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TWO A POSTERIORI ERROR ESTIMATES FOR ONE-DIMENSIONAL SCALAR CONSERVATION LAWS*

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Abstract. In this paper, we propose a posteriori local error estimates for numerical schemes in the context of one-dimensional scalar conservation laws. We first consider the schemes for which a consistent in-cell entropy inequality can be derived. Then we extend this result to second-order schemes written in viscous form satisfying weak entropy inequalities. As an illustration, we show several numerical tests on the Burgers equation and we propose an adaptive algorithm for the selection of the mesh.

Key words. conservation laws, entropy dissipation, a posteriori error estimates, adaptive finite difference method

AMS subject classifications. 65M15, 65M50

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1. Introduction.

We consider one-dimensional finite difference schemes for the approximation of the scalar conservation law:

$$u_t + f(u)_x = 0,\ x \in \mathbb{R}, t > 0$$

with initial data in the space of functions of bounded variation. In this paper, we propose a simple framework which is used to derive explicit a posteriori error estimates in the local L^1 norm for this class of numerical schemes. The idea is to use the classical Kruzkov-type estimates, [19], [20] in the form of [4], [17], starting from an approximate entropy inequality the approximate solution is supposed to satisfy. From a numerical point of view, this provides a way to convert any in-cell discrete entropy inequality into an error estimate. The a posteriori estimates derived in this work are local and have the form

$$\int_K |u(x,t) - v^h(x,t)|\, dx \le \mathcal{E}(v^h, t, K), \tag{1.1}$$

where v^h denotes the approximation of u obtained by an E-scheme or a (possibly) high-order scheme, K is a typical cell of the space partition, and $\mathcal{E}(v^h, t, K)$ is a computable quantity that depends on the data of the problem and on values of v^h in appropriate extended cones of dependence; cf. Theorems 3 and 4.

Error estimates of the form (1.1) are proposed for schemes satisfying two types of entropy inequalities. First we consider schemes that satisfy "strong" (consistent) entropy inequalities, Theorem 3. Then starting from general (high-order) schemes written in "viscous form" (cf. (1.7), (1.8)), we are able to derive a very weak entropy

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inequality which is not consistent in the sense of Lax and Wendroff [22]. In the final estimate this lack of consistency gives rise to an additional error term in the error estimate of Theorem 4. It is interesting to note the nice behavior of this additional term into the smooth regions and its relation to Oleinik's entropy condition at shocks; cf. Remark 1. Also, it seems that this term is in fact of higher order, if compared with the other terms in the estimate; cf. Remarks 1 and 3. Indeed, under stronger CFL conditions, this is shown in the case of convex fluxes, Remark 3. In the general case the numerical tests confirm that the contribution of this term is actually very small. On the other hand the estimator in Theorem 4 is at best $O(h^{1/2})$, when shocks are present; cf. Remark 2. Note that estimates of the form (1.1) are meaningful even for schemes for which we do not know if they are convergent or not. Indeed if our computed approximations are such that $\mathcal{E} \leq \tau$, τ given tolerance, then the error in (1.1) is at most τ. In particular these estimates can be used in computations with, e.g., MUSCL schemes even when the flux is not convex.

Estimates of the form (1.1) can be used to design adaptive algorithms. As an illustration, we present some numerical runs on the modified Lax–Friedrichs first-order scheme introduced by Tadmor [36] and on its second-order extension obtained by means of the piecewise constant viscosity modification proposed by Osher and Tadmor [32]. These tests involve some Riemann problems and the computation of a stationary shock for the inviscid Burgers equation and show that right-hand side of (1.1) tends to zero as $h \to 0$. In addition, we present some preliminary computations based on an adaptive algorithm using the estimate proposed in Theorem 4. A summary of the conclusions from our experiments is presented in the beginning of section 4.

As pointed out first in [8], [9] the approach of [20] used henceforth in the literature by many authors to obtain estimates for schemes approximating the entropy solution of the conservation law (cf., e.g., the references in [9]) is an a posteriori approach. In most of these works the focus was to obtain rates of convergence and not estimates of the form (1.1) suitable for computations. A posteriori estimates in L^1 were derived in [10] for high-order finite element schemes using the technique of Kuznetsov [20]. In [16] a posteriori estimates in a weighted space-time L^2 norm for finite element methods approximating the ε-viscous perturbation of one-dimensional systems of conservation laws are obtained. Estimates in a dual Lip_+ norm for convex fluxes were proved in [26]; see also [34] for similar estimates for finite element schemes. See also [23], [24], [2], [33] and the references therein for adaptive schemes for conservation laws. The results of this paper were announced in [14]. In [14] a posteriori estimates for monotone finite volume schemes using the same ideas were also derived; cf. also [18] for similar results in the space-time L^1 norm.

It is known that the second-order schemes considered in this paper converge to the entropy solution of the conservation law in the case where the flux is convex [32]. For the antidiffusion schemes considered in [7] convergence and error estimates were proved provided that appropriate discrete entropy inequalities hold; cf. also [28]. The in-cell entropy inequalities shown in the present paper are different and suffice to prove the estimates in Theorem 4.

This paper is organized as follows. In what follows, we introduce the schemes and our notation. We consider the incremental coefficients of Harten [15] (cf. (1.11)), which are of great use in this work, and we present a variation of the error theorem from [4], [17]. In section 2, we derive a local a posteriori error estimate for entropy consistent schemes using a strong in-cell discrete entropy inequality. In section 3, we extend this result to more general schemes written in viscous form by means of

a weaker discrete entropy inequality. To prove the main result of this section, some lemmas are needed and their proofs are forwarded in the appendix. Then, in section 4, we present some numerical computations using the modified Lax–Friedrichs scheme and its second-order extension. This provides a way to compare the estimated and the exact error for these numerical approximations. As a final illustration, we display some approximate solutions generated by an adaptive second-order algorithm based on our second-error estimate.

Preliminaries. We are interested in the Cauchy problem for the following partial differential equation:

$$\begin{array}{l} u_t + f(u)_x = 0 \text{ with } (x,t) \in \mathbb{R} \times \mathbb{R}_*^+, \\ u(.,0) = u_0 \in BV(\mathbb{R}), \end{array} \tag{1.2}$$

where $f \in C^2(\mathbb{R})$ and $BV(\mathbb{R})$ denotes the space of functions of bounded variation, [12]. Problem 1.2 admits a unique *entropy* solution, [19], i.e.,

$$\forall \text{ positive } \phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+), \int_{\mathbb{R} \times \mathbb{R}_*^+} [U(u)\phi_t + F(u)\phi_x] dxdt + \int_{\mathbb{R}} U(u_0)(x)\phi(x,0)dx \geq 0,$$

where $U \in C^2(\mathbb{R})$ is strictly convex and $F' = U'f'$. In the case of strictly convex fluxes, $f'' \geq a$, $a > 0$, the entropy solution u satisfies the following one-sided Lipschitz estimate (Oleinik [29]):

$$\begin{array}{l} \forall (t,s) \in (\mathbb{R}^+)^2, \\ t \geq s, \sup\limits_{x \neq y} \left(\dfrac{u(x,t) - u(y,t)}{x - y} \right)_+ \stackrel{def}{=} \|u(.,t)\|_{Lip^+(\mathbb{R})} \leq \dfrac{1}{\frac{1}{\|u(.,s)\|_{Lip^+(\mathbb{R})}} + a(t-s)}. \end{array} \tag{1.3}$$

For each $a \in \mathbb{R}$, the notation $(a)_+$ stands for: $\max(a, 0)$.

Numerical schemes and approximate solutions. Following, for instance, [12], we recall here the basic properties of the most usual finite difference schemes for (1.2). We introduce consequently a space-time discretization defined by a uniform time-step $\Delta t > 0$ and a countable collection of grid points $x_{j+\frac{1}{2}}$ for $j \in \mathbb{Z}$. Also let

$$h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}; 0 < \underline{h} = \inf_{j \in \mathbb{Z}} h_j; h = \sup_{j \in \mathbb{Z}} h_j. \tag{1.4}$$

Then we denote x_j the middle-points of the cells $K_j \stackrel{def}{=} [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$: $x_j = x_{j+\frac{1}{2}} - (h_j/2) = x_{j-\frac{1}{2}} + (h_j/2)$. Similarly, we define time intervals of the type: $I^n \stackrel{def}{=} [t^n, t^{n+1})$ where $t^n = n\Delta t$ for $n \in \mathbb{N}$. In all this work, we will consider piecewise constant numerical approximations v^h of u generated by *conservative* algorithms in the sense of Lax and Wendroff [22].

$$\begin{array}{rcl} v^h : \mathbb{R} \times \mathbb{R}^+ & \to & \mathbb{R}, \\ (x,t) & \mapsto & v^h(x,t) = v_j^n \text{ if } x \in K_j; t \in I^n. \end{array} \tag{1.5}$$

Let us define the following discretization for u_0: $\forall j \in \mathbb{Z}, v_j^0 = \frac{1}{h_j} \int_{K_j} u_0(x)dx$. Then we consider schemes that are constructed using a consistent "essentially 3-point"

numerical flux function g. In particular let g, $g : \mathbb{R}^{2p} \to \mathbb{R}$ be a Lipschitz continuous function such that

$$\begin{array}{rcl}(v^n_{j-p+1}, \dots, v^n_{j+p}) & \mapsto & g^n_{j+\frac{1}{2}} \stackrel{def}{=} g(v^n_{j-p+1}, \dots, v^n_{j+p}),\\ g(v, ..., v) & = & f(v).\end{array} \tag{1.6}$$

The *explicit* schemes we will consider in this section are written as

$$v^{n+1}_j = v^n_j - \frac{\Delta t}{h_j}(g^n_{j+\frac{1}{2}} - g^n_{j-\frac{1}{2}}). \tag{1.7}$$

Following [32], we write these (possibly high-order) numerical schemes in the equivalent form

$$v^{n+1}_j = v^n_j - \frac{\Delta t}{2h_j}(\mathbf{f}^n_{j+1} - \mathbf{f}^n_{j-1}) + \frac{\Delta t}{2h_j}(\mathbf{Q}^n_{j+\frac{1}{2}}(v^n_{j+1} - v^n_j) - \mathbf{Q}^n_{j-\frac{1}{2}}(v^n_j - v^n_{j-1})), \tag{1.8}$$

where the quantities $\mathbf{f}^n_j$ and $\mathbf{Q}^n_{j+\frac{1}{2}}$ are, respectively, called *modified flux* and *numerical viscosity* for the scheme. We assume that the modified flux is of the form

$$\begin{array}{l}\mathbf{f}^n_j = f(v^n_j) + \tilde{f}^n_j,\\ \tilde{f}^n_j \text{ depends on } v^n_{j-1}, v^n_j, v^n_{j+1}.\end{array} \tag{1.9}$$

In [32] it was shown that these schemes are SOR (second-order accurate in the smooth regions) as soon as the following requirement is ensured:

$$\begin{cases}\mathbf{Q}^n_{j+\frac{1}{2}} = \lambda(a^n_{j+\frac{1}{2}})^2 + \frac{\tilde{f}^n_{j+1} - \tilde{f}^n_j}{v^n_{j+1} - v^n_j} + O(|v^n_{j+1} - v^n_j|),\\ a^n_{j+\frac{1}{2}} = \frac{f(v^n_{j+1}) - f(v^n_j)}{v^n_{j+1} - v^n_j}.\end{cases} \tag{1.10}$$

More precisely, second-order accuracy is given up in the neighborhood of nonsonic extremal points. We mainly refer to [32] for detailed results.

