PROOF
Yokohama Publishers
http://www.ybook.co.jp

Stability and Convergence of Relaxation Finite Element Schemes for the Incompressible Navier-Stokes Equations

Theodoros Katsaounis, Charalambos Makridakis, and Chiara Simeoni

Abstract. We are interested in the numerical modelling of the incompressible Navier-Stokes equations, in two space dimensions. The approach considered consists in developing semi-discrete finite element schemes for appropriate relaxation models, which formally arise as hyperbolic approximations of the Navier-Stokes equations. Stability properties of the relaxation finite element schemes are derived from estimating suitable modifications of the standard energy functional, that is suggested by the presence of relaxation terms. These techniques are also applied to prove error estimates and deduce the convergence of relaxation finite element schemes to the (smooth) solutions of the incompressible Navier-Stokes equations.

1. Introduction

We consider the incompressible Navier-Stokes equations, in two space dimensions,

$$\begin{aligned} &\partial_t u + u \cdot \nabla u - \nu \, \Delta u + \nabla p \, = \, 0 \, , \\ &\nabla \cdot u \, = \, 0 \, , \quad t \in \mathbb{R}_+ \, , \; x \in \mathbb{T}^2, \end{aligned} \tag{1}$$

which describe the flow of an incompressible fluid with kinematic viscosity $\nu > 0$, where $u(t,x) \in \mathbb{R}^2$ and $p(t,x) \in \mathbb{R}$ are the velocity and the pressure field, respectively. For the sake of simplicity, we assume the fluid moves in a two-dimensional torus $\mathbb{T}^2 \subset \mathbb{R}^2$, to reduce to homogeneous boundary conditions.

We introduce the relaxation approximation of (1) considered in [4], for $\varepsilon > 0$,

$$\begin{aligned} &\left\{ \begin{aligned} &\partial_t u^\varepsilon + \nabla \cdot U^\varepsilon + \nabla p^\varepsilon \, = \, 0 \, , \\ &\partial_t U^\varepsilon + \tfrac{\nu}{\varepsilon} \, \nabla u^\varepsilon \, = \, -\tfrac{1}{\varepsilon} \, (U^\varepsilon - u^\varepsilon \otimes u^\varepsilon) \, , \end{aligned} \right. \\ &\nabla \cdot u^\varepsilon \, = \, 0 \, , \quad t \in \mathbb{R}_+ \, , \; x \in \mathbb{T}^2, \end{aligned} \tag{2}$$

which is obtained by singular perturbations of the nonlinear term in (1) through a supplementary (matrix valued) variable $U^\varepsilon(t,x) \in \mathbb{R}^2 \times \mathbb{R}^2$, or rather, by means of

diffusive scaling from standard relaxation approximations of the Euler equations, most in the spirit of [9]. We mention that similar relaxation models are also proposed in [2] for numerical purposes, as reduced discrete velocity kinetic models, and these approximations are systematically used to study the relaxation limits of hyperbolic conservation laws (see [6] and [12], for instance).
We recall the one-field equation associated with the relaxation system (2), namely

$$
\begin{aligned}
&\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \nu\, \Delta u^\varepsilon + \varepsilon\, \partial_{tt} u^\varepsilon + \nabla p^\varepsilon + \varepsilon\, \partial_t \nabla p^\varepsilon \;=\; 0\,, \\
&\nabla \cdot u^\varepsilon \;=\; 0\,, \quad t \in \mathbb{R}_+\,,\; x \in \mathbb{T}^2,
\end{aligned} \tag{3}
$$

which turns out to be a hyperbolic approximation of the Navier-Stokes equations (1) with a small dissipative correction (that is viewed as part of a wave operator).
We refer to [4] for the rigorous justification of the convergence, as $\varepsilon \to 0$, toward the incompressible Navier-Stokes equations, by the "modulated energy" method [3].

We deal with a numerical approach to the Navier-Stokes equations (1) based on standard finite element schemes for the relaxation approximation (3).
Let $\mathbb{T}_h$ be a (uniformly regular) partition of $\mathbb{R}^2$, with characteristic parameter $h > 0$. The discrete velocity u_h and pressure p_h are determined in finite element spaces denoted by $V_h \subset H^1(\mathbb{T}^2)$ and $\Pi_h \subset L^2(\mathbb{T}^2)$, respectively. These spaces are assumed to satisfy typical approximability properties, namely for V_h consisting of piecewise polynomials (globally continuous) on $\mathbb{T}_h$ such that

$$
\inf_{v \in V_h} \,[\, ||u - v||_{L^2} + h\, ||u - v||_{H^1}] \;\leq\; C\, h^k\, ||u||_{H^k}\,, \quad u \in H^k(\mathbb{T}^2)\,,\; k \geq 2\,. \tag{4}
$$

