u}_h)(t_\star - \tau), \mathbf{z}(t_\star - \tau)\rangle \\
&\quad - \langle (\mathbf{U} - \mathbf{u}_h)(0), \mathbf{z}(0)\rangle - \int_0^{t_\star-\tau} \langle (\mathbf{U} - \mathbf{u}_h), \mathbf{z}_t\rangle\, dt + \langle \mathbf{e}(0), \mathbf{z}(0)\rangle.
\end{aligned}$$

Using (2.13) we obtain

$$\|\mathbf{e}(t_\star)\|_{\mathbf{H}}^2 \le \Big[\int_{t_\star-\tau}^{t_\star} \|(\mathbf{U} - \mathbf{u}_h)_t\|_{\mathbf{H}}\, dt + (1 + \frac{1}{2})L_\tau \max_{0\le t\le t_\star-\tau} \|(\mathbf{U} - \mathbf{u}_h)(t)\|_{\mathbf{H}} + \|(\mathbf{u} - \mathbf{u}_h)(0)\|_{\mathbf{H}} \Big] \|\mathbf{e}(t_\star)\|_{\mathbf{H}}.$$

□

It is interesting to note that the term

$$\int_{t_\star-\tau}^{t_\star} \|(\mathbf{U}-\mathbf{u}_h)_t\|_{\mathbf{H}}\,dt$$

appearing in the above estimate can be bounded by $\max_{t_\star-\tau\le t\le t_\star}\|(\mathbf{U}-\mathbf{u}_h)(t)\|_{\mathbf{H}}$ in the fully discrete case, provided that τ is of the order of the time step. A similar phenomenon appears, of course, in the standard proofs by duality in the time-discrete and fully discrete cases, cf. [27, 14]. We refer to the forthcoming work [20] where the fully discrete case for parabolic problems is considered. Thus, assuming for the time being that this term is not present in our case also, ignoring the error of the initial condition and recalling Theorem 1.1 we see that the above arguments would lead to a bound of the form

$$\|(\mathbf{u}-\mathbf{u}_h)(t_\star)\|_{\mathbf{H}} \le \big(2+\frac{1}{2}L_\tau\big)\max_{0\le t\le t_\star}\ \mathcal{E}((\mathbf{u}_h,p_h),\mathbf{g}_h;\mathbf{H})\,.$$

3. Application: Crouzeix–Raviart finite element discretization

In this section we will apply the a posteriori estimates in the case of the classical two dimensional Crouzeix–Raviart spaces of the lowest order $\mathbf{V}_h\times\Pi_h$, [12]. We will need some further notation: For $D\subset\Omega$ and $s\ge 0$, integer, we denote by $H^s(D)$ the usual Sobolev spaces, and by $\mathbf{H}^s(D)$ their vector counterparts. Their norms and seminorms are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ respectively. We will denote the L^2 norm of vector or scalar functions, simply by $\|\cdot\|$. For a piecewise regular vector function $\mathbf{v}$ we define the discrete gradient as the L^2–matrix $\nabla_h\mathbf{v}|_K=\nabla(\mathbf{v}|_K)$, $K\in\mathcal{T}_h$.

We consider a family of *shape–regular* triangulations $\{\mathcal{T}_h\}_{0<h<1}$ of Ω, i.e., any two triangles in $\mathcal{T}_h$ share at most a vertex or an edge, where h is the maximum diameter of the triangles of $\mathcal{T}_h$ [5]. With $E_h(K)$ we denote the set of the edges of $K\in\mathcal{T}_h$. Also, let E_h^{in} be the edges of $\mathcal{T}_h$ that are not part of $\partial\Omega$, and define $E_h^{\text{in}}(K)$ in a similar way. In, addition h_K denotes the diameter of the triangle K, $|K|$ its area, and h_e the length of an edge $e\in E_h(K)$.

Next, let V_h be the Crouzeix–Raviart nonconforming finite element space, cf. [12], associated with $\mathcal{T}_h$ and $\mathbf{V}_h$ its vector counterpart. Recall that V_h consists of piecewise linear functions that are continuous at the midpoints of the elements of the triangulation $\mathcal{T}_h$. The pressure space is just

$$\Pi_h=\{\psi\in\Pi=L_0^2:\psi|_K\in\mathbb{P}_0,\ \forall\,K\in T_h\}.$$

The finite element approximation $(\mathbf{u}_h,p_h):[0,T]\to\mathbf{V}_h\times\Pi_h$ of semidiscrete problem is defined by (1.11). It is well known, cf., e.g., [12, §6], [26, Proposition 4.13] that the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, defined in the elementwise sence, satisfy (1.14) and (1.15). The analysis of a priori and a posteriori error estimation of finite element approximations to the stationary Stokes equations has been considered in, e.g., [12, 13, 17, 28, 15].

We will show how the abstract results of the previous sections can be applied when we consider residual type estimators for the Crouzeix–Raviart space discretization. We will adapt the estimators of [13] and note that other alternatives are possible, [28, 1]. To this end we let $\boldsymbol{\sigma} = \mathbf{U} - \mathbf{u}_h, \xi = P - p_h$ and recall that Lemma 1.1 implies that $(\mathbf{u}_h, p_h)$ is the stationary Stokes approximation of a problem with exact solution $(\mathbf{U}, P)$. Thus introducing the classical conforming space

$$\mathbf{X}_h = \mathbf{V}_h \cap \mathbf{H}_0^1(\Omega) \tag{3.1}$$

we have the orthogonality relation

$$a(\boldsymbol{\sigma}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, \xi) = 0 \quad \forall\, \boldsymbol{\varphi} \in \mathbf{X}_h. \tag{3.2}$$

