

A-POSTERIORI ERROR ESTIMATES FOR A FINITE VOLUME METHOD FOR THE STOKES PROBLEM IN TWO DIMENSIONS

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ABSTRACT. Two a-posteriori error estimators for a nonconforming finite volume discretization of the stationary Stokes equations are presented and analyzed. Numerical experiments are used to confirm the expected rate of convergence of the estimators.

1. INTRODUCTION

In recent years finite volume element methods have become very popular among the engineering community for flow computations. This is mainly due to their simplicity and to the local conservation properties that they enjoy. These properties are discrete versions of certain balance laws that characterize the problem. Indeed, in the finite volume setting, it is exactly these discrete balance laws that provide the necessary equations needed in order to define the numerical scheme. Because of this fact the stability and convergence analysis of the finite volume methods is involved. Although finite volume methods are in many cases the method of choice for reliable computations, their connection to adaptive mesh refinement techniques is still not well-developed, mainly due to the lack of appropriate a-posteriori estimates (see, however, [1], where a-posteriori estimates for a finite volume method for elliptic equations are discussed, and [3] where a-posteriori estimates are derived for a finite volume method for a class of convection-dominated elliptic equations). In this paper we show that a-posteriori estimates similar to the estimates valid in the finite element method are also valid for the finite volume schemes for the (simplest) model for incompressible flow computations, namely the linear stationary Stokes equations.

Let Ω be a bounded, convex polygonal domain in \mathbb{R}^2 . We consider the following boundary value problem for the stationary Stokes equation: We seek a vector function $\mathbf{u}: \Omega \rightarrow \mathbb{R}^2$ (the velocity field) and a scalar function $p: \Omega \rightarrow \mathbb{R}$ (the pressure) such that:

$$(1.1) \quad \begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0, & \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0}, & \text{on } \partial\Omega. \end{aligned}$$

Here $\mathbf{f} \in L^2(\Omega)^2$ is the vector field of the so-called external forces. Results concerning existence, uniqueness and regularity of solutions of (1.1) may be found in [16]. We shall make the assumption that the data and solution of (1.1) are regular enough to guarantee the estimates presented below.

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We consider the finite volume discretization based on the classical Crouzeix-Raviart spaces of the lowest order $\mathbf{V}_h \times Q_h$, [12]: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$(1.2) \quad - \int_{\partial b_e} \nabla \mathbf{u}_h \mathbf{n} + \int_{\partial b_e} p_h \mathbf{n} = \int_{b_e} \mathbf{f}, \quad \forall e \in E_h^{\text{in}},$$

$$(1.3) \quad \int_K \operatorname{div} \mathbf{u}_h = 0, \quad \forall K \in \mathcal{T}_h.$$

Here b_e denote the control volumes, K and e the elements and the edges of the triangulation respectively, see Section 3 for details. For a priori error analysis for the above method we refer to [10], but see also [1, 11, 20]. In this paper we prove residual based a-posteriori estimates for the resulting error in velocity and pressure. The estimators are very similar to the estimators that control the error in the classical finite element discretization of the Stokes equations with the Crouzeix-Raviart spaces, [13, 14, 18]. For a-posteriori error estimates of finite element methods for elliptic problems cf. e.g., [2, 4, 5, 17, 18].

The remaining of the paper is organized as follows: In Section 2 we introduce the necessary notation and in Section 3 we define our finite volume method for (1.1). Its variational formulation (3.9) is the main ingredient in our analysis, cf. Theorem 3.1, [8, 9, 10]. In Sections 4 and 5 we derive the a-posteriori estimates in H^1 and L^2 -norm in Theorems 4.1 and 5.1, respectively. In Section 6 we present the results of numerical experiments that confirm the expected rates of convergence of the estimators.

2. NOTATION AND PRELIMINARIES

For $D \subset \Omega$ and $s \geq 0$, integer, we denote by $H^s(D)$ and $L^2(D) \equiv H^0(D)$ the usual Sobolev and Lebesgue spaces equipped with the norm

$$\|v\|_{s,D} = \left(\sum_{|\alpha| \leq s} \int_D |\partial^\alpha v|^2 \right)^{1/2}$$

and the semi-norm

$$|v|_{s,D} = \left(\sum_{|\alpha|=s} \int_D |\partial^\alpha v|^2 \right)^{1/2},$$

for $v \in H^s(D)$. In particular, for $s = 1$, we shall denote by $H_0^1(\Omega)$ the space of weakly differentiable functions with zero trace. The inner product of $L^2(D)$ will be denoted by $(\cdot, \cdot)_D$. We shall also make use of the space $L_0^2(\Omega)$, the subspace of $L^2(\Omega)$ whose functions have zero integral mean.

It will also prove necessary to introduce notation for inner products and norms on subsets of the boundary. For $\Gamma \subset \partial\Omega$ with positive measure, $L^2(\Gamma)$ will denote the standard Hilbert space of square integrable functions on Γ . The inner product and norm on $L^2(\Gamma)$ will be denoted by $(\cdot, \cdot)_\Gamma$, $\|\cdot\|_{0,\Gamma}$, respectively.

To treat vector valued functions we introduce the spaces $\mathbf{H}^s(D) = H^s(D)^2$, $s \geq 0$, and use the same notation for the corresponding norms and inner products on $\mathbf{H}^s(D)$ and $\mathbf{L}^2(D)$, respectively. When $D = \Omega$ we shall omit the subscript D from the inner product and norm symbols. When $s = 0$, we also suppress the subscript s .

