Theoretical Micromagnetics

Lecture Series

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Lecture 2a. The magnetization vector

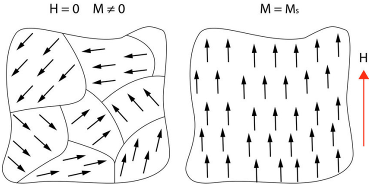
Consider a ferromagnet with aligned magnetic moments

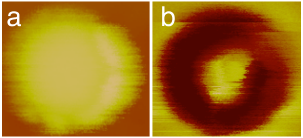
The magnetization is the density of magnetic moments $\boldsymbol{\mu}$ in a volume

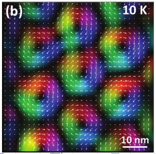
$$\mathbf{M} = \frac{\Delta \boldsymbol{\mu}}{\Delta V}, \qquad \Delta V \text{ is a small volume.}$$

By applying a strong magnetic field we may align all magnetic moments ("saturate" the magnetization) along the field direction and measure the saturation magnetization M_s.

Magnetic domains







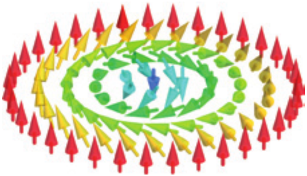
The Bloch sphere

The magnetization vector takes values on the Bloch sphere, $\mathbf{M}^2 = M_s^2$.

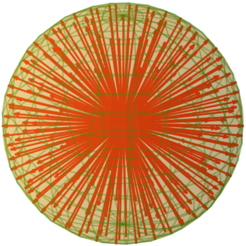
A ferromagnet is described by the magnetization vector $\mathbf{M} = \mathbf{M}(x,t)$ with (see, Landau, Lifshitz, Pitaevskii, "Statistical Physics II")

$$|\mathbf{M}| = M_s \ (=\text{const.}).$$

Magnetization configuration



Bloch sphere



A continuous spin variable

Let us assume a chain of spins which may not be perfectly aligned. The exchange energy depends on the neighbours of each spin $\mathbf{S}_\alpha$,

$$E_{\mathrm{ex}} = -J \sum \mathbf{S}_\alpha \cdot \mathbf{S}_{\alpha+1} = -\frac{J}{2} \sum \mathbf{S}_\alpha \cdot (\mathbf{S}_{\alpha+1} + \mathbf{S}_{\alpha-1}).$$

A continuum approximation

Consider a small parameter ϵ and define (ϵ can be defined in different ways)

- A space variable $x = \epsilon\alpha$ where α is an integer index (ϵ may be the spacing between atoms).
- A continuous field $\mathbf{S}(x)$ such that $\mathbf{S}_\alpha = \mathbf{S}(x)$ at the position of each spin α.

The continuous field $\mathbf{S}(x)$ is connecting the discrete spins (atoms) of the material.

Taylor expansion

The advantage of the continuous field is that we can make a

Taylor approximation

When the distance ϵ between spins is small, we have (Taylor expansion)

$$\mathbf{S}_{\alpha\pm1} \approx \mathbf{S} \pm \epsilon\partial_x\mathbf{S} + \frac{\epsilon^2}{2}\partial_x^2\mathbf{S}, \qquad \mathbf{S}_\alpha \to \mathbf{S}.$$

This assumes that

- There is a continuous field $\mathbf{S}(x)$.
- Neighbouring spins differ only a little.

Example (Use the Taylor approximation in the expression for the exchange energy)

$$E_{\rm ex} = -\frac{J}{2}\sum \mathbf{S}_\alpha \cdot (\mathbf{S}_{\alpha+1} + \mathbf{S}_{\alpha-1}) \approx \cdots$$

Exchange energy (continuum)

Exchange energy

Use the Taylor expansion in the exchange energy

$$E_{\rm ex} = -J \sum \left(|\mathbf{S}|^2 + \frac{\epsilon^2}{2} \mathbf{S} \cdot \partial_x^2 \mathbf{S} \right) \to -\frac{J}{2}\epsilon \int \mathbf{S} \cdot \partial_x^2 \mathbf{S}\, dx$$

Since $\mathbf{M} \sim \mathbf{S}$ we have $E_{\rm ex} \sim -\int \mathbf{M} \cdot \partial_x^2 \mathbf{M}\, dx$

and this gives, by a partial integration

$$E_{\rm ex} = \frac{A}{M_s^2} \int \partial_x \mathbf{M} \cdot \partial_x \mathbf{M}\, dx.$$

- A is the exchange constant (parameter).
- $E_{\rm ex}$ is non-negative.
- Its minimum (perfect alignment, $\partial_x \mathbf{M} = 0$) lies at zero.
- All directions in space, for $\mathbf{M}$, are equivalent.