Then as in [13] we decompose the velocity error $\nabla_h \boldsymbol{\sigma}$ as $\nabla_h \boldsymbol{\sigma} = \nabla \mathbf{r} - q\mathbf{I} + \operatorname{curl} \mathbf{s}$, where $q \in L_0^2(\Omega)$, $\mathbf{r} \in \mathbf{H}_0^1(\Omega)$ with $\operatorname{div} \mathbf{r} = 0$ and $\mathbf{s} \in \mathbf{H}^1(\Omega)$ and

$$|\mathbf{r}|_1 + |\mathbf{s}|_1 \leq C\, \|\nabla_h \mathbf{e}\|. \tag{3.3}$$

Next for two matrices B and D we denote their inner product by $B : D = \sum_{i,j=1}^{2} B_{ij} D_{ij}$. Then since $\int_\Omega \nabla_h \boldsymbol{\sigma} \colon \operatorname{curl} \boldsymbol{\psi} = 0, \quad \forall\, \boldsymbol{\psi} \in \mathbf{X}_h$, we finally get, [13],

$$\begin{aligned} \|\nabla_h \boldsymbol{\sigma}\|^2 = &\int_\Omega (\nabla_h \boldsymbol{\sigma} - \xi \mathbf{I}) \colon \nabla(\mathbf{r} - \boldsymbol{\chi}) \\ &+ \int_\Omega \nabla_h \boldsymbol{\sigma} \colon \operatorname{curl}(\mathbf{s} - \boldsymbol{\psi}), \quad \forall\, \boldsymbol{\chi},\, \boldsymbol{\psi} \in \mathbf{X}_h. \end{aligned}$$

We thus have

$$\begin{aligned} \|\nabla_h \boldsymbol{\sigma}\|^2 = &\sum_K \int_K \mathbf{g}_h \cdot (\mathbf{r} - \boldsymbol{\chi}) \\ &- \int_{\partial K} (\nabla_h \mathbf{u}_h - p_h \mathbf{I})\mathbf{n} \cdot (\mathbf{r} - \boldsymbol{\chi}) - \int_{\partial K} \nabla_h \mathbf{u}_h \boldsymbol{\tau} \cdot (\mathbf{s} - \boldsymbol{\psi}), \end{aligned} \tag{3.4}$$

for any $\boldsymbol{\chi}, \boldsymbol{\psi} \in \mathbf{X}_h$. Here $\boldsymbol{\tau} = (-n_2, n_1)^t$ is the tangent vector and $\mathbf{I}$ is the identity matrix. Note in addition the elementwise relation, (1.24),

$$\begin{aligned} \mathbf{g}_h|_K &= -\tilde{\Delta}_h \mathbf{u}_h - \mathbf{f}_h + \mathbf{f} = -\mathbf{u}_{h,t} + \mathbf{f} \\ &= -\left(\mathbf{u}_{h,t} - \Delta \mathbf{u}_h + \nabla p_h - \mathbf{f}\right) =: -\mathbf{R}_K\,. \end{aligned} \tag{3.5}$$

Obviously $\Delta \mathbf{u}_h = 0, \nabla p_h = 0$ in each element. Here $\mathbf{u}_{h,t} - \Delta \mathbf{u}_h + \nabla p_h - \mathbf{f} = \mathbf{R}_K$ denotes the inner residual. Note that the argument above works with higher polynomial degrees too. In that case the inner residual will appear in the estimator since in (3.4) above instead of $\mathbf{g}_h$ we would have $\mathbf{g}_h + \Delta \mathbf{u}_h - \nabla p_h$. Then still $\mathbf{g}_h + \Delta \mathbf{u}_h - \nabla p_h = -\mathbf{R}_K$. We proceed now to define the estimators. To this end, we first use the standard notation to define

$$[\![(\nabla \mathbf{u}_h - p_h \mathbf{I})\mathbf{n}_e]\!]_e = (\nabla \mathbf{u}_h|_{K^-} - p_h|_{K^-}\mathbf{I})\mathbf{n}_e - (\nabla \mathbf{u}_h|_{K^+} - p_h|_{K^+}\mathbf{I})\mathbf{n}_e,$$

and

$$[\![\nabla \mathbf{u}_h \,\boldsymbol{\tau}_e]\!]_e = (\nabla \mathbf{u}_h|_{K^-})\boldsymbol{\tau}_e - (\nabla \mathbf{u}_h|_{K^+})\boldsymbol{\tau}_e.$$

For all edges $e \in E_h$ we let

$$\mathbf{J}_{e,\mathbf{n}} = \begin{cases} [\![(\nabla \mathbf{u}_h - p_h \mathbf{I})\mathbf{n}_e]\!]_e, & \text{if } e \in E_h^{\text{in}}, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{J}_{e,\boldsymbol{\tau}} = \begin{cases} [\![\nabla \mathbf{u}_h \,\boldsymbol{\tau}_e]\!]_e, & \text{if } e \in E_h^{\text{in}}, \\ 2\nabla \mathbf{u}_h \boldsymbol{\tau}_e, & \text{otherwise}. \end{cases}$$

We thus define the local error estimators $\eta_{1,K}(\mathbf{u}_h)$, $K \in \mathcal{T}_h$, by

$$\eta_{1,K}(\mathbf{u}_h)^2 = h_K^2 \, \|\mathbf{R}_K\|_{0,K}^2 + \frac{1}{2} \sum_{e \in E_h(K)} h_e \big(\|\mathbf{J}_{e,\mathbf{n}}\|_{0,e}^2 + \|\mathbf{J}_{e,\boldsymbol{\tau}}\|_{0,e}^2 \big), \tag{3.6}$$

where the inner residual $\mathbf{R}_K = \mathbf{u}_{h,t} - \Delta \mathbf{u}_h + \nabla p_h - \mathbf{f}$. At this point it is useful to note that the term involving $\mathbf{J}_{e,\boldsymbol{\tau}}$ can be replaced by the jump of $\mathbf{u}_h$. Indeed, an elementary calculation shows that, [17],

$$\|[\![\mathbf{u}_h]\!]_e\|_{0,e}^2 = \frac{h_e^2}{12} \, \|\mathbf{J}_{e,\boldsymbol{\tau}}\|_{0,e}^2. \tag{3.7}$$