For $\mathbf{v} = (v_1, v_2)^t$ with appropriate regularity we define

$$\nabla \mathbf{v} = \begin{pmatrix} \partial v_1 / \partial x & \partial v_1 / \partial y \\ \partial v_2 / \partial x & \partial v_2 / \partial y \end{pmatrix}, \quad \operatorname{curl} \mathbf{v} = \begin{pmatrix} \partial v_1 / \partial y & -\partial v_1 / \partial x \\ \partial v_2 / \partial y & -\partial v_2 / \partial x \end{pmatrix},$$

and for a piecewise regular vector function \mathbf{v} we define the discrete gradient as the L^2 -matrix $\nabla_h \mathbf{v}|_K = \nabla(\mathbf{v}|_K)$, $K \in \mathcal{T}_h$. If A and B are two matrices we define the inner product

$$A : B = \sum_{i,j=1}^2 A_{ij} B_{ij}$$

and the divergence of A as the vector

$$\operatorname{div} A = \begin{pmatrix} \partial A_{11} / \partial x + \partial A_{12} / \partial y \\ \partial A_{21} / \partial x + \partial A_{22} / \partial y \end{pmatrix}.$$

For future reference we note (but see also [7, Chapter 9]) that

$$(2.1) \quad \int_D \operatorname{div} A \cdot \mathbf{w} = - \int_D A : \nabla \mathbf{w} + \int_{\partial D} \mathbf{A} \mathbf{n} \cdot \mathbf{w},$$

for any A with the appropriate regularity, and

$$(2.2) \quad \int_D \nabla \mathbf{v} : \operatorname{curl} \mathbf{w} = \int_{\partial D} \nabla \mathbf{v} \mathbf{t} \cdot \mathbf{w},$$

for any $\mathbf{v}, \mathbf{w} \in \mathbf{H}^1(D)$. Here, \mathbf{n} is the outward normal to ∂D and \mathbf{t} is the tangent vector to ∂D .

Finally, for $k \geq 0$, integer, we denote by \mathbb{P}_k the space of polynomials in two variables of degree at most k .

We consider a family of *shape-regular* triangulations $\{\mathcal{T}_h\}_{0 < h < 1}$ of Ω , i.e., any two triangles in \mathcal{T}_h share at most a vertex or an edge, where h is the maximum diameter of the triangles of \mathcal{T}_h . With $E_h(K)$ we denote the set of the edges of $K \in \mathcal{T}_h$. Also, let E_h^{in} be the edges of \mathcal{T}_h that are not part of $\partial\Omega$, and define $E_h^{\text{in}}(K)$ in a similar way. In addition, h_K denotes the diameter of the triangle K , $|K|$ its area, and $|e|$ the length of an edge $e \in E_h(K)$.

Next, let V_h be the Crouzeix–Raviart nonconforming finite element space, cf. [12], associated with \mathcal{T}_h and \mathbf{V}_h its vector counterpart. Based on the variational formulation

$$(2.3) \quad \begin{aligned} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \operatorname{div} \mathbf{v} &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ \int_{\Omega} q \operatorname{div} \mathbf{u} &= 0, \quad \forall q \in L_0^2(\Omega), \end{aligned}$$

of (1.1) and with

$$Q_h = \{\psi \in L_0^2 : \psi|_K \in \mathbb{P}_0, \forall K \in \mathcal{T}_h\},$$

the finite element approximation $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ of problem (1.1) is defined by

$$(2.4) \quad \begin{aligned} a(\mathbf{u}_h, \boldsymbol{\chi}) + b(p_h, \boldsymbol{\chi}) &= (\mathbf{f}, \boldsymbol{\chi}), \quad \forall \boldsymbol{\chi} \in \mathbf{V}_h, \\ b(\psi, \mathbf{u}) &= 0, \quad \forall \psi \in Q_h, \end{aligned}$$

where

$$a(\mathbf{v}, \mathbf{w}) = \sum_K \int_K \nabla \mathbf{v} : \nabla \mathbf{w}, \quad b(q, \mathbf{v}) = - \sum_K \int_K q \operatorname{div} \mathbf{v},$$

for $\mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$ and $q \in L^2(\Omega)$. It is well known, cf., e.g., [12, §6], [16, Proposition 4.13] that the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ satisfy the following coercivity and inf-sup conditions: there exist positive constants α and β , independent of h , such that

$$(2.5a) \quad \alpha |\chi|_{1,h}^2 \leq a(\chi, \chi), \quad \forall \chi \in \mathbf{V}_h,$$

$$(2.5b) \quad \|\psi\| \leq \beta \sup_{\substack{\mathbf{v}_h \in \mathbf{V}_h, \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{b(\psi, \mathbf{v}_h)}{|\mathbf{v}_h|_{1,h}}, \quad \forall \psi \in Q_h,$$

where $|\mathbf{v}_h|_{1,h}^2 = \sum_K |\mathbf{v}_h|_{1,K}^2$. The analysis of a-priori and a-posteriori error estimation of finite element approximations of (\mathbf{u}, p) has been extensively analyzed, cf., e.g., [12, 13, 14, 15]. Our approach will be based on this well-established analysis and the fact that we view the finite volume method as a variational crime of the corresponding finite element scheme.

In the analysis below, we shall also make use of the standard conforming finite element space $X_h = \{\chi \in H_0^1(\Omega) : \chi|_K \in \mathbb{P}_1, \forall K \in \mathcal{T}_h\}$, and denote by \mathbf{X}_h its vector counterpart. In addition, we consider a suitable interpolation operator $\mathbf{I}_h : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{X}_h$, satisfying, cf. e.g., [19], [7, (4.8.5)],

$$(2.6a) \quad \|\mathbf{v} - \mathbf{I}_h \mathbf{v}\|_{0,K} \leq C |K|^{1/2} |\mathbf{v}|_{1,\tilde{K}},$$

$$(2.6b) \quad |\mathbf{I}_h \mathbf{v}|_{1,K} \leq |\mathbf{v}|_{1,K} + |\mathbf{v} - \mathbf{I}_h \mathbf{v}|_{1,K} \leq C |\mathbf{v}|_{1,\tilde{K}},$$

$$(2.6c) \quad \|\mathbf{v} - \mathbf{I}_h \mathbf{v}\|_{0,e} \leq C |e|^{1/2} |\mathbf{v}|_{1,\tilde{K}}, \quad e \in E_h^{\text{in}},$$

where \tilde{K} is the union of all elements K' such that $\bar{K} \cap \bar{K}' \neq \emptyset$, with $K \in \mathcal{T}_h$. The last bound follows by the first two and a trace inequality:

$$(2.7) \quad \|\mathbf{v} - \mathbf{I}_h \mathbf{v}\|_{0,e}^2 \leq C \|\mathbf{v} - \mathbf{I}_h \mathbf{v}\|_{0,K} \|\mathbf{v} - \mathbf{I}_h \mathbf{v}\|_{1,K},$$

cf. e.g., [7, (1.6.6)], by applying a standard homogeneity argument. Note that

$$\begin{aligned} \|\mathbf{v} - \mathbf{I}_h \mathbf{v}\|_{1,K} &\leq \|\mathbf{v} - \mathbf{I}_h \mathbf{v}\|_{0,K} + |\mathbf{v} - \mathbf{I}_h \mathbf{v}|_{1,K} \\ &\leq C |K|^{1/2} |\mathbf{v}|_{1,\tilde{K}} + C |\mathbf{v}|_{1,\tilde{K}} \leq C |\mathbf{v}|_{1,\tilde{K}}. \end{aligned}$$