The nonlinear σ-model

A system with the energy $E_{\rm ex}$ and $\mathbf{M}^2 = \text{const.}$ is called the nonlinear σ-model.

Exercise ($O(3)$ invariance of $E_{\rm ex}$)

(a) Write $E_{\rm ex}$ using the components $\mathbf{M} = (M_1, M_2, M_3)$. (b) Consider a uniform rotation for $\mathbf{M}$ and show that $E_{\rm ex}$ remains invariant.

Magnetocrystalline anisotropy

Materials are anisotropic in a natural way, e.g., due to the crystal structure.
Anisotropic contributions come from relativistic effects.
Some types of anisotropy are simply modelled.

Easy-plane anisotropy

The energy term ($K > 0$ the anisotropy parameter)

$$E_{\mathrm{a}} = \frac{K}{M_s^2} \int (M_3)^2 \, dx$$

favours the states where $\mathbf{M}$ lies on the plane (12), i.e., $M_3 = 0$.

Easy-axis anisotropy

$$E_{\mathrm{a}} = \frac{K}{M_s^2} \int (M_s^2 - M_3^2) \, dx$$

favours the states where $\mathbf{M}$ is fully aligned along the third axis, i.e., $M_3 = \pm M_s$ or $\mathbf{M} = (0, 0, \pm M_s)$.

Examples. Minima of anisotropy energy.

Example (easy-plane anisotropy)

(a) For the easy-plane anisotropy, give all minimum energy solutions.

(b) Show that the energy is invariant with respect to rotations of the vector $\mathbf{M}$ in the (12) plane.

Example (easy-axis anisotropy)

(a) Write the easy-axis anisotropy formula in a manifestly non-negative form to show that $E_{\rm a} \geq 0$.

(b) Give all minimum energy solutions (based on that formula).

Energy and length scales

In three-dimensions (3D), we have the exchange energy

$$E_{\rm ex} = \frac{A}{M_s^2} \int \partial_\mu \mathbf{M} \cdot \partial_\mu \mathbf{M} \, d^3x, \quad \mu = 1, 2, 3.$$

Question (Write explicitly the exchange energy density)

Note that summation is implied for the repeated index μ.

Total energy

In a simple model we assume a ferromagnet with exchange and anisotropy energy. For a 3D magnet,

$$E = E_{\rm ex} + E_{\rm a} = \frac{A}{M_s^2} \int \partial_\mu \mathbf{M} \cdot \partial_\mu \mathbf{M} \, d^3x + \frac{K}{M_s^2} \int (M_s^2 - M_3^2) \, d^3x.$$

Units for the physical constants $A : \mathrm{J/m}$, $K : \mathrm{J/m^3}$.

Dimensional analysis

The length scale of this model

The two energy terms indicate a natural length scale

$$\ell_{\rm DW} = \sqrt{\frac{A}{K}}.$$

Example (For $A = 10^{-11}\,\mathrm{J/m}, M_s = 10^6\,\mathrm{A/m}, K = 4 \times 10^5\,\mathrm{J/m^3}$)

We calculate $\ell_{\rm DW} = 5 \times 10^{-9}\,\mathrm{m} = 5\,\mathrm{nm}$.

We define the dimensionless magnetization according to

$$\mathbf{m} = \frac{\mathbf{M}}{M_s}, \qquad \mathbf{m}^2 = 1$$

and we have the energy

$$E = A \int \frac{\partial \mathbf{m}}{\partial x_\mu} \cdot \frac{\partial \mathbf{m}}{\partial x_\mu}\, d^3x + K \int (1 - m_3^2)\, d^3x$$

$$= K \left[\ell_{\rm DW}^2 \int \frac{\partial \mathbf{m}}{\partial x_\mu} \cdot \frac{\partial \mathbf{m}}{\partial x_\mu}\, d^3x + \int (1 - m_3^2)\, d^3x \right]$$

Dimensionless form of energy

We define dimensionless space variables (i.e., scale space by $\ell_{\rm DW}$)

$$x_\mu = \xi_\mu \, \ell_{\rm DW}$$

and have the energy

$$E = (K\ell_{\rm DW}^3) \left[\int \partial_\mu \mathbf{m} \cdot \partial_\mu \mathbf{m} \, d^3\xi + \int (1 - m_3^2) \, d^3\xi \right].$$