In the sequel, for comparison's sake we will use estimators of the form (3.6). The estimator $\eta_1(\mathbf{u}_h)$ is defined by assembling the local estimators:

$$\eta_1(\mathbf{u}_h) = \big(\sum_K \; \eta_{1,K}(\mathbf{u}_h)^2 \; \big)^{1/2}. \tag{3.8}$$

The proof is completed by using standard arguments: Let $\mathbf{I}_h$ be a Clement–type interpolant onto $\mathbf{X}_h$ which is locally quasi-stable in H^1 [24, 5]. We choose $\boldsymbol{\chi} = \mathbf{I}_h \mathbf{r}$, $\boldsymbol{\psi} = \mathbf{I}_h \mathbf{s}$ in (3.4) and use the approximation properties of the interpolant to conclude that:

Lemma 3.1. *The following estimate holds:*

$$\mathcal{E}((\mathbf{u}_h(t), p_h(t)), \mathbf{g}_h(t); \mathbf{V}) \le C \, \eta_1 \left(\mathbf{u}_h(t) \right). \tag{3.9}$$

Next we need an estimate for $\mathcal{E}((\mathbf{u}_{h,t}(t), p_{h,t}(t)), \mathbf{g}_{h,t}(t); \mathbf{H})$.

It is standard to consider the dual problem: Find $(\mathbf{z}, s) \in (\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega) \times H^1(\Omega) \cap L_0^2(\Omega))$ such that

$$\begin{aligned} -\Delta \mathbf{z} - \nabla s = \boldsymbol{\sigma}_t & \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{z} = 0, & \quad \text{in } \Omega, \\ \mathbf{z} = \mathbf{0}, & \quad \text{on } \partial\Omega. \end{aligned} \tag{3.10}$$

Under known conditions on Ω the solution $(\mathbf{z}, s)$ satisfies the elliptic regularity estimate

$$\|\mathbf{z}\|_2 + \|s\|_1 \le C \|\boldsymbol{\sigma}_t\|. \tag{3.11}$$

Multiplying the first equation in (3.10) by $\boldsymbol{\sigma}_t$ and using integration by parts and the second equation in (3.10), we obtain

(3.12)
$$\|\boldsymbol{\sigma}_t\|^2 = \sum_K \Big\{ \int_K \big(\nabla \mathbf{z} : \nabla(\mathbf{U}_t - \mathbf{u}_{h,t}) + s\ \mathrm{div}(\mathbf{U}_t - \mathbf{u}_{h,t}) - (P_t - p_{h,t})\ \mathrm{div}\, \mathbf{z}\big) + \int_{\partial K} \nabla \mathbf{z}\, \mathbf{n} \cdot \mathbf{u}_{h,t} + \int_{\partial K} s\, \mathbf{n} \cdot \mathbf{u}_{h,t} \Big\}$$

Also, differentiating (3.2) with respect to t, we have

$$a(\boldsymbol{\sigma}_t, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, \xi_t) = 0, \quad \forall\, \boldsymbol{\varphi} \in \mathbf{X}_h\,.$$

In addition, $b(\mathbf{U}_t - \mathbf{u}_{h,t}, \psi) = 0,\ \psi \in \Pi_h$. Therefore

$$\begin{aligned} \|\boldsymbol{\sigma}_t\|^2 = &\sum_K \Big\{ \int_K \big(-\Delta \boldsymbol{\sigma}_t + \nabla \xi_t\big) \cdot (\mathbf{z} - \boldsymbol{\chi}) \\ &+ \int_K \big((s-\psi)\ \mathrm{div}(\boldsymbol{\sigma}_t) + \int_{\partial K} \big(\nabla(\boldsymbol{\sigma}_t)\mathbf{n} - \xi_t\, \mathbf{n}\big) \cdot (\mathbf{z} - \boldsymbol{\chi}) \\ &+ \int_{\partial K} \big(\nabla \mathbf{z}\, \mathbf{n} \cdot \mathbf{u}_{h,t} + s\, \mathbf{n} \cdot \mathbf{u}_{h,t}\big) \Big\}. \end{aligned}$$

Since in each element K we have $\mathbf{g}_{h,t} = -\partial_t \mathbf{R}_K = -\mathbf{R}_{K,t}$ we obtain

(3.13)
$$\begin{aligned} \|\boldsymbol{\sigma}_t\|^2 = &\sum_K \Big\{ \int_K (-\mathbf{R}_{K,t} \cdot (\mathbf{z} - \boldsymbol{\chi}) - (s-\psi)\ \mathrm{div}\, \mathbf{u}_{h,t}) \\ &+ \int_{\partial K} \big(\nabla \mathbf{u}_{h,t}\, \mathbf{n} - p_{h,t}\, \mathbf{n}\big) \cdot (\mathbf{z} - \boldsymbol{\chi}) + \int_{\partial K} \big(\nabla \mathbf{z}\, \mathbf{n} + s\, \mathbf{n}\big) \cdot \mathbf{u}_{h,t} \Big\}, \end{aligned}$$

for any $\boldsymbol{\chi} \in \mathbf{X}_h$ and $\psi \in \Pi_h$.

Define now the local error estimators $\eta_{0,K}$ by

$$\begin{aligned} \eta_{0,K}(\mathbf{u}_{h,t})^2 = &\, h_K^4 \|\mathbf{R}_{K,t}\|_{0,K}^2 + h_K^2\, \|\,\mathrm{div}\, \mathbf{u}_{h,t}\|_{0,K}^2 \\ &+ \frac{1}{2} \sum_{e \in E_h^{\mathrm{in}}(K)} \big(h_e^3\, \|\partial_t \mathbf{J}_{e,\mathbf{n}}\|_{0,e}^2 + h_e^3\, \|\partial_t \mathbf{J}_{e,\boldsymbol{\tau}}\|_{0,e}^2\big). \end{aligned}$$

The L^2 spatial estimator is then

$$\eta_0(\mathbf{u}_{h,t}) = \big(\sum_K \eta_{0,K}(\mathbf{u}_{h,t})^2\ \big)^{1/2}. \tag{3.14}$$