3. THE FINITE VOLUME SCHEME

Finite volume methods rely on a local conservation property of the differential equation. Thus, integrating (1.1) over a region $b \subset \Omega$ and using Green's formula, we obtain

$$(3.1) \quad - \int_{\partial b} \nabla \mathbf{u} \mathbf{n} + \int_{\partial b} p \mathbf{n} = \int_b \mathbf{f}.$$

In addition (1.1) gives $\int_K \text{div} \mathbf{u} = 0$, for any triangle $K \in \mathcal{T}_h$. Having these in mind we shall construct a finite volume scheme for the approximation of (1.1). The finite volume approximations will be in $\mathbf{V}_h \times Q_h$ and satisfy a relation similar to (3.1) over a finite collection of subregions of Ω , usually referred to as *control volumes*.

We construct these control volumes in the following way. Let z_K be an inner point of $K \in \mathcal{T}_h$. We connect z_K with line segments to the vertices of K , thus partitioning K into three subtriangles K_e , $e \in E_h(K)$. Then with each side $e \in E_h$ we associate a quadrilateral b_e , which consists of the union of the subregions K_e , (see Figure 1). Our finite volume method for (1.1) is then: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$

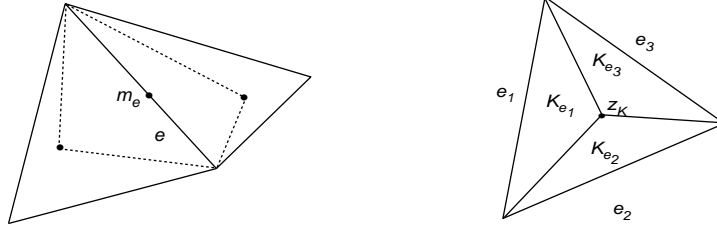


FIGURE 1. *Left:* Two triangles with common side e . With dotted lines the corresponding box b_e . *Right:* A triangle K partitioned into three subtriangles K_e .

satisfying

$$(3.2) \quad - \int_{\partial b_e} \nabla \mathbf{u}_h \mathbf{n} + \int_{\partial b_e} p_h \mathbf{n} = \int_{b_e} \mathbf{f}, \quad \forall e \in E_h^{\text{in}},$$

$$(3.3) \quad \int_K \operatorname{div} \mathbf{u}_h = 0, \quad \forall K \in \mathcal{T}_h.$$

Let $\bar{\mathbf{V}}_h = \{\chi \in \mathbf{L}^2(\Omega) : \chi|_{b_e} \in \mathbb{P}_0, \text{ if } e \in E_h^{\text{in}}; \chi|_{b_e} = 0, \text{ if } e \subset \partial\Omega\}$. The functions $\{\phi_e\}_{e \in E_h^{\text{in}}}$ defined by

$$\phi_e = \begin{cases} 1, & \text{on } b_e, \\ 0, & \text{elsewhere,} \end{cases}$$

form a basis of $\bar{\mathbf{V}}_h$. Define now the operator $\Lambda_h: C(\Omega)^2 + \mathbf{V}_h \rightarrow \bar{\mathbf{V}}_h$ by

$$(3.4) \quad \Lambda_h \mathbf{v} = \sum_{e \in E_h^{\text{in}}} \mathbf{v}(m_e) \phi_e,$$

where m_e is the midpoint of the edge e . If $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\chi \in \mathbf{V}_h$ it is easily seen that

$$(3.5) \quad (\mathbf{f}, \Lambda_h \chi) = \sum_{e \in E_h^{\text{in}}} \chi(m_e) \cdot \int_{b_e} \mathbf{f} \phi_e = \sum_{e \in E_h^{\text{in}}} \chi(m_e) \cdot \int_{b_e} \mathbf{f}.$$

In addition, since χ is a linear polynomial on K , there holds

$$(3.6) \quad \|\chi - \Lambda_h \chi\|_{0,K}^2 = \sum_{e \in E_h(K)} \|\chi - \Lambda_h \chi\|_{0,K_e}^2 \leq Ch_K^2 |\chi|_{1,K}^2, \quad \forall \chi \in \mathbf{V}_h,$$

$$(3.7) \quad \int_e \Lambda_h \chi = \int_e \chi, \quad \forall \chi \in \mathbf{V}_h, \quad \forall e \in E_h.$$

Indeed, with $\bar{\chi}$ denoting the average of χ on K_e , we have

$$(3.8) \quad \begin{aligned} \|\chi - \Lambda_h \chi\|_{0,K_e} &= \left(\int_{K_e} |\chi(x) - \chi(m_e)|^2 dx \right)^{1/2} \\ &= \left(\int_{K_e} |(\chi - \bar{\chi})(x) - (\chi - \bar{\chi})(m_e)|^2 dx \right)^{1/2} \\ &\leq Ch_K \|\chi - \bar{\chi}\|_{L^\infty(K_e)} \\ &\leq Ch_K^2 |\chi|_{W_1^\infty(K_e)} = Ch_K |\chi|_{1,K_e}. \end{aligned}$$

In (3.8) we have used a scaled version of Friedrichs' inequality, [7, (4.3.14)], and the fact that $\nabla \chi$ is constant on K_e . Alternatively the same bound follows by a simple integration of the formula

$$\varphi(x) - \varphi(m_e) = (x - m_e) \cdot \nabla \varphi$$

which holds for any linear function φ in K_e .

