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Finite volume schemes for Hamilton–Jacobi equations

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Summary. We introduce two classes of monotone finite volume schemes for Hamilton-Jacobi equations. The corresponding approximating functions are piecewise linear defined on a mesh consisting of triangles. The schemes are shown to converge to the viscosity solution of the Hamilton–Jacobi equation.

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0. Introduction

In this paper we consider finite volume schemes approximating the viscosity solution of the Hamilton–Jacobi equation

$$\begin{aligned} u_t + H(Du) &= 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \\ u &= u_0 \quad \text{on } \mathbb{R}^N \times \{0\}, \end{aligned} \tag{0.1}$$

where the Hamiltonian $H \in C^{0,1}(\mathbb{R}^N)$, the space of Lipschitz continuous functions in $\mathbb{R}^N$, and $u_0 \in BUC(\mathbb{R}^N)$, the space of bounded and uniformly continuous functions on $\mathbb{R}^N$.

A few comments about viscosity solutions are in order here. The class of viscosity solutions, which was introduced by Crandall and Lions in [CL1], is the "correct class" of weak solutions for Hamilton–Jacobi equations as well as fully nonlinear possibly degenerate second order elliptic and parabolic pde. We refer to the book of Barles [B] and the "User's Guide" of Crandall,

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Ishii and Lions [CIL] for a general overview of the scope of the theory and some of its applications.

We consider two different classes of finite volume schemes namely the co-volume and the edge-centered schemes. In both cases the mesh consists of triangles. In the case of co-volume schemes the approximating solution belongs to the space of continuous piecewise linear functions (the standard $\mathbb{P}_1$ finite element space), while in the case of the edge-centered schemes it belongs to the space of piecewise linear functions which are continuous at the midpoints of the triangles (the Crouzeix-Raviart space [CR]).

Both schemes are shown to be consistent and monotone. The general theory of Barles and Souganidis [BaS] (see also Crandall and Lions [CL2] and Souganidis [S]) provides the uniform convergence of the approximation as well as an error estimate. In particular we show here that if u_h is the approximating solution, h being the mesh size, then, as $h \to 0$,

$$u_h \to u \text{ uniformly in } \mathbb{R}^N \times [0,T]. \tag{0.2}$$

Moreover, if $u_0 \in C^{0,1}(\mathbb{R}^N) \cap BUC(\mathbb{R}^N)$, then, for there exists $C > 0$ such that

$$\|u - u_h\|_{L^\infty} \le Ch^{1/2}. \tag{0.3}$$

A different class of schemes defined on triangular meshes for (0.1) is proposed and analyzed by Abgrall [A]. The schemes of [A], however, do not appear to be easily extendable to second-order problems, as opposed, for example, to the results in [Ch], where a detailed study of various finite volume methods for linear second order problems is presented. Finite volumes are widely used in the numerical approximation of conservation laws, cf. e.g. [CCL] and the references therein. Co-volume methods for mean curvature equations were studied in [W]. Finally we refer to [KMS] where we consider finite volume schemes to second-order fully nonlinear problems.

The paper is organized as follows: In Sect. 1 we introduce the necessary notation, define the schemes, and state our results. In Sect. 2 we recall the abstract framework of [BaS] which is used to prove the convergence and the error estimates. In Sect. 3 we prove our results showing that the schemes under consideration satisfy the assumptions of the convergence result stated in Sect. 2. To simplify the presentation throughout the paper we only consider the case $N = 2$.

1. Preliminaries

We consider triangulations $\mathcal{T}_h$ of $\mathbb{R}^2$ into nonoverlapping, nonempty, open triangles T, with diameter h_T, such that $\mathbb{R}^2 = \bigcup_{T \in \mathcal{T}_h} T$. We assume that $\mathcal{T}_h$

satisfy the following conditions:

(1.1)
- (i) Intersecting triangles have either a common vertex or a common edge.
- (ii) No more than μ triangles have a common vertex.
- (iii) $h = \sup_{T \in \mathcal{T}_h} h_T < 1$,

and
- (iv) $\mathcal{T}_h$ is regular, i.e., there exists a constant γ independent of h such that if ρ_T is the diameter of the largest ball $B \subset T$, then, for all $T \in \mathcal{T}_h$, $h_T \leq \gamma \rho_T$.

The last assumption implies that if e is a face of the triangle T, then

(i) the length $m(e)$ of the side e and h_T are comparable,

(ii) the angles of the triangles are no smaller than $\theta_0 > 0$ (minimal angle condition), and

(iii) neighboring triangles T with T', $T \cap T' \neq \emptyset$ have comparable area, i.e., there exist positive constants c_1, c_2 such that $c_1 m(T') \leq m(T) \leq m(T')c_2$, where $m(T)$ denotes the area of the triangle T.