To conclude the estimate, one can show as in [17] that

$$\begin{aligned} \sum_K \sum_{e \in E_h(K)} \int_e \big(\nabla \mathbf{z}\, \mathbf{n} + s\, \mathbf{n}\big) \cdot \mathbf{u}_{h,t} \le &C \left(\sum_{e \in E_h^{\mathrm{in}}} h_e\, \|[\![\mathbf{u}_{h,t}]\!]_e\|_{0,e}^2 \right)^{1/2} \|\boldsymbol{\sigma}_t\| \\ \le &C \left(\sum_{e \in E_h^{\mathrm{in}}} h_e^3\, \|\partial_t \mathbf{J}_{e,\boldsymbol{\tau}}\|_{0,e}^2 \right)^{1/2} \|\boldsymbol{\sigma}_t\|. \end{aligned} \tag{3.15}$$

Here we used the fact that $\int_e [\![\mathbf{u}_{h,t}]\!]_e = 0$, the approximation properties of the $L^2(e)$ projection onto $\mathbb{P}_0(e)$ and (3.11).

Then, choosing $\boldsymbol{\chi}$ as the standard nodal interpolant of $\mathbf{z}$ and $\psi = I_h s$ we finally conclude that

Lemma 3.2. *The following estimate holds:*

$$\mathcal{E}((\mathbf{u}_{h,t}(t), p_{h,t}(t)), \mathbf{g}_{h,t}(t); \mathbf{H}) \le C\,\eta_1\,(\mathbf{u}_{h,t}(t)\,)\,. \tag{3.16}$$

Further, assuming that (3.11) *holds we have*

$$\mathcal{E}((\mathbf{u}_{h,t}(t), p_{h,t}(t)), \mathbf{g}_{h,t}(t); \mathbf{H}) \le C\,\eta_0\,(\mathbf{u}_{h,t}(t)\,)\,. \tag{3.17}$$

We can now apply Theorems 2.1 and 2.2 to obtain H^1 estimates in our case.

Theorem 3.1. (Residual $L^2(H^1)$ and $L^\infty(H^1)$–norm error estimates) *Assume that* $(\mathbf{u}, p)$ *is the solution of the time dependent Stokes Problem* (1.1) *and* $(\mathbf{u}_h, p_h)$ *is the Crouzeix–Raviart finite element approximation* (1.11). *Then the following a posteriori bounds hold for* $0 < t \le T$

$$\Big(\int_0^t \|(\mathbf{u}-\mathbf{u}_h)(s)\|_{\mathbf{V}}^2 ds\Big)^{1/2} \le \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{H}} + C\Big(\int_0^t \eta_1((\mathbf{u}_{h,t}(s)\,)^2 ds\Big)^{1/2}$$
$$+ C\eta_1((\mathbf{u}_h(0)\,) + C\Big(\int_0^t \eta_1((\mathbf{u}_h(s)\,)^2 ds\Big)^{1/2},$$

and

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\|_{\mathbf{H}} \le \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{V}} + C\Big(\int_0^t \eta_1((\mathbf{u}_{h,t}(s)\,)^2 ds\Big)^{1/2}$$
$$+ C\eta_1((\mathbf{u}_h(0)\,) + C\eta_1(\mathbf{u}_h(t)\,).$$

Remark 3.3. *In the above estimates we did not assume the elliptic regularity estimate* (3.11) *and we have used the (crude) first bound of Lemma 3.2. Still the estimates in Theorem 3.1 are of optimal order. In the case of e.g. convex polygonal domains where* (3.11) *holds the estimator* $\eta_1((\mathbf{u}_{h,t}(s)\,)$ *in these bounds should be replaced by* $\eta_0((\mathbf{u}_{h,t}(s)\,)$.

For the $L^\infty(L^2)$ estimate note that due to the fact that we use the Crouzeix–Raviart elements of lowest order the term $\mathcal{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{V}^\star)$ in Theorem 2.1 is simply bounded by $\mathcal{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{H})$.

In the estimate of the next theorem, as in the elliptic case, to gain a power of h in the order we have to use the elliptic regularity bound (3.11). Due to the abstract form of our estimators in Theorem 2.1 other finer choices can be made when (3.11) is not valid. To address this issue, detailed work related to the specific form of possible singularities of the exact solution is required. This case will not be considered in this paper.

We thus have

Theorem 3.2. (Residual $L^\infty(L^2)$–norm error estimates). *Assume that* $(\mathbf{u}, p)$ *is the solution of the time dependent Stokes Problem* (1.1) *and* $(\mathbf{u}_h, p_h)$ *is* *the Crouzeix–Raviart finite element approximation* (1.11). *Assume further that* (3.11) *holds. Then the following a posteriori bound holds for* $0 < t \le T$

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\|_{\mathbf{H}} \le \|\mathbf{u}_0-\mathbf{u}_h^0\|_{\mathbf{H}} + C\Big(\int_0^t \eta_0((\mathbf{u}_{h,t}(s)\,)^2 ds\Big)^{1/2} + C\eta_0(\mathbf{u}_h(0)\,) + C\eta_0(\mathbf{u}_h(t)\,).$$

4. Application: A finite volume scheme

Finite volume methods rely on local conservation properties of the differential equation. Thus, integrating (1.1) over a region $b \subset \Omega$ and using Green's formula, we obtain

$$\int_b \mathbf{u}_t - \int_{\partial b} \nabla \mathbf{u}\,\mathbf{n} + \int_{\partial b} p\,\mathbf{n} = \int_b \mathbf{f}. \tag{4.1}$$

In addition (1.1) gives $\int_{\mathscr{O}} \mathrm{div}\mathbf{u} = 0$, for appropriate domains $\mathscr{O}$. The finite volume scheme seeks approximations in the Crouzeix–Raviart couple $\mathbf{V}_h \times \Pi_h$ used in the previous section satisfying a local conservation property (4.1) over the *control volumes.* The number of these control volumes is equal to the dimension of $\mathbf{V}_h$. To fix notation we consider the following construction. Let z_K be an inner point of $K \in T_h$. We connect z_K with line segments to the vertices of K, thus partitioning K into three subtriangles K_e, $e \in E_h(K)$. Then with each side $e \in E_h$ we associate a quadrilateral b_e, which consists of the union of the subregions K_e.