We write $K\ell_{\rm DW}^3 = A\ell_{\rm DW}$, and re-instate the usual variable $\xi \to x$ to get

$$E = (2A\ell_{\rm DW}) \left[\tfrac{1}{2} \int \partial_\mu \mathbf{m} \cdot \partial_\mu \mathbf{m} \, d^3x + \tfrac{1}{2} \int (1 - m_3^2) \, d^3x \right].$$

The natural energy scale is $(2A\ell_{\rm DW})$

Remark

This scaled energy form has no free parameter.

Lecture 2b. Derivation of the time-independent equation

Static configurations of the magnetization

The magnetization $\mathbf{m}(\boldsymbol{x})$ of the material reduces to a configuration that minimizes the magnetic energy $E(\mathbf{m})$.

The equation for $\mathbf{m}(\boldsymbol{x})$ is obtained as the Euler-Lagrange equation for the minimization of the energy, with the constraint

$$\mathbf{m}^2(\boldsymbol{x}) = 1.$$

The constraint is imposed via a *Lagrange multiplier* $\lambda(\boldsymbol{x})$. [Raj, Sec. 3.3][FG, Sec. 12.2]
For a demonstration, we consider the exchange interaction and we have to extremize the functional

$$L[\mathbf{m}] = \int d^3x \underbrace{\left[\tfrac{1}{2}\partial_\mu \mathbf{m} \cdot \partial_\mu \mathbf{m} + \tfrac{\lambda(\boldsymbol{x})}{2}(1 - \mathbf{m}^2)\right]}_{\mathcal{L}}.$$

The Euler-Lagrange equation

The functional L is minimized for $\mathbf{m}(\boldsymbol{x})$ that satisfies the Euler-Lagrange equation

$$-\frac{\delta L}{\delta \mathbf{m}} = 0 \Rightarrow \frac{d}{dx_\mu}\left(\frac{\partial \mathcal{L}}{\partial_\mu \mathbf{m}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{m}} = 0.$$

We calculate

$$-\frac{\delta L}{\delta \mathbf{m}} = \frac{d}{dx_\mu}\left(\partial_\mu \mathbf{m}\right) + \lambda \mathbf{m} = \partial_\mu \partial_\mu \mathbf{m} + \lambda \mathbf{m} = 0$$

or

$$\Delta \mathbf{m} + \lambda \mathbf{m} = 0.$$

We multiply the above by $\mathbf{m}$ in order to obtain the Lagrange multiplier,

$$\mathbf{m} \cdot \Delta\mathbf{m} + \lambda \mathbf{m} \cdot \mathbf{m} = 0 \Rightarrow \lambda = -\mathbf{m} \cdot \Delta\mathbf{m}$$

and we use this to eliminate λ in the field equation

$$\Delta\mathbf{m} - (\mathbf{m} \cdot \Delta\mathbf{m})\mathbf{m} = 0 \Rightarrow \mathbf{m} \times (\mathbf{m} \times \Delta\mathbf{m}) = 0.$$

The latter is equivalent to

$$\mathbf{m} \times \Delta\mathbf{m} = 0.$$

Quiz

Equation for the minimization of the exchange energy. Give an example of solution for the 1D equation

$$\mathbf{m} \times \partial_x^2 \mathbf{m} = 0.$$

The Landau-Lifshitz (LL) equation - static sector

Form of the Landau-Lifshitz equation

Let us assume an energy functional $E(\mathbf{m})$. We find

$$\mathbf{m} \times \boldsymbol{f} = 0, \qquad \boldsymbol{f} = -\frac{\delta E}{\delta \mathbf{m}}.$$

- For $\boldsymbol{f} = \mathbf{h}$ we recover the standard equation of magnetism for a magnetic moment $\mathbf{m}$ in an external magnetic field bh.
- For $E = E_{\rm ex} = \frac{1}{2}\int \partial_x \mathbf{m} \cdot \partial_x \mathbf{m}\, dx$ we have $\boldsymbol{f} = -\frac{\delta E}{\delta \mathbf{m}} = \partial_x^2 \mathbf{m}$.
- Solutions are $\mathbf{m}$ such that $\mathbf{m} \parallel \boldsymbol{f}$.

Exercise (Static Landau-Lifshitz equation)

Assume an energy functional E and derive the static Landau-Lifshitz equation under the constraint $\mathbf{m}^2 = 1$.