1.1. Co-volume discretization

Given a triangulation $\mathcal{T}_h$ we construct a dual (non-triangular mesh) with vertices the circumcenters of the triangles and edges joining the circumcenters of triangles that have adjacent sides. We associate to each vertex $A \in V_h$, the set of all the vertices of the triangulation, the co-volume V_A bounded by the edges of the dual mesh (see Fig. 1). We denote by A_ℓ, $1 < \ell \leq \mu_A$, the vertices of the triangles different from the common one A enumerated in the counterclockwise direction, and by e_{A_ℓ} the line segment joining A and A_ℓ and by $e^{\perp}_{A_\ell}$ the edge of V_A that intersects perpendicularly e_{A_ℓ}. The triangle

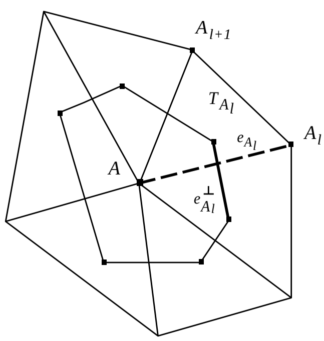

Fig. 1. The volume V_A

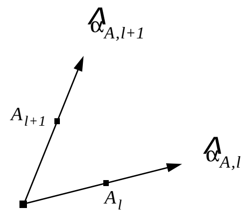

Fig. 2. The unit vectors $\widehat{\alpha}_{A,\ell}$ and $\widehat{\alpha}_{A,\ell+1}$

with vertices A, A_ℓ, $A_{\ell+1}$ is denoted by $T_{A,\ell}$, and the angle between e_{A_ℓ}, $e_{A_{\ell+1}}$ by $\theta_{A,\ell}$. We also use the notation $\widehat{\alpha}_{A,\ell}$ for the unit vector along e_{A_ℓ} directed towards A_ℓ and the notation $\widehat{\alpha}^{\perp}_{A,\ell}$ for the unit vector obtained by rotating $\widehat{\alpha}_{A,\ell}$ counterclockwise 90° (see Fig. 2). The part of $e^{\perp}_{A_\ell}$ that lies in $T_{A,\ell-1}$ is denoted by $e^{\perp,-}_{A_\ell}$ and the one that lies in $T_{A,\ell}$ is denoted by $e^{\perp,+}_{A_\ell}$. Finally b_{A_ℓ} is the line segment joining A and the vertex of V_A that lies in $T_{A,\ell}$.

We construct approximations $u^n_h \approx u(\,\cdot\,, t_n)$, of the viscosity solution u, where $\{t_n : \; n = 0, 1, \dots, N\}$, is a given partition of $[0, T]$ with (constant) time step $\tau = t_{n+1} - t_n$. Our construction can be easily generalized to nonuniform time steps. The approximating function u^n_h lies in the space of continuous piecewise linear functions defined on $\mathcal{T}_h$, i.e.,

$$u^n_h \in S_h = \{\varphi \in C(\mathbb{R}^2) : \varphi|_T \in \mathbb{P}_1(T) \;\text{ for }\; T \in \mathcal{T}_h\}.$$

Next we present a formal argument to motivate the choice of the scheme. In particular we add a small diffusion term to the Hamilton–Jacobi equation, integrate over the co-volume V_A and replace the time derivative by a first-order Euler approximation. We obtain

$$\begin{aligned} &\frac{1}{\tau}\int_{V_A} (u(t_{n+1}) - u(t_n)\,)dx \\ &\qquad + \int_{V_A} H(Du(t_n)\,)dx \approx \varepsilon_h \int_{\partial V_A} \frac{\partial u(t_n)}{\partial \nu} dS \end{aligned} \tag{1.2}$$

where we denote by ν the outward normal to the boundary ∂V_A of the co-volume.

If u^n_h is the approximating function at time t_n, we denote its value at A by u^n_A, with $u^n_A = u^n_h(A) \approx \frac{1}{m(V_A)} \int_{V_A} u(t_n)dx$. Then (1.2) takes the form

$$\begin{aligned} &m(V_A)\frac{u^{n+1}_A - u^n_A}{\tau} \\ &+ \sum_{V_A \cap T_{A,\ell}} m(T_{A,\ell} \cap V_A) H(Du^n_{T_{A,\ell}}) = \varepsilon_{h,A} \int_{\partial V_A} \frac{\partial u^n_h}{\partial \nu} dS, \end{aligned} \tag{1.3}$$

where $u^n_{T_{A,\ell}} = u^n_h\big|_{T_{A,\ell}}$ and $Du^n_{T_{A,\ell}} = Du^n_h\big|_{T_{A,\ell}}$. The diffusion coefficient $\varepsilon_{h,A}$ will be defined below. Since $u^n_h \in S_h$, u^n_h it is uniquely defined by its values at the vertices of the triangles; so, given u^n_h we compute u^{n+1}_h by (1.3).