In order to study the convergence properties of the schemes (1.7), (1.8), it is convenient to introduce their discrete spatial *total variation*:

$$TV(v^h)(t^n) = \sum_{j \in \mathbb{Z}} |v^n_{j+1} - v^n_j|.$$

Moreover, we will use the following notation: $\lambda \stackrel{def}{=} \Delta t/\underline{h}$. Following Harten [15], we now define the coefficients $\mathbf{m}^n_{j+\frac{1}{2}}, \mathbf{p}^n_{j-\frac{1}{2}}$ which are going to be widely used throughout this paper:

$$\begin{cases}\mathbf{m}^n_{j+\frac{1}{2}} = \mathbf{Q}^n_{j+\frac{1}{2}} - \frac{\mathbf{f}^n_{j+1} - \mathbf{f}^n_j}{v^n_{j+1} - v^n_j},\\ \mathbf{p}^n_{j-\frac{1}{2}} = \mathbf{Q}^n_{j-\frac{1}{2}} + \frac{\mathbf{f}^n_j - \mathbf{f}^n_{j-1}}{v^n_j - v^n_{j-1}},\end{cases} \quad (j,n) \in \mathbb{Z} \times \mathbb{N}. \tag{1.11}$$

These coefficients are clearly positive if the requirements of the following theorem are met.

THEOREM 1 (see [32], [12]). *Assume that the following CFL condition is met:*

$$\left|\frac{\mathbf{f}^n_{j+1} - \mathbf{f}^n_j}{v^n_{j+1} - v^n_j}\right| \le \mathbf{Q}^n_{j+\frac{1}{2}} \le \frac{1}{2\lambda}, \quad (j,n) \in \mathbb{Z} \times N. \tag{1.12}$$

Then the numerical approximation v^h is L^∞-stable and total-variation-diminishing (TVD):

$$\begin{aligned} &\text{(i) } \|v^h\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \le \|u_0\|_{L^\infty(\mathbb{R})},\\ &\text{(ii) } TV(v^h)(t^n) \le TV(v^h)(t^{n-1}) \le TV(u_0). \end{aligned} \tag{1.13}$$

To derive our a posteriori error estimates we will use the framework of [4], [17]. The following approximation lemma is a slight variant of the main result in [17] which was based on a result proposed by Bouchut and Perthame [4] (see also [13] for a version designed for inhomogeneous equations). In all that follows, we shall use the following notation:

$$\mathbb{R}^+_* = (0, +\infty).$$

THEOREM 2. *Let $v^h \in L^\infty(\mathbb{R}^+; BV(\mathbb{R}))$ be a numerical approximation of u the entropy solution of (1.2). Suppose further that v^h satisfies the hypotheses of Theorem 1 and the following approximate entropy inequality for every positive test-function with compact support $\varphi \in C^1_0(\mathbb{R}\times\mathbb{R}^+_*)$ and all $k \in \mathbb{R}$ such that $|k| \le \|u_0\|_{L^\infty(\mathbb{R})}$:*

$$\begin{aligned} &-\int_{\mathbb{R}\times\mathbb{R}^+_*} \big[|v^h(x,t)-k|\varphi_t(x,t) + sgn(v^h(x,t)-k)[f(v^h)(x,t)-f(k)]\varphi_x(x,t)\big]dxdt\\ &\qquad \le \int_{\mathbb{R}^+_*}\sum_{j\in\mathbb{Z}}\int_{K_j}\alpha(x,t)\left(\sup_{x'\in K_j}\varphi(x',t)\right)dxdt\\ &\qquad\qquad + \int_{\mathbb{R}\times\mathbb{R}^+_*}\Big(\beta(x,t)|\varphi_x(x,t)| + \gamma(x,t)|\varphi_t(x,t)|\Big)dxdt \end{aligned} \tag{1.14}$$

with α, β, γ some positive k-independent functions in $L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^+_*)$ and $\{K_j\}_{j\in\mathbb{Z}}$ is a partition of $\mathbb{R}$.
Let $(T, R, \Delta, \delta, \nu) \in (\mathbb{R}^+)^4 \times \mathbb{R}^+_*$ and $h_j \le \Delta$, then the following estimate holds:

$$\begin{aligned} \int_{|x|\le R}|u(s,T)-v^h(s,T)|ds &\le \int_{|x|\le R+MT+\Delta}|u(s,0)-v^h(s,0)|ds\\ &+ 2\mathcal{C}(\Delta + M\delta)TV(u_0)\\ &+ \int_0^T\int_{|x|\le R+M(T-t)+\Delta}\left\{\alpha(s,t) + \frac{4\mathcal{C}\beta(s,t)}{\Delta} + \mathcal{C}\left(\frac{2M}{\Delta}+\frac{1}{\delta}\right)\gamma(s,t)\right\}dsdt\\ &+ \mathcal{C}\sup_{0\le\bar t\le T+\nu}\int_{|x|\le R+M(T-\bar t)+\Delta}\gamma(s,\bar t)ds \end{aligned} \tag{1.15}$$

with $M = \max\{|f'(\xi)|, |\xi| \le \|u_0\|_{L^\infty(\mathbb{R})}\}$ and $\mathcal{C}$ an absolute constant such that $\mathcal{C} \ge 2$.

Proof. The proof of this result follows [19], [4], [17]. Considering two positive functions Φ, ζ in $C^1_0(\mathbb{R}\times\mathbb{R}^+_*)$, we set $\phi(x,t,y,s) = \Phi(x,t)\zeta(x-y,t-s)$. We sum both the entropy inequality for $v^h(x,t)$ and $u(y,s)$ and we integrate on $(\mathbb{R}\times\mathbb{R}^+_*)^2$:

$$\begin{aligned} &-\int_{(\mathbb{R}\times\mathbb{R}^+_*)^2}[|u(x,t)-v^h(y,s)|\Phi_t(x,t)\\ &+\operatorname{sgn}(u(x,t)-v^h(y,s))[f(u)(x,t)-f(v^h)(y,s)]\Phi_x(x,t)]\zeta(x-y,t-s)dxdtdyds\\ &\le A + B + C, \end{aligned}$$

where

$$\begin{cases} A = \displaystyle\int_{(\mathbb{R}\times\mathbb{R}_*^+)} \int_{\mathbb{R}_*^+} \sum_j \int_{K_j} \alpha(x,t) \sup_{x'\in K_j} \big(|\phi(x',t,y,s)|\big) dxdtdyds, \\ B = \displaystyle\int_{(\mathbb{R}\times\mathbb{R}_*^+)^2} \beta(x,t)|\phi_x(x,t,y,s)| dxdtdyds, \\ C = \displaystyle\int_{(\mathbb{R}\times\mathbb{R}_*^+)^2} \gamma(x,t)|\phi_t(x,t,y,s)| dxdtdyds. \end{cases}$$

We now choose the functions ζ and Φ: for any positive constants δ, Δ, we define $\zeta(x,t) = \zeta^x(x)\zeta^t(t)$, where ζ^x, ζ^t satisfy

$$\begin{aligned} &\int_{\mathbb{R}} \zeta^t(t)dt = 1, \int_{\mathbb{R}} \zeta^x(x)dx = 1, \\ &\zeta^t(t) = \zeta_1^t(t/\delta)/\delta \text{ and } \operatorname{supp}(\zeta_1^t) \subset (-1,0), \\ &\zeta^x(x) = \zeta_1^x(x/\Delta)/\Delta \text{ and } \operatorname{supp}(\zeta_1^x) \subset (-1/4,1/4). \end{aligned}$$

Then $|\zeta_1^t(t)| \le (1+m)$, $|\zeta_1^x(x)| \le 2(1+m)$ with $m \ge \eta > 0$. Similarly, we define a regularized Heaviside function Y_θ such that $Y_\theta(-\infty) = 0$, $Y_\theta'(t) = Y_1'(t/\theta)/\theta$, and $\int Y_1'(t)dt = 1$. With another parameter $\varepsilon > 0$, we set

$$0 \le \chi(t) = Y_\varepsilon(t) - Y_\varepsilon(t-T) \le 1 \in C^1(0, T+\varepsilon)$$

and, finally, we get the expression of $\Phi \in C_0^1(\mathbb{R}\times\mathbb{R}_*^+)$:

$$\Phi(x,t) = \chi(t)[1 - Y_\theta(|x| - R - \Delta/2 - M(T-t)] \stackrel{def}{=} \chi(t)\psi(x,t) \le 1$$

which remains smooth as long as $M\varepsilon \le R + \Delta/2$. We also deduce the corresponding bounds for the derivatives of Φ: $|\Phi_x(x,t)| \le \max_{\xi\in\mathbb{R}} |Y_1'(\xi)|/\theta$, $|\Phi_t(x,t)| \le |\chi'(t)| + M.\max_{\xi\in\mathbb{R}} |Y_1'(\xi)|/\theta$, and $|\chi'(t)| \le 2\max_{\xi\in\mathbb{R}} |Y_1'(\xi)/\varepsilon|$ where we have once again $|Y_1'(t)| \le (1+m)$ with $m \ge \eta > 0$. Because of the Lipschitz property coming from the flux function regularity and the uniform bound on the amplitude of the solutions, we get

$$-\int_{(\mathbb{R}\times\mathbb{R}_*^+)^2} |u(x,t) - v^h(y,s)|\chi'(t)\psi(x,t)\zeta(x-y,t-s)dxdtdyds \le A + B + C.$$