In the following theorem we establish the existence and uniqueness of the finite volume approximation:

Theorem 3.1. *There exists a unique solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ of the finite volume method (3.2)–(3.3) which satisfies*

$$(3.9) \quad \begin{aligned} a(\mathbf{u}_h, \chi) + b(p_h, \chi) &= (\mathbf{f}, \Lambda_h \chi), \quad \forall \chi \in \mathbf{V}_h, \\ b(\psi, \mathbf{u}_h) &= 0, \quad \forall \psi \in Q_h. \end{aligned}$$

Proof. We multiply (3.2) with $\chi(m_e)$ and then sum over all $e \in E_h^{\text{in}}$ so that

$$(3.10) \quad a_h(\mathbf{u}_h, \chi) + b_h(p_h, \chi) = (\mathbf{f}, \Lambda_h \chi), \quad \forall \chi \in \mathbf{V}_h,$$

where $a_h : (\mathbf{H}^1(\Omega) + \mathbf{V}_h) \times (\mathbf{H}^1(\Omega) + \mathbf{V}_h) \rightarrow \mathbb{R}$ and $b_h : L^2(\Omega) \times (\mathbf{H}^1(\Omega) + \mathbf{V}_h) \rightarrow \mathbb{R}$ are defined by

$$\begin{aligned} a_h(\mathbf{v}, \mathbf{w}) &= - \sum_{e \in E_h^{\text{in}}} \mathbf{w}(m_e) \cdot \int_{\partial b_e} \nabla \mathbf{v} \mathbf{n}, \\ b_h(q, \mathbf{v}) &= \sum_{e \in E_h^{\text{in}}} \mathbf{v}(m_e) \cdot \int_{\partial b_e} q \mathbf{n}. \end{aligned}$$

If we multiply now (3.3) with $\psi|_K$ and sum over all $K \in T_h$, we obtain

$$(3.11) \quad b(\psi, \mathbf{u}_h) = 0, \quad \forall \psi \in Q_h.$$

Also, in view of [8] and [10, Lemma 2.4] we have

$$(3.12) \quad \begin{aligned} a(\chi, \psi) &= a_h(\chi, \psi), \quad \forall \chi, \psi \in \mathbf{V}_h, \\ b(\psi, \chi) &= b_h(\psi, \chi), \quad \forall \chi \in \mathbf{V}_h, \psi \in Q_h. \end{aligned}$$

By combining (3.10)–(3.12) we have that (\mathbf{u}_h, p_h) satisfies (3.9). Moreover, the existence and uniqueness of the solution is now clear, in view of the coercivity and inf–sup conditions (2.5a), (2.5b). \square

Remark 3.1. Note that $\text{div } \mathbf{u}_h$ is piecewise constant and therefore by (3.3) \mathbf{u}_h is divergence free. The same holds for the corresponding finite element approximation but we have to observe first that $\text{div } \mathbf{u}_h \in L_0^2(\Omega)$. This is a consequence of the homogeneous boundary conditions and of the fact that $\mathbf{u}_h \in \mathbf{V}_h$.

4. H^1 -NORM ERROR ESTIMATORS

Given an interior edge e we chose an arbitrary normal direction $\mathbf{n}_e = (n_1, n_2)^t$ and denote by K^+ and K^- the two triangles sharing this edge, with K^+ the triangle for which \mathbf{n}_e is the outward normal. When e is a boundary edge we choose \mathbf{n}_e as the outward normal vector. We define

$$\llbracket (\nabla \mathbf{u}_h - p_h \mathbf{I}) \mathbf{n}_e \rrbracket_e = (\nabla \mathbf{u}_h|_{K^-} - p_h|_{K^-} \mathbf{I}) \mathbf{n}_e - (\nabla \mathbf{u}_h|_{K^+} - p_h|_{K^+} \mathbf{I}) \mathbf{n}_e,$$

and

$$\llbracket \nabla \mathbf{u}_h \mathbf{t}_e \rrbracket_e = (\nabla \mathbf{u}_h|_{K^-}) \mathbf{t}_e - (\nabla \mathbf{u}_h|_{K^+}) \mathbf{t}_e,$$

where $\mathbf{t}_e = (-n_2, n_1)^t$ is the tangent vector and \mathbf{I} is the identity matrix. For all edges $e \in E_h$ we let

$$\mathbf{J}_{e,\mathbf{n}} = \begin{cases} \llbracket (\nabla \mathbf{u}_h - p_h \mathbf{I}) \mathbf{n}_e \rrbracket_e, & \text{if } e \in E_h^{\text{in}}, \\ 0, & \text{otherwise,} \end{cases} \quad \mathbf{J}_{e,\mathbf{t}} = \begin{cases} \llbracket \nabla \mathbf{u}_h \mathbf{t}_e \rrbracket_e, & \text{if } e \in E_h^{\text{in}}, \\ 2 \nabla \mathbf{u}_h \mathbf{t}_e, & \text{otherwise,} \end{cases}$$

and introduce the local error estimators η_K , $K \in \mathcal{T}_h$,

$$(4.1) \quad \eta_K^2 = |K| \|\mathbf{f}\|_{0,K}^2 + \frac{1}{2} \sum_{e \in E_h(K)} |e|^2 (|\mathbf{J}_{e,\mathbf{n}}|^2 + |\mathbf{J}_{e,\mathbf{t}}|^2),$$

and the global H^1 -norm error estimator

$$(4.2) \quad \eta = \left(\sum_K \eta_K^2 \right)^{1/2}.$$

Let $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ and $\varepsilon = p - p_h$ denote the errors in the velocity and pressure, respectively. Subtracting equations (2.3) and (3.9) we obtain

$$(4.3) \quad a(\mathbf{e}, \boldsymbol{\chi}) + b(\varepsilon, \boldsymbol{\chi}) = (\mathbf{f}, \boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\chi}), \quad \forall \boldsymbol{\chi} \in \mathbf{X}_h \equiv \mathbf{H}_0^1(\Omega) \cap \mathbf{V}_h.$$

In the following lemma we estimate $\|\varepsilon\|$ in terms of the global error estimator (4.2) and $\|\nabla_h \mathbf{e}\|$ (see also [13, Lemma 3.1]).