We may also write the co-volume scheme as

$$u^{n+1}_A = u^n_A - \tau H_A(Du^n_{T_{A,1}}, \dots, Du^n_{T_{A,\mu_A}}), \tag{1.4}$$

where the discrete Hamiltonian H_A is given by

$$\begin{aligned} &H_A(Du_{T^n_{A,1}}, \dots , Du^n_{T_{A,\mu_A}}) \\ &\quad = \frac{1}{m(V_A)} \sum_{T_{A,\ell}\cap V_A \neq \emptyset} m(V_A \cap T_{A,\ell}) H(Du^n_{T_{A,\ell}}) \\ &\qquad - \frac{\varepsilon_{h,A}}{m(V_A)} \int_{\partial V_A} \frac{\partial u^n_h}{\partial \nu} dS. \end{aligned} \tag{1.5}$$

The approximate solution u^n_h is reconstructed on each triangle T by interpolating the values at the edges, i.e.,

$$\begin{aligned} &u^n_h(x)\big|_T = I_T[u^n_A] \in \mathbb{P}_1(T), \\ &\qquad \text{where} \quad I_T[u^n_A](B) = u^n_B,\ B \in V_h(T),\ T \in \mathcal{T}_h, \end{aligned} \tag{1.6}$$

and $V_h(T)$ denotes the set of the vertices of the triangle T. Finally u^0_h can be selected, e.g., simply by interpolating the initial condition u_0.

Our result is:

Theorem 1. *Assume that $H \in C^{0,1}(\mathbb{R}^2)$, let T_h be a triangulation of $\mathbb{R}^2$ satisfying (1.1) and consider the scheme defined by (1.4) and (1.5). There exist positive constants K and C_{CFL} such that if $\varepsilon_{h,A} = \varepsilon_h(A) = K \max_\ell h_{T_{A_\ell}}$ in (1.3) and*

$$\max_\ell \frac{\tau}{m(e_{A_\ell})} \le C_{\mathrm{CFL}}, \tag{1.7}$$

then for all $u_0 \in BUC(\mathbb{R}^2)$, as $h \to 0$ and $n \to \infty$,

$$\sup_{A\in V_h,\, 0\le n\le N} |u(A, t_n) - u^n_h(A)| \to 0.$$

If in addition $u_0 \in C^{0,1}(\mathbb{R}^2)$, then there exists a constant $C = C(\gamma, \|Du_0\|, \|D_pH\|) > 0$ such that

$$\sup_{A\in V_h,\, 0\le n\le N} |u(A, t_n) - u^n_h(A)| \le Ch^{1/2}.$$

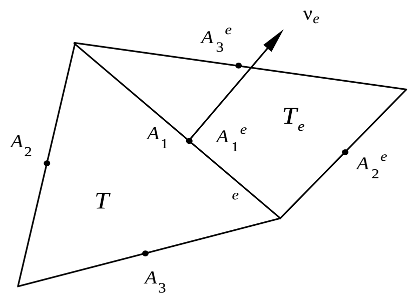

Fig. 3. The control volume for the edge-centered discretization

Remark 1.1 One can obtain explicit bounds for K and C_{CFL} by keeping track of the constants in the proof of Lemma 3.1 below. An estimate for C can be found in [CL2] and [S].

Remark 1.2 The error estimates in [CL2] and [S] for monotone finite difference schemes are obtained for locally Lipschitz Hamiltonians. Indeed such schemes are shown to be Lipschitz continuous since they commute with spatial translations. This is not true in the case at hand. Actually it is typical for finite volume schemes on unstructured meshes not to commute with spatial translations. Hence there is no control in general on the numerical gradients. The special finite volume schemes proposed in [A] are in fact designed to be intrinsic and therefore to provide control on the numerical gradients.

1.2. Edge-centered discretization

Given a triangle T of $\mathcal{T}_h$ we denote by $e_\ell(T)$, $\ell = 1, 2, 3$, the edges of the triangle and by T_e the neighbouring triangle that shares the edge e with T. The middle points of the edges of T, T_e will be denoted by A_ℓ and A_ℓ^e respectively, $\ell = 1, 2, 3$, named in the counterclockwise direction. The common middle points are the A_1, A_1^e. The unit normal vector ν_e to the common edge is directed towards T_e. The set consisting of all the middle points of our triangulation will be denoted by M_h and the set of the middle points of the triangle T by $M_h(T)$.

We will construct approximations $u_h^n \approx u(\,\cdot\,, t_n)$, of the viscosity solution u, where $\{t_n,\ n = 0, 1, \ldots, N\}$, is a given partition of the time domain with (constant) time step $\tau = t_{n+1} - t_n$. The approximating function u_h^n will lie in the space h of nonconforming piecewise linear functions defined on $\mathcal{T}_h$ introduced in [CR], i.e.,

$$u_h^n \in X_h = \{\varphi\ : \varphi|_T \in \mathbb{P}_1(T) \text{ for } T \in \mathcal{T}_h \quad \text{and} \\ \varphi \text{ is continuous at every } A \in M_h\}.$$