Taking $\varepsilon \to 0$ in the above relation and using the bounds

$$\begin{aligned} A &\le \int_{|x|\le R+M(T-t)+\Delta} \alpha(x,t)dxdt, \\ B &\le \left(\frac{\|Y_1'\|_{L^\infty}}{\theta} + 2\frac{\|\zeta_1^x\|_{L^\infty}}{\Delta}\right) \int_{|x|\le R+M(T-t)+\Delta} \beta(x,t)dxdt, \\ C &\le \left(M\frac{\|Y_1'\|_{L^\infty}}{\theta} + 2\frac{\|\zeta_1^t\|_{L^\infty}}{\delta}\right) \int_{|x|\le R+M(T-t)+\Delta} \gamma(x,t)dxdt \\ &\quad + 2\|Y_1'\|_{L^\infty} \left\{ \sup_{0\le\bar t\le T+\nu} \int_{|x|\le R+M(T-\bar t)+\Delta} \gamma(x,\bar t)dx \right\} \end{aligned}$$

we derive the desired bound following the arguments in [4], [17]. □

2. The case of strongly entropy consistent numerical schemes. In this section, we consider the particular case of numerical schemes which are endowed with a strong consistency relation with the continuous problem (1.2). More precisely, this means that for all entropy functions U, there exists a consistent and Lipschitz continuous *numerical entropy flux* G such that the following inequality holds:

$$U(v_j^{n+1}) - U(v_j^n) + \frac{\Delta t}{h_j}(G_{j+\frac{1}{2}}^n - G_{j-\frac{1}{2}}^n) \leq 0 \tag{2.1}$$

with the same notation conventions as before. This requirement (2.1) is met in the case of E-schemes [30, 32] or monotone schemes [11] (which are special E-schemes). We also know that this class of algorithms is most of the time limited to (formal) first-order accuracy, so we can restrict ourselves to numerical approximations generated by a 3-points scheme corresponding to the value $p = 1$ in (1.6):

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{h_j}[g(v_j^n, v_{j+1}^n) - g(v_{j-1}^n, v_j^n)]. \tag{2.2}$$

Thanks to the regularity requirements, it is possible to rewrite them under a unique *viscous form* determined by the *viscosity coefficients* denoted $Q_{j+\frac{1}{2}}^n$:

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{2h_j}(f(v_{j+1}^n) - f(v_{j-1}^n)) + \frac{\Delta t}{2h_j}(Q_{j+\frac{1}{2}}^n(v_{j+1}^n - v_j^n) - Q_{j-\frac{1}{2}}^n(v_j^n - v_{j-1}^n)). \tag{2.3}$$

In this case, any modified flux used in (1.8) boils down to the continuous one. We can also simply define in a unique way $Q_{j+\frac{1}{2}}^n = [f(v_j^n) + f(v_{j+1}^n) - 2g_{j+\frac{1}{2}}^n]/(v_{j+1}^n - v_j^n)$ as a function of the two adjacent values (v_j^n, v_{j+1}^n). It is possible to derive an analytical expression of these coefficients at least for the formal first-order schemes like Godunov, Lax–Friedrichs, Murman–Roe, Engquist–Osher [32]. Of course, provided we use a sufficiently small time-step, we are in position to use Theorem 1 to bound the resulting numerical approximation v^h for which one gets a somehow simplified form for the CFL restriction and the incremental coefficients:

$$\left|\frac{f(v_{j+1}^n) - f(v_j^n)}{v_{j+1}^n - v_j^n}\right| \leq Q_{j+\frac{1}{2}}^n \leq \frac{1}{2\lambda} \text{ and } \begin{cases} m_{j+\frac{1}{2}}^n = Q_{j+\frac{1}{2}}^n - \frac{f(v_{j+1}^n) - f(v_j^n)}{v_{j+1}^n - v_j^n}, \\ p_{j-\frac{1}{2}}^n = Q_{j-\frac{1}{2}}^n + \frac{f(v_j^n) - f(v_{j-1}^n)}{v_j^n - v_{j-1}^n}, \end{cases} \quad (j,n) \in \mathbb{Z}\times\mathbb{N}.$$

Derivation of the error estimate. The main purpose of this paragraph is to prove the following result.

THEOREM 3. *Assume that the restrictions of Theorem* 1 *are met for the scheme* (2.2) *which is supposed to satisfy* (2.1) *for all Lipschitz continuous entropies* U. *Then we have for all* $T > 0$

$$\begin{aligned} \int_{K_j} |v^h(x,T) - u(x,T)|dx &\leq \int_{x_{j-\frac{1}{2}}-MT-\Delta}^{x_{j+\frac{1}{2}}+MT+\Delta} |v^h(x,0) - u(x,0)|dx \\ &+ 2\mathcal{C}(3\sqrt{2}+1)\sqrt{TV(u_0)}\left(\sum_{n=0}^{N}\Delta t\mathcal{X}_V^n\right)^{\frac{1}{2}} + \mathcal{C}\lambda \sup_{n=0,\dots,N+1}\{\mathcal{X}_V^n\}, \end{aligned} \tag{2.4}$$

where

(2.5)
$$\mathcal{X}_V^n = \sum_{\ell\pm 1\in J^n_{j,\Delta}} h_\ell Q^n_{\ell+\frac12}|v^n_{\ell+1}-v^n_\ell| \text{ with}$$
$$J^n_{j,\Delta} = \{\ell\pm 1 \ : x_\ell \in B(x_j, h_j/2 + M(T-t)+\Delta)\},$$
$$B(x_j, h_j/2+M(T-t)+\Delta) = [x_{j-\frac12} - M(T-t) - \Delta, x_{j+\frac12} + M(T-t)+\Delta],$$
$$\Delta = 3\sqrt{MTh/2},\ \mathcal{C}=2,\ \lambda = \Delta t/\underline{h},$$
$$M = \sup\{|f'(\xi)| \text{ with } |\xi|\le \|u_0\|_{L^\infty(\mathbb{R})}\}; N = T/\Delta t.$$

Proof. We have to compute the following quantity for all $k\in\mathbb{R}$ and all positive test-functions φ in $C^1_0(\mathbb{R}\times\mathbb{R}^+_*)$:

$$\mathcal{I} = -\int_{(\mathbb{R}\times\mathbb{R}^+_*)} [|v^h-k|\varphi_t + \operatorname{sgn}(v^h-k)(f(v^h)-f(k))\varphi_x](x,t)dxdt. \tag{2.6}$$

It is therefore sufficient to consider only the one-parameter family of Kruzkov's entropies: $U^k(u)=|u-k|$, $F^k(u)=\operatorname{sgn}(u-k)(f(u)-f(k))$. Since v^h is a piecewise constant function, we can rewrite $\mathcal{I}$ as

(2.7)
$$\mathcal{I} = \sum_{j,n}[U^k(v_j^{n+1})-U^k(v_j^n)] \underbrace{\int_{K_j}\varphi(x,t^{n+1})dx}_{\bar\varphi^{n+1}_{K_j}} + [F^k(v^n_{j+1})-F^k(v^n_j)] \underbrace{\int_{I^n}\varphi(x_{j+\frac12},t).dt}_{\bar\varphi^{I^n}_{j+\frac12}}$$

Thanks to the strong entropy requirement (2.1), there exists a consistent numerical entropy flux G^k, $(G^k(v^n_j,v^n_j)=F^k(v^n_j))$ associated with each U^k. Thus one can show that

$$\begin{aligned}\mathcal{I} \le \sum_{j,n}&[U^k(v_j^{n+1})-U^k(v_j^n)](\bar\varphi^{n+1}_{K_j} - \bar\varphi^n_j/\Delta t)\\ &+[G^k(v^n_j,v^n_{j+1})-G^k(v^n_j,v^n_j)](\bar\varphi^{I^n}_{j+\frac12}-\bar\varphi^n_j/h_j)\\ &+[G^k(v^n_j,v^n_j)-G^k(v^n_{j-1},v^n_j)](\bar\varphi^{I^n}_{j-\frac12}-\bar\varphi^n_j/h_j).\end{aligned} \tag{2.8}$$

To estimate the right-hand side of (2.8) we proceed as follows:

$$\begin{aligned}|U^k(v_j^{n+1})-U^k(v_j^n)| &\le |v_j^{n+1}-v_j^n|\\ &\le (\lambda/2)\underbrace{\left[Q^n_{j+\frac12} - \frac{f(v^n_{j+1})-f(v^n_j)}{v^n_{j+1}-v^n_j}\right]}_{m^n_{j+\frac12}\ge 0}|v^n_{j+1}-v^n_j|\\ &\quad + (\lambda/2)\underbrace{\left[Q^n_{j-\frac12} + \frac{f(v^n_j)-f(v^n_{j-1})}{v^n_j-v^n_{j-1}}\right]}_{p^n_{j-\frac12}\ge 0}|v^n_j-v^n_{j-1}|,\end{aligned} \tag{2.9}$$

and

$$|G^k(v_j^n,v_{j+1}^n)-G^k(v_j^n,v_j^n)|+|G^k(v_j^n,v_j^n)-G^k(v_{j-1}^n,v_j^n)|\le M[|v_{j+1}^n-v_j^n|+|v_j^n-v_{j-1}^n|].$$

Moreover, $|\bar\varphi_{K_j}^{n+1}-\bar\varphi_j^n/\Delta t|\le\int_{K_j\times I^n}|\varphi_t(x,t)|dxdt$ and $|\bar\varphi_{j\pm\frac12}^{I^n}-\bar\varphi_j^n/h_j|\le\int_{K_j\times I^n}|\varphi_x(x,t)|dxdt$. At this point, we can define the coefficients α,β,γ used in Theorem 2 for $(x,t)\in K_j\times I^n$:

$$\begin{aligned}&\alpha(x,t)\equiv 0,\\ &\beta(x,t)=M[|v_{j+1}^n-v_j^n|+|v_j^n-v_{j-1}^n|],\\ &\gamma(x,t)=(\lambda/2)[m_{j+\frac12}^n|v_{j+1}^n-v_j^n|+p_{j-\frac12}^n|v_j^n-v_{j-1}^n|].\end{aligned}$$

According to this theorem, we have the following error estimate for all $T>0$:

$$\begin{aligned}\int_{|x|\le R}|u(s,T)-v^h(s,T)|ds&\le\int_{|x|\le R+MT+\Delta}|u(s,0)-v^h(s,0)|ds\\ &+2\mathcal{C}(\Delta+M\delta)TV(u_0)+\int_0^T\int_{B(x_j,h_j/2+M(T-t)+\Delta)}\left\{\alpha(s,t)\right.\\ &\left.+\frac{4\mathcal{C}\beta(s,t)}{\Delta}+\mathcal{C}\left(\frac{2M}{\Delta}+\frac1\delta\right)\gamma(s,t)\right\}dsdt\\ &+\mathcal{C}\sup_{0\le\bar t\le T+\nu}\int_{B(x_j,h_j/2+M(T-\bar t)+\Delta)}\gamma(s,\bar t)ds\end{aligned}$$

with $B(x_j,h_j/2+M(T-t)+\Delta)=[x_{j-\frac12}-M(T-t)-\Delta,x_{j+\frac12}+M(T-t)+\Delta]\ \forall t\in[0,T]$. Since the functions β,γ are piecewise constant, the right-hand side can be rewritten as a discrete summation. In fact, we denote by N the value $T/\Delta t$ and we consider the following two terms:

$$\begin{aligned}E_1&=2\mathcal{C}\left\{\Delta TV(u_0)+\frac{M}{\Delta}\sum_{n=0}^N\Delta t\sum_{\ell\in J_{j,\Delta}^n}(4+\lambda Q_{\ell+\frac12}^n)\,h_\ell|v_{\ell+1}^n-v_\ell^n|\right\},\\ E_2&=2\mathcal{C}M\delta TV(u_0)+\frac{\mathcal{C}}{\delta}\sum_{n=0}^N\Delta t\sum_{\ell\in J_{j,\Delta}^n}\lambda Q_{\ell+\frac12}^n\,h_\ell\,|v_{\ell+1}^n-v_\ell^n|.\end{aligned}$$