Lemma 4.1. *The following estimate holds:*

$$(4.4) \quad \|\varepsilon\| \leq C(\eta + \|\nabla_h \mathbf{e}\|).$$

Proof. From the inf-sup condition (2.5b) and the fact that $\varepsilon \in L_0^2(\Omega)$, there exists $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ such that

$$(4.5) \quad \|\varepsilon\| \leq C \frac{\int_{\Omega} \varepsilon \operatorname{div} \mathbf{v}}{|\mathbf{v}|_1},$$

where the constant C is independent of ε and \mathbf{v} , cf., e.g., [15, Section 5.1]. Using the error equation (4.3) we obtain

$$\begin{aligned} \int_{\Omega} \varepsilon \operatorname{div} \mathbf{v} &= \int_{\Omega} \varepsilon \operatorname{div}(\mathbf{v} - \boldsymbol{\chi}) + \int_{\Omega} \varepsilon \operatorname{div} \boldsymbol{\chi} \\ &= \int_{\Omega} \varepsilon \operatorname{div}(\mathbf{v} - \boldsymbol{\chi}) - a(\mathbf{e}, \mathbf{v} - \boldsymbol{\chi}) + a(\mathbf{e}, \mathbf{v}) + (\mathbf{f}, \boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\chi}), \end{aligned}$$

for any $\boldsymbol{\chi} \in \mathbf{X}_h$. Integrating by parts on each element K the relation above yields

$$\begin{aligned} \int_{\Omega} \varepsilon \operatorname{div} \mathbf{v} &= \sum_K \left\{ \int_K (-\nabla p + \Delta \mathbf{u}) \cdot (\mathbf{v} - \boldsymbol{\chi}) + \int_{\partial K} (\nabla_h \mathbf{u}_h - p_h \mathbf{I}) \mathbf{n} \cdot (\mathbf{v} - \boldsymbol{\chi}) \right\} \\ &\quad + a(\mathbf{e}, \mathbf{v}) + (\mathbf{f}, \boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\chi}) \\ &= \sum_K \left\{ - \int_K \mathbf{f} \cdot (\mathbf{v} - \boldsymbol{\chi}) + \int_K \mathbf{f} \cdot (\boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\chi}) \right. \\ &\quad \left. - \frac{1}{2} \sum_{e \in E_h(K)} \int_e \mathbf{J}_{e,\mathbf{n}} \cdot (\mathbf{v} - \boldsymbol{\chi}) \right\} + a(\mathbf{e}, \mathbf{v}). \end{aligned}$$

Choosing now $\boldsymbol{\chi} = \mathbf{I}_h \mathbf{v}$ and using the Cauchy–Schwarz inequality, (3.6) and (2.6) we have

$$\begin{aligned} \int_{\Omega} \varepsilon \operatorname{div} \mathbf{v} &= \sum_K \left\{ |K|^{1/2} \|\mathbf{f}\|_{0,K} |\mathbf{v}|_{1,\tilde{K}} + \frac{1}{2} \sum_{e \in E_h(K)} |e|^{1/2} |\mathbf{J}_{e,\mathbf{n}}| |\mathbf{v}|_{1,\tilde{K}} \right\} + \|\nabla_h \mathbf{e}\| |\mathbf{v}|_1 \\ &\leq C \left\{ \eta + \|\nabla_h \mathbf{e}\| \right\} |\mathbf{v}|_1, \end{aligned}$$

since,

$$\|\mathbf{I}_h \mathbf{v} - \boldsymbol{\Lambda}_h(\mathbf{I}_h \mathbf{v})\|_{0,K} \leq C |K|^{1/2} \|\mathbf{I}_h \mathbf{v}\|_{1,K} \leq C |K|^{1/2} |\mathbf{v}|_{1,\tilde{K}}.$$

Thus, using (4.5) we conclude the proof. \square

In the next lemma we show that $\|\nabla_h \mathbf{e}\|$ is dominated by the global error estimator (4.2) (see also [13, Lemma 3.2]).

Lemma 4.2. *The following estimate holds:*

$$\|\nabla_h \mathbf{e}\| \leq C \eta.$$

Proof. Following [13] we decompose the velocity error $\nabla_h \mathbf{e}$ as $\nabla_h \mathbf{e} = \nabla \mathbf{r} - q \mathbf{I} + \operatorname{curl} \mathbf{s}$, where $q \in L_0^2(\Omega)$, $\mathbf{r} \in \mathbf{H}_0^1(\Omega)$ with $\operatorname{div} \mathbf{r} = 0$ and $\mathbf{s} \in \mathbf{H}^1(\Omega)$ satisfying

$$(4.6) \quad |\mathbf{r}|_1 + |\mathbf{s}|_1 \leq C \|\nabla_h \mathbf{e}\|.$$

Using the error equation (4.3), the fact that $\operatorname{div} \mathbf{r} = 0$ and the orthogonality relation (cf. [7, Chapter 9] and (2.2))

$$\int_{\Omega} \nabla_h \mathbf{e} : \operatorname{curl} \boldsymbol{\psi} = 0, \quad \forall \boldsymbol{\psi} \in \mathbf{X}_h,$$

we have

$$\begin{aligned} \|\nabla_h \mathbf{e}\|^2 &= \int_{\Omega} (\nabla_h \mathbf{e} - \varepsilon \mathbf{I}) : \nabla(\mathbf{r} - \boldsymbol{\chi}) + \int_{\Omega} \mathbf{f} \cdot (\boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\chi}) \\ &\quad + \int_{\Omega} \nabla_h \mathbf{e} : \operatorname{curl}(\mathbf{s} - \boldsymbol{\psi}), \quad \forall \boldsymbol{\chi}, \boldsymbol{\psi} \in \mathbf{X}_h. \end{aligned}$$

Integrating then by parts in each element we have

$$\begin{aligned} \|\nabla_h \mathbf{e}\|^2 &= \sum_K \int_K \mathbf{f} \cdot (\mathbf{r} - \boldsymbol{\chi}) + \int_K \mathbf{f} \cdot (\boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\chi}) \\ &\quad - \int_{\partial K} (\nabla_h \mathbf{u}_h - p_h \mathbf{I}) \cdot (\mathbf{r} - \boldsymbol{\chi}) - \int_{\partial K} \nabla_h \mathbf{u}_h \mathbf{t} \cdot (\mathbf{s} - \boldsymbol{\psi}), \end{aligned}$$

for any $\boldsymbol{\chi}, \boldsymbol{\psi} \in \mathbf{X}_h$. Finally, choosing $\boldsymbol{\chi} = \mathbf{I}_h \mathbf{r}$, $\boldsymbol{\psi} = \mathbf{I}_h \mathbf{s}$, applying the Cauchy–Schwarz inequality and using (2.6), (3.6) and (4.6) we obtain the desired estimate. \square