Adding a small diffusion term to the Hamilton–Jacobi equation and averaging over the union of two neighbouring triangles T and T_e, we are lead to the following scheme, yielding approximate values of the solution at the middle point $A_1 \equiv A_1^e$ of the common edge e of T and T_e,

$$u_{A_1}^{n+1} = u_{A_1}^n - \tau g_{A_1}(Du_T^n, Du_{T_e}^n)\,, \tag{1.8}$$

where the numerical Hamiltonian is defined

$$\begin{aligned} g_{A_1}(Du_T^n, Du_{T_e}^n) &= g_{A_1}(Du_T^n, Du_{T_e}^n, \nu_e) \\ &= H\left(\frac{1}{m(T)+m(T_e)}[m(T)Du_T^n + m(T_e)Du_{T_e}^n]\right) \\ &\quad - \theta_h[Du_{T_e}^n - Du_T^n]\cdot \nu_e. \end{aligned} \tag{1.9}$$

The approximate solution u_h^n is reconstructed by its values at the middle points by interpolating

$$\begin{aligned} &u_h^n(x)|_T = I_T[u_A^n] \in \mathbb{P}_1(T), \\ &\qquad \text{where } I_T[u_A^n](B) = u_B^n,\ B \in M_h(T),\ T \in \mathcal{T}_h. \end{aligned} \tag{1.10}$$

Also, u_h^0 can be chosen to be the element of X_h that interpolates the initial condition u_0 at the midle points $A \in M_h$.

Our theorem is:

Theorem 2. *Assume that $H \in C^{0,1}(\mathbb{R}^2)$, $u_0 \in BUC(\mathbb{R}^2)$ and let $\mathcal{T}_h$ be a triangulation satisfying (1.1). Consider the scheme defined by (1.8) and (1.9) with θ_h given by (3.15). If the* CFL *condition (3.14) holds, then, as $h \to 0$ and $n \to \infty$,*

$$\sup_{A\in M_h, 0\le n\le N} |u(A,t_n) - u_h^n(A)| \to 0.$$

If, in addition, $u_0 \in C^{0,1}(\mathbb{R}^2)$, then there exists a constant $C = C(\gamma, \|D_pH\|_{L^\infty}, \|Du_0\|) > 0$ such that

$$\sup_{A\in M_h, 0\le n\le N} |u(A,t_n) - u_h^n(A)| \le Ch^{1/2}.$$

2. An abstract formulation

Here we borrow from [BaS] to present an abstract result which yields, after checking its assumptions, Theorems 1 and 2. As stated in [BaS] this result, which applies also to fully nonlinear second-order equations, yields only local uniform convergence and no error estimates. A straightforward modification of its proof, however, yields uniform convergence and error estimates for Hamilton–Jacobi equations. We refer to [CL2] and [S] where such results were obtained for finite difference schemes.

For $\rho > 0$ let $S(\rho) : B(\mathbb{R}^N) \to B(\mathbb{R}^N)$, where $B(D)$ is the space of bounded functions defined on D – we use $B(\mathbb{R}^N)$ instead of $L^\infty(\mathbb{R}^N)$ to point out that no measure theory is involved in this framework – be such that the following conditions hold:

(2.1) monotonicity, i.e., if $u \geq v$, then $S(\rho)u \geq S(\rho)v$,
(2.2) invariance under translations with constants, i.e., $S(\rho)(u+k) = S(\rho)u + k, \quad k \in \mathbb{R}$,

and

(2.3) consistency, i.e, for all $\phi \in C^\infty(\mathbb{R}^N)$,

$$\frac{\phi - S(\rho)\phi}{\rho} \to H(D\phi) \quad \text{as } \rho \to 0.$$

Given such an S and a positive integer M define $u_M : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ by
(2.4)

$$u_M(x,t) = \begin{cases} S\Big(t - i\frac{T}{M}\Big)u_M\Big(\cdot, i\frac{T}{M}\Big)(x), & \text{if } t \in \Big(i\frac{T}{M}, (i+1)\frac{T}{M}\Big], \\ u_0(x), & \text{if } t = 0. \end{cases}$$

We have:

Theorem 2.1 *Assume (2.1), (2.2), (2.3) and $H \in C(\mathbb{R}^N)$, $u_0 \in BUC(\mathbb{R}^N)$. If $u \in BUC(\mathbb{R}^N \times [0,T])$ is the viscosity solution of (1.1), then as $M \to \infty$,*

$$u_M \to u \quad \textit{uniformly on} \quad \mathbb{R}^N \times [0,T].$$

To obtain an error estimate, the consistency condition needs to be strengthened to

$$\Big|\frac{\phi - S(\rho)\phi}{\rho} - H(D\phi)\Big| \leq O(\rho(\|D\phi\| + \|D^2\phi\|)). \tag{2.5}$$

We have:

Theorem 2.2 *Assume (2.1), (2.2), (2.5), $H \in C^{0,1}(\mathbb{R}^N)$ and $u_0 \in C^{0,1}(\mathbb{R}^N) \cap BUC(\mathbb{R}^N)$. If $u \in C^{0,1}(\mathbb{R}^N \times [0,T])$ is the viscosity solution of (1.1) and u_M is defined by (2.4), there exists a positive constant $C = C(\|Du_0\|)$ such that*

$$\|u_M - u\|_\infty \leq CM^{-1/2}.$$

3. The proofs

3.1. Co-volume methods

For $\tau > 0$ we define $S(\tau) : B(\mathbb{R}^2) \to B(\mathbb{R}^2)$ as follows: For $v \in B(\mathbb{R}^2)$, let $I_h v \in S_h$ be its piecewise interpolant. Then $S(\tau)v$ is given by applying the scheme (1.4-6) to $I_h v$ over one time step.