We focus on spatial discretizations of (3), the so-called "semi-discrete schemes" (for the computational issues, a semi-implicit discretization in time is eventually considered in [1], for which some analogue of the estimates below can also be proved). In a first attempt, we only address the question of establishing error estimates for the velocity field, therefore we introduce the subspace $Z_h \subset V_h$ of approximate divergence-free functions, given by

$$
Z_h \;=\; \{\, v \in V_h \;;\; \langle\, \nabla \cdot v\,,\, q\,\rangle = 0\,,\; \forall\, q \in \Pi_h \,\}\,, \tag{5}
$$

where the inner-product $\langle .\,, . \rangle$ is defined for the L^2-norm.

According to [5], we take the variational formulation of (3) into account and the corresponding finite element scheme, for the numerical solution $u_h \in Z_h$, reads

$$
\langle \partial_t u_h\,, v\rangle + \langle N(u_h, u_h), v\rangle + \nu\, \langle \nabla u_h\,, \nabla v\rangle + \varepsilon\, \langle \partial_{tt} u_h\,, v\rangle \;=\; 0\,, \quad \forall\, v \in Z_h\,, \tag{6}
$$

$$
N(u_h, u_h) \;=\; u_h \cdot \nabla u_h + \frac{1}{2}\, (\nabla \cdot u_h)\, u_h\,, \tag{7}
$$

and the modified numerical nonlinear term (7) is introduced to keep the conservation property $\langle N(u_h, u_h), u_h\rangle = 0$ in the finite element framework (we remark that the divergence-free constraint holds only weakly in the finite element space Z_h).

2. "Sub-characteristic" Condition and Stability Estimates

The qualitative analysis of finite element schemes is performed by applying similar techniques as for the corresponding continuous problems (we refer to [7], [8] and [11] for standard finite element approximations of the Navier-Stokes equations).

To deduce stability properties of (6)-(7), we study an appropriate modification of the standard L^2-energy functional, whose form is suggested by the presence of the relaxation terms (see [1], [4] and [12], for instance).

Lemma 2.1. *Under the following assumption on the numerical solution,*

$$\|u_h\|_{L^\infty} \leq \sqrt{\frac{2\,\nu}{5\,\varepsilon}}\,, \quad \forall\, t \in \mathbb{R}_+\,, \tag{8}$$

the relaxation finite element scheme (6)-(7) *satisfies the dissipation estimate*

$$\frac{1}{2}\,\|u_h + \varepsilon\,\partial_t u_h\|^2_{L^2} + \frac{1}{2}\,\varepsilon^2\,\|\partial_t u_h\|^2_{L^2} + \varepsilon\,\nu\,\|\nabla u_h\|^2_{L^2} \leq C(u_h^0)\,, \tag{9}$$

for some constant $C(u_h^0)$ only depending on the approximation of initial data of (1)*.*

Proof. We take $v = u_h \in Z_h$ in (6) to obtain

$$\frac{1}{2}\,\frac{d}{dt}\|u_h\|^2_{L^2} + \nu\,\|\nabla u_h\|^2_{L^2} + \varepsilon\,\frac{d}{dt}(\partial_t u_h\,, u_h) - \varepsilon\,\|\partial_t u_h\|^2_{L^2} = 0\,. \tag{10}$$

To estimate the time derivative, we take $v = \partial_t u_h \in Z_h$ in (6) and we obtain

$$\|\partial_t u_h\|^2_{L^2} + \langle N(u_h, u_h), \partial_t u_h\rangle + \frac{1}{2}\,\nu\,\frac{d}{dt}\|\nabla u_h\|^2_{L^2} + \frac{1}{2}\,\varepsilon\,\frac{d}{dt}\|\partial_t u_h\|^2_{L^2} = 0\,. \tag{11}$$

Then, we multiply (11) by $2\,\varepsilon$ (as a correction by lower order effects) and we combine with (10), rearranging the resulting equality as follows,

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\|u_h + \varepsilon\,\partial_t u_h\|^2_{L^2} + \frac{1}{2}\,\varepsilon^2\,\frac{d}{dt}\|\partial_t u_h\|^2_{L^2} + \varepsilon\,\nu\,\frac{d}{dt}\|\nabla u_h\|^2_{L^2} \\
&= -\,\nu\,\|\nabla u_h\|^2_{L^2} - \varepsilon\,\|\partial_t u_h\|^2_{L^2} - 2\,\varepsilon\,\langle N(u_h, u_h), \partial_t u_h\rangle \\
&= -\,\varepsilon\,\|\partial_t u_h + N(u_h, u_h)\|^2_{L^2} - \nu\,\|\nabla u_h\|^2_{L^2} + \varepsilon\,\|N(u_h, u_h)\|^2_{L^2}\,.
\end{aligned} \tag{12}$$