The corresponding finite volume method for the time dependent Stokes problem is: Seek $(\mathbf{u}_h, p_h) : [0,T] \to \mathbf{V}_h \times \Pi_h$ satisfying

$$\int_{b_e} \mathbf{u}_{h,t} - \int_{\partial b_e} \nabla \mathbf{u}_h\,\mathbf{n} + \int_{\partial b_e} p_h\,\mathbf{n} = \int_{b_e} \mathbf{f}, \quad \forall\, e \in E_h^{\mathrm{in}}, \tag{4.2}$$

$$\int_K \mathrm{div}\,\mathbf{u}_h = 0, \quad \forall\, K \in \mathcal{T}_h. \tag{4.3}$$

We refer to the works [10, 11] for a priori and to [8] for a posteriori estimates for finite volume methods for the stationary Stokes problem.

It is important for the sequel to note that the finite volume scheme admits a variational formulation similar to the one of finite element case, [8, 6, 7].

Lemma 4.1. *There exists a unique solution* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \Pi_h$ *of the finite volume method* (4.2)–(4.3) *which satisfies*

$$\begin{aligned} \langle \mathbf{u}_{h,t}, \boldsymbol{\Lambda}_h \boldsymbol{\varphi}\rangle + a(\mathbf{u}_h, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, p_h) = \langle \mathbf{f}, \boldsymbol{\Lambda}_h \boldsymbol{\varphi}\rangle, \quad \forall\, \boldsymbol{\varphi} \in \mathbf{V}_h,\\ b(\mathbf{u}_h, \psi) = 0, \quad \forall\, \psi \in \Pi_h, \end{aligned} \tag{4.4}$$

where the operator $\boldsymbol{\Lambda}_h : C(\Omega)^2 + \mathbf{V}_h \to \mathbf{V}_h$ *is defined by*

$$\boldsymbol{\Lambda}_h \mathbf{v} = \sum_{e \in E_h^{in}} \mathbf{v}(m_e) \chi_{b_e}, \tag{4.5}$$

χ_{b_e} *being the characteristic function of* b_e *and* m_e *the midpoint of the edge* e.

For future reference we list the main properties of the operator $\mathbf{\Lambda}_h$ [8, 6]

$$\|\boldsymbol{\varphi}-\mathbf{\Lambda}_h\boldsymbol{\varphi}\|_{0,K}^2=\sum_{e\in E_h(K)}\|\boldsymbol{\varphi}-\mathbf{\Lambda}_h\boldsymbol{\varphi}\|_{0,K_e}^2\le Ch_K^2|\boldsymbol{\varphi}|_{1,K}^2,\quad\forall\,\boldsymbol{\varphi}\in\mathbf{V}_h,\tag{4.6}$$

$$\int_e\mathbf{\Lambda}_h\boldsymbol{\varphi}=\int_e\boldsymbol{\varphi},\quad\forall\,\boldsymbol{\varphi}\in\mathbf{V}_h,\ \forall\,e\in E_h.\tag{4.7}$$

We now introduce two operators that will be useful in the sequel; $\mathscr{L}_h : L^2(\Omega)^2\to\mathbf{V}_h$ and $\tilde{\mathscr{L}}_h : L^2(\Omega)^2\to\mathbf{Z}_h$ by

$$\begin{aligned}\langle\mathscr{L}_h\mathbf{v},\boldsymbol{\varphi}\rangle&=\langle\mathbf{v},\mathbf{\Lambda}_h\boldsymbol{\varphi}\rangle\quad\forall\,\boldsymbol{\varphi}\in\mathbf{V}_h,\\ \langle\tilde{\mathscr{L}}_h\mathbf{v},\boldsymbol{\omega}\rangle&=\langle\mathbf{v},\mathbf{\Lambda}_h\boldsymbol{\omega}\rangle\quad\forall\,\boldsymbol{\omega}\in\mathbf{Z}_h\,.\end{aligned}\tag{4.8}$$

Obviously $\mathscr{L}_h$ and $\tilde{\mathscr{L}}_h$ are well defined. In view of the above definitions and Lemma 4.1 we conclude that the solution of the finite volume scheme satisfies

$$\tilde{\mathscr{L}}_h\mathbf{u}_{h,t}-\tilde{\Delta}_h\mathbf{u}_h-\tilde{\mathscr{L}}_h\mathbf{f}=0.\tag{4.9}$$

The derivation of the a posteriori estimates follows the lines of the abstract analysis in Section 1, but certain modifications are required. We start by redefining the Stokes reconstruction. Let $(\mathbf{U},P)\in\mathbf{V}\times\Pi$ be the solution of the stationary Stokes problem

$$\begin{aligned}a(\mathbf{U},\mathbf{v})+b(\mathbf{v},P)&=\langle\mathbf{g}_h,\mathbf{v}\rangle\quad\text{for all }\mathbf{v}\in\mathbf{V},\\ b(\mathbf{U},q)&=0\quad\text{for all }q\in\Pi,\end{aligned}\tag{4.10}$$

where

$$\mathbf{g}_h:=-\tilde{\Delta}_h\mathbf{u}_h-\tilde{\mathscr{L}}_h\mathbf{f}+\mathbf{f}+\tilde{\mathscr{L}}_h\mathbf{u}_{h,t}-\mathbf{u}_{h,t}.\tag{4.11}$$

According to the definition of $\mathbf{g}_h$ and (4.9) we get

$$\mathbf{g}_h=\mathbf{f}-\mathbf{u}_{h,t}.\tag{4.12}$$

One of the reasons motivating the definition of $\mathbf{g}_h$ is that in view of (4.12) the error equation for $\mathbf{e}=\mathbf{U}-\mathbf{u}$ and $\varepsilon=P-p$ is the one in Theorem (1.1):

$$\begin{aligned}\mathbf{e}_t-\Delta\mathbf{e}+\nabla\varepsilon&=(\mathbf{U}-\mathbf{u}_h)_t\\ \operatorname{div}\mathbf{e}&=0\,.\end{aligned}\tag{4.13}$$