The best values for the Δ,δ constants are implicitly given by

$$\begin{aligned}\Delta&=\sqrt{\frac{M}{TV(u_0)}}\left\{\sum_{n=0}^N\Delta t\sum_{\ell\in J_{j,\Delta}^n}(4+\lambda Q_{\ell+\frac12}^n)\,h_\ell\,|v_{\ell+1}^n-v_\ell^n|\right\}^{\frac12},\\ \delta&=\sqrt{\frac{1}{2MTV(u_0)}}\left\{\sum_{n=0}^N\Delta t\sum_{\ell\in J_{j,\Delta}^n}\lambda Q_{\ell+\frac12}^n\,h_\ell\,|v_{\ell+1}^n-v_\ell^n|\right\}^{\frac12}.\end{aligned}$$

According to the CFL condition (**??**) which implies the TVD property, an upper bound for Δ is given by

$$\Delta\le\sqrt{\frac{Mh}{TV(u_0)}}\left[\sum_{n=0}^N\Delta t\sum_{\ell\in\mathbb{Z}}\left(4+\frac12\right)|v_{\ell+1}^0-v_\ell^0|\right]^{\frac12}\le 3\Big(\frac{MT}{2}\,h\Big)^{\frac12}.$$

So, for this choice of Δ, δ, the terms E_1 and E_2 are bounded by

$$4\mathcal{C}\sqrt{TV(u_0)}\left[\sum_{n=0}^{N}\Delta t\sum_{\ell\in J^n_{j,\Delta}}\frac{9}{2}Q^n_{\ell+\frac12}\,h_\ell\,|v^n_{\ell+1}-v^n_\ell|\right]^{\frac12},$$

and

$$2\mathcal{C}\sqrt{TV(u_0)}\left[\sum_{n=0}^{N}\Delta t\sum_{\ell\in J^n_{j,\Delta}}Q^n_{\ell+\frac12}\,h_\ell\,|v^n_{\ell+1}-v^n_\ell|\right]^{\frac12},$$

respectively (where in $J^n_{j,\Delta}$, Δ is replaced by its upper bound $3(\frac{MT}{2}\,h)^{\frac12}$). It remains only to add the first-order time term and thus to complete the proof. □

3. Error estimates for second-order resolution (SOR) schemes. In this section, we will derive estimates for SOR schemes of the form (1.7). Given any "essentially 3-point" conservative discretization of the type (1.7), one is always able to define a viscous form (1.8) considering a modified flux $\mathbf{f}^n_j$ and a modified numerical viscosity $\mathbf{Q}^n_{j+\frac12}=\mathbf{m}^n_{j+\frac12}+\mathbf{p}^n_{j+\frac12}$ for which the following analogue of the first-order formula holds:

$$\mathbf{Q}^n_{j+\frac12}=\frac{\mathbf{f}^n_j+\mathbf{f}^n_{j+1}-2g^n_{j+\frac12}}{v^n_{j+1}-v^n_j}.$$

We will use also a discrete one-sided Lipschitz seminorm of the numerical approximations as another indicator in the error estimate:

$$\|v^h(.,t^n)\|_{lip^+(\mathbb{R})}=\sup_{j\in\mathbb{Z}}\left(\frac{v^n_{j+1}-v^n_j}{h_j}\right)_+. \tag{3.1}$$

A numerical scheme is said to be lip^+ -stable if the quantity (3.1) remains bounded as $h\to 0\ \forall t^n>0$ (see, e.g., [5], [26], [27], [35], [37] for details).

A weaker in-cell entropy inequality. In this paragraph we derive a weak in-cell entropy inequality for general numerical schemes written in viscous form (1.8). This will be useful to once again use Theorem 2.

LEMMA 1. *Assume that the scheme* (1.8) *satisfies the stability conditions of Theorem* 1, *then, for all entropies* $U^k(v)=|v-k|$, $k\in\mathbb{R}$ *and all* $(j,n)\in\mathbb{Z}\times\mathbb{N}$, *we have the following in-cell inequality:*

$$U^k(v^{n+1}_j)-U^k(v^n_j)-\frac{\Delta t}{2h_j}[\mathbf{m}^n_{j+\frac12}(U^k(v^n_{j+1})-U^k(v^n_j))-\mathbf{p}^n_{j-\frac12}(U^k(v^n_j)-U^k(v^n_{j-1}))]\le 0. \tag{3.2}$$

Proof. Multiplying the viscous form (1.8) by $\operatorname{sgn}(v^{n+1}_j-k)$, one gets for every $k\in\mathbb{R}$:

$$\begin{aligned}|v^{n+1}_j-k|&-\operatorname{sgn}(v^{n+1}_j-k)(v^n_j-k)\\&-\frac{\Delta t}{2h_j}\mathbf{m}_{j+\frac12}\operatorname{sgn}(v^{n+1}_j-k)(v^n_{j+1}-v^n_j)\\&+\frac{\Delta t}{2h_j}\mathbf{p}_{j-\frac12}\operatorname{sgn}(v^{n+1}_j-k)(v^n_j-v^n_{j-1})=0.\end{aligned}$$

Since $\operatorname{sgn}(v^{n+1}_j-k)(v^n_j-k)\le|v^n_j-k|$, using the CFL condition of Theorem 3, we get

$$|v_j^{n+1}-k|-|v_j^n-k|\left(1-\frac{\Delta t}{2h_j}(\mathbf{m}_{j+\frac12}+\mathbf{p}_{j-\frac12})\right)$$
$$-\frac{\Delta t}{2h_j}\mathbf{m}_{j+\frac12}|v_{j+1}^n-k|-\frac{\Delta t}{2h_j}\mathbf{p}_{j-\frac12}|v_{j-1}^n-k|\le 0.$$

This yields the desired result. □

An approximate entropy inequality for v^h. As in the previous section, we now want to bound the following quantity for all $k\in\mathbb{R}$ and all positive test-functions φ in $C_0^1(\mathbb{R}\times\mathbb{R}_*^+)$. The main point will be to compensate the lack of the entropy consistency of (3.2).

$$\mathcal{I}=-\int_{(\mathbb{R}\times\mathbb{R}_*^+)}[\underbrace{|v^h-k|}_{U^k(v^h)}\varphi_t+\underbrace{\mathrm{sgn}(v^h-k)(f(v^h)-f(k))}_{F^k(v^h)}\varphi_x](x,t)dxdt.$$

Since the approximate solution v^h is piecewise constant,

$$\begin{aligned}\mathcal{I}&=\sum_{j,n}\bar\varphi_{K_j}^{n+1}[U^k(v_j^{n+1})-U^k(v_j^n)]+\bar\varphi_{j+\frac12}^{I^n}\int_{v_j^n}^{v_{j+1}^n}\mathrm{sgn}(v-k)\frac{\mathbf{f}_{j+1}^n-\mathbf{f}_j^n}{v_{j+1}^n-v_j^n}dv\\&\quad+\bar\varphi_{j+\frac12}^{I^n}\int_{v_j^n}^{v_{j+1}^n}\mathrm{sgn}(v-k)\left[f'(v)-\frac{\mathbf{f}_{j+1}^n-\mathbf{f}_j^n}{v_{j+1}^n-v_j^n}\right]dv\\&\overset{def}{=}\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3,\end{aligned}\tag{3.3}$$

where $\bar\varphi_{K_j}^{n+1}=\int_{K_j}\varphi(x,t^{n+1})dx$, $\bar\varphi_{j+\frac12}^{I^n}=\int_{I^n}\varphi(x_{j+\frac12},t)dt$.

We now have the following estimate.

LEMMA 2.

$$\begin{aligned}\mathcal{I}_1+\mathcal{I}_2\le\sum_{j,n}\frac12\left[\mathbf{m}_{j+\frac12}^n|v_{j+1}^n-v_j^n|+\mathbf{p}_{j-\frac12}^n|v_j^n-v_{j-1}^n|\right]\\\times\left(\int_{K_j\times I^n}\lambda|\varphi_t(x,t)|+|\varphi_x(x,t)|dxdt\right).\end{aligned}\tag{3.4}$$

Proof. We have

$$\begin{aligned}\mathcal{I}_1+\mathcal{I}_2&=\sum_{j,n}\bar\varphi_{K_j}^{n+1}[U^k(v_j^{n+1})-U^k(v_j^n)]\\&\quad+\frac12\left[\bar\varphi_{j+\frac12}^{I^n}\int_{v_j^n}^{v_{j+1}^n}\mathrm{sgn}(v-k)\frac{\mathbf{f}_{j+1}^n-\mathbf{f}_j^n}{v_{j+1}^n-v_j^n}.dv\right.\\&\qquad\left.+\bar\varphi_{j-\frac12}^{I^n}\int_{v_{j-1}^n}^{v_j^n}\mathrm{sgn}(v-k)\frac{\mathbf{f}_j^n-\mathbf{f}_{j-1}^n}{v_j^n-v_{j-1}^n}.dv\right].\end{aligned}$$

Using the weak entropy inequality (3.2) multiplied by $\bar\varphi_j^n=\int_{I^n}\int_{K_j}\varphi(x,t)dxdt\ge 0$

we finally obtain

$$\begin{aligned}\mathcal{I}_1+\mathcal{I}_2 \le & \sum_{j,n}(\bar\varphi_{K_j}^{n+1}-\bar\varphi_j^n/\Delta t)[U^k(v_j^{n+1})-U^k(v_j^n)]\\ &+\sum_{j,n}\bar\varphi_j^n/(2h_j)[\mathbf{Q}_{j+\frac12}^n(U^k(v_{j+1}^n)-U^k(v_j^n))\\ &\qquad -\mathbf{Q}_{j-\frac12}^n(U^k(v_j^n)-U^k(v_{j-1}^n))]\\ &+\frac12\sum_{j,n}(\bar\varphi_{j+\frac12}^{I^n}-\bar\varphi_j^n/h_j)\int_{v_j^n}^{v_{j+1}^n}\operatorname{sgn}(v-k)\frac{\mathbf{f}_{j+1}^n-\mathbf{f}_j^n}{v_{j+1}^n-v_j^n}dv\\ &+\frac12\sum_{j,n}(\bar\varphi_{j-\frac12}^{I^n}-\bar\varphi_j^n/h_j)\int_{v_{j-1}^n}^{v_j^n}\operatorname{sgn}(v-k)\frac{\mathbf{f}_j^n-\mathbf{f}_{j-1}^n}{v_j^n-v_{j-1}^n}dv.\end{aligned}$$