The LL equation - exchange and uniaxial anisotropy

Energy (exchange and easy-axis anisotropy)

$$E = \int \epsilon \, dx = \frac{1}{2} \int \partial_x \mathbf{m} \cdot \partial_x \mathbf{m} \, dx + \frac{1}{2} \int (1 - m_3^2) dx.$$

The variational derivative

$$\boldsymbol{f} = -\frac{\delta E}{\delta \mathbf{m}} = \frac{d}{dx} \left(\frac{\partial \epsilon}{\partial(\partial_x \mathbf{m})} \right) - \frac{\partial \epsilon}{\partial \mathbf{m}} = \partial_x^2 \mathbf{m} + m_3 \hat{\mathbf{e}}_3.$$

Landau-Lifshitz equation for exchange and easy-axis anisotropy

$$\mathbf{m} \times \underbrace{(\Delta \mathbf{m} + m_3 \hat{\mathbf{e}}_3)}_{\boldsymbol{f}} = 0.$$

Quiz

on the model with exchange and anisotropy.

Question (Find the uniform solutions)

- Uniform solutions are those for space-independent $\mathbf{m}$.

In this case, $\boldsymbol{f} = m_3 \hat{\boldsymbol{e}}_3$. In order to have a solution, we need $\mathbf{m} \parallel \boldsymbol{f}$, that is, $\mathbf{m} \parallel \hat{\boldsymbol{e}}_3 \Rightarrow \mathbf{m} = \pm \hat{\boldsymbol{e}}_3$ (pointing in the north or south pole).

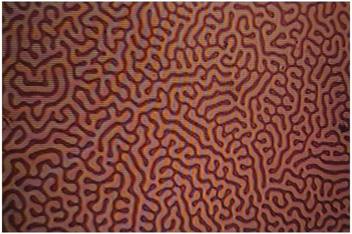
- The uniform solution is called the *ferromagnetic state*.

Question (easy-plane anisotropy)

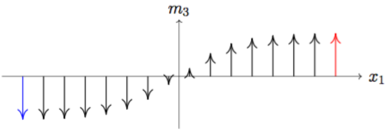
What is the static Landau-Lifshitz equation for easy-plane anisotropy?

Lecture 2c. The magnetic domain wall (DW)

Domain pattern



Sketch of domain wall



A transition layer

- Magnetic domains are regions where the magnetization is almost uniform.
- A domain wall is the magnetization configuration between two uniform states with different magnetization.

The spherical parametrization for the magnetization

The spherical angles Θ, Φ

We can explicitly resolve the constraint $\mathbf{m}^2 = 1$,

$$m_1 = \sin\Theta\cos\Phi, \quad m_2 = \sin\Theta\sin\Phi, \quad m_3 = \cos\Theta.$$

In a model with easy-axis anisotropy, we have two ground states, $\mathbf{m} = (0, 0, \pm 1)$, or $\Theta = 0, \pi$ (north and south pole of the sphere).

We look for a topological soliton connecting the north and the south pole

We confine ourselves to the one-dimensional case $\mathbf{m} = \mathbf{m}(x)$.
We try the simplest possibility of a meridian on the Bloch sphere

$$\Theta = \Theta(x), \quad \Phi = \phi_0 : \text{const}.$$

A magnetic domain wall on the Bloch sphere

Example (Bloch wall)

For $\phi_0 = \pi/2$ we have

$$m_1 = 0, \quad m_2(x) = \sin\Theta(x), \quad m_3(x) = \cos\Theta(x).$$

- Draw a DW on the Bloch sphere.
- Consider the variation of the vector $\mathbf{m}$ in the space variable x.

A magnetic domain wall (DW)

The Landau-Lifshitz equation for $\mathbf{m}(x)$ (for exchange and easy-axis anisotropy with anisotropy parameter k^2)

$$\mathbf{m} \times \left(\mathbf{m}'' + k^2 m_3 \hat{\boldsymbol{e}}_3\right) = 0 \Rightarrow \begin{cases} m_2 m_3'' - m_3 m_2'' + k^2 m_2 m_3 = 0 \\ m_3 m_1'' - m_1 m_3'' - k^2 m_1 m_3 = 0 \\ m_1 m_2'' - m_2 m_1'' = 0 \end{cases}$$