The fact that S satisfies (2.2) is now immediate. In view of the fact (see [BrS] for example)

$$||D(I_h\varphi - \varphi)|| \leq Ch\|D^2\varphi\|,$$

and of the stability of I_h in the max norm, it suffices to check (2.5) only at the vertices. Since

$$\frac{\varepsilon_h}{m(V_A)} \int_{\partial V_A} p \cdot \nu = 0, \tag{3.2}$$

the consistency condition (2.5) follows using the fact that, for all $p \in \mathbb{R}^2$,

$$H_A(p, \ldots, p) = H(p). \tag{3.3}$$

The monotonicity condition (2.1) follows from the obvious fact that I_h is monotone and the following lemma:

Lemma 3.1 *There exist constants K and $C_{\mathrm{CFL}} = C(K, \gamma, \|D_p H\|_{L^\infty})$ such that, if*
$\varepsilon_h(A) = K \max_\ell h_{T_{A,\ell}}$ *for a sufficiently large constant K and*

$$\max_\ell \frac{\tau}{m(e_{A_\ell})} \leq C_{\mathrm{CFL}}, \tag{3.4}$$

then the scheme defined by (1.4) is monotone, i.e., the function

$$G_A(u_A, u_{A_1}, \ldots u_{A_{\mu_A}}) = u_A - \tau H_A(Du_{T_{A,1}}, \ldots, Du_{T_{A,\mu_A}}) \tag{3.5}$$

is a nondecreasing function of each argument.

Remark 3.1 Explicit bounds for K and C_{CFL} can be obtained by keeping track to the constants in the proof of Lemma 3.1. (See (3.10–12) below.)

Proof. To prove the claim we need to find the explicit dependence of G_A on u_{A_j}. We begin by expressing $Du_{T_{A_\ell}}$ in terms of u_{A_j}. We have

$$
\begin{aligned}
(Du_{T_{A_\ell}}, \widehat{e}_j) = \Big[&\frac{1}{1-(\widehat{\alpha}_{A_\ell},\widehat{\alpha}_{A_{\ell+1}})^2}(Du_{T_{A_\ell}},\widehat{\alpha}_{A_\ell}) \\
&- \frac{(\widehat{\alpha}_{A_\ell},\widehat{\alpha}_{A_{\ell+1}})}{1-(\widehat{\alpha}_{A_\ell},\widehat{\alpha}_{A_{\ell+1}})^2}(Du_{T_{A_\ell}},\widehat{\alpha}_{A_{\ell+1}})\Big](\widehat{\alpha}_{A_\ell},\widehat{e}_j) \\
&+ \Big[\frac{1}{1-(\widehat{\alpha}_{A_\ell},\widehat{\alpha}_{A_{\ell+1}})^2}(Du_{T_{A_\ell}},\widehat{\alpha}_{A_{\ell+1}}) \\
&- \frac{(\widehat{\alpha}_{A_\ell},\widehat{\alpha}_{A_{\ell+1}})}{1-(\widehat{\alpha}_{A_\ell},\widehat{\alpha}_{A_{\ell+1}})^2}(Du_{T_{A_\ell}},\widehat{\alpha}_{A_\ell})\Big](\widehat{\alpha}_{A_{\ell+1}},\widehat{e}_j),
\end{aligned}
\tag{3.6}
$$

where $\widehat{e}_1$, $\widehat{e}_2$ are the standard unit vectors,

$$
\begin{aligned}
(Du_{T_{A_\ell}},\widehat{\alpha}_{A_\ell}) &= \frac{u_{A_\ell}-u_A}{m(e_{A_\ell})} \quad \text{and} \\
(Du_{T_{A_\ell}},\widehat{\alpha}_{A_{\ell+1}}) &= \frac{u_{A_{\ell+1}}-u_A}{m(e_{A_{\ell+1}})}\,.
\end{aligned}
\tag{3.7}
$$

Also

$$
\frac{\varepsilon_h}{m(V_A)}\int_{\partial V_A}\frac{\partial u_h}{\partial \nu} = \frac{\varepsilon_h}{m(V_A)}\sum_{\ell}\frac{u_{A_\ell}-u_A}{m(e_{A_\ell})}m(e^{\perp}_{A_\ell})\,.
\tag{3.8}
$$