Thus estimating $\mathbf{e}=\mathbf{U}-\mathbf{u}$ and $\varepsilon=P-p$ is done as in Section 2 provided that we know how to handle $\mathbf{U}-\mathbf{u}_h$ and $(\mathbf{U}-\mathbf{u}_h)_t$. In the remaining part of this section we show that although Lemma 1.1 is no longer valid as such, $\mathbf{U}-\mathbf{u}_h$ satisfies the necessary orthogonality relations needed to estimate $\mathbf{U}-\mathbf{u}_h$ and $(\mathbf{U}-\mathbf{u}_h)_t$ by applying the stationary a posteriori theory for the finite volume scheme [8]. In fact it is interesting that $(\mathbf{u}_h,p_h)$ *is the stationary finite volume solution to problem* (4.10):

Lemma 4.2. *Assume that* $(\mathbf{U}, P)$ *is the unique solution of the stationary Stokes problem* (4.10) *and* $(\mathbf{U}_h, P_h)$ *its finite volume solution, namely*

$$\begin{aligned} a(\mathbf{U}_h, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, P_h) = \langle \mathbf{g}_h, \boldsymbol{\Lambda}_h \boldsymbol{\varphi} \rangle \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{V}_h, \\ b(\mathbf{U}_h, q) = 0 \quad \text{for all } q \in \Pi_h. \end{aligned} \tag{4.14}$$

Then

$$\mathbf{U}_h(t) = \mathbf{u}_h(t) \quad \text{and } P_h(t) = p_h(t), \tag{4.15}$$

where $(\mathbf{u}_h, p_h)$ *is the solution of (4.4).*

Proof. Let $\boldsymbol{\varphi} \in \mathbf{Z}_h$, then $b(\boldsymbol{\varphi}, P_h) = 0$ and $a(\mathbf{U}_h, \boldsymbol{\varphi}) = \langle \mathbf{g}_h, \boldsymbol{\Lambda}_h \boldsymbol{\varphi} \rangle$. Now, in view of (4.4), (4.12) we have

$$a(\mathbf{U}_h, \boldsymbol{\varphi}) = \langle \mathbf{g}_h, \boldsymbol{\Lambda}_h \boldsymbol{\varphi} \rangle = \langle \mathbf{f} - \mathbf{u}_{h,t}, \boldsymbol{\Lambda}_h \boldsymbol{\varphi} \rangle = a(\mathbf{u}_h, \boldsymbol{\varphi}), \tag{4.16}$$

i.e.,

$$a(\mathbf{U}_h - \mathbf{u}_h, \boldsymbol{\varphi}) = 0 \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{Z}_h. \tag{4.17}$$

Since $\mathbf{U}_h,\ \mathbf{u}_h \in \mathbf{Z}_h$, we get $\mathbf{U}_h(t) = \mathbf{u}_h(t)$. Subtracting (4.14) from (4.4)

$$\begin{aligned} b(\boldsymbol{\varphi}, p_h) - b(\boldsymbol{\varphi}, P_h) &= -a(\mathbf{u}_h, \boldsymbol{\varphi}) - \langle \mathbf{u}_{h,t}, \boldsymbol{\Lambda}_h \boldsymbol{\varphi} \rangle + \langle \mathbf{f}, \boldsymbol{\Lambda}_h \boldsymbol{\varphi} \rangle - \\ &\quad - [-a(\mathbf{U}_h, \boldsymbol{\varphi}) + \langle \mathbf{g}_h, \boldsymbol{\Lambda}_h \boldsymbol{\varphi} \rangle] \\ &= a(\mathbf{U}_h - \mathbf{u}_h, \boldsymbol{\varphi}) - \langle \mathbf{u}_{h,t} - \mathbf{f} + \mathbf{g}_h, \boldsymbol{\Lambda}_h \boldsymbol{\varphi} \rangle, \end{aligned} \tag{4.18}$$

for all $\boldsymbol{\varphi} \in \mathbf{V}_h$. According to (4.12) and the fact that $\mathbf{U}_h = \mathbf{u}_h$ we get

$$b(\boldsymbol{\varphi}, p_h - P_h) = 0 \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{V}_h.$$

Due to the discrete inf–sup assumption (1.15) we obtain $p_h = P_h$. Therefore, $(\mathbf{u}_h, p_h) \in \mathbf{Z}_h \times \Pi_h$ is the finite volume solution of the stationary Stokes equation whose exact solution is $(\mathbf{U}, P)$. □

We now turn to the estimate $\mathbf{U} - \mathbf{u}_h$. As in the previous section, $\boldsymbol{\sigma} = \mathbf{U} - \mathbf{u}_h, \xi = P - p_h$. Note that Lemma 4.1 implies that $(\mathbf{u}_h, p_h)$ satisfies

$$a(\mathbf{u}_h, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, p_h) = \langle \mathscr{L}_h \mathbf{f}, \boldsymbol{\varphi} \rangle - \langle \mathscr{L}_h \mathbf{u}_{h,t}, \boldsymbol{\varphi} \rangle \quad \forall\, \boldsymbol{\varphi} \in \mathbf{V}_h. \tag{4.19}$$

Therefore in view of the definition of $(\mathbf{U}, P)$ and of (4.12) we have the orthogonality relation on the conforming space $\mathbf{X}_h$

$$a(\boldsymbol{\sigma}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, \xi) = \langle \mathbf{f} - \mathscr{L}_h \mathbf{f}, \boldsymbol{\varphi} \rangle - \langle \mathbf{u}_{h,t} - \mathscr{L}_h \mathbf{u}_{h,t}, \boldsymbol{\varphi} \rangle \quad \forall\, \boldsymbol{\varphi} \in \mathbf{X}_h. \tag{4.20}$$