To estimate the space terms of the right-hand side of this inequality, first we observe that $\int_{v_j^n}^{v_{j+1}^n}\operatorname{sgn}(v-k)\frac{\mathbf{f}_{j+1}^n-\mathbf{f}_j^n}{v_{j+1}^n-v_j^n}dv=\frac{\mathbf{f}_{j+1}^n-\mathbf{f}_j^n}{v_{j+1}^n-v_j^n}(U^k(v_{j+1}^n)-U^k(v_j^n))$. So the quantity

$$\begin{aligned}\mathcal{R}=&\frac12\sum_{j,n}\bar\varphi_j^n/h_j[\mathbf{Q}_{j+\frac12}^n(U^k(v_{j+1}^n)-U^k(v_j^n))-\mathbf{Q}_{j-\frac12}^n(U^k(v_j^n)-U^k(v_{j-1}^n))]\\ &+\frac12\sum_{j,n}(\bar\varphi_{j+\frac12}^{I^n}-\bar\varphi_j^n/h_j)\frac{\mathbf{f}_{j+1}^n-\mathbf{f}_j^n}{v_{j+1}^n-v_j^n}[U^k(v_{j+1}^n)-U^k(v_j^n)]\\ &+\frac12\sum_{j,n}(\bar\varphi_{j-\frac12}^{I^n}-\bar\varphi_j^n/h_j)\frac{\mathbf{f}_j^n-\mathbf{f}_{j-1}^n}{v_j^n-v_{j-1}^n}[U^k(v_j^n)-U^k(v_{j-1}^n)]\end{aligned}$$

becomes, after a summation by parts,

$$\begin{aligned}\mathcal{R}=&\frac12\sum_{j,n}\mathbf{Q}_{j-\frac12}^n[U^k(v_j^n)-U^k(v_{j-1}^n)][(\bar\varphi_{j-\frac12}^{I^n}-\bar\varphi_j^n/h_j)-(\bar\varphi_{j-\frac12}^{I^n}-\varphi_{j-1}^n/h_{j-1})]\\ &+\frac12\sum_{j,n}\frac{\mathbf{f}_j^n-\mathbf{f}_{j-1}^n}{v_j^n-v_{j-1}^n}[U^k(v_j^n)-U^k(v_{j-1}^n)][(\bar\varphi_{j-\frac12}^{I^n}-\bar\varphi_j^n/h_j)\\ &\qquad +(\bar\varphi_{j-\frac12}^{I^n}-\varphi_{j-1}^n/h_{j-1})].\end{aligned}$$

Therefore,

$$\mathcal{R}=\frac12\sum_{j,n}[U^k(v_j^n)-U^k(v_{j-1}^n)]\left[\mathbf{p}_{j-\frac12}^n(\bar\varphi_{j-\frac12}^{I^n}-\bar\varphi_j^n/h_j)-\mathbf{m}_{j-\frac12}^n(\bar\varphi_{j-\frac12}^{I^n}-\varphi_{j-1}^n/h_{j-1})\right].$$

And we have

$$\begin{aligned}\mathcal{I}_1+\mathcal{I}_2\le&\sum_{j,n}(\bar\varphi_{K_j}^{n+1}-\bar\varphi_j^n/\Delta t)[U^k(v_j^{n+1})-U^k(v_j^n)]\\ &+\frac12\sum_{j,n}[U^k(v_j^n)-U^k(v_{j-1}^n)]\left[\mathbf{p}_{j-\frac12}^n(\bar\varphi_{j-\frac12}^{I^n}-\bar\varphi_j^n/h_j)\right.\\ &\qquad\left.-\mathbf{m}_{j-\frac12}^n(\bar\varphi_{j-\frac12}^{I^n}-\varphi_{j-1}^n/h_{j-1})\right].\end{aligned}$$

The proof of the lemma is complete by observing that

$$\Big|\sum_{j,n}(\bar\varphi_{K_j}^{n+1}-\bar\varphi_j^n/\Delta t)[U^k(v_j^{n+1})-U^k(v_j^n)]\Big| \\ \leq \frac{\lambda}{2}\sum_{j,n}\Big[\mathbf{m}_{j+\frac12}^n|v_{j+1}^n-v_j^n|+\mathbf{p}_{j-\frac12}^n|v_j^n-v_{j-1}^n|\Big]\int_{K_j\times I^n}|\varphi_t(x,t)|dxdt$$

and

$$\frac12\left|\sum_{j,n}[U^k(v_j^n)-U^k(v_{j-1}^n)]\Big[\mathbf{p}_{j-\frac12}^n(\bar\varphi_{j-\frac12}^{I^n}-\bar\varphi_j^n/h_j)-\mathbf{m}_{j-\frac12}^n(\bar\varphi_{j-\frac12}^{I^n}-\varphi_{j-1}^n/h_{j-1})\Big]\right| \\ \leq \frac12\sum_{j,n}\Big[\mathbf{m}_{j+\frac12}^n|v_{j+1}^n-v_j^n|+\mathbf{p}_{j-\frac12}^n|v_j^n-v_{j-1}^n|\Big]\int_{K_j\times I^n}|\varphi_x(x,t)|dxdt. \qquad \square$$

The proofs of the next two lemmas are forwarded to the appendix.

LEMMA 3. *For every $(a,b)\in\mathbb{R}^2$ and $\theta\in[0,1]$, let us define the following quantity:*

$$A(a,b;\theta)\overset{def}{=}2\big[(1-\theta)f(a)+\theta f(b)-f\big((1-\theta)a+\theta b\big)\big].$$

Then we have

$$\mathcal{I}_3\leq\sum_{j,n}\underbrace{(\omega_{j+\frac12}^n+\nu_{j+\frac12}^n)}_{\mathbf{q}_{j+\frac12}^n}\int_{K_j\times I^n}\sup_{x'\in K_j}\frac{\varphi(x',t)}{h_j}dxdt, \tag{3.5}$$

where with $\mathbf{1}_S$ indicating the characteristic function of S,

$$\omega_{j+\frac12}^n=\left[(\mathbf{1}_{v_j^n\leq v_{j+1}^n}-\mathbf{1}_{v_j^n\geq v_{j+1}^n})\max_{\theta\in[0,1]}A(v_j^n,v_{j+1}^n;\theta)\right]_+ \quad\text{and}\quad \nu_{j+\frac12}^n=|\tilde f_{j+1}^n-\tilde f_j^n|.$$

Remark 1. Since the flux function f is smooth, we have $\omega_{j+\frac12}^n=O(|v_{j+1}^n-v_j^n|^2)$. In addition, the difference of two adjacent correction terms [32] is known to be such that $\nu_{j+\frac12}^n=O(|v_{j+1}^n-v_j^n|^2)$. Consequently, the contribution of $\mathcal{I}_3$ will be actually very small in the areas of smoothness of the entropy solution of (1.2). Moreover, one notices that Oleinik's entropy condition [29] implies that if the values v_j^n,v_{j+1}^n are connected by a single shock, then $\omega_{j+\frac12}^n=0$.

Assuming that f is convex, one can refine the expression of $\omega_{j+\frac12}^n$.

LEMMA 4. *In the case where f is convex there holds*

$$\forall j\in\mathbb{Z},\ \omega_{j+\frac12}^n\leq\max_{\xi\in[v_j^n,v_{j+1}^n]}f''(\xi)|v_{j+1}^n-v_j^n|(v_{j+1}^n-v_j^n)_+. \tag{3.6}$$

Summarizing, under the assumptions of Theorem 1, v^h satisfies the following

approximate entropy inequality:

(3.7)
$$
\begin{aligned}
-\int_{(\mathbb{R}\times\mathbb{R}_*^+)} &[|v^h-k|\varphi_t+\operatorname{sgn}(v^h-k)(f(v^h)-f(k))\varphi_x](x,t)dxdt\\
&\le \sum_{j,n}\mathbf{q}^n_{j+\frac12}\,\frac{1}{h_j}\int_{K_j\times I^n}\sup_{x'\in K_j}\varphi(x',t)dxdt\\
&+\sum_{j,n}\frac{\lambda}{2}\big[\mathbf{m}^n_{j+\frac12}|v^n_{j+1}-v^n_j|+\mathbf{p}^n_{j-\frac12}|v^n_j-v^n_{j-1}|\big]\int_{K_j\times I^n}|\varphi_t(x,t)|dxdt\\
&+\sum_{j,n}\frac{1}{2}\big[\mathbf{m}^n_{j+\frac12}|v^n_{j+1}-v^n_j|+\mathbf{p}^n_{j-\frac12}|v^n_j-v^n_{j-1}|\big]\int_{K_j\times I^n}|\varphi_x(x,t)|dxdt.
\end{aligned}
$$

Now we can use the framework provided by Theorem 2 starting from the approximate inequality (3.7).

THEOREM 4. *Assume that the restrictions of Theorem* 1 *are met for the scheme* (1.8) *which satisfies* (3.2) *for all Lipschitz continuous entropies* U^k, $k\in\mathbb{R}$. *Then we have for all* $T>0$:

(3.8)
$$
\begin{aligned}
\int_{K_j}|v^h(x,T)-u(x,T)|dx &\le \int_{x_{j-\frac12}-MT-\Delta}^{x_{j+\frac12}+MT+\Delta}|v^h(x,0)-u(x,0)|dx\\
&+2\mathcal{C}(\sqrt{10}+1)\sqrt{TV(u_0)}\left\{\sum_{n=0}^{N}\Delta t\mathcal{X}^n_V\right\}^{1/2}+\sum_{n=0}^{N}\Delta t\,\mathcal{X}^n_E+\mathcal{C}\lambda\sup_{n=0,\dots,N+1}\{\mathcal{X}^n_V\},
\end{aligned}
$$

where

(3.9)
$$
\begin{aligned}
\mathcal{X}^n_V &= \sum_{\ell\in J^n_{j,\Delta}} h_\ell\,\mathbf{Q}^n_{\ell+\frac12}|v^n_{\ell+1}-v^n_\ell|,\\
\mathcal{X}^n_E &= \sum_{\ell\in J^n_{j,\Delta}}\mathbf{q}^n_{\ell+\frac12}\ \textit{with}\ \mathbf{q}^n_{\ell+\frac12}=\omega^n_{\ell+\frac12}+\nu^n_{\ell+\frac12},\\
J^n_{j,\Delta} &= \{\ell\pm1\ :x_\ell\in B(x_j,h_j/2+M(T-t^n)+\Delta)\},\ \Delta=\sqrt{\frac{5Th}{4\lambda}},\\
M &= \sup\{|f'(\xi)|\ \textit{with}\ |\xi|\le\|u_0\|_{L^\infty(\mathbb{R})}\},\ \mathcal{C}=2,\ N=T/\Delta t.
\end{aligned}
$$

Remark 2. Note that the error bound in (3.8) is at best $O(h^{1/2})$. This is also observed computationally; cf. section 4. This means that this estimator "cannot see" the possible extra accuracy of an SOR scheme. Up to the authors' knowledge, there are no known asymptotic estimates of higher order for high resolution schemes; cf. also section 4.