Choose the case

$$m_1 = 0, \quad m_2 = \sin\Theta, \quad m_3 = \cos\Theta.$$

thus

$$m_2'' = -\sin\Theta\,\Theta'^2 + \cos\Theta\,\Theta'', \quad m_3'' = -\cos\Theta\,\Theta'^2 - \sin\Theta\,\Theta''.$$

The first equation gives (the other two are trivially satisfied)

$$\Theta'' - k^2 \sin\Theta\cos\Theta = 0.$$

Multiply by $2\Theta'$

$$\left[(\Theta')^2 - k^2 \sin^2\Theta\right]' = 0 \Rightarrow (\Theta')^2 - k^2 \sin^2\Theta = C.$$

There are many solutions for the one-dimensional equation

We are only interested in *localized solutions*. We consider uniform domains for $|x| > 0$, therefore, we require $\Theta = 0, \pi$ and $\Theta' = 0$ at $x = \pm\infty$.

From the condition at $x \to \pm\infty$ we get $C = 0$ and we have

$$\Theta' = \pm k \sin\Theta.$$

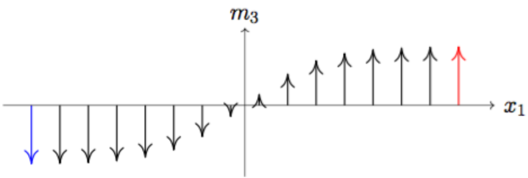
The solution of the latter is

$$e^{\pm kx} = \pm \tan\left(\frac{\Theta}{2}\right).$$

Check that (for the plus signs)

- For $x \to -\infty$ we have $\Theta = 0$ (north pole).
- For $x \to \infty$ we have $\Theta = \pi$ (south pole).

Domain wall details



Static domain wall (DW)

Use trigonometric identities (for the half angle)

$$m_1 = \frac{1}{\cosh(kx)} \cos\phi_0, \quad m_2 = \frac{1}{\cosh(kx)} \sin\phi_0, \quad m_3 = \tanh(kx).$$

That is valid for boundary conditions $\mathbf{m}(x = \pm\infty) = (0, 0, \pm 1)$.

The figure shows $m_3(x)$ for a domain wall with $\phi_0 = \pm\pi/2$.
The width of the domain wall can be considered to be $1/k$, i.e.,

$$\ell_{\mathrm{DW}} = \sqrt{A/K}.$$

Bloch and Néel walls

There are many domain wall solutions

We get a different domain wall solution for every $0 \leq \phi_0 < 2\pi$. Within this model, the energy is the same for all walls.

Bloch wall, choose $\phi_0 = \pm\pi/2$

$$m_1 = 0, \quad m_2 = \pm\frac{1}{\cosh(kx)}, \quad m_3 = \tanh(kx).$$

Néel wall, choose $\phi_0 = 0, \pi$

$$m_1 = \pm\frac{1}{\cosh(kx)}, \quad m_2 = 0, \quad m_3 = \tanh(kx).$$

Lecture 2e. Magnetostatic field

Maxwell's equations

A ferromagnet produces a magnetic field $\mathbf{H}_m$. For static configurations $\mathbf{M}$, this is given by Maxwell's equations omitting time derivatives

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{H}_m = 0, \qquad \mathbf{B} = \mu_0(\mathbf{H}_m + \mathbf{M}).$$

Apply the normalization

$$\mathbf{h}_m = \frac{\mathbf{H}_m}{M_s}$$

and write

$$\boldsymbol{\nabla} \cdot (\mathbf{h}_m + \mathbf{m}) = 0, \qquad \boldsymbol{\nabla} \times \mathbf{h}_m = 0.$$

This is called the magnetostatic field $\mathbf{h}_m$, because time derivatives have been neglected in Maxwell's equations.

Source of a magnetostatic field

Magnetic field due to $\mathbf{m}$

Write Maxwell's equations as

$$\boldsymbol{\nabla} \cdot \mathbf{h}_m = -\boldsymbol{\nabla} \cdot \mathbf{m}, \qquad \boldsymbol{\nabla} \times \mathbf{h}_m = 0.$$

Thus, the magnetic field source is $-\boldsymbol{\nabla} \cdot \mathbf{m}$.

Note the similarity between the equations for the magnetostatic field

with those for the field $\boldsymbol{E}$ of a charge density ρ in electrostatics. They are identical under the correspondence

- $\rho \to -\boldsymbol{\nabla} \cdot \mathbf{m}$.
- $\boldsymbol{E} \to \mathbf{h}_m$.

Quiz. Examples in simple geometries.