Using the two lemmas above and arguments entirely analogous to those in [13, Theorem 3.2] we can also derive the lower bound. Thus we have

Theorem 4.1. *There exist two positive constants C_1 and C_2 such that*

$$(4.7) \quad \|\varepsilon\| + \|\nabla_h \mathbf{e}\| \leq C_1 \eta,$$

$$(4.8) \quad C_2 \eta \leq \|\varepsilon\| + \|\nabla_h \mathbf{e}\| + \left(\sum_K h_K^2 \|\mathbf{f} - \bar{\mathbf{f}}_K\|_{0,K} \right)^{1/2},$$

where $\bar{\mathbf{f}}_K$ is the mean value of \mathbf{f} on K .

5. L^2 -NORM ERROR ESTIMATORS

In this section we introduce and analyze an a-posteriori estimator for the L^2 -norm error of the velocity. The estimator is similar to the one derived in [14] for the corresponding finite element method and the derivation follows that of the aforementioned paper.

We introduce the local error estimator $\tilde{\eta}_K$ defined by

$$\begin{aligned} \tilde{\eta}_K^2 &= |K|^2 \|\mathbf{f}\|_{0,K}^2 + |K| \|\mathbf{f} - \bar{\mathbf{f}}_K\|_{0,K}^2 + |K| \|\operatorname{div} \mathbf{u}_h\|_{0,K}^2 \\ &\quad + \frac{1}{2} \sum_{e \in E_h^{\text{in}}(K)} (|e|^3 |\mathbf{J}_{e,\mathbf{n}}|^2 + |e| \|[\![\mathbf{u}_h]\!]_e\|_{0,e}^2), \end{aligned}$$

where $[\![\mathbf{u}_h]\!]_e$ denotes the jump of \mathbf{u}_h across the edge e , and the global one

$$(5.1) \quad \tilde{\eta} = \left(\sum_K \tilde{\eta}_K^2 \right)^{1/2}.$$

We consider the scalar version of the interpolation operator $I_h: H^1(\Omega) \rightarrow X_h$ of nonsmooth functions of [19] satisfying, cf. (2.6),

$$(5.2) \quad \|q - I_h q\|_{0,K} \leq C |K|^{1/2} |q|_{1,\tilde{K}}, \quad \forall q \in H^1(\Omega).$$

In addition we denote by $\tilde{\mathbf{I}}_h: \mathbf{H}^2(\Omega) \rightarrow \mathbf{X}_h$ the standard nodal interpolation operator, [7]. We shall use the known approximation properties

$$(5.3) \quad \begin{aligned} \|\mathbf{v} - \tilde{\mathbf{I}}_h \mathbf{v}\|_{0,K} &\leq C |K| |\mathbf{v}|_{2,K}, \quad \forall \mathbf{v} \in \mathbf{H}^2(\Omega), \\ \|\mathbf{v} - \tilde{\mathbf{I}}_h \mathbf{v}\|_{0,e} &\leq C |e|^{3/2} |\mathbf{v}|_{2,K}, \quad \forall e \in E_h^{\text{in}}, \quad \forall \mathbf{v} \in \mathbf{H}^2(\Omega), \end{aligned}$$

and the stability bound

$$(5.4) \quad |\tilde{\mathbf{I}}_h \mathbf{v}|_{1,K} \leq C \|\mathbf{v}\|_{2,K}, \quad \forall \mathbf{v} \in \mathbf{H}^2(\Omega),$$

where \tilde{K} is the union of all elements K' such that $\bar{K} \cap \bar{K}' \neq \emptyset$, with $K \in \mathcal{T}_h$. For the last estimate note that

$$\begin{aligned} |\tilde{\mathbf{I}}_h \mathbf{v}|_{1,K} &\leq |\mathbf{v} - \tilde{\mathbf{I}}_h \mathbf{v}|_{1,K} + |\mathbf{v}|_{1,K} \\ &\leq Ch_K |\mathbf{v}|_{2,K} + |\mathbf{v}|_{1,K} \leq C \|\mathbf{v}\|_{2,K}. \end{aligned}$$

In addition, we shall assume that in the construction of the control volumes b_e , the point z_K is chosen as the barycenter of K . Hence,

$$(5.5) \quad \int_K (\chi - \mathbf{\Lambda}_h \chi) = 0, \quad \forall \chi \in \mathbf{V}_h.$$

Theorem 5.1. *The following estimate holds:*

$$\|\mathbf{u} - \mathbf{u}_h\| \leq C \tilde{\eta}.$$

Proof. We consider the following auxiliary problem: Find $(\mathbf{z}, s) \in (\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)) \times H^1(\Omega) \cap L_0^2(\Omega)$ such that

$$(5.6) \quad \begin{aligned} -\Delta \mathbf{z} - \nabla s &= \mathbf{u} - \mathbf{u}_h, & \text{in } \Omega, \\ \operatorname{div} \mathbf{z} &= 0, & \text{in } \Omega, \\ \mathbf{z} &= \mathbf{0}, & \text{on } \partial\Omega. \end{aligned}$$

For the solution pair (\mathbf{z}, s) we have the following stability estimate (cf. [12])

$$(5.7) \quad \|\mathbf{z}\|_2 + \|s\|_1 \leq C \|\mathbf{u} - \mathbf{u}_h\|.$$

Multiplying the first equation in (5.6) by $\mathbf{u} - \mathbf{u}_h$ and using integration by parts and the second equation in (5.6), we obtain

$$(5.8) \quad \begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|^2 = & \sum_K \left\{ \int_K (\nabla \mathbf{z} : \nabla(\mathbf{u} - \mathbf{u}_h) + s \operatorname{div}(\mathbf{u} - \mathbf{u}_h) - (p - p_h) \operatorname{div} \mathbf{z}) \right. \\ & \left. + \int_{\partial K} \nabla \mathbf{z} \mathbf{n} \cdot \mathbf{u}_h + \int_{\partial K} s \mathbf{n} \cdot \mathbf{u}_h \right\} \end{aligned}$$