Remark 3. It seems that the new terms $\mathcal{X}^n_E$ which measure the lack of the entropy consistency of the scheme are of higher order if compared to the other terms in the estimate. Indeed, first note that in the case where $\mathbf{f}^n_j=f(v^n_j)$ and the flux is convex, then this term is $O(h)$ while the other terms in the estimate are $O(h^{1/2})$. This is a consequence of the lip^+ stability of these schemes proved by Tadmor. In the general case this is supported by heuristic arguments based on Remark 1 and on the fact that

the correction term $\tilde{f}_j^n$ should be "close" to $h(Q^n_{j+\frac{1}{2}} - \lambda(a^n_{j+\frac{1}{2}})^2)/2$; cf. [12]. We will see the existence of numerical evidence of the nice behavior of this indicator in the tests displayed in the forthcoming section.

Proof. Starting from (3.7), we define the following piecewise constant functions for $(x,t) \in K_j \times I^n$:

$$\begin{aligned} \alpha(x,t) &= \mathbf{q}^n_{j+\frac{1}{2}} / h_j, \\ \beta(x,t) &= \frac{1}{2}\left(\mathbf{m}^n_{j+\frac{1}{2}} |v^n_{j+1} - v^n_j| + \mathbf{p}^n_{j-\frac{1}{2}} |v^n_j - v^n_{j-1}|\right), \\ \gamma(x,t) &= \frac{\lambda}{2}\left(\mathbf{m}^n_{j+\frac{1}{2}} |v^n_{j+1} - v^n_j| + \mathbf{p}^n_{j-\frac{1}{2}} |v^n_j - v^n_{j-1}|\right). \end{aligned} \tag{3.10}$$

Then Theorem 2 implies

$$\begin{aligned} &\int_{|x|\le R} |u(s,T) - v^h(s,T)|.ds \\ &\quad \le \int_{|x|\le R+MT+\Delta} |u(s,0) - v^h(s,0)|.ds + 2\mathcal{C}(\Delta + M\delta)TV(u_0) \\ &\quad + \int_0^T \int_{|x|\le R+M(T-t)+\Delta} \left\{\alpha(s,t) + \frac{4\mathcal{C}\beta(s,t)}{\Delta} + \mathcal{C}\left(\frac{2M}{\Delta} + \frac{1}{\delta}\right)\gamma(s,t)\right\} ds.dt \\ &\quad + \mathcal{C} \sup_{0\le \bar{t}\le T+\nu} \int_{|x|\le R+M(T-\bar{t})+\Delta} \gamma(s,\bar{t}) ds \end{aligned}$$

with $B(x_j, h_j/2 + M(T-t) + \Delta) = [x_{j-\frac{1}{2}} - M(T-t) - \Delta, x_{j+\frac{1}{2}} + M(T-t) + \Delta]\ \forall t \in [0,T]$. By the definition of $J^n_{j,\Delta}$ we have

$$\int_{B(x_j, h_j/2+M(T-t)+\Delta)} \alpha(s,t) ds.dt \le \left(\sum_{n=0}^N \Delta t \sum_{\ell \in J^n_{j,\Delta}} \mathbf{q}^n_{\ell+\frac{1}{2}}\right). \tag{3.11}$$

Similar bounds hold for the other terms. These terms are bounded by $E_1 + E_2$ where

$$\begin{aligned} E_1 &= \mathcal{C}\left[2\Delta TV(u_0) + \frac{1}{\Delta}\sum_{n=0}^N \Delta t \sum_{\ell \in J^n_{j,\Delta}} (4 + 2M\lambda)\mathbf{Q}^n_{\ell+\frac{1}{2}}\, h_\ell\, |v^n_{\ell+1} - v^n_\ell|\right], \\ E_2 &= \mathcal{C}\left[2M\delta TV(u_0) + \frac{1}{\delta}\sum_{n=0}^N \Delta t \sum_{\ell \in J^n_{j,\Delta}} \lambda\mathbf{Q}^n_{\ell+\frac{1}{2}}\, h_\ell\, |v^n_{\ell+1} - v^n_\ell|\right]. \end{aligned}$$

Using in this expression the CFL condition $2M\lambda \le 1$ yields the optimal value of Δ:

$$\Delta = \frac{1}{\sqrt{2TV(u_0)}}\left[\sum_{n=0}^N \Delta t \sum_{\ell \in J^n_{j,\Delta}} 5\mathbf{Q}^n_{\ell+\frac{1}{2}}\, h_\ell\, |v^n_{\ell+1} - v^n_\ell|\right]^{1/2}.$$

The TVD property and $\lambda \sup_{j,n} \mathbf{Q}^n_{j+\frac{1}{2}} \le \frac{1}{2}$, imply that $\Delta \le \sqrt{\frac{5}{4\lambda} Th}$. We choose also the following value for δ:

$$\delta = \frac{1}{\sqrt{2MTV(u_0)}}\left[\sum_{n=0}^N \Delta t \sum_{\ell \in J^n_{j,\Delta}} \lambda\mathbf{Q}^n_{\ell+\frac{1}{2}}\, h_\ell\, |v^n_{\ell+1} - v^n_\ell|\right]^{1/2}.$$

Then, replacing in $J^n_{j,\Delta}\ \Delta$ by its upper bound $\left(\frac{5}{4\lambda}Th\right)^{1/2}$, we conclude that E_1 and E_2 are bounded, respectively, by

$$2\mathcal{C}\sqrt{TV(u_0)}\left[\sum_{n=0}^{N}\Delta t\sum_{\ell\in J^n_{j,\Delta}}10\mathbf{Q}^n_{\ell+\frac{1}{2}}\,h_\ell\,|v^n_{\ell+1}-v^n_\ell|\right]^{1/2}$$

and

$$2\mathcal{C}\sqrt{TV(u_0)}\left[\sum_{n=0}^{N}\Delta t\sum_{\ell\in J^n_{j,\Delta}}\mathbf{Q}^n_{\ell+\frac{1}{2}}\,h_\ell\,|v^n_{\ell+1}-v^n_\ell|\right]^{1/2}$$

and thus the proof is complete. □

4. Numerical results. In this section, we display some numerical experiments on the inviscid Burgers equation:

$$\begin{cases} u_t+\left(\frac{u^2}{2}\right)_x=0,\\ u_0\in BV(\mathbb{R}). \end{cases} \tag{4.1}$$

We will restrict ourselves to sufficiently simple initial data in order to be able to derive the exact solution and to compare the two estimates (2.4), (3.8) with the *real* error of the numerical schemes.

In sections 4.1 and 4.2 below we compare the estimators (2.4), (3.8) with the real error as $h\to 0$. These runs use uniform mesh. Our first conclusion is that both estimators approach 0 as we refine the mesh. In addition it is verified numerically that the estimators control the error. On the other hand, the experiments show (for the case of second-order schemes, see also Remark 2 and the discussion below) that they are quite pessimistic. Of course, asymptotical exactness (cf., e.g., [1]) is probably asking for too much in the context of a nonlinear problem with such a singular behavior as (1.2), especially since convergence of "reasonable" schemes towards its entropy solution is a subtle issue. It is clear, however, that the qualitative behavior of our estimators is the right one. Also, in the case of schemes for which the theory of this paper can be applied, we have rigorous control of the error whether or not there is any underlying available convergence theory. For the runs in section 4.1 one should remark also that the test problem has only one shock, while the estimator works under no assumptions on the number of shocks (which can be arbitrary big) or the genuine nonlinearity of the flux function f. The rate observed for the estimator is $O(h^{1/2})$ which is known to be optimal in the general case.

In the case of second-order schemes considered in section 4.2 the estimator has similar behavior. An important conclusion is that the estimator (3.8) is justified as a reliable test for convergence. On the other hand, its main drawback is probably that it cannot see the extra accuracy of a second-order scheme and it decreases still with a rate $O(h^{1/2})$. This fact is probably coming from the method of proof and it is related to the open problem in theory of proving higher order (than $O(h^{1/2})$) estimates for higher order schemes. (It is to be noted that most of the known results for high-order schemes, in the general case, provide orders of convergence less than $O(h^{1/2})$, or there are not convergence results at all—e.g., in the case of [25] for nonconvex fluxes).

In section 4.3 we use the estimates (2.4), (3.8) to design an adaptive code. Our purpose is to show that it is indeed possible to gain in accuracy and to add points in the right places by using these estimators. This is clear from the runs in section 4.3. On the other hand it should be noted that the design of an efficient adaptive algorithm based on the information gained by our estimates is an independent problem that we do not claim to address in this paper. This will be the subject of further investigation.

4.1. Two Riemann problems for the modified Lax–Friedrichs scheme. Here we consider the *modified* Lax–Friedrichs scheme introduced by Tadmor [36]. It corresponds to the upper bound for the numerical viscosity coefficient in Theorem 1: $Q^n_{j+\frac{1}{2}} \equiv \frac{1}{2\lambda}$. We only used regular grids for which $h_j \equiv \Delta x \ \forall j \in \mathbb{Z}$. That means in particular that we have simply $\lambda = \Delta t/\Delta x$. We performed a numerical test on two Riemann problems for (4.1):

$$u_0(x) = \left| \begin{array}{l} 1 \text{ for } x < 1 \\ 0 \text{ for } x > 1 \end{array} \right. ; \ u_0(x) = \left| \begin{array}{l} 0 \text{ for } x < 1 \\ 1 \text{ for } x > 1 \, . \end{array} \right. \tag{4.2}$$

Since the numerical scheme is proved to be endowed with a strong in-cell entropy inequality, we are in position to use both of the estimates (2.4), (3.8). We compare also these values with the real global L^1 error. The results for $T = 0.4$ are displayed in Figure 4.1 for the shock and for the expansive wave. We used $\lambda = \frac{1}{2}$ and 16 values for Δx between $\Delta x = 0.05$ and $\Delta x = 0.000276$. The computational domain is the interval $[0, 2]$.