The proof rests on applying again the argument in [13] as in the previous section and taking into account the additional error term resulting from the finite volume discretization. Thus, with the same notation as in Section 3, we use the decomposition $\nabla_h \boldsymbol{\sigma} = \nabla \mathbf{r} - q\mathbf{I} + \operatorname{curl} \mathbf{s}$, where $q \in L_0^2(\Omega)$, $\mathbf{r} \in \mathbf{H}_0^1(\Omega)$ with $\operatorname{div} \mathbf{r} = 0$ and $\mathbf{s} \in \mathbf{H}^1(\Omega)$. Thus, cf. Section 3, using (4.20) and the definition of $\mathscr{L}_h$ we finally get

(4.21)
$$\|\nabla_h \boldsymbol{\sigma}\|^2 = \sum_K \int_K \mathbf{g}_h \cdot (\mathbf{r} - \boldsymbol{\chi}) + \int_K (\mathbf{f} - \mathbf{u}_{h,t}) \cdot (\boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\chi}) - \int_{\partial K} (\nabla_h \mathbf{u}_h - p_h \mathbf{I}) \mathbf{n} \cdot (\mathbf{r} - \boldsymbol{\chi}) - \int_{\partial K} \nabla_h \mathbf{u}_h \boldsymbol{\tau} \cdot (\mathbf{s} - \boldsymbol{\psi}),$$

for any $\boldsymbol{\chi}$, $\boldsymbol{\psi} \in \mathbf{X}_h$. Still we denote by $\mathbf{R}_K$ the inner elementwise residual $\mathbf{R}_K = \mathbf{u}_{h,t} - \Delta \mathbf{u}_h + \nabla p_h - \mathbf{f}$. Then $\mathbf{g}_h|_K = -\mathbf{u}_{h,t} + \mathbf{f} = -\mathbf{R}_K$. Consider now the local error estimators $\eta_{1,K}(\mathbf{u}_h)$, defined in (3.6) and the global estimator $\eta_1(\mathbf{u}_h)$ defined by (3.8). Then we choose $\boldsymbol{\chi} = \mathbf{I}_h \mathbf{r}$, $\boldsymbol{\psi} = \mathbf{I}_h \mathbf{s}$ in (4.21), $\mathbf{I}_h$ being a Clement-type interpolant onto $\mathbf{X}_h$. Using the approximation properties of the interpolant and of the operator $\boldsymbol{\Lambda}_h$, (4.6), we conclude as in [8] that:

Lemma 4.3. *The following estimate holds:*

$$\|\mathbf{U} - \mathbf{u}_h\|_{\mathbf{V}} \leq C\, \eta_1\,(\mathbf{u}_h(t)\,)\,. \tag{4.22}$$

Next we will provide an a posteriori estimator for the L^2-norm error of the velocity. From the a priori analysis of finite volume methods, it is known, [6], that in order to get $\mathcal{O}(h^2)$ convergence in L^2, z_K is has to be chosen as the barycenter of K. Therefore in the sequel we assume that in the construction of the control volumes b_e, the point z_K is chosen to be the barycenter of K. In this case we will have

$$\int_K (\boldsymbol{\varphi} - \boldsymbol{\Lambda}_h \boldsymbol{\varphi}) = 0, \quad \forall\, \boldsymbol{\varphi} \in \mathbf{V}_h. \tag{4.23}$$

We consider again the dual problem (3.10). Then (3.12) is still valid. In addition, differentiating (4.20) with respect to t, we obtain, since all operators commute with time differentiation,

$$a(\boldsymbol{\sigma}_t, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, \xi_t) = \langle \mathbf{f}_t - \mathscr{L}_h \mathbf{f}_t, \boldsymbol{\varphi} \rangle - \langle \mathbf{u}_{h,tt} - \mathscr{L}_h \mathbf{u}_{h,tt}, \boldsymbol{\varphi} \rangle \quad \forall\, \boldsymbol{\varphi} \in \mathbf{X}_h.$$

In the next equation we also use the fact that $b(\mathbf{U}_t - \mathbf{u}_{h,t}, \psi) = 0, \psi \in \Pi_h$. Therefore

$$\begin{aligned} \|\boldsymbol{\sigma}_t\|^2 = \sum_K \Big\{ &\int_K \big(-\Delta \boldsymbol{\sigma}_t + \nabla \xi_t\big) \cdot (\mathbf{z} - \boldsymbol{\chi}) \\ &+ \int_K \big((s - \psi)\ \mathrm{div}(\boldsymbol{\sigma}_t) + \int_{\partial K} \big(\nabla(\boldsymbol{\sigma}_t)\, \mathbf{n} - \xi_t\, \mathbf{n}\big) \cdot (\mathbf{z} - \boldsymbol{\chi}) \\ &+ \int_K (\mathbf{f}_t - \mathbf{u}_{h,tt}) \cdot (\boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\varphi})\big) \\ &+ \int_{\partial K} \big(\nabla \mathbf{z}\, \mathbf{n} \cdot \mathbf{u}_{h,t} + s\, \mathbf{n} \cdot \mathbf{u}_{h,t}\big) \Big\}. \end{aligned}$$

Using the fact that $\mathbf{g}_{h,t} = -\partial_t \mathbf{R}_K = -\mathbf{R}_{K,t}$ and (4.23), we obtain

(4.24)
$$\begin{aligned}\|\boldsymbol{\sigma}_t\|^2 = \sum_K \Big\{ \int_K & (-\mathbf{R}_{K,t} \cdot (\mathbf{z} - \boldsymbol{\chi}) - (s - \psi) \operatorname{div} \mathbf{u}_{h,t}) \\ & - \int_K (\mathbf{R}_{K,t} - \overline{\mathbf{R}_{K,t}}) \cdot (\boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\chi})\big) \\ & + \int_{\partial K} \big(\nabla \mathbf{u}_{h,t}\, \mathbf{n} - p_{h,t}\, \mathbf{n}\big) \cdot (\mathbf{z} - \boldsymbol{\chi}) + \int_{\partial K} \big(\nabla \mathbf{z}\, \mathbf{n} + s\, \mathbf{n}\big) \cdot \mathbf{u}_{h,t} \Big\},\end{aligned}$$