Also, in view of (2.3), (3.9) and (4.3), we have

$$a(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\chi}) + b(p - p_h, \boldsymbol{\chi}) + b(\psi, \mathbf{u} - \mathbf{u}_h) = (\mathbf{f}, \boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\chi}), \quad \forall \boldsymbol{\chi} \in \mathbf{X}_h, \forall \psi \in Q_h.$$

Combining now (5.8) with the relation above we have

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|^2 = & \sum_K \left\{ \int_K (-\Delta(\mathbf{u} - \mathbf{u}_h) + \nabla(p - p_h)) \cdot (\mathbf{z} - \boldsymbol{\chi}) \right. \\ & + \int_K ((s - \psi) \operatorname{div}(\mathbf{u} - \mathbf{u}_h) + \mathbf{f} \cdot (\boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\chi})) \\ & + \int_{\partial K} (\nabla(\mathbf{u} - \mathbf{u}_h) \mathbf{n} - (p - p_h) \mathbf{n}) \cdot (\mathbf{z} - \boldsymbol{\chi}) \\ & \left. + \int_{\partial K} (\nabla \mathbf{z} \mathbf{n} \cdot \mathbf{u}_h + s \mathbf{n} \cdot \mathbf{u}_h) \right\}, \end{aligned}$$

and upon using (5.5) we obtain

$$(5.9) \quad \begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|^2 = & \sum_K \left\{ \int_K (\mathbf{f} \cdot (\mathbf{z} - \boldsymbol{\chi}) - (s - \psi) \operatorname{div} \mathbf{u}_h + (\mathbf{f} - \bar{\mathbf{f}}_K) \cdot (\boldsymbol{\chi} - \boldsymbol{\Lambda}_h \boldsymbol{\chi})) \right. \\ & \left. + \int_{\partial K} (\nabla \mathbf{u}_h \mathbf{n} - p_h \mathbf{n}) \cdot (\mathbf{z} - \boldsymbol{\chi}) + \int_{\partial K} (\nabla \mathbf{z} \mathbf{n} + s \mathbf{n}) \cdot \mathbf{u}_h \right\}, \end{aligned}$$

for any $\boldsymbol{\chi} \in \mathbf{X}_h$ and $\psi \in Q_h$. In view of (5.7), the last term of (5.9) may be estimated by

$$(5.10) \quad \sum_K \sum_{e \in E_h(K)} \int_e (\nabla \mathbf{z} \mathbf{n} + s \mathbf{n}) \cdot \mathbf{u}_h \leq C \left(\sum_{e \in E_h^{in}} |e| \|\llbracket \mathbf{u}_h \rrbracket_e\|_{0,e}^2 \right)^{1/2} \|\mathbf{u} - \mathbf{u}_h\|.$$

Then, choosing $\boldsymbol{\chi} = \tilde{\mathbf{I}}_h \mathbf{z}$ and $\psi = I_h s$ in (5.9) and using (5.2), (5.3), (3.6), (5.4) and (5.10), we finally obtain

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\| \leq & C \left(\sum_K |K|^2 \|\mathbf{f}\|_{0,K}^2 + |K| \|\mathbf{f} - \bar{\mathbf{f}}_K\|_{0,K}^2 + |K| \|\operatorname{div} \mathbf{u}_h\|_{0,K}^2 \right. \\ & \left. + \frac{1}{2} \sum_{e \in E_h^{in}} |e|^3 \|\llbracket \nabla \mathbf{u}_h \mathbf{n}_e - p_h \mathbf{I} \mathbf{n}_e \rrbracket_e\|_{0,e}^2 + |e| \|\llbracket \mathbf{u}_h \rrbracket_e\|_{0,e}^2 \right)^{1/2}. \end{aligned}$$

In particular we have used the following estimate, cf. (3.6), (5.4),

$$\begin{aligned} \int_K (\mathbf{f} - \bar{\mathbf{f}}_K) \cdot (\tilde{\mathbf{I}}_h \mathbf{z} - \boldsymbol{\Lambda}_h \tilde{\mathbf{I}}_h \mathbf{z}) & \leq \|\mathbf{f} - \bar{\mathbf{f}}_K\|_{0,K} \|\tilde{\mathbf{I}}_h \mathbf{z} - \boldsymbol{\Lambda}_h \tilde{\mathbf{I}}_h \mathbf{z}\|_{0,K} \\ & \leq C |K|^{1/2} \|\mathbf{f} - \bar{\mathbf{f}}_K\|_{0,K} |\tilde{\mathbf{I}}_h \mathbf{z}|_{1,K} \\ & \leq C |K|^{1/2} \|\mathbf{f} - \bar{\mathbf{f}}_K\|_{0,K} \|\mathbf{z}\|_{2,K}. \end{aligned}$$

The proof is thus complete. \square

Remark 5.1. The two terms in the estimator containing \mathbf{f} are of the same order $O(h^2)$ if $\mathbf{f} \in \mathbf{H}^1$. This is compatible with the a priori L^2 estimate of finite volume methods for scalar elliptic problems

$$\|u - u_h\| \leq Ch^2(\|u\|_2 + \|f\|_1)$$

proved in [8, 9], in the case where the point z_K is chosen as the barycenter of K . The proof of this a priori bound can be extended to cover the finite volume schemes for the Stokes problem considered in this paper.

Remark 5.2. Note that the term $\|[\mathbf{u}_h]_e\|_{0,e}$ in the estimator $\tilde{\eta}$ may be replaced by $|e|^2 \|\mathbf{J}_{e,\mathbf{t}}\|_{0,e}$, since (cf. [14])

$$\|[\mathbf{u}_h]_e\|_{0,e} = \frac{|e|^2}{12} \|\mathbf{J}_{e,\mathbf{t}}\|_{0,e}.$$

6. NUMERICAL EXPERIMENTS

In this section we present some numerical experiments which confirm experimentally that our error estimators are of the correct order and thus may be used as the basis of an adaptive mesh refinement process.