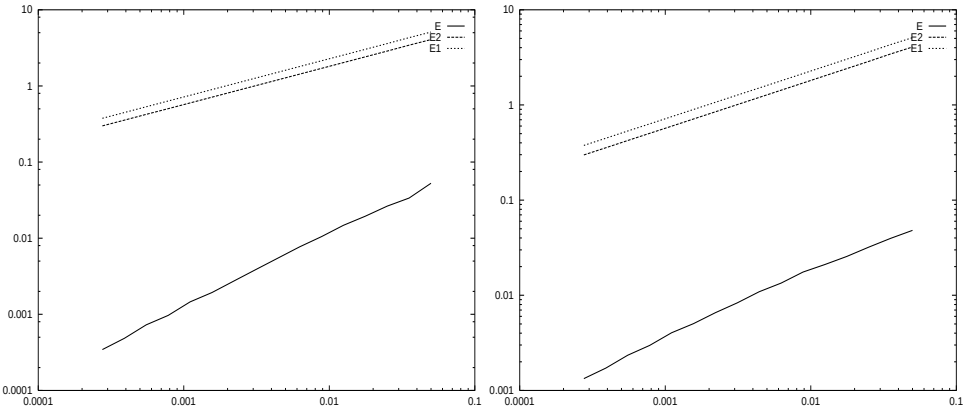

FIG. 4.1. *Comparison between the error estimates* (2.4) *(upper dotted lines-E*1*),* (3.8) *(lower dotted lines-E*2*) and the real error (solid lines-E). Left: shock wave. Right: rarefaction wave.*

The solid line corresponds to the real error of the numerical scheme on the whole computational domain. It shows a nearly first-order convergence in both cases regardless of the smoothness of the exact solution. The two dotted lines display the estimates given by (2.4), (3.8) on the same interval.

4.2. An experiment with its SOR extension. We present now some results obtained by a second-order extension of the former Lax–Friedrichs scheme. This SOR algorithm will be derived by means of the piecewise constant viscosity modification introduced by Osher and Tadmor. To carry this out, we have to choose convenient values for $\tilde{f}^n_j$ and $\mathbf{Q}^n_{j+\frac{1}{2}}$ in order to use them in (1.8). One possible choice suggested

in [32] is given by

$$\tilde{f}_j^n = \frac{\operatorname{sgn}(v_{j+1}^n - v_j^n) + \operatorname{sgn}(v_j^n - v_{j-1}^n)}{4} \times \min\left\{(Q_{j+\frac{1}{2}}^n - \lambda(a_{j+\frac{1}{2}}^n)^2)|v_{j+1}^n - v_j^n|, (Q_{j-\frac{1}{2}}^n - \lambda(a_{j-\frac{1}{2}}^n)^2)|v_j^n - v_{j-1}^n|\right\},$$

$$\mathbf{Q}_{j+\frac{1}{2}}^n = Q_{j+\frac{1}{2}}^n + \left|\frac{\tilde{f}_{j+1}^n - \tilde{f}_j^n}{v_{j+1}^n - v_j^n}\right| \text{ with } Q_{j+\frac{1}{2}}^n = \frac{1}{2\lambda} \text{ and } a_{j+\frac{1}{2}}^n = \frac{f(v_{j+1}^n) - f(v_j^n)}{v_{j+1}^n - v_j^n}.$$

These values satisfy the second-order requirement (1.10). Moreover, the numerical solution v^h will be endowed with the TVD property under the following CFL condition: $\left|\frac{f(v_{j+1}^n) - f(v_j^n)}{v_{j+1}^n - v_j^n}\right| \leq \frac{2}{3\lambda}$. Consequently, we will consider the following initial data for (4.1):

$$u_0(x) = \sin(\pi x) \text{ for } x \in [0, 2]. \tag{4.3}$$

Its exact solution develops a standing shock at $x = 1$ and can be computed accurately using the method of characteristics. This provides a way to compare the real global error of the numerical scheme with the estimate (3.8). We used once again a uniform grid and also the same range of parameters as in the preceding paragraph. The results at time $T = 0.4$ are displayed in Figure 4.2 where the solid line represents the exact errors and the dotted one the estimated ones. For the sake of completeness, we also present some results before the formation of the stationary shock at time $T = 0.15$. The second-order accuracy for this smooth solution is clearly noticeable. On the contrary, the estimated errors for the SOR modification are extremely close to the ones obtained for the former first-order modified Lax–Friedrichs scheme; cf. section 3, Remark 2. In addition the contribution of the $\mathcal{X}_E^n$ terms in the estimate is indeed very small in this case.

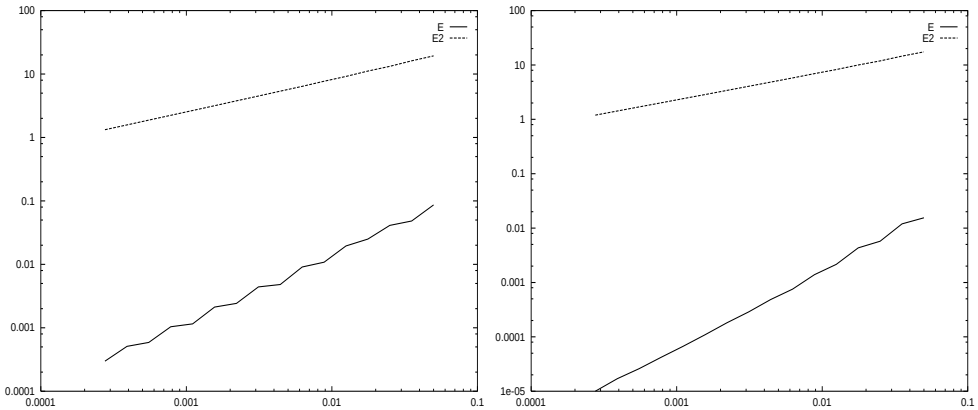

FIG. 4.2. *Comparison between the error estimate* (3.8) *(dotted lines-E2) and the real error (solid lines-E). Left: standing shock at* $T = 0.4$. *Right: smooth solution at* $T = 0.15$.

4.3. An example of adaptive algorithm built on these estimates. As a last illustration we present preliminary computational experiments on the inviscid Burgers equation (4.1) with an SOR adaptive numerical scheme using as a building block the former extension of the Lax–Friedrichs scheme. The algorithm consists of

solving this PDE on a variable grid whose cells are denoted $(K_j)_{j\in\mathbb{Z}}$. At each time t^n, the local estimate (3.8) is used to compute an upper bound of the error $e_{K_j}(t^n)$ in each mesh cell. Given a certain tolerance $\epsilon(t^n) > 0$, we adjust the computational grid in such a way that the following requirement is fulfilled:

$$\forall j, \frac{\epsilon(t^n)}{20} \leq e_{K_j}(t^n) \leq \epsilon(t^n). \tag{4.4}$$

This is achieved by refining and derefining the grid for each time step. (The factor 20 in (4.4) is not very important. One should remark only that a factor close to 2, say, is not the right choice. This is because if the solution has nearly "flat areas" (e.g., in a Riemann problem) $e_{K_j}(t^n)$ will be very small there and respecting a criterion with a factor 2 will result in very coarse mesh. This eventually will destroy the solution.)

The test problem we considered consists of solving (4.1) associated with the following Lipschitz continuous initial data:

$$u_0(x) = \left| \begin{array}{ccc} 1 & \text{for} & x < 0, \\ (1-x) & \text{for} & 0 \leq x \leq 2, \\ -1 & \text{for} & x > 2. \end{array} \right.$$

Its exact solution develops a stationary shock at $x = 1$ for $t \geq 1$. Everywhere else, the solution is constant: it is therefore relevant to try to optimize the computational grid on such a problem since it is merely in the transition zone that a higher number of points are needed to accurately describe the solution. Consequently, we display in Figures 4.3 and 4.4 the numerical results and the space discretization obtained at time $T = 0.8$ before the formation of the shock. The CFL number is fixed $\lambda \equiv 0.45$. Then the grid is adjusted to fulfill the requirement (4.4) for each time step $0 < t^n \leq T$.

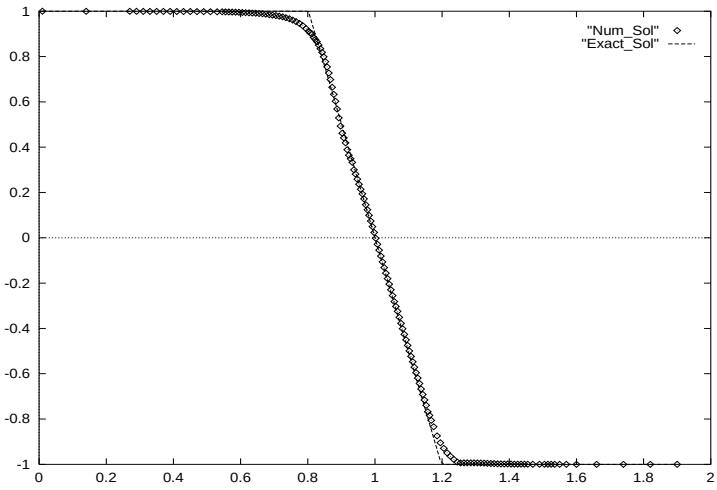

FIG. 4.3. *Numerical and exact solutions at time* $T = 0.8$.

We turn now to the time $T = 1.2$ for which the stationary shock is clearly noticeable on Figure 4.5. The computational grid fits quite well with the structure of this steady-state solution; see Figure 4.6. This run was carried out using the same parameters as the preceding one. We can display the evolution in time of the exact error of the considered numerical scheme together with the estimated one in Figure

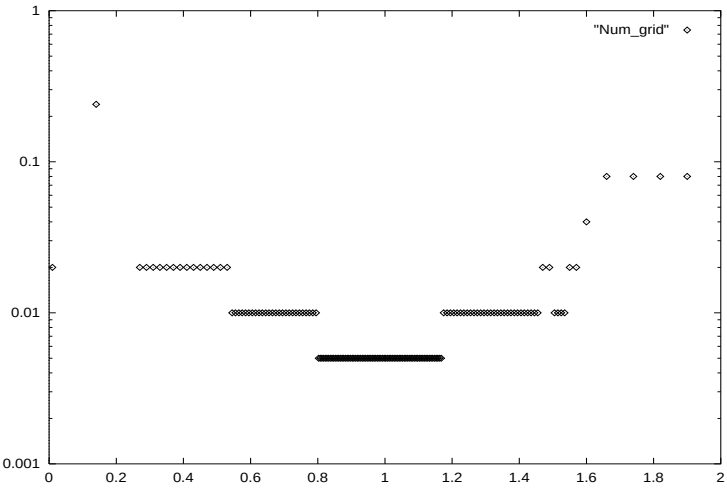

FIG. 4.4. *Size of the cells in the computational grid at time* $T = 0.8$.