Using (1.5), (3.7) and (3.8) we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial u_A}G_A \\
&= 1 - \frac{\tau}{m(V_A)}\sum_{T_{A,\ell}\cap V_A\neq\emptyset} m(V_A\cap T_{A,\ell})H_{p_k}(Du_{T_{A,\ell}}) \\
&\quad \times\Big\{\frac{1}{1-(\widehat{\alpha}_{A_\ell},\widehat{\alpha}_{A_{\ell+1}})^2}\left[\frac{(\widehat{\alpha}_{A_\ell},\widehat{\alpha}_{A_{\ell+1}})}{m(e_{A_{\ell+1}})}-\frac{1}{m(e_{A_\ell})}\right](\widehat{\alpha}_{A_\ell},\widehat{e}_k) \\
&\quad + \frac{1}{1-(\widehat{\alpha}_{A_\ell},\widehat{\alpha}_{A_{\ell+1}})^2}\left[\frac{(\widehat{\alpha}_{A_\ell},\widehat{\alpha}_{A_{\ell+1}})}{m(e_{A_\ell})}-\frac{1}{m(e_{A_{\ell+1}})}\right](\widehat{\alpha}_{A_{\ell+1}},\widehat{e}_k)\Big\} \\
&\quad - \frac{\tau}{m(V_A)}\varepsilon_h\sum_{\ell}\frac{m(e^{\perp}_{A_\ell})}{m(e_{A_\ell})} = 1 - B_1 - B_2,
\end{aligned}
$$

where we follow the summation convention for the repeated index k.

Our assumptions on the mesh imply that

(i) $m(e_{A_\ell})$ and $m(e_{A_{\ell+1}})$ are comparable with a constant independent of the partition,
(ii) $1-(\widehat{\alpha}_{A_\ell},\widehat{\alpha}_{A_{\ell+1}})^2 \geq C_0 > 0$,

and

(iii) $m(e^{\perp}_{A_\ell})/m(e_{A_\ell})$ is bounded from above.

Hence there exist positive constants K_1, K_2 independent of the partition $\mathcal{T}_h$ such that

$$|B_1| \leq \frac{\tau}{m(V_A)} \sum_{T_{A,\ell} \cap V_A \neq \emptyset} m(V_A \cap T_{A,\ell}) \frac{C}{m(e_{A_\ell})} \|D_p H\|_{L^\infty}$$
$$\leq K_1 \|D_p H\|_{L^\infty} \max_\ell \frac{\tau}{m(e_{A_\ell})}\,,$$

and

$$|B_2| \leq \frac{\tau}{m(V_A)} C_1 \varepsilon_h(A) \leq \max_\ell \frac{\tau}{m(e_{A_\ell})} \frac{1}{\min_\ell h_{T_{A,\ell}}} C_1 \varepsilon_h(A)$$
$$\leq K_2\, K \max_\ell \frac{\tau}{m(e_{A_\ell})}\,.$$

It is the last step above where we used the regularity of $\mathcal{T}_h$ and the fact that the viscosity coefficient ε_h is given by

$$\varepsilon_h(A) = K \max_\ell h_{T_{A,\ell}} \tag{3.9}$$

Therefore

$$\frac{\partial}{\partial u_A} G_A \geq 0$$

provided that

$$1 - \max_\ell \frac{\tau}{m(e_{A_\ell})} \Big[K_1 \|D_p H\|_{L^\infty} + K\, K_2 \Big] \geq 0,$$

i.e.,

$$\max_\ell \frac{\tau}{m(e_{A_\ell})} \leq \frac{1}{K_1 \|D_p H\|_{L^\infty} + K\, K_2}. \tag{3.10}$$

We finally compute

$$
\begin{aligned}
&\frac{\partial}{\partial u_{A_\ell}} G_A \\
&= -\frac{\tau}{m(V_A)} m(V_A \cap T_{A,\ell-1}) \\
&\qquad \times \sum_{k=1}^{2} \Big[\frac{(\widehat{\alpha}_{A_\ell}, \widehat{e}_k) - (\widehat{\alpha}_{A_{\ell-1}}, \widehat{\alpha}_{A_\ell})(\widehat{\alpha}_{A_{\ell-1}}, \widehat{e}_k)}{m(e_{A_\ell})(1 - (\widehat{\alpha}_{A_{\ell-1}}, \widehat{\alpha}_{A_\ell})^2)} \Big] H_{p_k}(Du_{T_{A,\ell-1}}) \\
&\quad - \frac{\tau}{m(V_A)} m(V_A \cap T_{A,\ell}) \\
&\qquad \times \sum_{k=1}^{2} \Big[\frac{(\widehat{\alpha}_{A_\ell}, \widehat{e}_k) - (\widehat{\alpha}_{A_{\ell+1}}, \widehat{\alpha}_{A_\ell})(\widehat{\alpha}_{A_{\ell+1}}, \widehat{e}_k)}{m(e_{A_\ell})(1 - (\widehat{\alpha}_{A_{\ell+1}}, \widehat{\alpha}_{A_\ell})^2)} \Big] H_{p_k}(Du_{T_{A,\ell}}) \\
&\quad + \varepsilon_h \frac{\tau}{m(V_A)} \frac{m(e^{\perp}_{A_\ell})}{m(e_{A_\ell})}.
\end{aligned}
$$