Our test problem is (1.1) on the unit square, with the external forces \mathbf{f} chosen so that the exact solution (\mathbf{u}, p) is

$$\begin{aligned} u_1(x, y) &= (x^4 - 2x^3 + x^2)(4y^3 - 6y^2 + 2y), \\ u_2(x, y) &= -(4x^3 - 6x^2 + 2x)(y^4 - 2y^3 + y^2), \\ p(x, y) &= x^5 + y^5. \end{aligned}$$

It is easy to check that \mathbf{u} as defined above satisfies the incompressibility condition and vanishes on the boundary of the unit square.

Let us first describe our algorithm for the numerical solution of (3.9). Given a regular triangulation $\mathcal{T}_h = \{K_i : i = 1, \dots, N_T\}$ of $\Omega = [0, 1]^2$, we let $\{\psi_j\}_{j=1}^{N_T}$ denote a basis of P_h , the space of $L^2(\Omega)$ -functions which reduce to constants on each element of the triangulation \mathcal{T}_h . We enumerate the interior edges of \mathcal{T}_h as e_j , $j = 1, \dots, N_E$ and denote by $\{\phi_j\}_{j=1}^{N_E}$ a basis of the (scalar) Crouzeix–Raviart finite element space. Then the functions $\phi_j^{(1)} = (\phi_j, 0)^t$, $\phi_j^{(2)} = (0, \phi_j)^t$, $j = 1, \dots, N_E$, form a basis of \mathbf{V}_h . Determining the solution of (3.9) leads to the indefinite system

$$(6.1) \quad \begin{aligned} SU + B^t P &= F \\ BU &= 0, \end{aligned}$$

where $S \in \mathbb{R}^{2N_E \times 2N_E}$, $B \in \mathbb{R}^{N_T \times 2N_E}$ with $\dim \ker B^t = 1$, and $U, F \in \mathbb{R}^{2N_E}$, $P \in \mathbb{R}^{N_T}$. Here,

$$S = \begin{bmatrix} \tilde{S} & 0 \\ 0 & \tilde{S} \end{bmatrix}, \quad B = [B^{(1)} \quad B^{(2)}],$$

where $\tilde{S}_{ij} = a(\phi_i^{(1)}, \phi_j^{(1)}) = a(\phi_i^{(2)}, \phi_j^{(2)})$, $i, j = 1, \dots, N_E$ and $B_{ij}^{(\ell)} = b(\psi_i, \phi_j^{(\ell)})$, $i = 1, \dots, N_T$, $j = 1, \dots, N_E$, $\ell = 1, 2$, are the *stiffness* and *pressure* matrices, respectively.

We solved (6.1) using Uzawa's method (cf., e.g., [16], [6]). In our numerical experiments we used the variation of Uzawa's algorithm equivalent to applying the conjugate gradient method to the so-called *reduced equation* $BS^{-1}B^tP = BS^{-1}F$.

To compensate for the fact that the matrix B is not of full rank, we specified the pressure at one element and modified the right-hand side accordingly.

To validate the method and the code, we solved our test problem numerically and measured the error in the velocity and the pressure. The results are shown in Table 1 and Table 2, along with the experimental rates of convergence, which clearly agree with the theoretical ones derived in [10] and confirm that the error estimators (4.2) and (5.1) are of the correct order. The initial triangulation of the unit square shown in Figure 2 was successively refined to produce triangulations consisting of 32, 64, 128, 256, 512, 1024 triangles, respectively.

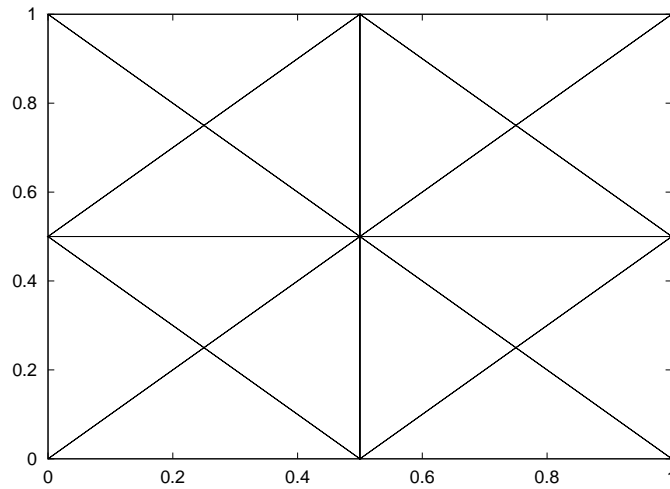


FIGURE 2. Initial triangulation of the unit square.

TABLE 1. True error and rates of convergence for (3.9)

N_T	$\ \mathbf{u} - \mathbf{u}_h\ _1 + \ p - p_h\ _0$	Rate	$\ \mathbf{u} - \mathbf{u}_h\ _0$	Rate
32	2.71947(-1)		1.16684(-2)	
64	1.95491(-1)	0.952	5.89109(-3)	1.972
128	1.39835(-1)	0.967	2.95885(-3)	1.987
256	9.94622(-2)	0.983	1.48404(-3)	1.991
512	7.03978(-2)	0.997	7.43309(-4)	1.995
1024	4.96486(-2)	1.008	3.70625(-4)	2.008

TABLE 2. Estimated error and rates of convergence for (3.9)

N_T	η	Rate	$\tilde{\eta}$	Rate
32	4.31386(-1)		2.95761(-2)	
64	3.12960(-1)	0.926	1.50831(-3)	1.943
128	2.24774(-1)	0.955	7.64153(-3)	1.962
256	1.76920(-1)	0.971	3.85535(-3)	1.974
512	1.25971(-1)	0.980	1.94176(-3)	1.979
1024	8.93223(-2)	0.992	9.75264(-4)	1.987

7. CONCLUSIONS

In this paper we have undertaken the task of proving a-posteriori error estimates for a finite volume scheme for the linear Stokes equation, based on the classical Crouzeix–Raviart finite element. The key idea in establishing the results is that of viewing the finite volume discretization as a perturbed finite element discretization. Then the estimates follow by applying known techniques of the a-posteriori estimation of finite element methods together with appropriate estimation of the new error terms. This is a rather general approach that may be applied to other finite volume methods as well. By means of numerical experiments we have confirmed that the estimators are of the correct order and have the expected rate of convergence.

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