for any $\boldsymbol{\chi} \in \mathbf{X}_h$ and $\psi \in \Pi_h$. Here $\overline{\mathbf{R}_{K,t}}$ denotes the average of $\mathbf{R}_{K,t}$ over K. The local error estimators $\widetilde{\eta}_{0,K}$ in the finite volume case are slightly different than the corresponding in the finite element case:

$$\begin{aligned}\widetilde{\eta}_{0,K}(\mathbf{u}_{h,t})^2 = & \, h_K^4 \|\mathbf{R}_{K,t}\|_{0,K}^2 + h_K^2 \, \| \operatorname{div} \mathbf{u}_{h,t}\|_{0,K}^2 \\ & + h_K^2 \|\mathbf{R}_{K,t} - \overline{\mathbf{R}_{K,t}}\|_{0,K}^2 \\ & + \frac{1}{2} \sum_{e \in E_h^{\text{in}}(K)} \big(h_e^3 \, \|\partial_t \mathbf{J}_{e,\mathbf{n}}\|_{0,e}^2 + h_e^3 \, \|\partial_t \mathbf{J}_{e,\boldsymbol{\tau}}\|_{0,e}^2\big).\end{aligned}$$

The L^2 estimator is defined by

$$\widetilde{\eta}_0(\mathbf{u}_{h,t}\,) = \Big(\sum_K \; \widetilde{\eta}_{0,K}(\mathbf{u}_{h,t})^2 \; \Big)^{1/2}. \tag{4.25}$$

Then, we choose χ as the standard nodal interpolant of $\mathbf{z}$, denoted by $\mathbf{I}_{h,\mathscr{N}}\mathbf{z}$ and $\psi = I_h s$. Following [8] we observe

$$\begin{aligned}\int_K \big(\mathbf{R}_{K,t} - \overline{\mathbf{R}_{K,t}}\big) \cdot \big(\mathbf{I}_{h,\mathscr{N}}\mathbf{z} - \boldsymbol{\Lambda}_h \mathbf{I}_{h,\mathscr{N}}\mathbf{z})\big) & \le \|\mathbf{R}_{K,t} - \overline{\mathbf{R}_{K,t}}\|_{0,K} \, \|\mathbf{I}_{h,\mathscr{N}}\mathbf{z} - \boldsymbol{\Lambda}_h \tilde{\mathbf{I}}_h \mathbf{z}\|_{0,K} \\ & \le C|K|^{1/2} \|\mathbf{R}_{K,t} - \overline{\mathbf{R}_{K,t}}\|_{0,K} \, |\mathbf{I}_{h,\mathscr{N}}\mathbf{z}|_{1,K} \\ & \le C|K|^{1/2} \|\mathbf{R}_{K,t} - \overline{\mathbf{R}_{K,t}}\|_{0,K} \, \|\mathbf{z}\|_{2,K}\,.\end{aligned}$$

The proof is thus complete.

We have thus proved

Lemma 4.4. *The following estimate holds:*

$$\|\mathbf{u}_{h,t} - \mathbf{U}_t\|_{\mathbf{H}} \le C\, \eta_1\, (\mathbf{u}_{h,t}(t)\,)\,. \tag{4.26}$$

Further, assuming that (3.11) *and* (4.23) *hold we have*

$$\|\mathbf{u}_{h,t} - \mathbf{U}_t\|_{\mathbf{H}} \le C\, \widetilde{\eta}_0\, (\mathbf{u}_{h,t}(t)\,)\,. \tag{4.27}$$

We can now apply Theorem 2.2 to obtain H^1 estimates in our case.

Theorem 4.1. (Residual $L^2(H^1)$ and $L^\infty(H^1)$–norm error estimates) *Assume that* $(\mathbf{u}, p)$ *is the solution of the time dependent Stokes Problem* (1.1) *and*

$(\mathbf{u}_h, p_h)$ *is the finite volume approximation* (4.2), (4.3). *Then the following a posteriori bounds hold for* $0 < t \leq T$

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\|_{\mathbf{H}} \leq \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{V}} + C\Big(\int_0^t \eta_1((\mathbf{u}_{h,t}(s)\,)^2 ds\Big)^{1/2} + C\eta_1((\mathbf{u}_h(0)\,) + C\eta_1(\mathbf{u}_h(t)\,).$$

and

$$\Big(\int_0^t \|(\mathbf{u}-\mathbf{u}_h)(s)\|_{\mathbf{V}}^2 ds\Big)^{1/2} \leq \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{H}} + C\Big(\int_0^t \eta_1((\mathbf{u}_{h,t}(s)\,)^2 ds\Big)^{1/2} + C\eta_1((\mathbf{u}_h(0)\,) + C\Big(\int_0^t \eta_1((\mathbf{u}_h(s)\,)^2 ds\Big)^{1/2}.$$

Remark 4.5. *In the case where* (3.11) *and* (4.23) *hold the estimator* $\eta_1((\mathbf{u}_{h,t}(s)\,)$ *in these bounds should be replaced by* $\widetilde{\eta}_0((\mathbf{u}_{h,t}(s)\,)$.

Further we prove the following $L^\infty(L^2)$ a posteriori estimate. The same remarks preceding Theorem 3.2 apply also here regarding the assumption of the elliptic regularity bound (3.11).

Theorem 4.2. (Residual L^2–norm error estimates) *Assume that* $(\mathbf{u}, p)$ *is the solution of the time dependent Stokes Problem* (1.1) *and* $(\mathbf{u}_h, p_h)$ *is the finite volume approximation* (4.2), (4.3). *Assume further that* (3.11) *and* (4.23) *hold. Then the following a posteriori bound holds for* $0 < t \leq T$

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\|_{\mathbf{H}} \leq \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{H}} + C\Big(\int_0^t \widetilde{\eta}_0((\mathbf{u}_{h,t}(s)\,)^2 ds\Big)^{1/2} + C\widetilde{\eta}_0(\mathbf{u}_h(0)\,) + C\widetilde{\eta}_0(\mathbf{u}_h(t)\,).$$

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