The assumptions on the regularity of the mesh yield $m(e^{\perp}_{A_\ell})/m(e_{A_\ell}) \geq C_0^-$, therefore, as before, for some constant M independent of h,

$$
\frac{\partial}{\partial u_{A_\ell}} G_A \geq \frac{\tau}{m(V_A)} \Big[\varepsilon_h(A) C_0^- - M \max_{\ell} h_{T_{A_\ell}} \|D_p H\|_{L^\infty} \Big].
$$

In view of (3.9) we see that

$$
\frac{\partial}{\partial u_{A_\ell}} G_A \geq 0,
$$

provided

$$
K = \frac{M \, \|D_p H\|_{L^\infty}}{C_0^-} \, . \tag{3.11}
$$

If (3.11) holds, and

$$
\max_{\ell} \frac{\tau}{m(e_{A_\ell})} \leq \frac{1}{\|D_p H\|_{L^\infty} (K_1 + K_2 \, M / C_0^-)}, \tag{3.12}
$$

then (3.10) is also satisfied and the scheme is monotone. □

The proof of Theorem 1 follows now from Theorems 2.1 and 2.2.

3.2. Edge-centered schemes

For $\tau > 0$ we define $S(\tau) : B(\mathbb{R}^2) \to B(\mathbb{R}^2)$ as follows. Let $\mathcal{T}_h'$ be a dual to $\mathcal{T}_h$ partition of $\mathbb{R}^2$ consisting of the rectangles that are created if we connect the vertices of each triangle of $\mathcal{T}_h$ with its barycenter and elliminate the edges of the triangles of $\mathcal{T}_h$. For $v \in B(\mathbb{R}^2)$, let $\widetilde{I}_h v \in X_h$ be its piecewise interpolant. Let further, $v_h \in X_h$ to be the function we are getting by applying the scheme (1.8–9) to $\widetilde{I}_h v$ over one time step. Since $\widetilde{I}_h$ is not monotone, we introduce the interpolant Π_h to the set of the piecewise constant functions on $\mathcal{T}_h'$, i.e.,

$$\Pi_h \varphi|_{R_M} = \varphi(M), \quad M \in M_h, R_M \in \mathcal{T}_h'.$$

Note that there is a one-to-one correspondence between the set of the middle points M_h of $\mathcal{T}_h$ and of $\mathcal{T}_h'$. Then we set $S(\tau)v = \Pi_h v_h$.

The fact that S satisfies (2.2) is immediate. Since, for $x \in R_M$ and φ smooth,

$$|H(D\varphi(M)) - H(D\varphi(x))| \le Ch\|D^2\varphi\|, \quad M \in M_h,\ R_M \in \mathcal{T}_h',$$

it suffices to check (2.5) only at the middle points. But then the consistency condition with the error estimate (2.5) follows from the observation that, for all $p \in \mathbb{R}^2$,

$$g_A(p,p) = H(p), \tag{3.14}$$

and the approximation properties of $\widetilde{I}_h$, (cf. [CR], [BrS]).

The monotonicity of S follows from the monotonicity of Π_h and the following lemma.

Lemma 3.2 *Assume that all the angles ω of our triangulation $\mathcal{T}_h$ satisfy $\omega \le \omega_0 < \pi/2$, where ω_0 is independent of h. Choose θ_h in (1.9) such that*

$$\theta_h(A_1) = \frac{\|D_P H\|_{L^\infty}}{\cos\omega_0} \, \frac{m(T)}{m(T) + m(T_e)} \,. \tag{3.15}$$

If the CFL condition

$$\max_e \tau \frac{m(e)}{m(T_e)} \le \frac{\cos\omega_0}{\|D_P H\|_{L^\infty}} \tag{3.16}$$

is satisfied then the scheme defined (1.8) is monotone, i.e., if, for $A_1 \in M_h$, the function

$$G_{A_1}(u_{A_1}, u_{A_2}, u_{A_3}, u_{A_2^e}, u_{A_3^e}) = u_{A_1} - \tau g_{A_1}(Du_T, Du_{T_e})$$

is a nondecreasing function of each argument.