Recalling (7), the last two terms of (12) are bounded as follows,

$$-\,\nu\,\|\nabla u_h\|^2_{L^2} + \varepsilon\,\|N(u_h, u_h)\|^2_{L^2} \leq -\left(\nu - \frac{5}{2}\,\varepsilon\,\|u_h\|^2_{L^\infty}\right)\|\nabla u_h\|^2_{L^2}\,. \tag{13}$$

The result (9) is thus achieved, after integrating (12) with respect to time, by using the uniform bound (8) to conclude the estimate of (13). □

We remark that the a priori L^∞-bound (8) is equivalent to the "sub-characteristic condition" required in [1] to derive stability estimates for relaxation finite element approximations of hyperbolic conservation laws, that reveals the regularization role of the wave operator in (3).

The "sub-characteristic condition" in (8) is actually satisfied for the relaxation finite element scheme (6)-(7), because of some "inverse-inequalities" which hold in the finite element spaces, namely, in two space dimensions,

$$\|u_h\|_{L^\infty} \leq C\, h^{-1} \|u_h\|_{L^2}\,, \quad \forall\, t \in \mathbb{R}_+\,. \tag{14}$$

Such inequalities, obviously degenerate with respect to the characteristic parameter h, are proved for several classes of finite element spaces (refer to [5] and [7], for instance), also including the framework (5).
Therefore, it follows from (14) that a sufficient condition for (8) to be satisfied is

$$\|u_h\|_{L^2} \leq C\, h \sqrt{\frac{2\,\nu}{5\,\varepsilon}}\,, \quad \forall\, t \in \mathbb{R}_+\,, \tag{15}$$

that is guaranteed by the stability estimate (9), provided that (15) holds for an approximation of the initial data of (1), and with (at least) $\varepsilon = O(h^2)$ as the ratio between the diffusion parameters.

3. Error Estimate for the Velocity

We consider standard arguments to derive error estimates for finite element schemes. According to [7], the technique is based on splitting the error between the solution to the Navier-Stokes equations (1) and the numerical solution obtained by applying the relaxation finite element scheme (6)-(7) into two parts, as follows,

$$u - u_h \;=\; (u - \mathbb{P}_h u) + (\mathbb{P}_h u - u_h)\;, \tag{16}$$

where $\mathbb{P}_h$ is some appropriate projection operator into the finite element space (5).

The consistency error $\eta(u) = (u - \mathbb{P}_h u)$ is interpreted in terms of approximability properties of the finite element spaces, according to the definition of $\mathbb{P}_h u \in Z_h$.
In a first attempt, we introduce the "elliptic projection" associated with the linear stationary part of the Navier-Stokes equations (1), namely

$$\langle \nabla\, \mathbb{P}_h u\,, \nabla v\rangle \;=\; \langle \nabla u\,, \nabla v\rangle\;, \quad \forall\, v \in Z_h\,, \tag{17}$$

for which some analogue of (4) in the framework (5) also holds (see [5], for instance).
We thus deduce that

$$\|\eta(u)\|_{L^2} \;\leq\; C\, h^k\, ||u||_{H^k}\,, \quad u \in H^k(\mathbb{T}^2)\,,\; k \geq 2\,. \tag{18}$$

Taking the variational formulation of (1) and (6) into account, together with (17), we establish a variational equation for the stability error $\xi(u, u_h) = (\mathbb{P}_h u - u_h) \in Z_h$ in (16), namely

$$\begin{aligned} &\langle \partial_t \xi\,, v\rangle + \langle u\cdot\nabla u - N(u_h, u_h)\,, v\rangle + \nu\; \langle \nabla \xi\,, \nabla v\rangle + \varepsilon\; \langle \partial_{tt}\xi, v\rangle \\ &= -\,\langle \partial_t \eta\,, \phi\rangle + \varepsilon\; \langle \partial_{tt}\mathbb{P}_h u\,, \phi\rangle - \langle \nabla p, v\rangle\,, \quad \forall\, v \in Z_h\,. \end{aligned} \tag{19}$$

We make similar computations as for the proof of the L^2-stability estimate (9) in Lemma 2.1 and, in the right-hand side of (19), we use the consistency properties (18) of the elliptic projection, to deduce the following convergence result.