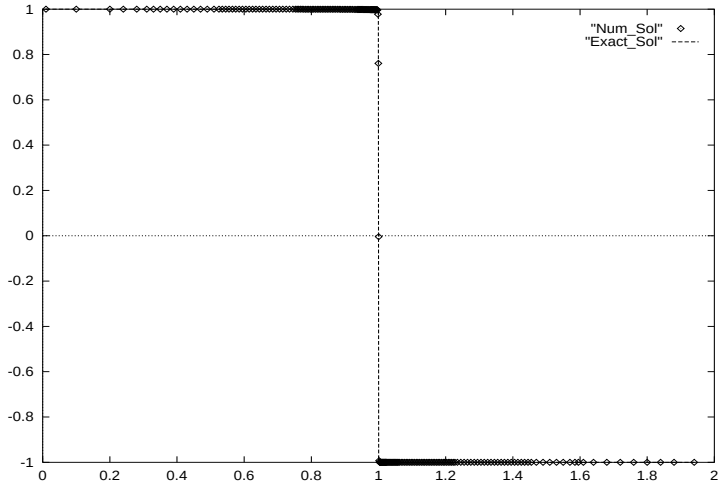

FIG. 4.5. *Numerical and exact solutions at time* $T = 1.2$.

4.7. One can see the slight increase of the exact error in $T = 1$ which corresponds to the shock appearance in the entropy solution.

Appendix A. Estimates on $\mathcal{I}_3$.

Proof of Lemma 3. We want to estimate for $k \in \mathbb{R}$:

$$\mathcal{I}_3 = \sum_{j,n} \bar{\varphi}^{I^n}_{j+\frac{1}{2}} \left(\int_{v^n_j}^{v^n_{j+1}} \operatorname{sgn}(v-k) \left(f'(v) - \frac{f(v^n_{j+1}) - f(v^n_j)}{v^n_{j+1} - v^n_j} \right) .dv \right.$$
$$\left. - \int_{v^n_j}^{v^n_{j+1}} \operatorname{sgn}(v-k) \frac{\tilde{f}^n_{j+1} - \tilde{f}^n_j}{v^n_{j+1} - v^n_j} .dv \right)$$

If $k \notin (a,b)$, the first integral is zero. So the desired result follows by observing that $\left| \int_{v^n_j}^{v^n_{j+1}} \operatorname{sgn}(v-k) \frac{\tilde{f}^n_{j+1} - \tilde{f}^n_j}{v^n_{j+1} - v^n_j} dv \right| \leq |\tilde{f}^n_{j+1} - \tilde{f}^n_j|$. On the contrary, if k lies in (a,b), we have

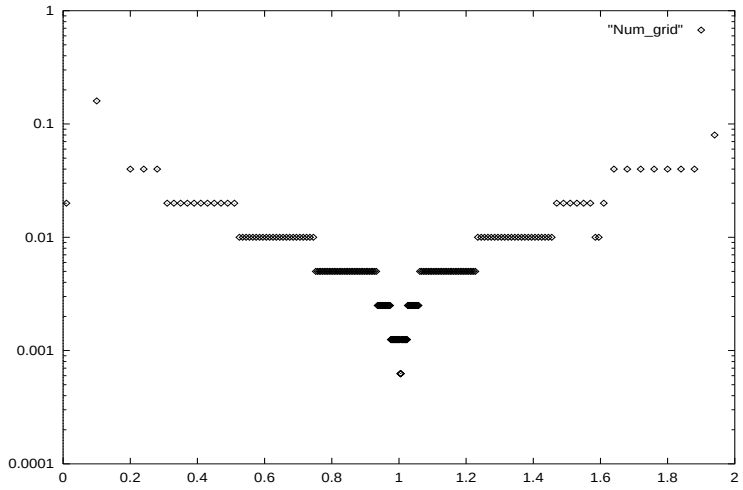

FIG. 4.6. *Size of the cells in the computational grid at time* $T = 1.2$.

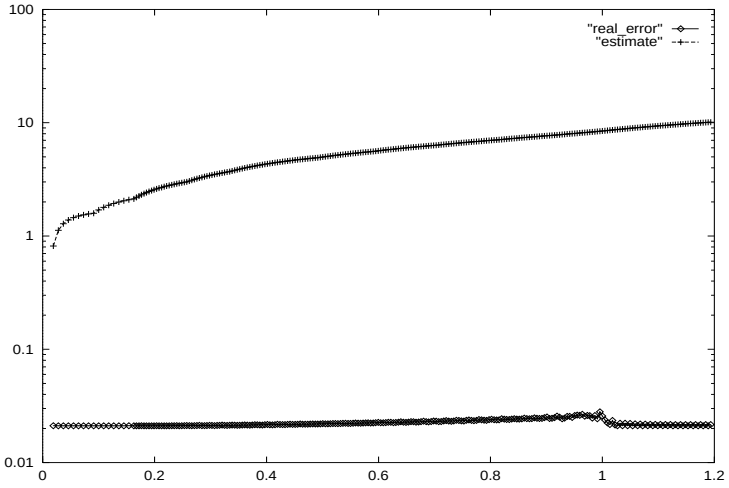

FIG. 4.7. *History of the real and estimated errors up to time* $T = 1.2$.

(i) $b < k < a$.
We can write $k = \theta b + (1-\theta)a$ for $\theta \in (0,1)$:

$$\begin{aligned}\int_a^b \operatorname{sgn}(v-k)\left(f'(v) - \frac{f(b)-f(a)}{b-a}\right)dv \\ &= 2f(k) - f(a) - f(b) - (2k-(a+b))\frac{f(b)-f(a)}{b-a} \\ &= 2f(k) - f(a) - f(b) + (1-2\theta)(f(b)-f(a)).\end{aligned}$$

(ii) $a < k < b$.

In this case, we can write $k = \theta a + (1-\theta)b$ for $\theta \in (0,1)$:

$$\begin{aligned}\int_a^b \operatorname{sgn}(v-k)\left(f'(v) - \frac{f(b)-f(a)}{b-a}\right)dv \\ &= f(a)+f(b)-2f(k)-((a+b)-2k)\frac{f(b)-f(a)}{b-a} \\ &= -2f(k)+f(a)+f(b)+(1-2\theta)(f(b)-f(a)). \qquad \square\end{aligned}$$

Proof of Lemma 4. The goal is to get an estimate on the following term, provided that f is convex and $k \in \mathbb{R}$:

$$\omega^n_{j+\frac{1}{2}} = \int_{v^n_j}^{v^n_{j+1}} \operatorname{sgn}(v-k)\left(f'(v) - \frac{f(v^n_{j+1})-f(v^n_j)}{v^n_{j+1}-v^n_j}\right)dv.$$

First of all, it is useful to find the sign of $\int_a^b \operatorname{sgn}(v-k)\left(f'(v) - \frac{f(b)-f(a)}{b-a}\right)dv$, $(a,b,k) \in \mathbb{R}^3$. If $k \notin (a,b)$, this integral is zero. So we look at the two other cases:

(i) $b < k < a$.

We can write $k = \theta b + (1-\theta)a$ for $\theta \in (0,1)$:

$$\begin{aligned}\int_a^b \operatorname{sgn}(v-k)\left(f'(v) - \frac{f(b)-f(a)}{b-a}\right)dv \\ &= 2f(k)-f(a)-f(b)-(2k-(a+b))\frac{f(b)-f(a)}{b-a} \\ &= 2f(k)-f(a)-f(b)+(1-2\theta)(f(b)-f(a)) \\ &\le -(1-2\theta)f(b)+(2-2\theta)f(a)-f(a)+(1-2\theta)(f(b)-f(a)) \\ &\le 0.\end{aligned}$$

We used Jensen's inequality: $f(\theta b+(1-\theta)a) \le \theta f(b)+(1-\theta)f(a)$.

(ii) $a < k < b$.

In this case, we can write $k = \theta a + (1-\theta)b$ for $\theta \in (0,1)$:

$$\begin{aligned}\int_a^b \operatorname{sgn}(v-k)\left(f'(v) - \frac{f(b)-f(a)}{b-a}\right)dv \\ &= f(a)+f(b)-2f(k)-((a+b)-2k)\frac{f(b)-f(a)}{b-a} \\ &= -2f(k)+f(a)+f(b)+(1-2\theta)(f(b)-f(a)) \\ &\ge -(1-2\theta)[f(b)-f(a)]+(1-2\theta)(f(b)-f(a)) \\ &\ge 0.\end{aligned}$$

So we already have that

$$\begin{aligned}\int_a^b \operatorname{sgn}(v-k)\left(f'(v) - \frac{f(b)-f(a)}{b-a}\right)dv \\ &\le \mathbf{1}_{a<k<b}\underbrace{\left(f(a)+f(b)-2f(k)-((a+b)-2k)\frac{f(b)-f(a)}{b-a}\right)}_{A}.\end{aligned}$$

Now, since f is convex and $a < k < b$, the slopes are increasing:

$$\frac{f(k)-f(a)}{k-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(k)}{b-k}. \tag{A.1}$$

(i) We multiply (A.1) by $(a+b-2k)$ assuming that this quantity is positive. We get

$$[f(k)-f(a)]\left(-1+\frac{b-k}{k-a}\right) \le \frac{f(b)-f(a)}{b-a}(a+b-2k) \le [f(b)-f(k)]\left(1+\frac{a-k}{b-k}\right).$$

So $-\frac{f(b)-f(a)}{b-a}(a+b-2k) \le [f(k)-f(a)]\left(1-\frac{b-k}{k-a}\right)$, and we have by the mean-value theorem for $\zeta_1 \in [k,b], \zeta_2 \in [a,k]$:

$$\begin{aligned} A &\le f(b)-f(k)-(b-k)\frac{f(k)-f(a)}{k-a} \\ &\le [f'(\zeta_1)-f'(\zeta_2)](b-k). \end{aligned}$$

(ii) Now, if $(a+b-2k)$ is negative, we get from (A.1)

$$[f(b)-f(k)]\left(1+\frac{a-k}{b-k}\right) \le \frac{f(b)-f(a)}{b-a}(a+b-2k) \le [f(k)-f(a)]\left(-1+\frac{b-k}{k-a}\right).$$

This means that

$$-\frac{f(b)-f(a)}{b-a}(a+b-2k) \le [f(k)-f(b)]\left(1+\frac{a-k}{b-k}\right).$$

Consequently, we get as before for $\zeta_3 \in [a,k], \zeta_4 \in [k,b]$

$$\begin{aligned} A &\le f(a)-f(k)+(a-k)\frac{f(k)-f(b)}{b-k} \\ &\le [f'(\zeta_3)-f'(\zeta_4)](a-k). \end{aligned}$$

Finally we have

$$A \le \max_{\xi \in]a,b[} f''(\xi)\left[(b-a)_+\right]^2.$$

This completes the proof of the lemma. ∎

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