Example (Magnetic field of an infinite cylinder)

Consider an infinite cylinder that is uniformly magnetized along its axis ($\mathbf{m} = \hat{\boldsymbol{e}}_3$). What is the magnetic field produced?

Example (Magnetic field in a thin film)

Consider a thin film uniformly magnetized perpendicular to the film plane ($\mathbf{m} = \hat{\boldsymbol{e}}_3$). What is the magnetic field produced?

Magnetostatic field: examples

Example (Infinitely elongated cylinder uniformly magnetized along its axis)

The magnetostatic field is (note that $\nabla \cdot \mathbf{m} = 0$)

$$\mathbf{h}_m = 0.$$

Example (Thin film uniformly magnetized)

Consider an infinite thin film in the xy plane uniformly magnetized perpendicular to the plane, $\mathbf{m} = \hat{\boldsymbol{e}}_z$. The magnetostatic field is

$$\mathbf{h}_m = -\mathbf{m} = -\hat{\boldsymbol{e}}_z.$$

Solutions of the latter type appear in examples in textbooks, e.g., in the case of the field in an ideal capacitor.

Quiz. Magnetostatic field of a wall.

Bloch wall, choose $\phi_0 = \pm\pi/2$

$$m_1 = 0, \quad m_2 = \pm\frac{1}{\cosh(kx)}, \quad m_3 = \pm\tanh(kx).$$

This gives $\boldsymbol{\nabla}\cdot\mathbf{m} = 0$ and thus produces no magnetic field. It minimizes the magnetostatic energy (not included in our model so far).

Néel wall, choose $\phi_0 = 0, \pi$

$$m_1 = \pm\frac{1}{\cosh(kx)}, \quad m_2 = 0, \quad m_3 = \pm\tanh(kx).$$

This gives $\boldsymbol{\nabla}\cdot\mathbf{m} = m_1' \neq 0$ and thus magnetic field *is* produced. This is added to the domain wall energy.

A nontrivial example with a simple solution

Example

In the case of $\mathbf{m} = \mathbf{m}(x)$, depending only on one space variable, we have the solution

$$\mathbf{h}_m(x) = -m_x(x)\hat{\boldsymbol{e}}_x.$$

This is because

- The equation $\boldsymbol{\nabla} \cdot \mathbf{h}_m = -\boldsymbol{\nabla} \cdot \mathbf{m}$ reduces to the 1D form $\partial_x h_x = -\partial_x m_x$ and it is satisfied.
- We assume $m_x(x) = 0$ at $x \to \pm\infty$, thus $\mathbf{h}_m$ satisfies the boundary condition $\mathbf{h}_m(\pm\infty) = 0$.
- At $y, z \to \pm\infty$ we do not impose a particular boundary condition (we only assume that $\mathbf{h}_m$ does not depend on y, z).

The exchange length

The magnetostatic energy

$$E_{\mathrm{m}} = \frac{1}{2}\mu_0 \int \mathbf{M} \cdot \mathbf{H}_m \, d^3x.$$

A typical energy density is $\frac{1}{2}\mu_0 M_s^2$ (in $\mathrm{J/m^3}$).
Comparison of exchange and magnetostatic energy gives rise to the definition of the exchange length

$$\ell_{\mathrm{ex}} = \sqrt{\frac{2A}{\mu_0 M_s^2}}.$$

Example (Exchange length for Permalloy)

For Permalloy, $A = 1.3 \times 10^{-11}\,\mathrm{J/m}$, $M_s = 0.69 \times 10^6\,\mathrm{A/m}$. We find $\ell_{\mathrm{ex}} = 6.59\,\mathrm{nm}$.

Rationalise using the exchange length

We may define dimensionless variables according to

$$x \to x\,\ell_{\rm ex}, \qquad \mathbf{h}_m = \frac{\mathbf{H}_m}{M_s}.$$

We get the energy, in units of $2A\ell_{\rm ex}$,

$$E = \frac{1}{2}\int \partial_\mu \mathbf{m}\cdot\partial_\mu \mathbf{m}\, d^3x + \frac{k^2}{2}\int (1-m_3^2)\, d^3x + \frac{1}{2}\int \mathbf{m}\cdot\mathbf{h}_m\, d^3x$$

where we defined (the "quality factor")

$$k^2 = \frac{2K}{\mu_0 M_s^2}.$$

Remark

This form of the energy has only one parameter k^2, the scaled (dimensionless) anisotropy.