Theorem 3.1. *Under the assumption* (8) *on the numerical solution, the relaxation finite element scheme* (6)-(7) *satisfies the following error estimate, for* $k \geq 2$,

$$\|u - u_h\|_{L^2} \leq C \exp\left(\frac{1}{\nu}\right) h^k , \quad \forall\, t \in \mathbb{R}_+ , \tag{20}$$

with some constant C *depending on smoothness properties of the solution to* (1).

We remark that is the treatment of the incompressibility condition which leads to the singular dependence of the error constants in (20) with respect to the diffusion parameters. That issue is particular important in the numerical analysis of the incompressible Navier-Stokes equations, as the question of dealing with the divergence-free condition rigorously is still an open problem.
The main idea of relaxation finite element schemes for the incompressible Navier-Stokes equations is to achieve stability also in the limit of small viscosity, when the transport terms become dominant and we recover the framework of hyperbolic conservation laws (we actually consider relaxation approximations of the incompressible Euler equations), for which these techniques have been successfully applied in [1]. In that sense, further works are in progress for some different relaxation models (see [6], for instance).

We refer to [10] for the details of the proof of Theorem (3.1) and the more general statement of the subjects covered in these notes.

Acknowledgments

The authors would like to thank Prof. Athanasios E. Tzavaras and Prof. P.E. Jabin for several helpful discussions.
This work was partially supported by the HYKE European Network HPRN-CT-2002-00282 and by the MCWAVE Marie Curie Host Fellowship at the I.A.C.M. - F.O.R.T.H. Heraklion, Crete (Greece).

References

[1] Ch. Arvanitis, Ch. Makridakis, A.E. Tzavaras, Stability and convergence of a class of finite element schemes for hyperbolic systems of conservation laws, to appear in *SIAM J. Numer. Anal.* (2004)

[2] M.K. Banda, A. Klar, L. Pareschi, M. Seaïd, Compressible and incompressible limits for hyperbolic systems with relaxation, *J. Comput. Appl. Math.* **168** (2004), no. 1-2, 41-52

[3] Y. Brenier, Convergence of the Vlasov-Poisson system to the incompressible Euler equations, *Comm. Partial Differential Equations* **25** (2000), no. 3-4, 737-754

[4] Y. Brenier, R. Natalini, M. Puel, On a relaxation approximation of the incompressible Navier-Stokes equations, *Proc. Amer. Math. Soc.*, **132** (2004), no. 4, 1021-1028

[5] S.C. Brenner, L.R. Scott, *The mathematical theory of finite element methods* Second Edition, Texts in Applied Mathematics **15**, Springer-Verlag, New York, 2002

[6] D. Donatelli, P. Marcati, Convergence of singular limits for multi-D semilinear hyperbolic systems to parabolic systems, *Trans. Amer. Math. Soc.* **356** (2004), no. 5, 2093-2121

[7] V. Girault, P.A. Raviart, *Finite element methods for Navier-Stokes equations. Theory and algorithms*, Springer Series in Computational Mathematics **5**, Springer-Verlag, Berlin, 1986

[8] J.G. Heywood, R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization, *SIAM J. Numer. Anal.* **19** (1982), no. 2, 275-311
[9] S. Jin, Z.P. Xin, The relaxation schemes for systems of conservation laws in arbitrary space dimensions, *Comm. Pure Appl. Math.* **48** (1995), no. 3, 235-276
[10] Th. Katsaounis, Ch. Makridakis, C. Simeoni, Relaxation methods and finite element schemes for the incompressible Navier-Stokes equations, *preprint*
[11] R. Temam, *Navier-Stokes equations. Theory and numerical analysis* (Reprint of the 1984 edition), AMS Chelsea Publishing, Providence, 2001
[12] A.E. Tzavaras, Materials with internal variables and relaxation to conservation laws, *Arch. Ration. Mech. Anal.* **146** (1999), no. 2, 129-155

Theodoros Katsaounis
Department of Applied Mathematics - University of Crete
P.O. Box 2208 - GR71409 Heraklion, Crete - Greece
E-mail address: thodoros@tem.uoc.gr

Charalambos Makridakis
Institute of Applied and Computational Mathematics - F.O.R.T.H.
P.O. Box 1527 - GR71110 Heraklion, Crete - Greece
E-mail address: makr@tem.uoc.gr

Chiara Simeoni
Laboratoire J.A. Dieudonné UMR CNRS 6621
Université de Nice - Sophia Antipolis
Parc Valrose - FR06108 Nice Cedex 2 - France
E-mail address: simeoni@math.unice.fr