Proof. Let $\{\Phi_\ell\}_{\ell=1}^3$ and $\{\Phi_\ell^e\}_{\ell=1}^3$, to be the Lagrange polynomials corresponding to $\{A_\ell\}_{\ell=1}^3$ and $\{A_\ell^e\}_{\ell=1}^3$ respectively, i.e., Φ_ℓ, $\Phi_\ell^e \in \mathbb{P}_1$ and $\Phi_\ell(A_k) = \delta_{\ell k}$, $\Phi_\ell^e(A_k^e) = \delta_{\ell k}$. Then

$$u_h(x) = \begin{cases} \displaystyle\sum_{\ell=1}^3 \Phi_\ell(x) u_{A_\ell}, & \text{if } x \in T, \\ \displaystyle\sum_{\ell=1}^3 \Phi_\ell^e(x) u_{A_\ell^e}, & \text{if } x \in T_e. \end{cases}$$

We have:

$$\begin{aligned} G_{A_1} &= u_{A_1} - \tau H \\ &\times \left(\frac{1}{m(T)+m(T_e)} \left(m(T) \sum_{\ell=1}^3 D\Phi_\ell(x) u_{A_\ell} + m(T_e) \sum_{\ell=1}^3 D\Phi_\ell^e(x) u_{A_\ell^e} \right) \right) \\ &+ \theta_h \sum_{\ell=1}^3 [D\Phi_\ell^e(x) u_{A_\ell^e} - D\Phi_\ell(x) u_{A_\ell}] \cdot \nu_e \, . \end{aligned}$$

Setting $B_e = \frac{1}{m(T)+m(T_e)}(m(T)Du_T + m(T_e)Du_{T_e})$, we compute

$$\begin{aligned} \frac{\partial}{\partial u_{A_1}} G_{A_1} = 1 - \frac{\tau}{m(T)+m(T_e)} &\Big\{ H_{p_1}(B_e)[m(T)\partial_{x_1}\Phi_1 + m(T_e)\partial_{x_1}\Phi_1^e] \\ &+ H_{p_2}(B_e)[m(T)\partial_{x_2}\Phi_1 + m(T_e)\partial_{x_2}\Phi_1^e] \Big\} \\ &+ \tau\theta_h (D\Phi_1^e - D\Phi_1) \cdot \nu_e \, . \end{aligned}$$

Since

$$A_2A_3 // e, \ A_2^eA_3^e // e, \ D\Phi_1, \ D\Phi_1^e \perp e_{A_1},$$

we have

$$D\Phi_1 = \frac{1}{\text{dist } (A_2A_3, e)} \nu_e = \frac{m(e)}{m(T)} \nu_e, \tag{3.17}$$
$$D\Phi_1^e = -\frac{1}{\text{dist } (A_2^eA_3^e, e)} \nu_e = -\frac{m(e)}{m(T_e)} \nu_e.$$

Therefore,

$$m(T)\partial_{x_i}\Phi_1 + m(T_e)\partial_{x_i}\Phi_1^e = 0, \qquad i = 1, 2.$$

We conclude observing that, in view of (3.16),

$$\frac{\partial}{\partial u(A_1)} G_{A_1}(u_T, u_{T_e}) = 1 - \tau\theta_h \left(\frac{m(e)}{m(T_e)} + \frac{m(e)}{m(T)} \right) > 0. \tag{3.18}$$

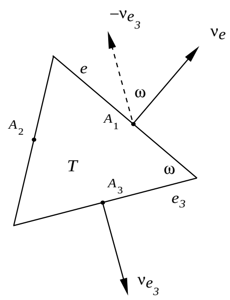

Fig. 4.

We also compute, for $\ell = 2, 3$,

$$\frac{\partial}{\partial u_{A_\ell}} G_{A_1}(u_T, u_{T_e}) = -\frac{\tau}{m(T)+m(T_e)}\{H_{p_1}(B_e)(m(T)\partial_{x_1}\Phi_\ell) + H_{p_2}(B_e)(m(T)\partial_{x_2}\Phi_\ell)\} - \tau\theta_h D\Phi_\ell \cdot \nu_e.$$

As before

$$D\Phi_\ell = \frac{m(e_{A_\ell})}{m(T)}\nu_{e_{A_\ell}}.$$

By our assumption for the angles of the triangles we have $-\nu_{e_{A_\ell}} \cdot \nu_e \geq \cos\omega_0$, (cf. Fig. 4). Therefore (3.17) yields

$$\begin{aligned}\frac{\partial}{\partial u_{A_\ell}} G_{A_1} \geq& -\tau\theta_h \frac{m(e_{A_\ell})}{m(T)}\nu_e \cdot \nu_{e_{A_\ell}} - \tau\|D_P H\|_{L^\infty}\frac{m(e_{A_\ell})}{m(T)+m(T_e)} \\ \geq& \tau\theta_h \frac{m(e_{A_\ell})}{m(T)}\cos\omega_0 - \tau\|D_P H\|_{L^\infty}\frac{m(e_{A_\ell})}{m(T)+m(T_e)}.\end{aligned}$$

If

$$\delta_h = \delta_h(A_1) = \frac{\|D_P H\|_{L^\infty}}{\cos\omega_0}\,\frac{m(T)}{m(T)+m(T_e)},$$

then

$$\frac{\partial}{\partial u_{A_\ell}} G^n_{A_1} \geq 0,$$

and the proof is complete. □

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